Pearson
SUCCESS
COMPANION

JEE
ADVANCED
MATHEMATICS

100
010
001
3,14
2+2=4
√MATH
$(X+Y)^2$

Coordinate Geometry

Pearson

Om Sharma

About Pearson

Pearson is the world's learning company, with presence across 70 countries worldwide. Our unique insights and world-class expertise comes from a long history of working closely with renowned teachers, authors and thought leaders, as a result of which, we have emerged as the preferred choice for millions of teachers and learners across the world.

We believe learning opens up opportunities, creates fulfilling careers and hence better lives. We hence collaborate with the best of minds to deliver you class-leading products, spread across the Higher Education and K12 spectrum.

Superior learning experience and improved outcomes are at the heart of everything we do. This product is the result of one such effort.

Your feedback plays a critical role in the evolution of our products and you can contact us - reachus@pearson.com. *We look forward to it.*

This page is intentionally left blank

2+2=4
3,14
I ♥
√MATH
(X+Y)²
y
x

JEE
ADVANCED

MATHEMATICS

Coordinate Geometry

This page is intentionally left blank

3,14
2+2=4
√MATH
(X+Y)²

JEE
ADVANCED
MATHEMATICS

Coordinate Geometry

Pearson

Om Sharma

Copyright © 2021 Pearson India Education Services Pvt. Ltd

Although the author and publisher have made every effort to ensure that the information in this book was correct at the time of editing and printing, the author and publisher do not assume and hereby disclaim any liability to any party for any loss or damage arising out of the use of this book caused by errors or omissions, whether such errors or omissions result from negligence, accident or any other cause. Further, names, pictures, images, characters, businesses, places, events and incidents are either the products of the author's imagination or used in a fictitious manner. Any resemblance to actual persons, living or dead or actual events is purely coincidental and do not intend to hurt sentiments of any individual, community, sect or religion.

In case of binding mistake, misprints or missing pages etc., the publisher's entire liability and your exclusive remedy is replacement of this book within reasonable time of purchase by similar edition/reprint of the book.

Published by Pearson India Education Services Pvt. Ltd, CIN: U72200TN2005PTC057128.

No part of this eBook may be used or reproduced in any manner whatsoever without the publisher's prior written consent.

This eBook may or may not include all assets that were part of the print version.
The publisher reserves the right to remove any material in this eBook at any time.

ISBN 978-93-539-4675-3
eISBN: 978-93-903-9415-9

Head Office: 15th Floor, Tower-B, World Trade Tower, Plot No. 1, Block-C, Sector 16,
Noida 201 301, Uttar Pradesh, India.
Registered Office: The HIVE, 3rd Floor, Pillaiyar Koil Street, Jawaharlal Nehru Road,
Anna Nagar, Chennai 600 040, Tamil Nadu, India.
Phone: 044-66540100
website: in.pearson.com, Email: companysecretary.india@pearson.com

I dedicate this book to my respected parents
Mr. Giriraj Prasad and Sampati Sharma
and family for their love, endless support,
encouragement and sacrifices

This page is intentionally left blank

CONTENTS

Chapter Insight xv

Preface xx

About the Author xxi

CHAPTER 1

POINT AND CARTESIAN SYSTEM 1.1

Introduction of Coordinate Geometry 1.1
Cartesian Co-ordinates 1.1
Distance Formula and its Applications. 1.2
Polar Coordinates 1.6
Reflection of a Point 1.9
Transformation of Axes 1.9
Section Formula 1.11
Area of Triangle 1.16
Area of a Quadrilateral 1.16
Area of a Plane Polygon. 1.17
Slope of a Line 1.21
Condition for Collinearity of Three Points 1.22
Coordinates of Some Particular Points 1.24
Locus of Point 1.29
Solved Problems *1.32*
Practice Exercises *1.36*
    Single Correct Choice Type Questions *1.36*
    Multiple Correct Choice Type Questions. *1.39*
    Linked Comprehension Type Questions *1.40*
    Matrix Match/Column Match Type Questions. *1.41*
    Integer/Numerical Answer Type Questions *1.42*
    Archive: JEE Main *1.43*
    Archive: JEE Advanced *1.44*
Answer Keys–Practice Exercises *1.45*

CHAPTER 2

STRAIGHT LINE 2.1

Definition of a Straight Line 2.1
Equation of Straight Line Parallel to Axes. 2.1
Slope of a Line 2.1
Different Forms of the Equation of a Straight Line 2.2
Intersection of Two Lines 2.8
Angle Between Two Lines 2.11
Position of Two Points with Respect to a Given Line 2.17
Parametric Form of Straight Line 2.18
Length of Perpendicular, Foot of Perpendicular and Image of a Point Relative to a Straight Line 2.26
Distance Between Two Parallel Lines 2.30
Concurrency of Lines. 2.33
Family of Straight Lines 2.35
Angle Bisectors of Two Intersecting Straight Lines 2.40
Pair of Straight Lines 2.43
General Equation of Second Degree Curve 2.46
Homogenisation of General Equation 2.47
Solved Problems *2.52*
Practice Exercises *2.61*
  Single Correct Choice Type Questions *2.61*
  Multiple Correct Choice Type Questions *2.69*
  Linked Comprehension Type Questions *2.73*
  Matrix Match/Column Match Type Questions *2.78*
  Integer/Numerical Answer Type Questions *2.81*
  Archive: JEE Main *2.82*
  Archive: JEE Advanced *2.85*
Answer Keys–Practice Exercises *2.87*

CHAPTER 3

CIRCLE 3.1

Definition of Circle 3.1
Cyclic Quadrilateral 3.2
Equation of Circle 3.3
General Equation of Circle 3.3
Point Circle 3.3
Intercepts of Circle on Co-ordinate Axes 3.5

Equation of Concentric Circle . . . 3.7
Circle in Diameter Form. . . . 3.7
Position of Point . . . 3.11
Tangent/Pair of Tangents on Circle . . . 3.16
Tangent for Standard Circle . . . 3.17
Geometrical Properties of Circle Related to Tangents and Chords . . . 3.19
Chord of Contact . . . 3.23
Director Circle . . . 3.28
Pole and Polar . . . 3.29
Chord with Given Middle Point . . . 3.30
Diameter of Circle . . . 3.30
Relative Position of Two Circles . . . 3.34
Length of Direct Tangent . . . 3.36
Length of Indirect Tangent . . . 3.36
Angle Between Two Intersecting Circles . . . 3.42
Radical Axis . . . 3.44
Radical Centre . . . 3.48
Coaxial System of Circles . . . 3.50
Family of Circles . . . 3.50
Application of Point Circle and Miscellaneous Cases . . . 3.53
Solved Problems . . . *3.57*
Practice Exercises . . . *3.62*
  Single Correct Choice Type Questions . . . *3.62*
  Multiple Correct Choice Type Questions. . . . *3.71*
  Linked Comprehension Type Questions . . . *3.75*
  Matrix Match/Column Match Type Questions. . . . *3.80*
  Integer/Numerical Answer Type Questions. . . . *3.84*
  Archive: JEE Main . . . *3.87*
  Archive: JEE Advanced . . . *3.90*
Answer Keys–Practice Exercises. . . . *3.93*

CHAPTER 4 **PARABOLA . . . 4.1**

Conic Section . . . 4.1
Equation of Conic Section . . . 4.2
Parabola . . . 4.4
Equation of Parabola Using Axis and Tangent at Vertex . . . 4.10

Focal Distance . . . 4.11
Parametric Equation of Parabola . . . 4.12
Position of a Point Relative to Parabola . . . 4.13
Position of Straight Line Relative to Parabola . . . 4.14
Condition of Tangency . . . 4.15
Length of Chord . . . 4.15
Focal Chord . . . 4.15
Equation of Tangent . . . 4.18
Chord of Contact . . . 4.23
Director Circle . . . 4.29
Pole and Polar of Parabola . . . 4.30
Chord with Given Middle Point . . . 4.31
Diameter of Parabola . . . 4.31
Normal of Parabola . . . 4.34
Normal from External Point (h, k) . . . 4.38
Highlights of Parabola . . . 4.44
Solved Problems . . . 4.48
Practice Exercises . . . 4.54
    Single Correct Choice Type Questions . . . 4.54
    Multiple Correct Choice Type Questions . . . 4.62
    Linked Comprehension Type Questions . . . 4.66
    Matrix Match/Column Match Type Questions . . . 4.70
    Integer/Numerical Answer Type Questions . . . 4.74
    Archive: JEE Main . . . 4.75
    Archive: JEE Advanced . . . 4.78
Answer Keys–Practice Exercises . . . 4.82

CHAPTER 5 **ELLIPSE . . . 5.1**

Introduction . . . 5.1
Standard Ellipse . . . 5.2
Equation of Ellipse Using Principal Axes . . . 5.8
Focal Distance . . . 5.9
Auxiliary Circle . . . 5.11
Parametric Equation of Ellipse . . . 5.11
Position of Point Relative to Ellipse . . . 5.14

Position of Line Relative to Ellipse . . . 5.15
Chord of Ellipse . . . 5.15
Tangent of Ellipse . . . 5.17
Chord of Contact . . . 5.24
Director Circle . . . 5.27
Pole and Polar . . . 5.29
Diameter of Ellipse . . . 5.32
Normal of an Ellipse . . . 5.35
Solved Problems . . . 5.43
Practice Exercises . . . 5.48
  Single Correct Choice Type Questions . . . 5.48
  Multiple Correct Choice Type Questions . . . 5.55
  Linked Comprehension Type Questions . . . 5.59
  Matrix Match/Column Match Type Questions . . . 5.65
  Integer/Numerical Answer Type Questions . . . 5.68
  Archive: JEE Main . . . 5.70
  Archive: JEE Advanced . . . 5.73
Answer Keys–Practice Exercises . . . 5.76

CHAPTER 6 **HYPERBOLA** . . . 6.1

Definition of Hyperbola . . . 6.2
Standard Hyperbola . . . 6.3
Conjugate Hyperbola . . . 6.6
Equation of Hyperbola Using Principal Axes . . . 6.9
Auxiliary Circle . . . 6.11
Focal Distance . . . 6.11
Location of Point . . . 6.11
Position of Line. . . . 6.12
Asymptotes . . . 6.20
Properties of Hyperbola and Asymptotes . . . 6.22
Rectangular Hyperbola: (Equilateral Hyperbola) . . . 6.26
Rectangular Hyperbola $xy = c^2$ . . . 6.26
Solved Problems . . . 6.33
Practice Exercises . . . 6.37

Single Correct Choice Type Questions *6.37*
Multiple Correct Choice Type Questions. *6.45*
Linked Comprehension Type Questions *6.47*
Matrix Match/Column Match Type Questions. *6.52*
Integer/Numerical Answer Type Questions. *6.55*
Archive: JEE Main *6.57*
Archive: JEE Advanced *6.58*
Answer Keys–Practice Exercises *6.62*

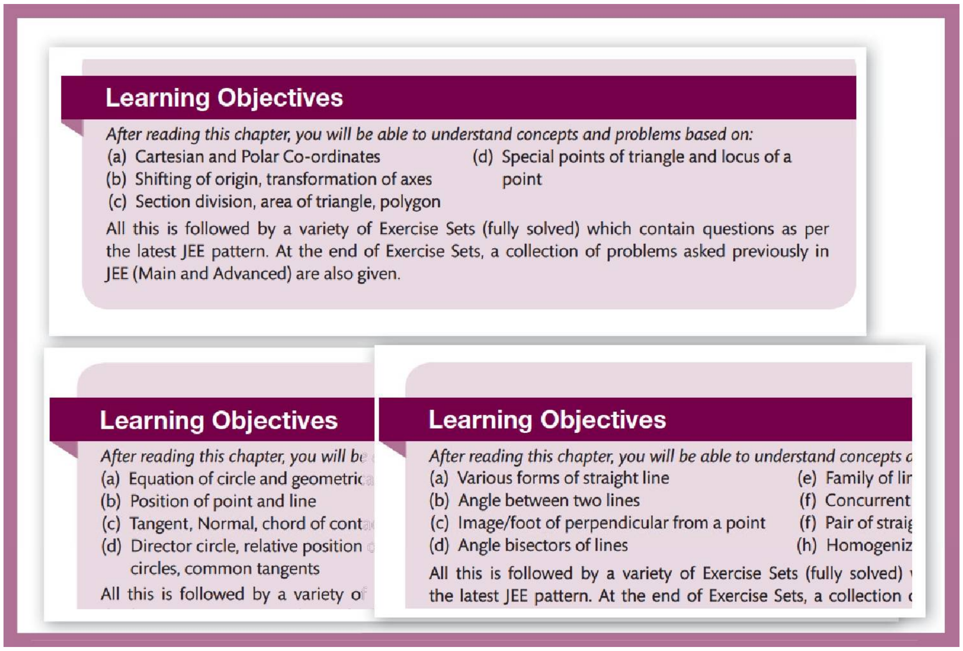
HINTS AND EXPLANATIONS

Chapter 1: Point and Cartesian System **H.3**
Chapter 2: Straight Line **H.23**
Chapter 3: Circle **H.77**
Chapter 4: Parabola **H.140**
Chapter 5: Ellipse **H.195**
Chapter 6: Hyperbola **H.245**

CHAPTER INSIGHT

Learning Objectives

Help the students set an aim to achieve the major take-aways from a particular chapter.



Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Cartesian and Polar Co-ordinates
(b) Shifting of origin, transformation of axes
(c) Section division, area of triangle, polygon
(d) Special points of triangle and locus of a point

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

Learning Objectives

After reading this chapter, you will be

(a) Equation of circle and geometric
(b) Position of point and line
(c) Tangent, Normal, chord of cont
(d) Director circle, relative position circles, common tangents

All this is followed by a variety o

Learning Objectives

After reading this chapter, you will be able to understand concepts

(a) Various forms of straight line
(b) Angle between two lines
(c) Image/foot of perpendicular from a point
(d) Angle bisectors of lines
(e) Family of lir
(f) Concurrent
(f) Pair of straig
(h) Homogeniz

All this is followed by a variety of Exercise Sets (fully solved)
the latest JEE pattern. At the end of Exercise Sets, a collection

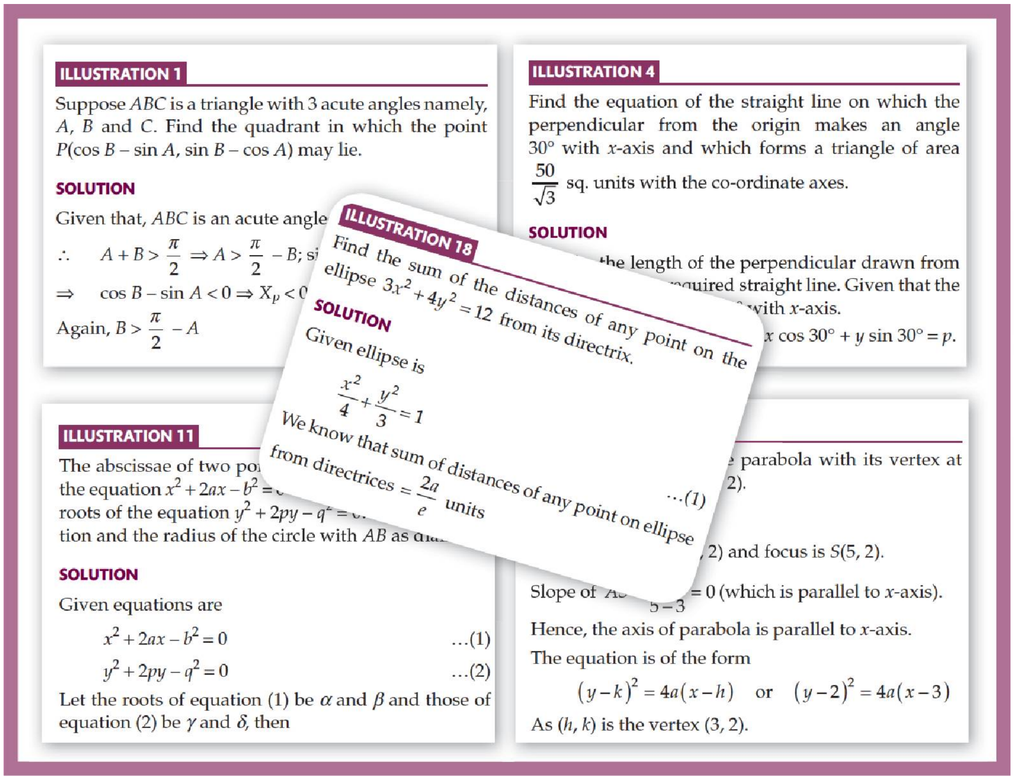


ILLUSTRATION 1

Suppose ABC is a triangle with 3 acute angles namely, A, B and C. Find the quadrant in which the point $P(\cos B - \sin A, \sin B - \cos A)$ may lie.

SOLUTION

Given that, ABC is an acute angle

$\therefore \quad A + B > \frac{\pi}{2} \Rightarrow A > \frac{\pi}{2} - B$; si

$\Rightarrow \quad \cos B - \sin A < 0 \Rightarrow X_p < 0$

Again, $B > \frac{\pi}{2} - A$

ILLUSTRATION 4

Find the equation of the straight line on which the perpendicular from the origin makes an angle 30° with x-axis and which forms a triangle of area $\frac{50}{\sqrt{3}}$ sq. units with the co-ordinate axes.

SOLUTION

the length of the perpendicular drawn from
quired straight line. Given that the
with x-axis.
$x \cos 30° + y \sin 30° = p$.

ILLUSTRATION 18

Find the sum of the distances of any point on the ellipse $3x^2 + 4y^2 = 12$ from its directrix.

SOLUTION

Given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

We know that sum of distances of any point on ellipse from directrices $= \frac{2a}{e}$ units ...(1)

ILLUSTRATION 11

The abscissae of two po
the equation $x^2 + 2ax - b^2 =$
roots of the equation $y^2 + 2py - q^2 =$
tion and the radius of the circle with AB as dia

SOLUTION

Given equations are

$$x^2 + 2ax - b^2 = 0 \qquad ...(1)$$

$$y^2 + 2py - q^2 = 0 \qquad ...(2)$$

Let the roots of equation (1) be α and β and those of equation (2) be γ and δ, then

parabola with its vertex at
2).

2) and focus is $S(5, 2)$.

Slope of

5 − 3

= 0 (which is parallel to x-axis).

Hence, the axis of parabola is parallel to x-axis.

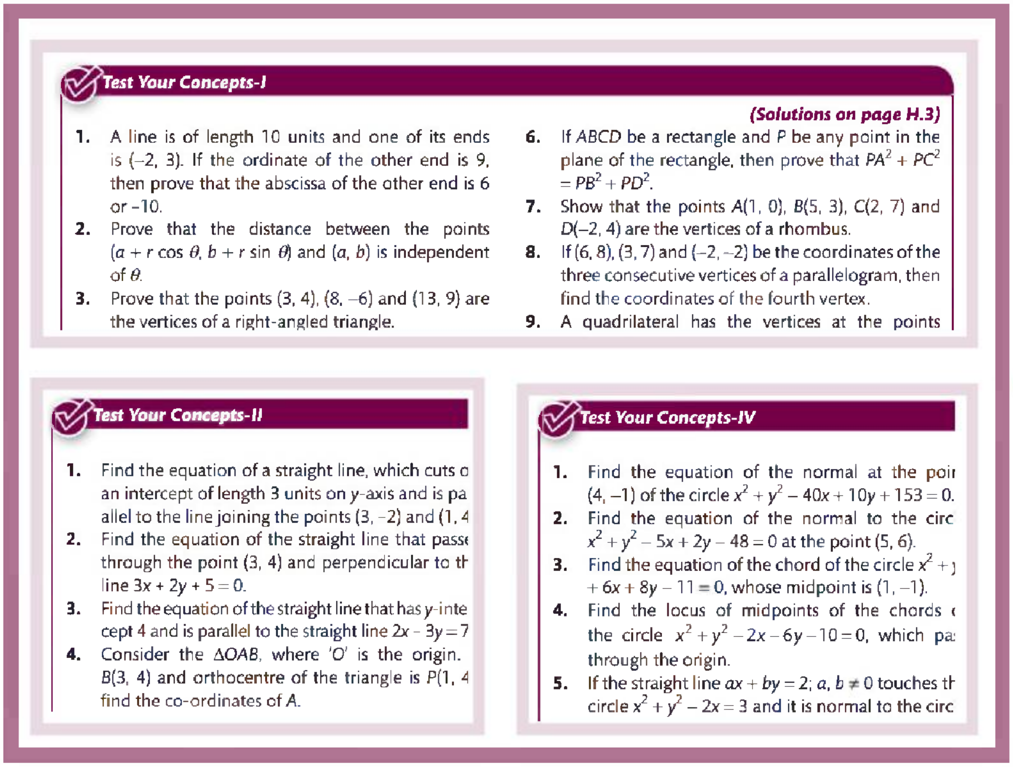
The equation is of the form

$$(y-k)^2 = 4a(x-h) \quad \text{or} \quad (y-2)^2 = 4a(x-3)$$

As (h, k) is the vertex (3, 2).

Theory with Illustrations

Elaborative and simple theory helps the students to understand the illustrations supporting the theory. Please note that theory and problem solving techniques are based on simple learning program IF → THEN → ELSE. I would suggest you not to attempt the illustrations without going through the theory of that section.



Test Your Concepts-I

(Solutions on page H.3)

1. A line is of length 10 units and one of its ends is (–2, 3). If the ordinate of the other end is 9, then prove that the abscissa of the other end is 6 or –10.
2. Prove that the distance between the points $(a + r\cos\theta, b + r\sin\theta)$ and (a, b) is independent of θ.
3. Prove that the points (3, 4), (8, –6) and (13, 9) are the vertices of a right-angled triangle.
6. If $ABCD$ be a rectangle and P be any point in the plane of the rectangle, then prove that $PA^2 + PC^2 = PB^2 + PD^2$.
7. Show that the points $A(1, 0)$, $B(5, 3)$, $C(2, 7)$ and $D(-2, 4)$ are the vertices of a rhombus.
8. If (6, 8), (3, 7) and (–2, –2) be the coordinates of the three consecutive vertices of a parallelogram, then find the coordinates of the fourth vertex.
9. A quadrilateral has the vertices at the points

Test Your Concepts-II

1. Find the equation of a straight line, which cuts o an intercept of length 3 units on y-axis and is pa allel to the line joining the points (3, –2) and (1, 4
2. Find the equation of the straight line that passe through the point (3, 4) and perpendicular to th line $3x + 2y + 5 = 0$.
3. Find the equation of the straight line that has y-inte cept 4 and is parallel to the straight line $2x - 3y = 7$
4. Consider the ΔOAB, where 'O' is the origin. $B(3, 4)$ and orthocentre of the triangle is $P(1, 4$ find the co-ordinates of A.

Test Your Concepts-IV

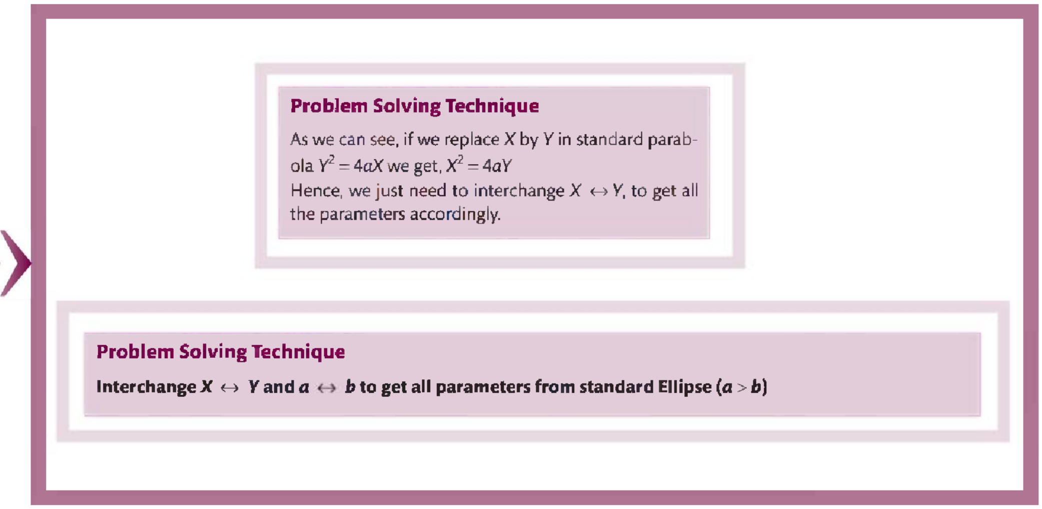
1. Find the equation of the normal at the poir (4, –1) of the circle $x^2 + y^2 - 40x + 10y + 153 = 0$.
2. Find the equation of the normal to the circ $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).
3. Find the equation of the chord of the circle $x^2 +$) $+ 6x + 8y - 11 = 0$, whose midpoint is (1, –1).
4. Find the locus of midpoints of the chords (the circle $x^2 + y^2 - 2x - 6y - 10 = 0$, which pa: through the origin.
5. If the straight line $ax + by = 2$; $a, b \neq 0$ touches th circle $x^2 + y^2 - 2x = 3$ and it is normal to the circ

Test Your Concepts

These topic based exercise sets are based on simple, single concept classification technique. These are meant for students practice after they study a particular topic and want to practice more on that topic learnt. Finally, in case of any difficulty they can refer to the hints and solutions to these exercise sets given at the end of the book.

Problem Solving Techniques

These techniques ensure that students become capable enough to solve a variety of problems in an easy and quick manner.



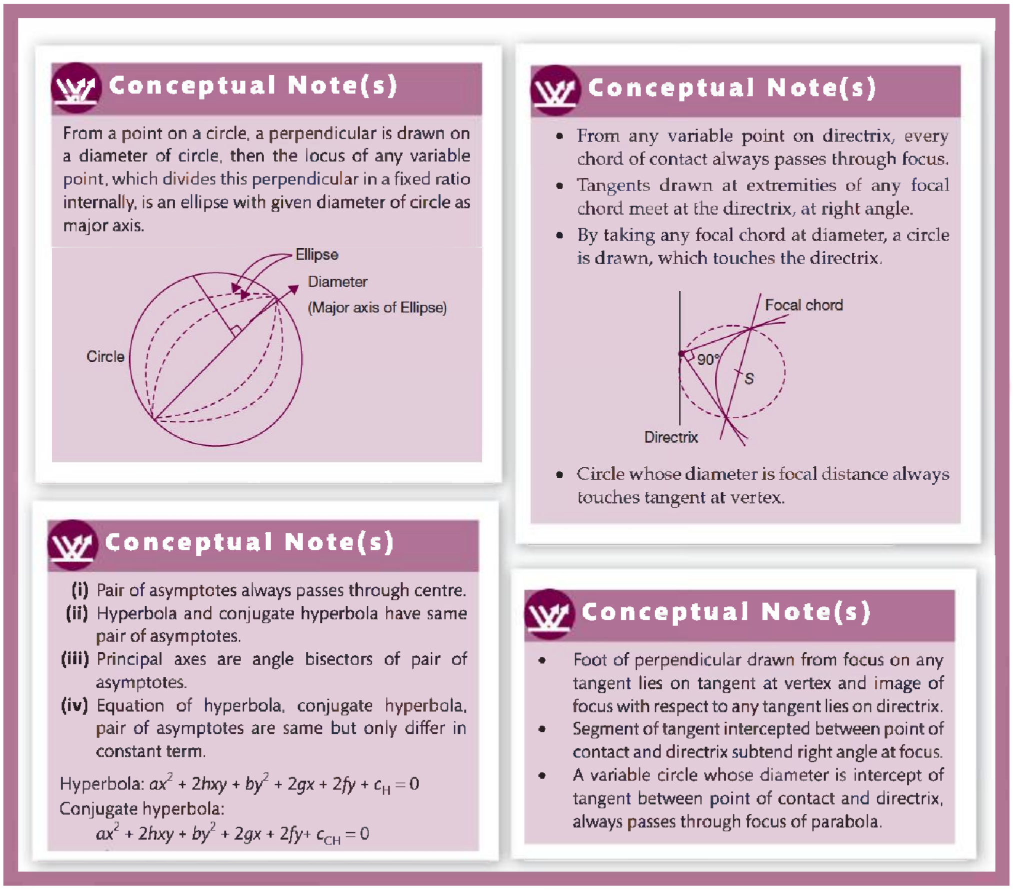
Problem Solving Technique

As we can see, if we replace X by Y in standard parabola $Y^2 = 4aX$ we get, $X^2 = 4aY$

Hence, we just need to interchange $X \leftrightarrow Y$, to get all the parameters accordingly.

Problem Solving Technique

Interchange $X \leftrightarrow Y$ and $a \leftrightarrow b$ to get all parameters from standard Ellipse ($a > b$)



Conceptual Note(s)

From a point on a circle, a perpendicular is drawn on a diameter of circle, then the locus of any variable point, which divides this perpendicular in a fixed ratio internally, is an ellipse with given diameter of circle as major axis.

Ellipse
Diameter
(Major axis of Ellipse)
Circle

Conceptual Note(s)

- From any variable point on directrix, every chord of contact always passes through focus.
- Tangents drawn at extremities of any focal chord meet at the directrix, at right angle.
- By taking any focal chord at diameter, a circle is drawn, which touches the directrix.

- Circle whose diameter is focal distance always touches tangent at vertex.

Conceptual Note(s)

(i) Pair of asymptotes always passes through centre.
(ii) Hyperbola and conjugate hyperbola have same pair of asymptotes.
(iii) Principal axes are angle bisectors of pair of asymptotes.
(iv) Equation of hyperbola, conjugate hyperbola, pair of asymptotes are same but only differ in constant term.

Hyperbola: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c_H = 0$
Conjugate hyperbola:
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c_{CH} = 0$

Conceptual Note(s)

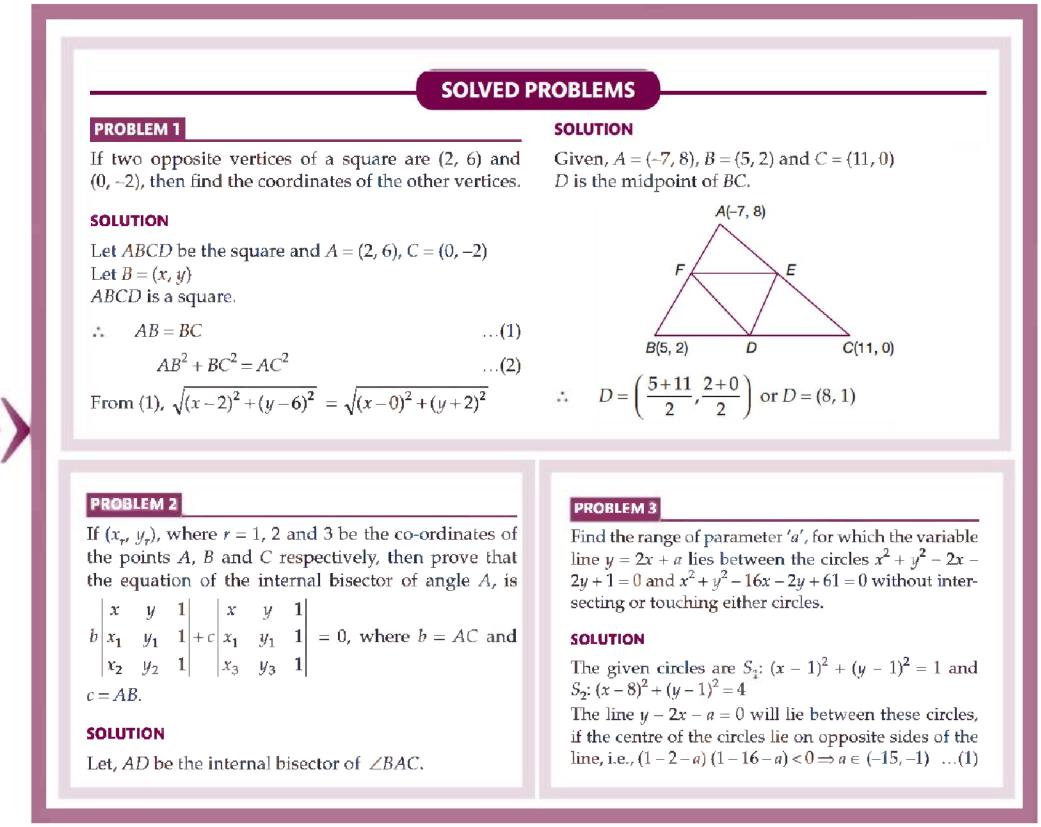
- Foot of perpendicular drawn from focus on any tangent lies on tangent at vertex and image of focus with respect to any tangent lies on directrix.
- Segment of tangent intercepted between point of contact and directrix subtend right angle at focus.
- A variable circle whose diameter is intercept of tangent between point of contact and directrix, always passes through focus of parabola.

Conceptual Notes

The Conceptual Notes, Remarks, Words of Advice, Misconception Removals provide warnings to the students about common errors and help them avoid falling for conceptual pitfalls.

Chapter End Solved Problems

These are based on multiple concept usage in a single problem approach so as to expose a student's brain to the ultimate throttle required to take the JEE examination.



SOLVED PROBLEMS

PROBLEM 1

If two opposite vertices of a square are (2, 6) and (0, –2), then find the coordinates of the other vertices.

SOLUTION

Let $ABCD$ be the square and $A = (2, 6)$, $C = (0, -2)$
Let $B = (x, y)$
$ABCD$ is a square.

$$\therefore \quad AB = BC \qquad \ldots(1)$$

$$AB^2 + BC^2 = AC^2 \qquad \ldots(2)$$

From (1), $\sqrt{(x-2)^2+(y-6)^2} = \sqrt{(x-0)^2+(y+2)^2}$

SOLUTION

Given, $A = (-7, 8)$, $B = (5, 2)$ and $C = (11, 0)$
D is the midpoint of BC.

A(–7, 8)
F
E
B(5, 2)
D
C(11, 0)

$$\therefore \quad D = \left(\frac{5+11}{2}, \frac{2+0}{2}\right) \text{ or } D = (8, 1)$$

PROBLEM 2

If (x_r, y_r), where $r = 1, 2$ and 3 be the co-ordinates of the points A, B and C respectively, then prove that the equation of the internal bisector of angle A, is

$$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0, \text{ where } b = AC \text{ and}$$

$c = AB$.

SOLUTION

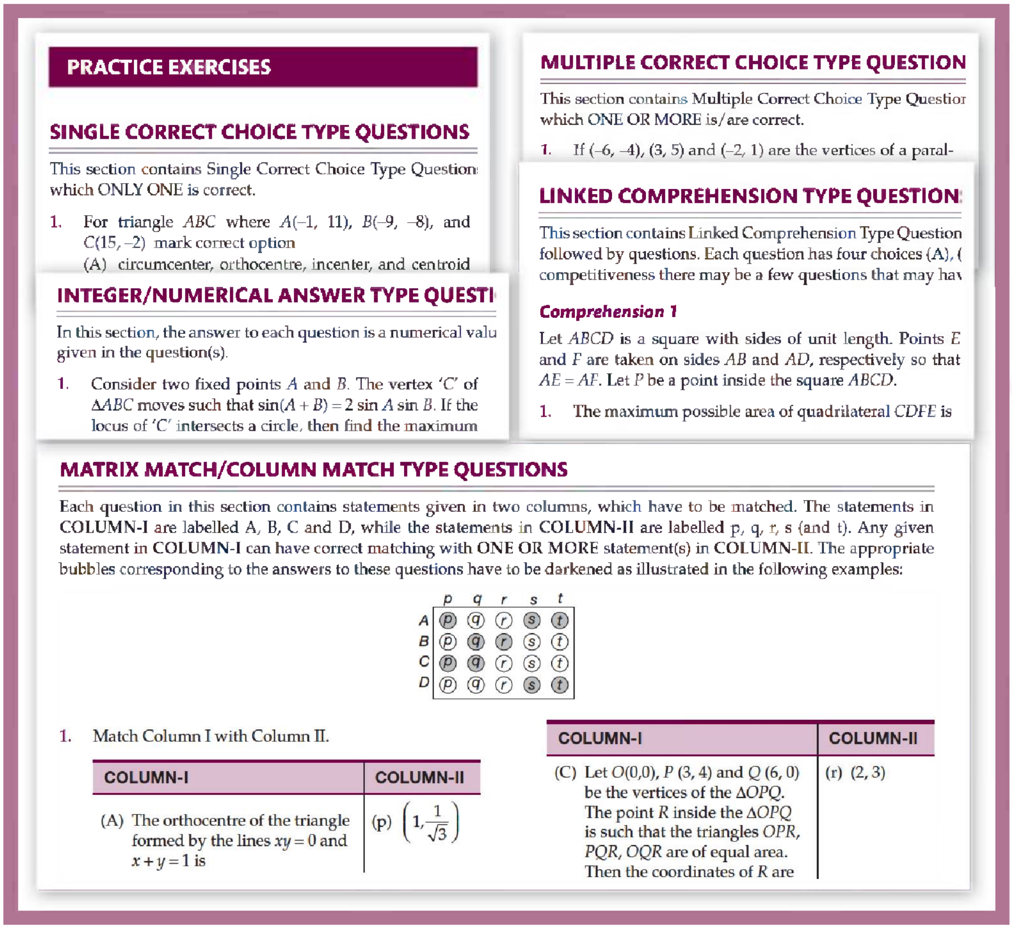
Let, AD be the internal bisector of $\angle BAC$.

PROBLEM 3

Find the range of parameter 'a', for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circles.

SOLUTION

The given circles are S_1: $(x - 1)^2 + (y - 1)^2 = 1$ and S_2: $(x - 8)^2 + (y - 1)^2 = 4$
The line $y - 2x - a = 0$ will lie between these circles, if the centre of the circles lie on opposite sides of the line, i.e., $(1 - 2 - a)(1 - 16 - a) < 0 \Rightarrow a \in (-15, -1)$ …(1)



PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Question which ONLY ONE is correct.

1. For triangle ABC where $A(-1, 11)$, $B(-9, -8)$, and $C(15, -2)$ mark correct option
(A) circumcenter, orthocentre, incenter, and centroid

INTEGER/NUMERICAL ANSWER TYPE QUESTI

In this section, the answer to each question is a numerical valu given in the question(s).

1. Consider two fixed points A and B. The vertex 'C' of ΔABC moves such that $\sin(A+B) = 2\sin A \sin B$. If the locus of '$C$' intersects a circle, then find the maximum

MULTIPLE CORRECT CHOICE TYPE QUESTION

This section contains Multiple Correct Choice Type Questior which ONE OR MORE is/are correct.

1. If $(-6, -4)$, $(3, 5)$ and $(-2, 1)$ are the vertices of a paral-

LINKED COMPREHENSION TYPE QUESTION

This section contains Linked Comprehension Type Question followed by questions. Each question has four choices (A), (competitiveness there may be a few questions that may hav

Comprehension 1

Let $ABCD$ is a square with sides of unit length. Points E and F are taken on sides AB and AD, respectively so that $AE = AF$. Let P be a point inside the square $ABCD$.

1. The maximum possible area of quadrilateral $CDFE$ is

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in COLUMN-I are labelled A, B, C and D, while the statements in COLUMN-II are labelled p, q, r, s (and t). Any given statement in COLUMN-I can have correct matching with ONE OR MORE statement(s) in COLUMN-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

1. Match Column I with Column II.

COLUMN-I	COLUMN-II
(A) The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is	(p) $\left(1, \frac{1}{\sqrt{3}}\right)$

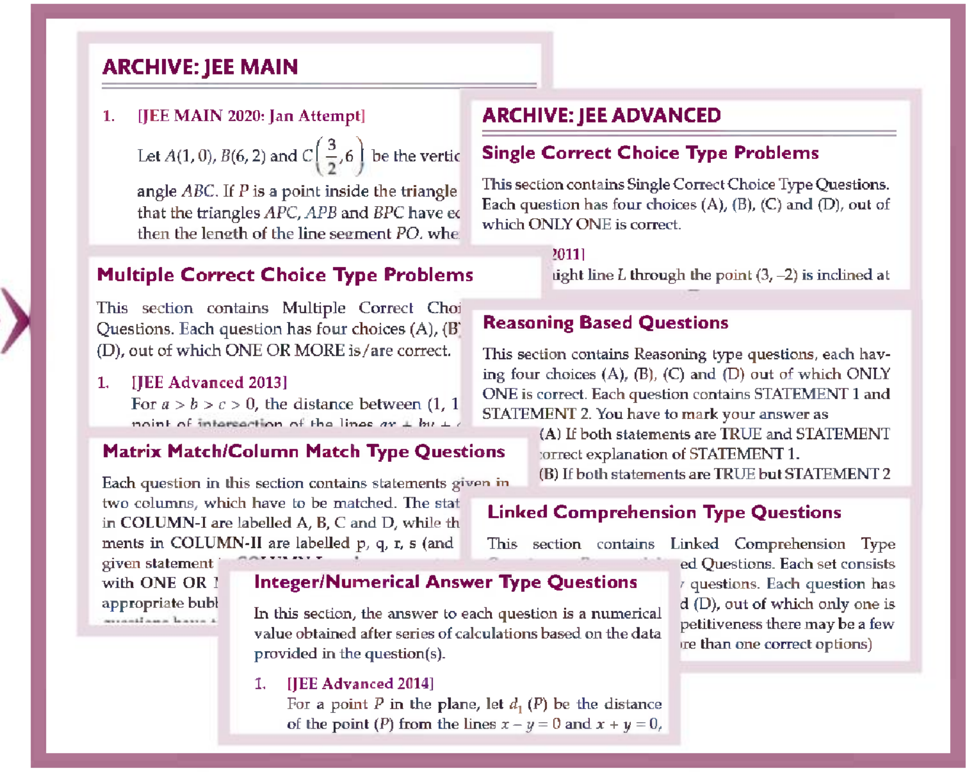
COLUMN-I	COLUMN-II
(C) Let $O(0,0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of the ΔOPQ. The point R inside the ΔOPQ is such that the triangles OPR, PQR, OQR are of equal area. Then the coordinates of R are	(r) $(2, 3)$

Practice Exercise

Inclusion of all types of questions asked in JEE Advanced in adequate numbers helps you with enough practice

Archive JEE Main and Advanced

From this fully updated section, students get to know the actual pattern of the problems asked in the past examinations.



ARCHIVE: JEE MAIN

1. [JEE MAIN 2020: Jan Attempt]

Let $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertic angle ABC. If P is a point inside the triangle that the triangles APC, APB and BPC have eq then the length of the line segment PQ, whe

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

2011]

ight line L through the point $(3, -2)$ is inclined at

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choi Questions. Each question has four choices (A), (B) (D), out of which ONE OR MORE is/are correct.

1. [JEE Advanced 2013]

For $a > b > c > 0$, the distance between (1, 1 point of intersection of the lines $ax + by + c$

Reasoning Based Questions

This section contains Reasoning type questions, each having four choices (A), (B), (C) and (D) out of which ONLY ONE is correct. Each question contains STATEMENT 1 and STATEMENT 2. You have to mark your answer as

(A) If both statements are TRUE and STATEMENT orrect explanation of STATEMENT 1.

(B) If both statements are TRUE but STATEMENT 2

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The stat in COLUMN-I are labelled A, B, C and D, while th ments in COLUMN-II are labelled p, q, r, s (and given statement with ONE OR appropriate bubb

Linked Comprehension Type Questions

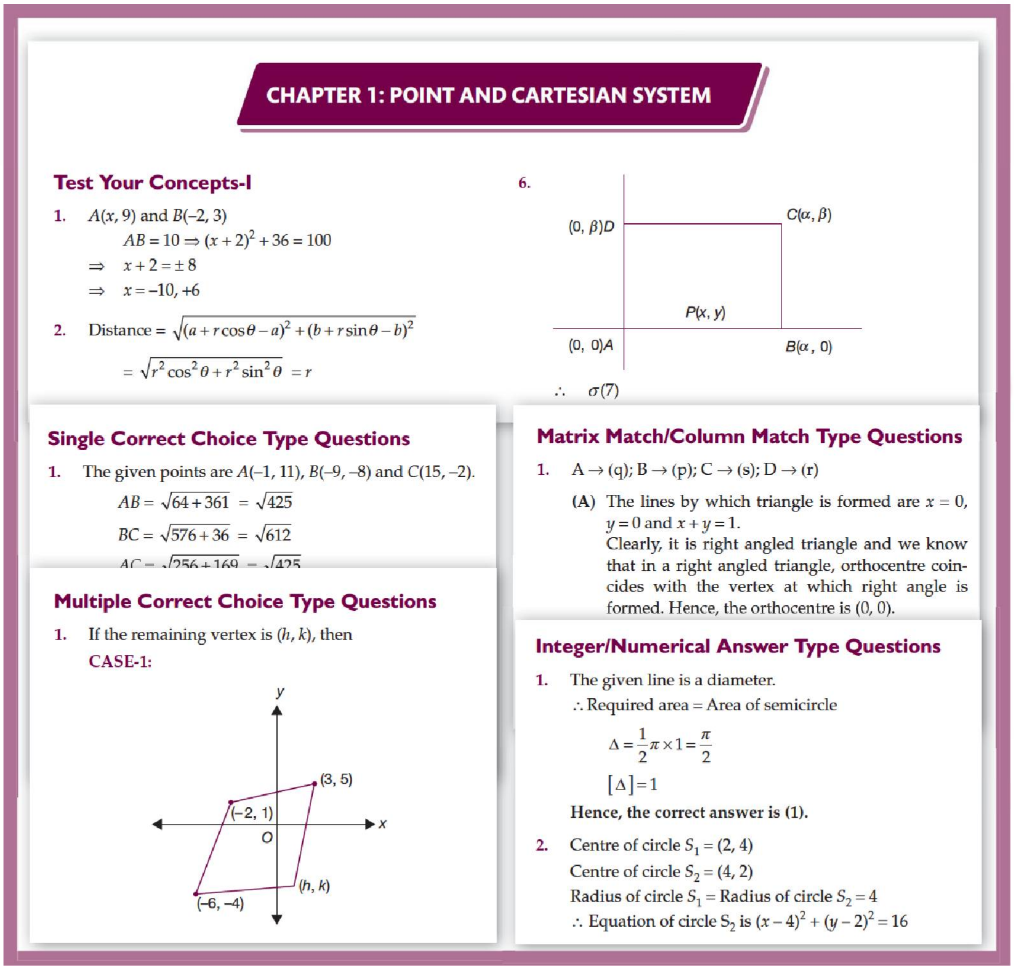
This section contains Linked Comprehension Type ed Questions. Each set consists questions. Each question has d (D), out of which only one is petitiveness there may be a few re than one correct options)

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [JEE Advanced 2014]

For a point P in the plane, let $d_1(P)$ be the distance of the point (P) from the lines $x - y = 0$ and $x + y = 0$,



CHAPTER 1: POINT AND CARTESIAN SYSTEM

Test Your Concepts-I

1. $A(x, 9)$ and $B(-2, 3)$

$$AB = 10 \Rightarrow (x+2)^2 + 36 = 100$$
$$\Rightarrow \quad x + 2 = \pm 8$$
$$\Rightarrow \quad x = -10, +6$$

2. Distance $= \sqrt{(a + r\cos\theta - a)^2 + (b + r\sin\theta - b)^2}$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

6.

$\therefore \quad \sigma(7)$

Single Correct Choice Type Questions

1. The given points are $A(-1, 11)$, $B(-9, -8)$ and $C(15, -2)$.

$$AB = \sqrt{64 + 361} = \sqrt{425}$$
$$BC = \sqrt{576 + 36} = \sqrt{612}$$
$$AC = \sqrt{256 + 169} = \sqrt{425}$$

Multiple Correct Choice Type Questions

1. If the remaining vertex is (h, k), then

CASE-1:

Matrix Match/Column Match Type Questions

1. A → (q); B → (p); C → (s); D → (r)

(A) The lines by which triangle is formed are $x = 0$, $y = 0$ and $x + y = 1$.
Clearly, it is right angled triangle and we know that in a right angled triangle, orthocentre coincides with the vertex at which right angle is formed. Hence, the orthocentre is (0, 0).

Integer/Numerical Answer Type Questions

1. The given line is a diameter.

$\therefore$ Required area = Area of semicircle

$$\Delta = \frac{1}{2}\pi \times 1 = \frac{\pi}{2}$$
$$[\Delta] = 1$$

Hence, the correct answer is (1).

2. Centre of circle $S_1 = (2, 4)$

Centre of circle $S_2 = (4, 2)$

Radius of circle S_1 = Radius of circle $S_2 = 4$

$\therefore$ Equation of circle S_2 is $(x-4)^2 + (y-2)^2 = 16$

Hints and Explanations

Exhaustive solutions with shortcuts (where ever needed), help students enhance their problem-solving skills.

PREFACE

In the past few years, the IIT-JEE has evolved as an examination designed to check a candidate's true scientific skills. The examination pattern needs one to see those little details which others fail to see. These details tell us how much in-depth we should know to explain a concept in the right direction. Keeping the present-day scenario in mind, this series is written for students, to allow them not only to learn the tools but also to see why they work so nicely in explaining the beauty of ideas behind the subject.

The central goal of this series is to help the students develop a thorough understanding of Mathematics as a subject. This series stresses on building a rock-solid technical knowledge based on firm foundation of the fundamental principles followed by a large collection of formulae. The primary philosophy of this book is to guide the aspirants towards detailed groundwork for strong conceptual understanding and development of problem-solving skills like mature and experienced mathematicians.

Each book in this new Mathematics series is to help the aspirants prepare for both Advanced and Main levels of JEE conducted for IITs and other elite engineering institutions in India. The books in this series is to strengthen the basic understanding of the concepts and their applications. The books will be equally useful for the students preparing for Mathematics Olympiads.

This book is enriched with detailed exhaustive theory that introduces the concepts of Mathematics in a clear, concise, thorough and easy-to-understand language. A large collection of relevant problems is provided in eight major categories (including updated archive for JEE Advanced and JEE Main), for which the solutions are demonstrated in a logical and stepwise manner. These not only make students exam-ready but also helps them gain analytical skills.

We have carefully divided the series into five parts to make the learning of different topics seamless for the students. These parts are

- Coordinate Geometry
- Algebra
- Trigonometry
- Calculus
- 3-D Vector

Finally, I would like to thank all my teacher friends who had been a guiding source of light throughout the entire journey of writing this book.

I apologise in advance for the errors (if any) that might have inadvertently crept in the text. I would be grateful to the readers who bring errors of any kind to my attention. I truly welcome all comments, critiques and suggestions. I hope this book will nourish you with the concepts involved so that you get a great rank at JEE.

Wish you all the best !

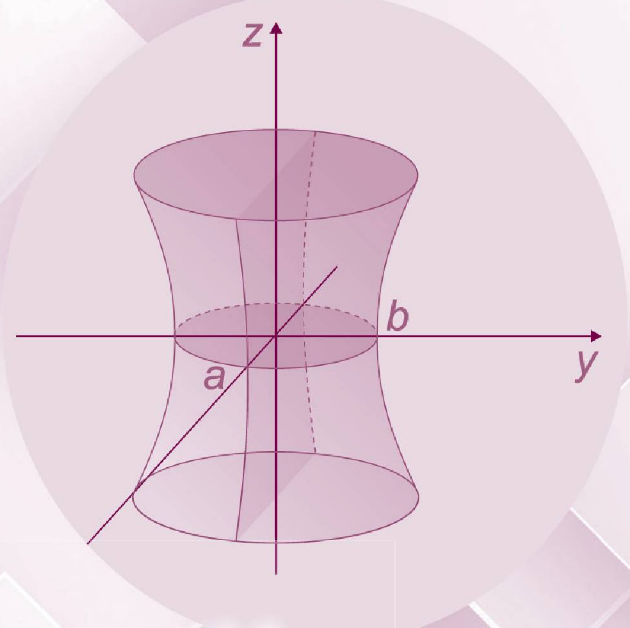
Om Sharma

ABOUT THE AUTHOR

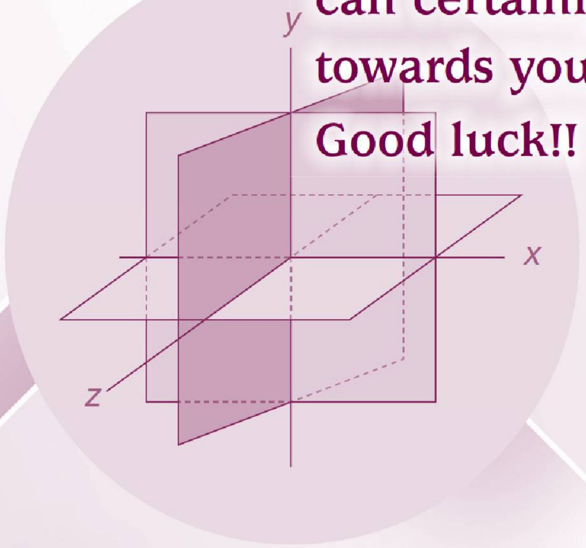
Om Sharma, an ex Research Fellow at BARC (Bhabha Atomic Research Centre), Mumbai, is an IIT Bombay (2007) Graduate. He has been a mathematics expert in the field of IIT JEE Mains and Advanced for more than 13 years from "Shiksha Ki Kaashi - Kota", Rajasthan. He has great exposure to teaching students in the offline and online mode of education.

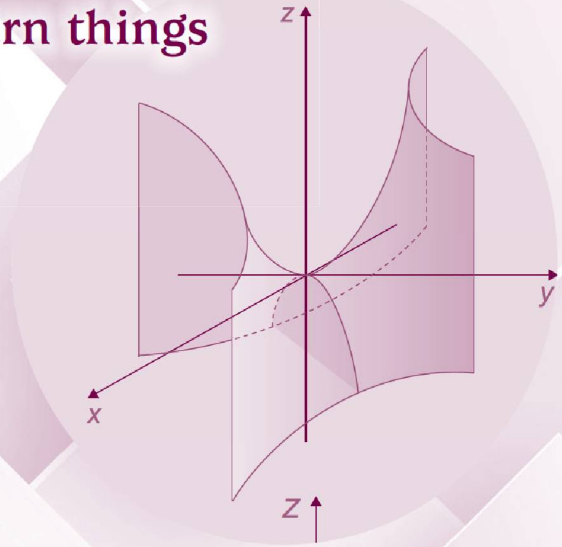
He is also a motivational speaker and exceptional mentor, having skills to motivate students and ignite the spark in them for achieving success in targeting their goal. Throughout this journey, by the Grace of God, under his guidance and mentorship, many of his students have secured top ranks and become successful engineers and technocrats.

From the Author's Desk



> Dear learners,
>
> Coordinate Geometry is an essential branch of mathematics and it helps students to score good marks in JEE Mains and Advanced. First chapter talks about basics of Coordinate Geometry such as section formula, shift of origin, area of triangle, polygon, locus problems and special points of triangle.
>
> It is advisable to practice adequate questions to increase problem-solving skills in prescribed time frame and that can certainly help you to turn things towards your favour.
>
> Good luck!!





CHAPTER

1 Point and Cartesian System

www.JEE S.IN

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Cartesian and Polar Co-ordinates
(b) Shifting of origin, transformation of axes
(c) Section division, area of triangle, polygon
(d) Special points of triangle and locus of a point

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

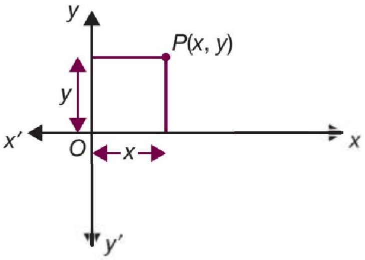
INTRODUCTION OF COORDINATE GEOMETRY

Coordinate geometry is a combination of algebra and geometry. A French philosopher, mathematician and scientist named, René Descartes was the first person, who carried out a systematic study of geometry by the use of algebra. The resulting combination of analysis and geometry is referred as **analytical geometry**.

CARTESIAN CO-ORDINATES

In two-dimensional coordinate system, two mutually perpendicular lines are used as axes, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system.

Segment (OX) along the x-axis to the right of the origin is taken as positive and the other segment OX' at the left of the origin is taken as negative. Segment OY along the y-axis above the origin is considered as positive and the segment OY' below the origin is taken as negative. The position of a point anywhere in the plane can be specified by two numbers, i.e., the coordinates of the point and it is denoted by $P(x, y)$.



The x-coordinate (or **abscissa**) is the algebraic distance of the point from the y-axis in a direction parallel to the x-axis (i.e., horizontally).

The y-coordinate (or **ordinate**) is the algebraic distance from the x-axis in a direction parallel to the y-axis (i.e., vertically). The origin O is the point (0, 0). Sign of point (x, y) located in different quadrants is given in the following table:

Location of Point *P*	Sign of *x*-coordinate (Abscissa)	Sign of *y*-coordinate (Ordinate)	Sign of (*x*, *y*)
1st quadrant	+	+	(+, +)
2nd quadrant	–	+	(–, +)
3rd quadrant	–	–	(–, –)
4th quadrant	+	–	(+, –)

The Cartesian plane xy is divided into four quadrants, each subtends right angle at origin.

xoy is Ist quadrant, $x'oy$ is IInd quadrant, $x'oy'$ is IIIrd quadrant and xoy' is IVth quadrant.

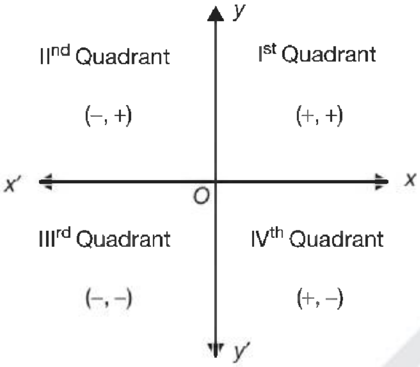


ILLUSTRATION 1

Suppose ABC is a triangle with 3 acute angles namely, A, B and C. Find the quadrant in which the point $P(\cos B - \sin A, \sin B - \cos A)$ may lie.

SOLUTION

Given that, ABC is an acute angled triangle.

$$\therefore \quad A + B > \frac{\pi}{2} \Rightarrow A > \frac{\pi}{2} - B; \sin A - \cos B > 0$$

$$\Rightarrow \quad \cos B - \sin A < 0 \Rightarrow X_P < 0$$

Again, $B > \dfrac{\pi}{2} - A$

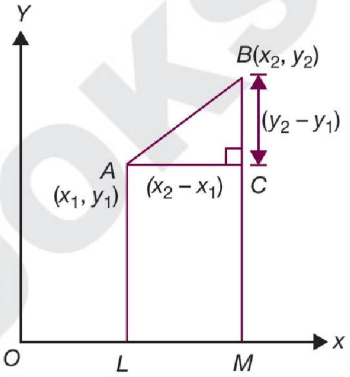
$$\therefore \quad \sin B > \cos A; \sin B - \cos A > 0 \Rightarrow Y_P > 0$$

$\therefore$ Point $P(\cos B - \sin A, \sin B - \cos A)$ lies in IInd quadrant only.

DISTANCE FORMULA AND ITS APPLICATIONS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points:

From A and B, AL and BM are drawn perpendicular to x-axis. Join AB and from A, draw AC perpendicular to BM.



If A and B are two points whose coordinates are (x_1, y_1) and (x_2, y_2), respectively.

Then the distance between point A and B is given by,

$$AB = \sqrt{AC^2 + BC^2}$$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\text{difference in abscissae})^2 + (\text{difference in ordinates})^2}$$

COROLLARY:

(a) The distance of the point (x, y) from the origin is given by,

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

(b) When the line segment AB is parallel to y-axis, it means $x_1 = x_2$.

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(y_1 - y_2)^2}$$

$$= |y_1 - y_2| \qquad (\because \sqrt{x^2} = |x|)$$

(c) When the line segment AB is parallel to x-axis, it means $y_1 = y_2$.

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2}$$

$$= |x_1 - x_2|$$

ILLUSTRATION 2

Let $A(6, -1)$, $B(1, 3)$ and $C(x, 8)$ be three points such that $AB = BC$. Find the value of x.

SOLUTION

Given $AB = BC$

$\therefore \quad AB^2 = BC^2$

$\therefore \quad (6-1)^2 + (-1-3)^2 = (1-x)^2 + (3-8)^2$

or $\quad 41 = (1-x)^2 + 25$

or $\quad (1-x)^2 = 16$

or $\quad 1 - x = \pm 4$

$\therefore \quad x = 1 \pm 4 = -3, 5$

ILLUSTRATION 3

The length of a line segment AB is 10 units. If the coordinates of one extremity are $(2, -3)$ and the abscissa of the other extremity is 10, then find all the possible values of the ordinate of the other extremity.

SOLUTION

Let $A(2, -3)$ and $B(10, k)$

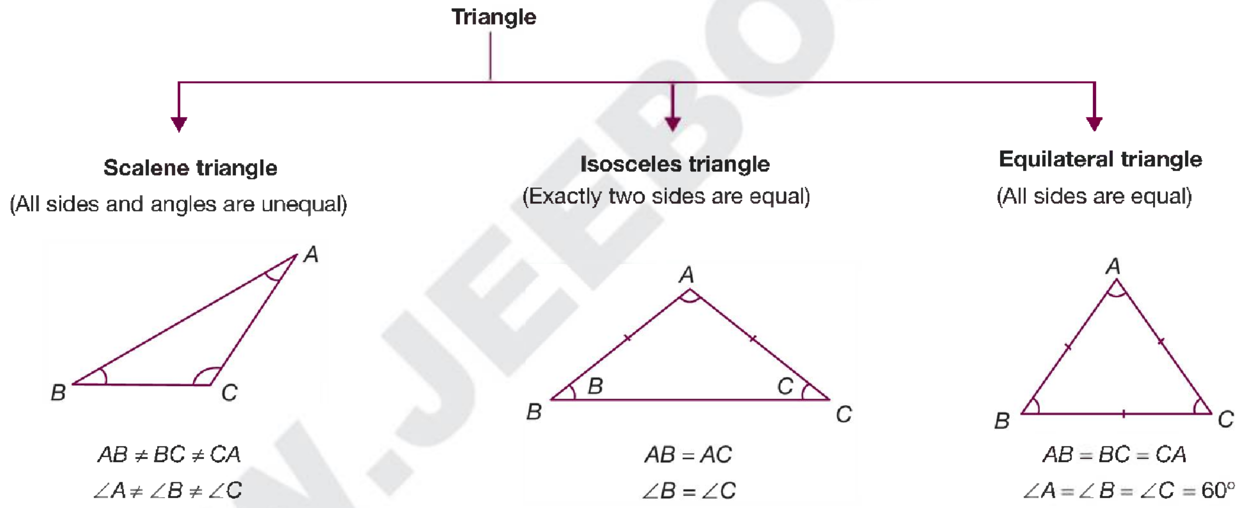
Now $AB = 10$

$\Rightarrow \quad 64 + (k+3)^2 = 100$

$\Rightarrow \quad (k+3)^2 = 36$

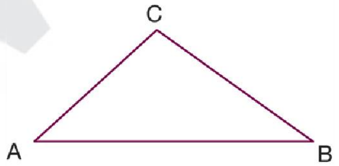
$\Rightarrow \quad k + 3 = 6 \text{ or } -6$

$\Rightarrow \quad k = +3 \text{ or } -9$


Triangle
Scalene triangle
(All sides and angles are unequal)
A
B
C
AB ≠ BC ≠ CA
∠A ≠ ∠B ≠ ∠C
Isosceles triangle
(Exactly two sides are equal)
A
B
C
B
C
AB = AC
∠B = ∠C
Equilateral triangle
(All sides are equal)
A
B
C
AB = BC = CA
∠A = ∠B = ∠C = 60°

Acute angled triangle

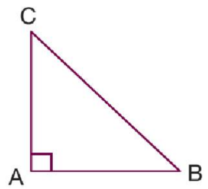
(All three angles are less than 90°)



$AB^2 + AC^2 > BC^2$

Right angled triangle

(One angle is 90°)

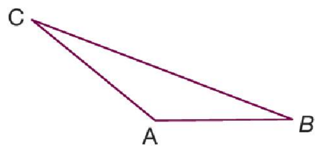


$AB^2 + AC^2 = BC^2$

(Pythagoras theorem)

Obtuse angled triangle

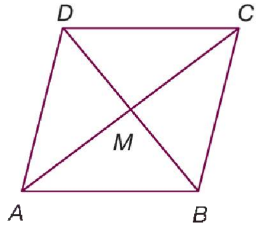
(Here angle A is more than 90°)



$AB^2 + AC^2 < BC^2$

Parallelogram

Opposite sides are parallel, equal and the diagonals bisect.



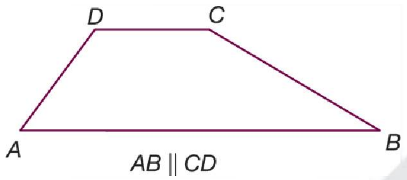
$$AB = CD,\ AB \,||\, CD$$

$$AD = BC,\ AD \,||\, BC$$

M is the midpoint of diagonals AC and BD.

Trapezium

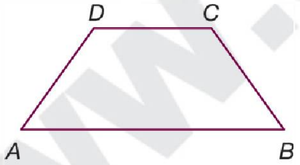
Two sides are parallel and the other two sides are non-parallel.



$AB \,||\, CD$

Isosceles Trapezium

Two sides are parallel and the other two non-parallel sides make equal angles with base.

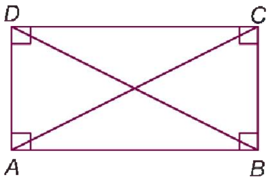


$AB \,||\, CD$

$AD = BC$

Rectangle

- Parallelogram whose sides enclosed 90° at every vertex.
- Diagonals are of equal length.

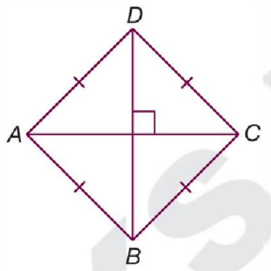


$AC = BD$

$AD \perp AB$

Rhombus

- Parallelogram whose all sides are of equal length.
- Diagonals are at right angle.

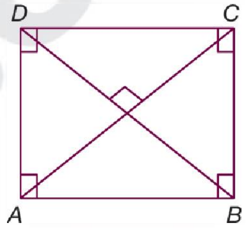


$$AB = BC = CD = DA$$

$$AC \perp BD$$

Square

- All sides are equal and perpendicular.
- Diagonals are equal and perpendicular.



$AB = BC = CD = DA$

$AC = BD$

$AB \perp AD;\ AC \perp BD$

Remark(s)

- Every square is a rectangle but converse is not true.
- Every square is a rhombus but converse is not true.
- Every rhombus is a parallelogram but converse is not true.
- Every rectangle is a parallelogram but converse is not true.

ILLUSTRATION 4

Prove that the points $A(2a, 4a)$, $B(2a, 6a)$ and $C(2a + \sqrt{3}a, 5a)(a > 0)$ are the vertices of an equilateral triangle.

SOLUTION

$$AB = \sqrt{(2a-2a)^2 + (4a-6a)^2} = 2a$$

$$BC = \sqrt{(\sqrt{3}\,a)^2 + a^2} = 2a$$

and $CA = \sqrt{(\sqrt{3}\,a)^2 + a^2} = 2a$

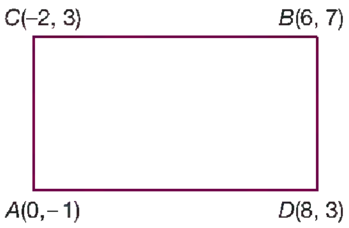
Since $AB = BC = CA$, the triangle is equilateral.

ILLUSTRATION 5

Show that the four points (0, –1), (6, 7), (–2, 3) and (8, 3) are the vertices of a rectangle.

SOLUTION

Let $A(0, -1)$, $B(6, 7)$, $C(-2, 3)$ and $D(8, 3)$ be the given points. Then,



$$AD = \sqrt{(8-0)^2+(3+1)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$BC = \sqrt{(6+2)^2+(7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2+(3+1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

and $BD = \sqrt{(8-6)^2+(3-7)^2} = \sqrt{4+16} = 2\sqrt{5}$

$\therefore AD = BC$ and $AC = BD$. So, $ADBC$ is a parallelogram.

Now, the diagonal $AB = \sqrt{(6-0)^2+(7+1)^2}$

$$= \sqrt{36+64} = 10$$

and $CD = \sqrt{(8+2)^2+(3-3)^2} = 10.$

Clearly,

$$AB^2 = AD^2 + DB^2 \text{ and } CD^2 = CB^2 + BD^2.$$

Hence, $ADBC$ is a rectangle.

ILLUSTRATION 6

If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then find a and b.

SOLUTION

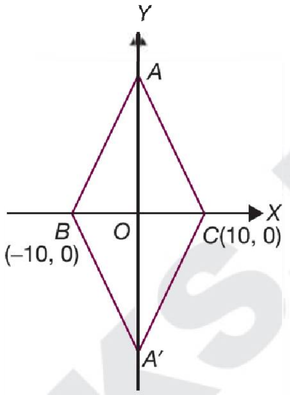
As diagonals of a parallelogram bisects each other,

$$\frac{4+a}{2} = 3 \Rightarrow a = 2 \text{ and } \frac{6+b}{2} = \frac{9}{2} \Rightarrow b = 3$$

ILLUSTRATION 7

The base of an equilateral triangle with side 20 cm lies x-axis such that the midpoint of the base is at the origin. Find the vertices of the triangle.

SOLUTION



Let BC be the base of the equilateral triangle.

$$\therefore \quad OC = OB = \frac{20}{2} \text{ cm} = 10 \text{ cm}$$

$\therefore$ Coordinates of B and C are (–10, 0) and (10, 0) respectively.

Two equilateral triangles with base BC are possible.

For ΔABC, we have, $AB = BC = CA = 20$ cm.

$$\therefore \quad OA = \sqrt{AC^2 - OC^2} = \sqrt{20^2 - 10^2} = 10\sqrt{3}$$

$$\therefore \quad A(0,\ 10\sqrt{3}\,)$$

For $\Delta A'BC$, we have, $A'B = A'C = BC = 20$ cm

$$\therefore \quad OA' = \sqrt{A'C^2 - OC^2} = \sqrt{20^2 - 10^2} = 10\sqrt{3}$$

$$\therefore \quad A'(0, -10\sqrt{3}\,).$$

ILLUSTRATION 8

If the segments joining the points $A(a, b)$ and $B(c, d)$ subtend an angle θ at the origin, then prove that

$$\cos\theta = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

SOLUTION

Let O be the origin. Then $OA^2 = a^2 + b^2$, $OB^2 = c^2 + d^2$ and

$$AB^2 = (c-a)^2 + (d-b)^2$$

Using cosine formula in ΔOAB, we have,

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos\theta$$

$$\Rightarrow \quad (c-a)^2 + (d-b)^2 = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$$

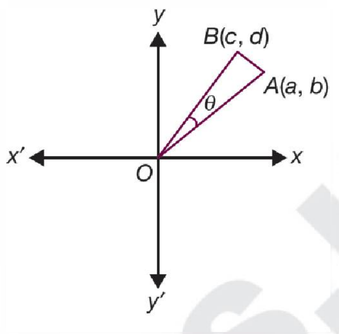
$$\Rightarrow \quad c^2 + a^2 - 2ac + d^2 + b^2 - 2bd$$

$$= a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$$

$$\Rightarrow \quad 2(ac+bd) = 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

Hence proved.



Test Your Concepts-I

(Solutions on page H.3)

1. A line is of length 10 units and one of its ends is (–2, 3). If the ordinate of the other end is 9, then prove that the abscissa of the other end is 6 or –10.
2. Prove that the distance between the points $(a + r\cos\theta, b + r\sin\theta)$ and (a, b) is independent of θ.
3. Prove that the points (3, 4), (8, –6) and (13, 9) are the vertices of a right-angled triangle.
4. If the line segment joining the points $A(a, b)$ and $B(c, d)$ subtends a right angle at the origin, show that $ac + bd = 0$.
5. In any ΔABC, prove that $AB^2 + AC^2 = 2(AD^2 + BD^2)$, where D is the midpoint of BC.
6. If $ABCD$ be a rectangle and P be any point in the plane of the rectangle, then prove that $PA^2 + PC^2 = PB^2 + PD^2$.
7. Show that the points $A(1, 0)$, $B(5, 3)$, $C(2, 7)$ and $D(-2, 4)$ are the vertices of a rhombus.
8. If (6, 8), (3, 7) and (–2, –2) be the coordinates of the three consecutive vertices of a parallelogram, then find the coordinates of the fourth vertex.
9. A quadrilateral has the vertices at the points (–4, 2), (2, 6), (8, 5) and (9, –7). Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.
10. Prove that the line joining the midpoints of the two sides of a triangle is parallel to the third side.

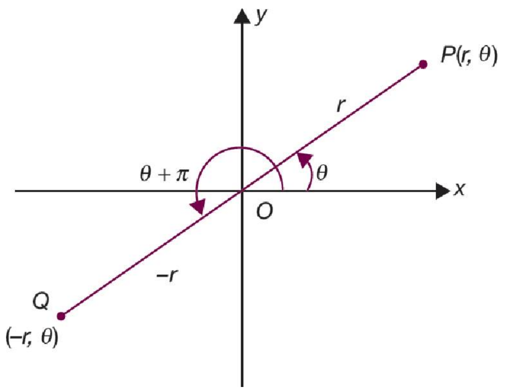
Answer Keys

8. (1, –1)

POLAR COORDINATES

A coordinate system is the one, where the position of a point is determined by the length of a line segment from a fixed origin and with the angle that the line segment OP makes with positive x-axis. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the x-axis and the radius vector OP is called the vectorial angle. By convention, **positive values** of θ are measured in an **anticlockwise sense**, negative values in clockwise sense. The coordinates of the point P are then specified as (r, θ).



Coordinate of a point *P*

The point $P(x, y)$ is known as Cartesian coordinate and the point $P(r, \theta)$ is known as polar coordinate.

$$x = r\cos\theta,\ y = r\sin\theta, \text{ where } |r| = \sqrt{x^2+y^2},$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Remark(s)

Point $P(r, \theta)$ can also be written as $(-r, \theta+\pi)$ or $(-r, \theta-\pi)$.

Point $Q(-r, \theta)$ can also be written as $(r, \theta+\pi)$ or $(r, \theta-\pi)$.

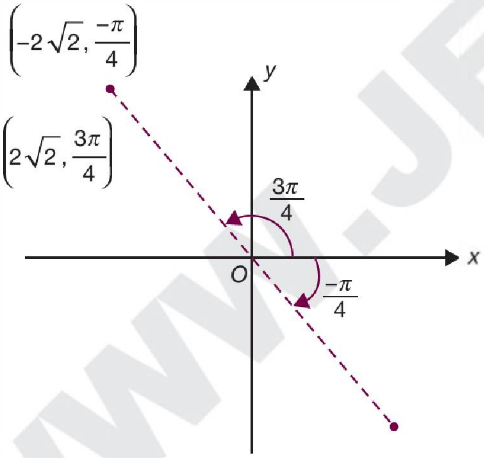
ILLUSTRATION 9

Express the point $P(-2, 2)$ in polar form.

SOLUTION

Point $P(-2, 2)$ lies in II$^{\text{nd}}$ quadrant.

$$\tan\theta = \frac{y}{x} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4}$$



$\left(-2\sqrt{2}, \frac{-\pi}{4}\right)$
$\left(2\sqrt{2}, \frac{3\pi}{4}\right)$
y
$\frac{3\pi}{4}$
x
O
$\frac{-\pi}{4}$

Using positive r, we have,

$$r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(2)^2} = 2\sqrt{2}$$

So, one set of polar coordinates is $(r, \theta) = P\left(2\sqrt{2}, \frac{3\pi}{4}\right)$.

Using negative r, we have coordinates $= P\left(-2\sqrt{2}, \frac{-\pi}{4}\right)$.

Both are the representation of the point $P(-2, 2)$.

ILLUSTRATION 10

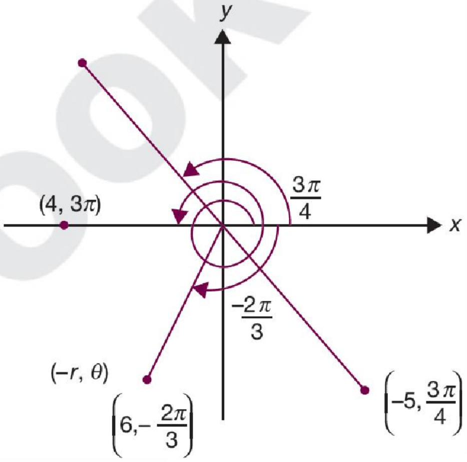
Plot the points, whose polar coordinates are given below.

(i) $(4, 3\pi)$

(ii) $\left(6, \frac{-2\pi}{3}\right)$

(iii) $\left(-5, \frac{3\pi}{4}\right)$

SOLUTION



y
$\frac{3\pi}{4}$
(4, 3π)
x
$\frac{-2\pi}{3}$
(−r, θ)
$\left(6, -\frac{2\pi}{3}\right)$
$\left(-5, \frac{3\pi}{4}\right)$

ILLUSTRATION 11

Convert the following points from polar coordinates to their corresponding Cartesian coordinates.

(i) $\left(2, \frac{-\pi}{3}\right)$

(ii) $\left(-2\sqrt{2}, \frac{-3\pi}{4}\right)$

SOLUTION

(i) $\left(2, \frac{-\pi}{3}\right) = (r, \theta)$

We have, $x = r\cos\theta = 2\cos\left(\frac{-\pi}{3}\right) = 1$

and $y = r\sin\theta = 2\sin\left(\frac{-\pi}{3}\right) = -\sqrt{3}$

Therefore, the point is $\left(1, -\sqrt{3}\right)$ in the Cartesian coordinates.

(ii) $x = -2\sqrt{2}\cos\left(\frac{-3\pi}{4}\right) = -2\sqrt{2}\left(\frac{-1}{\sqrt{2}}\right) = 2$

$$y = -2\sqrt{2}\sin\left(\frac{-3\pi}{4}\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$$

So, the equivalent Cartesian coordinates of the given polar coordinates are (2, 2).

ILLUSTRATION 12

Express the polar equation $r = 2\sin\theta$ in rectangular coordinates.

SOLUTION

We use the formulae, $r^2 = x^2 + y^2$ and $y = r\sin\theta$.
Given, $r = 2\sin\theta$
Therefore,

$$r^2 = 2r\sin\theta$$

$$\Rightarrow \quad x^2 + y^2 = 2y$$

$$\Rightarrow \quad x^2 - 2y + 1 + y^2 = 1$$

$$\Rightarrow \quad x^2 + (y-1)^2 = 1$$

ILLUSTRATION 13

Convert $x^2 - y^2 = 4$ into a polar equation.

SOLUTION

Given, $x^2 - y^2 = 4$

$$\Rightarrow \quad (r\cos\theta)^2 - (r\sin\theta)^2 = 4$$

$$\Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 4$$

$$\Rightarrow \quad r^2\cos 2\theta = 4$$

ILLUSTRATION 14

Convert the following equations to Cartesian coordinates.

(a) $r = a\sin^2\theta$

(b) $\sqrt{r} = \sqrt{a}\cos\frac{\theta}{2}$

SOLUTION

(a) $r = a\sin^2\theta$

By multiplying by r^2, we get,

$$r^3 = ar^2\sin^2\theta$$

We know that, $x = r\cos\theta, y = r\sin\theta, r = \sqrt{x^2+y^2}$,

Then the equation becomes

$$\left(x^2+y^2\right)^{\frac{3}{2}} = ay^2 \Rightarrow \left(x^2+y^2\right)^3 = a^2y^4$$

(b) $\sqrt{r} = \sqrt{a}\cos\frac{\theta}{2}$

Squaring the equation, we get,

$$r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1+\cos\theta)$$

Now by using the relation between Cartesian and polar coordinate, the equation becomes,

$$2r^2 = ar + ar\cos\theta$$

i.e., $2(x^2+y^2) = a\sqrt{x^2+y^2} + ax$

i.e., $(2x^2 + 2y^2 - ax)^2 = a^2(x^2+y^2)$

ILLUSTRATION 15

Convert $r\sin^2\theta = r\cos^2\theta + \frac{4}{r}$ into its equivalent Cartesian equation.

SOLUTION

Given, $r\sin^2\theta = r\cos^2\theta + \frac{4}{r} \Rightarrow r^2\sin^2\theta = r^2\cos^2\theta + 4$

We know that $x = r\cos\theta, y = r\sin\theta$

$$\Rightarrow \quad y^2 = x^2 + 4$$

ILLUSTRATION 16

Find the minimum distance of any point on the curve $x^2 + 2y^2 - 2xy = 1$ from the origin.

SOLUTION

Let the polar coordinates of any point on the curve be $P(r, \theta)$.
Then its Cartesian coordinates are $P(r\cos\theta, r\sin\theta)$, which lies on given curve

$$r^2\cos^2\theta + 2r^2\sin^2\theta - 2r^2\sin\theta\cos\theta = 1.$$

$$\text{or} \quad r^2 = \frac{1}{\cos^2\theta + 2\sin^2\theta - \sin 2\theta}$$

$$= \frac{1}{\sin^2\theta + 1 - \sin 2\theta}$$

$$= \frac{2}{1 - \cos 2\theta + 2 - 2\sin 2\theta}$$

$$= \frac{2}{3 - \cos 2\theta - 2\sin 2\theta}$$

Now, $-\sqrt{5} \le -\cos 2\theta - 2\sin 2\theta \le \sqrt{5}$

$$\Rightarrow \quad 3 - \sqrt{5} \le 3 - \cos 2\theta - 2\sin 2\theta \le 3 + \sqrt{5}$$

$$\Rightarrow \quad r^2_{\min} = \frac{2}{3+\sqrt{5}}$$

$$\Rightarrow \quad r_{\min} = \frac{\sqrt{2}}{\sqrt{3+\sqrt{5}}}$$

REFLECTION OF A POINT

Let $P(x, y)$ be any point, then its image with respect to

(a) x-axis is $Q(x, -y)$
(b) y-axis is $R(-x, y)$
(c) Origin is $S(-x, -y)$
(d) Line $y = x$ is $T(y, x)$

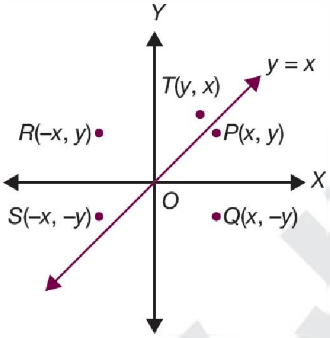
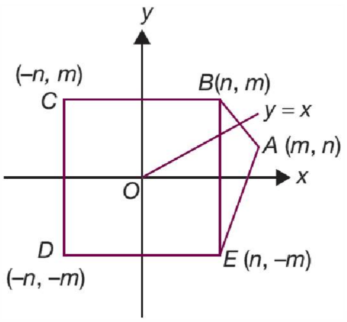


ILLUSTRATION 17

If m, n are integers with $0 < n < m$ and A is the point (m, n) on the Cartesian plane. B is the reflection of A in the line $y = x$. C is the reflection of B in the y-axis, D is the reflection of C in the x-axis and E is the reflection of D in the y-axis, then find the area of the pentagon $ABCDE$.

SOLUTION



Area of rectangle $BCDE = 4\,mn$

$$\text{Area of } ABC = \frac{2m(m-n)}{2} = m^2 - mn$$

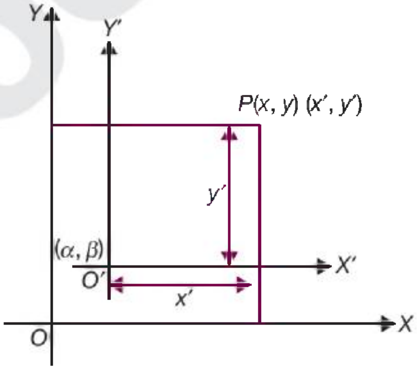
$\therefore$ Area of pentagon $= 4mn + m^2 - mn = m^2 + 3mn$

TRANSFORMATION OF AXES

(a) Shifting of Origin without Rotation of Axes

Let $P = (x, y)$ with respect to axes OX and OY. Let $O' = (\alpha, \beta)$ is origin with respect to axes $O'X'$ and $O'Y'$ and let $P = (x', y')$ with respect to axes $O'X'$ and $O'Y'$, where OX and $O'X'$ are parallel and OY and $O'Y'$ are parallel.

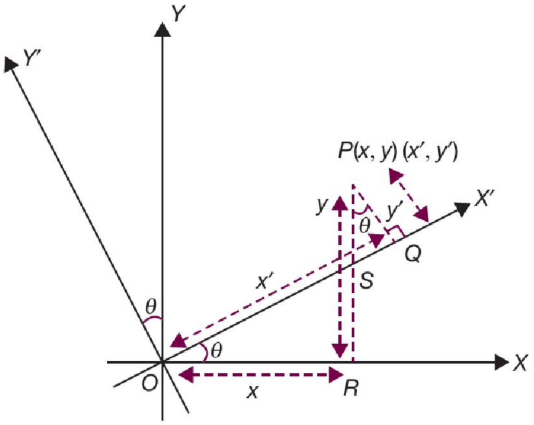
Then, $\quad x = x' + \alpha, y = y' + \beta$
or $\quad x' = x - \alpha, y' = y - \beta$



Thus, if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by substituting $x + \alpha$ in the place of x and $y + \beta$ in the place of y.

(b) Rotation of Axes without Shifting the Origin

Let O be the origin. Let $P = (x, y)$ with respect to axes OX and OY and let $P = (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$.



In the figure,

$SR = x \tan\theta,\ OS = x\sec\theta$

$\therefore\ PS = y - x\tan\theta$

and $SQ = x' - x\sec\theta$

Now in ΔPQS,

$$\sin\theta = \frac{SQ}{PS} = \frac{x' - x\sec\theta}{y - x\tan\theta}$$

$$\Rightarrow\ x' = y\sin\theta - x\frac{\sin\theta}{\cos\theta} + \frac{x}{\cos\theta}$$

$$= y\sin\theta + x\left(\frac{1-\sin^2\theta}{\cos\theta}\right)$$

$$\Rightarrow\ x' = x\cos\theta + y\sin\theta \qquad \ldots(1)$$

$$\text{Also, } \cos\theta = \frac{PQ}{PS} = \frac{y'}{y - x\tan\theta}$$

$$\Rightarrow\ y' = -x\sin\theta + y\sin\theta \qquad \ldots(2)$$

From (1) and (2),

$$\Rightarrow\ x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

Working Rule

xy coordinates axes are rotated by θ in anti-clockwise direction, where new axes are x' and y'.

$x' = x\cos\theta + y\sin\theta$ $y' = -x\sin\theta + y\cos\theta$	$x = x'\cos\theta - y'\sin\theta$ $y = x'\sin\theta + y'\cos\theta$

ILLUSTRATION 18

At what point should the origin be shifted, if the coordinates of a point (4, 5) become (–3, 9)?

SOLUTION

Let (h, k) be the point to which the origin is shifted. Then, $x = 4, y = 5, X = -3, Y = 9$

$\therefore\quad x = X + h$ and $y = Y + k$

or $\quad 4 = -3 + h$ and $5 = 9 + k$

or $\quad h = 7$ and $k = -4$

Hence, the origin must be shifted to (7, –4).

ILLUSTRATION 19

If the origin is shifted to the point (1, –2) without the rotation of the axes, then find the new equation of $y^2 - 4x + 4y + 8 = 0$ after shifting of origin.

SOLUTION

Substituting $x = X + 1$ and $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0$$

or $\quad Y^2 = 4X$

ILLUSTRATION 20

If the origin is shifted to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain a term in y and the constant term.

SOLUTION

Let the origin be shifted to (h, k). Then,

$$x = X + h \text{ and } y = Y + k.$$

Substituting $x = X + h$ and $y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get,

$$(Y+k)^2 + 4(Y+k) + 8(X+h) - 2 = 0$$

or $\quad Y^2 + (4 + 2k)Y + 8X + (k^2 + 4k + 8h - 2) = 0$

For this equation to be free from the term contacting Y and the constant term, we must have,

$$4 + 2k = 0 \text{ and } k^2 + 4k + 8h - 2 = 0$$

$$\Rightarrow\quad k = -2 \text{ and } h = \frac{3}{4}$$

Hence, the origin is shifted at the point $\left(\frac{3}{4}, -2\right)$.

ILLUSTRATION 21

The equation of a pair of straight lines is $ax^2 + 2hxy + by^2 = 0$. By what angle must the axes be rotated so that the term containing xy in the equation may be removed?

SOLUTION

Here the origin remains fixed. Let the axes be turned about the fixed origin through an angle ϕ in the anti-clockwise sense and new coordinates of the point (x, y) become (x', y').

Then the equations of transformation will be,

$$x = x' \cos\phi - y' \sin\phi$$

$$y = x' \sin\phi + y' \cos\phi$$

$\therefore$ The changed equation will be,

$$a(x' \cos\phi - y' \sin\phi)^2 + 2h(x' \cos\phi - y' \sin\phi)(x' \sin\phi + y' \cos\phi) + b(x' \sin\phi + y' \cos\phi)^2 = 0$$

or $$(a \cos^2\phi + 2h \sin\phi \cos\phi + b \sin^2\phi)\, x'^2 + (-a \sin 2\phi + 2h \cos 2\phi + b \sin 2\phi)\, x' y' + (a \sin^2\phi - 2h \sin\phi \cos\phi + b \cos^2\phi)\, y'^2 = 0$$

Now the transformed equation of the pair of lines for the new axes should not have xy term so coefficient of $x' y' = 0$

$$(b - a) \sin 2\phi + 2h \cos 2\phi = 0$$

or $$\tan 2\phi = \frac{2h}{a - b}$$

$\therefore$ $\phi = \dfrac{1}{2} \tan^{-1} \dfrac{2h}{a - b}$ is the required angle.

Test Your Concepts-II

(Solutions on page H.4)

1. Convert the Cartesian coordinates (2, –3) to their corresponding polar coordinates using positive r and negative r.

2. Convert the Cartesian equation, $y = 10$ into its polar equation.

3. Convert the following polar coordinates to their equivalent Cartesian coordinates.

(i) $(2, \pi)$

(ii) $\left(\sqrt{3}, \dfrac{\pi}{6}\right)$

4. Convert the following Cartesian coordinates to their corresponding polar coordinates using positive r.

(i) (1, -1)

(ii) (-3, -4)

5. Convert $2x^2 + 3y^2 = 6$ into its polar equation.

6. Convert $r = 4 \tan\theta \sec\theta$ into its equivalent Cartesian equation.

7. Find the minimum distance of any point on the line $3x + 4y - 10 = 0$ from the origin using polar coordinates.

8. If the axes are rotated through an angle of 30° in the anticlockwise direction, then find the coordinates of point $(4, 2\sqrt{3})$ with respect to new axis.

9. Reflecting the point (2, –1) about Y-axis, if the coordinate axes are rotated at 45° angle in negative direction without shifting the origin, then find the new coordinates of the point.

10. Having coordinate axes parallel to xy axes, the origin is shifted to a point (1, –2). Then find the transformed equation of $x^2 + y^2 = 2$.

Answer Keys

1. $\left(\sqrt{13}, -\tan^{-1}\dfrac{3}{2}\right)\left(-\sqrt{13}, \pi - \tan^{-1}\dfrac{3}{2}\right)$

2. $r = 10 \operatorname{cosec}\theta$

3. (i) $(-2, 0)$ (ii) $\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$

4. (i) $\left(\sqrt{2}, \dfrac{-\pi}{4}\right)$ or $\left(\sqrt{2}, \dfrac{7\pi}{4}\right)$ (ii) $\left(5, \pi + \tan^{-1}\dfrac{4}{3}\right)$

5. $r^2(2 + \sin^2\theta) = 6$

6. $x^2 = 4y$

7. 2 units

8. $(\sqrt{3}, -5)$

9. $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right)$

10. $x^2 + y^2 + 2x - 4y + 3 = 0$

SECTION FORMULA

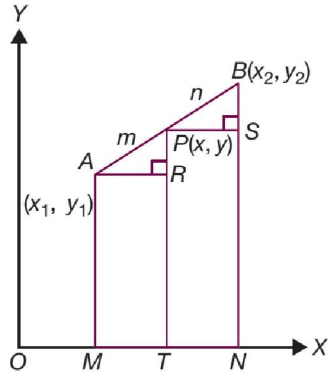
Internal Division

Let $P(x, y)$ be the point, which divides the line segment joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the given ratio $m{:}n$.

From A, B and P, draw AM, BN and PT perpendicular to x-axis. From A and P, draw AR and PS parallel to x-axis.

Then, $AR = x - x_1$, $PS = x_2 - x$

$$PR = y - y_1,\ BS = y_2 - y$$



From similar ΔAPR and PBS,

$$\frac{AR}{PS}=\frac{PR}{BS}=\frac{AP}{PB} \qquad (\because\ AP:PB=m:n)$$

or $$\frac{x-x_1}{x_2-x}=\frac{y-y_1}{y_2-y}=\frac{m}{n} \qquad \ldots(1)$$

From first and third term in relation (1), we have

$$\frac{x-x_1}{x_2-x}=\frac{m}{n} \text{ on solving, } x=\frac{mx_2+nx_1}{m+n}$$

Similarly, from second and third term in relation (1), we have $\dfrac{y-y_1}{y_2-y}=\dfrac{m}{n}$

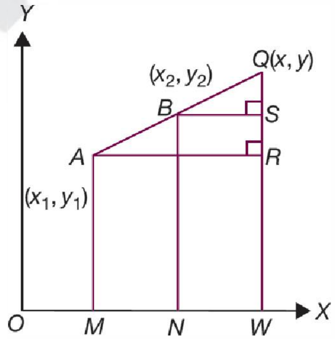
On solving, $y=\dfrac{my_2+ny_1}{m+n}$

Thus, the coordinates of $P(x, y)$

$$=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$

External Division

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points and let $Q(x, y)$ be the point, which divides AB externally in the ratio $m:n$. So that, $\dfrac{AQ}{QB}=\dfrac{m}{n}$.



From A, B and Q, draw AM, BN and QW perpendiculars on x-axis. Also, from A and B, draw AR and BS parallel to x-axis.

Then, from similar ΔAPR and BPS,

$$\frac{AQ}{QB}=\frac{AR}{BS}=\frac{QR}{QS}$$

or $$\frac{m}{n}=\frac{x-x_1}{x-x_2}=\frac{y-y_1}{y-y_2} \qquad \ldots(2)$$

From first and second term in relation (2), we have,

$$\frac{m}{n}=\frac{x-x_1}{x-x_2}$$

$$\Rightarrow \quad mx-mx_2=nx-nx_1 \text{ or } (m-n)x=mx_2-nx_1$$

$$\therefore \quad x=\frac{mx_2-nx_1}{m-n}$$

From first and third term in relation (2), we have,

$$\frac{m}{n}=\frac{y-y_1}{y-y_2}$$

$$\Rightarrow \quad (m-n)y=my_2-ny_1 \qquad \therefore\ y=\frac{my_2-ny_1}{m-n}$$

Thus, the coordinates of Q are (x, y)

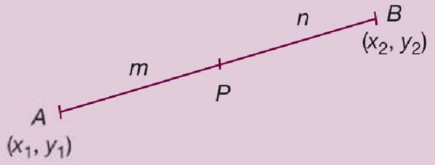
$$=\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)$$

Remember

Internal Division

Coordinates of a point, which divides the line segment joining two points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ in the ratio $m:n$ internally is given by,

$$P\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$



Corollary

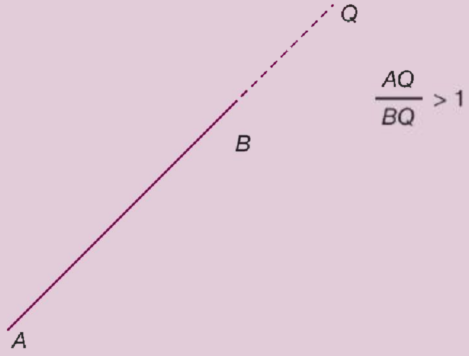
Coordinates of middle point: If P be the middle point of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then P will divide the line segment AB in the ratio 1:1.

$\therefore$ Coordinates of $P\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

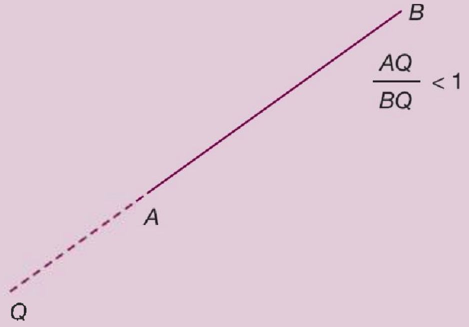
External Division

Coordinates of a point, which divides the line segment joining two points $A\,(x_1, y_1)$ and $B\,(x_2, y_2)$ in the ratio $m{:}n$ externally $Q\left(\dfrac{mx_2-nx_1}{m-n}, \dfrac{my_2-ny_1}{m-n}\right)$

$A-B-Q$ or $Q-A-B \Rightarrow Q$ divides line segment AB, externally.



$\dfrac{(AQ)}{(BQ)} > 1 \Rightarrow Q$ lies on the right of B



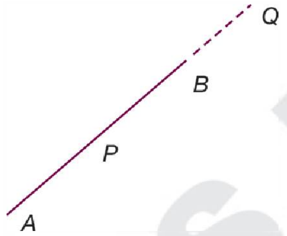
$\dfrac{(AQ)}{(BQ)} < 1 \Rightarrow Q$ lies on the left of A

Harmonic Conjugate

If P divides AB internally in the ratio $m{:}n$ and Q divides AB externally in the ratio $m{:}n$, then P and Q are said to be harmonic conjugate of each other with respect to AB.

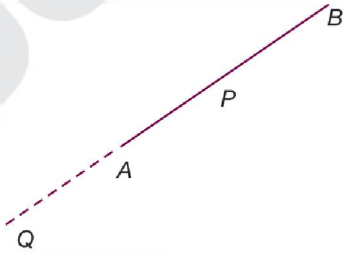
CASE-1: (when Q lies right of B)
Mathematically, AP, AB and AQ are in harmonic progression.

i.e., $\dfrac{2}{AB} = \dfrac{1}{AP} + \dfrac{1}{AQ}$



CASE-2: (when Q lies left of A)
Mathematically, BP, BA and BQ are in harmonic progression.

i.e., $\dfrac{2}{BA} = \dfrac{1}{BP} + \dfrac{1}{BQ}$



Working Rule

Coordinates of any point P on the line segment which divide the line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio of $\lambda{:}1$ is given by $P\left(\dfrac{x_1+\lambda x_2}{\lambda+1}, \dfrac{y_1+\lambda y_2}{\lambda+1}\right)$, $(\lambda \neq -1)$.

Line formed by joining points (x_1, y_1) and (x_2, y_2) is divided by

(a) X-axis in the ratio, make y-coordinate of point P as zero $\Leftrightarrow \lambda = \dfrac{-y_1}{y_2}$

(b) Y-axis in the ratio, then make x-coordinate of point P as zero $\Leftrightarrow \lambda = \dfrac{-x_1}{x_2}$

(c) Line $Ax+By+C=0$ in the ratio, where the point P satisfy with given line equation,

$$\lambda = -\left(\frac{Ax_1+By_1+C}{Ax_2+By_2+C}\right)$$

If λ is positive, then point P divides them internally and if λ is negative, then it divides them externally.

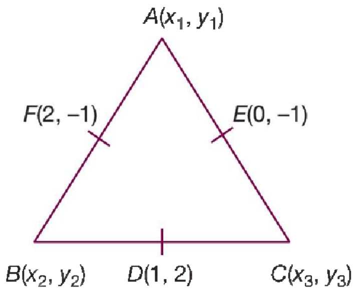
EBOOKS.IN

ILLUSTRATION 22

If the coordinates of the midpoints of the sides of a triangle are (1, 2), (0, –1) and (2, –1), then find the coordinates of its vertices.

SOLUTION

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC. Let $D(1, 2)$, $E(0, -1)$ and $F(2, -1)$ be the midpoints of sides BC, CA and AB, respectively.



Since D is the midpoint of BC,

$$\frac{x_2+x_3}{2}=1 \text{ and } \frac{y_2+y_3}{2}=2$$

$$\Rightarrow \quad x_2+x_3=2 \text{ and } y_2+y_3=4 \qquad \ldots(1)$$

Similarly, E and F are the midpoints of CA and AB, respectively.

$$\therefore \quad \frac{x_1+x_3}{2}=0 \text{ and } \frac{y_1+y_3}{2}=-1$$

$$\Rightarrow \quad x_1+x_3=0 \text{ and } y_1+y_3=-2 \qquad \ldots(2)$$

$$\text{and } \frac{x_1+x_2}{2}=2 \text{ and } \frac{y_1+y_2}{2}=-1$$

$$\Rightarrow \quad x_1+x_2=4 \text{ and } y_1+y_2=-2 \qquad \ldots(3)$$

From (1), (2) and (3), we get,

$$(x_2+x_3)+(x_1+x_3)+(x_1+x_2)=2+0+4$$

$$\text{and } (y_2+y_3)+(y_1+y_3)+(y_1+y_2)=4-2-2$$

$$\Rightarrow \quad x_1+x_2+x_3=3 \text{ and } y_1+y_2+y_3=0 \qquad \ldots(4)$$

On solving with equation (1), (2) and (3), we get,

$x_1 = 1$ and $y_1 = -4$; $x_2 = 3$ and $y_2 = 2$; $x_3 = -1$ and $y_3 = 2$

Hence, the vertices of the ΔABC are $A(1, -4)$, $B(3, 2)$ and $C(-1, 2)$.

ILLUSTRATION 23

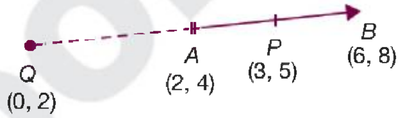
Find the coordinates of the point, which divides the line segment joining (2, 4) and (6, 8) in the ratio 1:3 internally and externally and verify their harmonic conjugate relation.

SOLUTION

Let given points are $A(2, 4)$ and $B(6, 8)$

Internal division $\quad P\left(\frac{6+6}{1+3},\frac{8+12}{1+3}\right)=P(3,5)$

External division $\quad Q\left(\frac{6-6}{1+3},\frac{8-12}{1-3}\right)=Q(0,2)$



Q lies left of A because $\frac{QA}{QB}=\frac{1}{3}<1$

$$BA = 4\sqrt{2}$$

$$BP = 3\sqrt{2}$$

$$BQ = 6\sqrt{2}$$

$$\frac{1}{BP}+\frac{1}{BQ}=\frac{1}{3\sqrt{2}}+\frac{1}{6\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

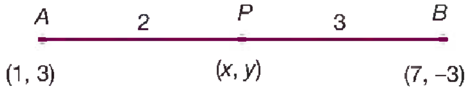
$$=\frac{2}{BA}$$

$\therefore$ BP, BA, BQ are in Harmonic Progression.

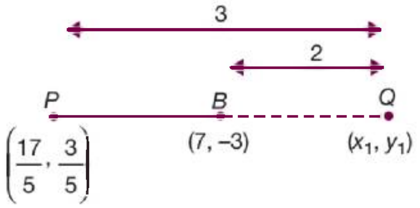
ILLUSTRATION 24

Given $A(1, 3)$ and $B(7, -3)$ points on xy-plane. A point P is taken on AB, which divides it internally in the ratio 2:3. A point Q divides PB externally in the ratio 3:2, then find the coordinates of point Q.

SOLUTION



For point P, $x=\frac{14+3}{5}=\frac{17}{5}$; $y=\frac{-6+9}{5}=\frac{3}{5}$



$$x_1=\frac{3\times 7-2\times\frac{17}{5}}{3-2}=\frac{105-34}{5};\ x_1=\frac{71}{5}$$

$$y_1=\frac{3\times(-3)-2\left(\frac{3}{5}\right)}{3-2};\ y_1=-9-\frac{6}{5}\Rightarrow y_1=\frac{-51}{5}$$

Hence, $Q\left(\frac{71}{5},\frac{-51}{5}\right)$

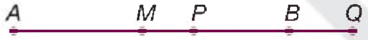
ILLUSTRATION 25

A line segment AB is divided internally and externally in the same ratio at P and Q respectively and M is the midpoint of AB. Then prove that MP, MB, MQ are in G.P.

SOLUTION

Let Q lie right of point A then

Clearly, we can say AP, AB and AQ are in H.P.



$$\frac{2}{AB}=\frac{1}{AP}+\frac{1}{AQ}$$

$(AP = AM + MP = MB + MP$ and

$AQ = AM + MQ = MB + MQ)$

$$=\frac{1}{MB+MP}+\frac{1}{MQ+MB}\qquad (AB=2MB)$$

$$\Rightarrow\quad \frac{1}{MB}-\frac{1}{MB+MP}=\frac{1}{MQ+MB}$$

$$\Rightarrow\quad MP(MQ+MB)=MB(MB+MP)$$

$$\Rightarrow\quad (MP)\,(MQ)=(MB)^2$$

Hence, MP, MB, MQ are in G.P.

ILLUSTRATION 26

If A and B are the points (–3, 4) and (2, 1), then find coordinates of the point C on AB produced such that $AC = 2BC$.

SOLUTION

Let the coordinates of point C be (p, q).

C divides AB externally in the ratio 2:1.

$$\therefore\quad p=\frac{2\times 2-(-3)(1)}{2-1}=7 \text{ and } q=\frac{2\times 1-(1)(4)}{1}=-2$$

Hence, the coordinates of points $C(7, -2)$.

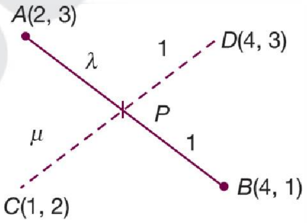
ILLUSTRATION 27

In what ratio is the line joining the point (2, 3) and (4, 1) divides the segment joining the points (1, 2) and (4, 3).

SOLUTION

Let the points be $A(2, 3)$, $B(4, 1)$, $C(1, 2)$ and $D(4, 3)$.

Let P divides AB in λ:1 and CD in μ:1.



Then the coordinates of P are

$$\left(\frac{4\lambda+2}{\lambda+1},\frac{\lambda+3}{\lambda+1}\right)=\left(\frac{4\mu+1}{\mu+1},\frac{3\mu+2}{\mu+1}\right)$$

$$\frac{4\lambda+2}{\lambda+1}=\frac{4\mu+1}{\mu+1}\ \text{and}\ \frac{\lambda+3}{\lambda+1}=\frac{3\mu+2}{\mu+1}$$

$$2+\frac{2\lambda}{\lambda+1}=1+\frac{3\mu}{\mu+1}\ \text{and}\ 1+\frac{2}{\lambda+1}=2+\frac{\mu}{\mu+1}$$

Add both relations we get,

$$3+2=3+\frac{4\mu}{\mu+1}\Rightarrow\mu=1$$

Hence, AB divided CD in 1:1 ratio.

Alternate method

Let P divides CD in μ:1.

$$P\left(\frac{4\mu+1}{\mu+1},\frac{3\mu+2}{\mu+1}\right)$$

Clearly P lies on AB $x + y - 5 = 0$ (it is the equation of AB, we will learn it in later section)

$$\therefore\quad \frac{4\mu+1}{\mu+1}+\frac{3\mu+2}{\mu+1}-5=0\Rightarrow\mu=1$$

Hence, the required ratio is μ:1, i.e., 1:1.

ILLUSTRATION 28

Two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are chosen on the graph of $f(x) = \ell n\, x$ with $0 < x_1 < x_2$. The points C and D trisect the line segment AB with $AC < CB$. Through C, a horizontal line is drawn to cut the curve at $E(x_3, y_3)$. If $x_1 = 1$ and $x_2 = 1000$, then find the value of x_3.

SOLUTION

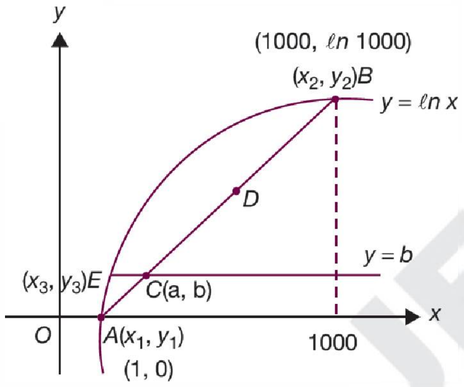
Using section formula,

$$a = \frac{2\cdot 1 + 1\cdot 1000}{3} = 334$$

$$b = \frac{2\cdot 0 + 1\cdot \ell n 1000}{3} \Rightarrow b = \frac{\ell n 1000}{3} \qquad \ldots(1)$$

Now the line $y = b$ intersects the curve $y = \ell n\, x$.

$$\therefore \quad b = \ell n\, x \qquad \ldots(2)$$



From (1) and (2),

$$\frac{\ell n 1000}{3} = \ell n\, x$$

$$\Rightarrow \quad \ell n\,(1000)^{\frac{1}{3}} = \ell n\, x$$

$$\therefore \quad x = (1000)^{\frac{1}{3}} = 10$$

AREA OF TRIANGLE

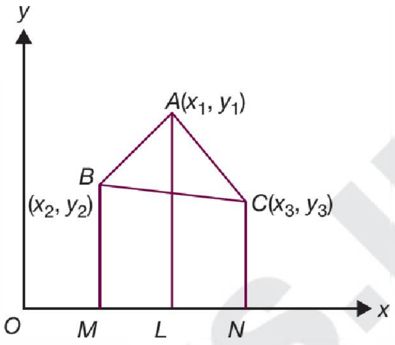
Let the vertices of the triangle be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Through A, B and C draw AL, BM and CN perpendiculars on x-axis.

Then,

$$ML = OL - OM = x_1 - x_2$$

$$LN = ON - OL = x_3 - x_1$$

$$MN = ON - OM = x_3 - x_2$$



Now, area of ΔABC

= Area of quadrilateral $BMLA$ + Area of quadrilateral $ALNC$ – Area of quadrilateral $BMNC$

$$= \frac{1}{2}(BM + AL)\cdot ML + \frac{1}{2}(AL + CN)\cdot LN - \frac{1}{2}(BM + CN)\cdot MN$$

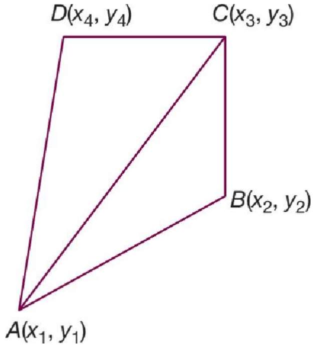
$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

AREA OF A QUADRILATERAL

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ be the vertices of a quadrilateral $ABCD$ taken in anticlockwise order.



Area of quadrilateral $ABCD$

$$= \text{Area of } \Delta ABC + \text{Area of } \Delta ACD$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$+ \frac{1}{2}[x_1(y_3 - y_4) + x_3(y_4 - y_1) + x_4(y_1 - y_3)]$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

$$+ \frac{1}{2}(x_1y_3 - x_3y_1 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4)$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4$$

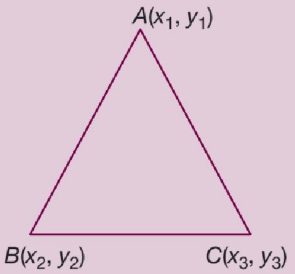
$$- x_4y_3 + x_4y_1 - x_1y_4)$$

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}\right]$$

Remember

Area of Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle.



$$\text{Then, area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

Alternate method

To remember the above formula, take the help of the **Stair Method**.

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$= \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

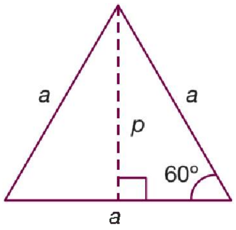
Remarks:

(i) If the area of triangle joining three points is zero, then the points are collinear.

Area of Equilateral Triangle

(ii) If 'a' be the side of equilateral triangle, then its area $= \frac{\sqrt{3}}{4}a^2$

PROOF:



$$\text{Area of } \Delta = \frac{1}{2}a \cdot a \sin 60° = \frac{\sqrt{3}}{4}a^2$$

(iii) If altitude of any equilateral triangle is p, then its area $= \frac{p^2}{\sqrt{3}}$

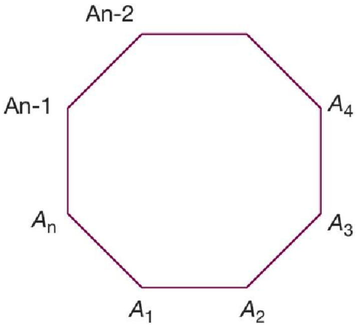
PROOF:

$p = a \sin 60°$

$$\text{Area of } \Delta = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\left(\frac{2p}{\sqrt{3}}\right)^2 = \frac{p^2}{\sqrt{3}}$$

AREA OF A PLANE POLYGON

Let there are n points. In order to find out the area of polygon formed by these n points, **first locate them on xy-plane** and then mark them as let A_1, A_2 .... A_n.



Now area of polygon is given by

$$=\left|\frac{1}{2}\left(\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}+\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}+\dots\right.\right.$$

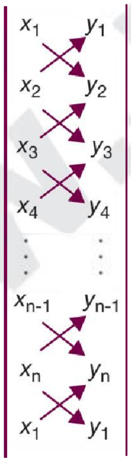
$$\left.\left.+\begin{vmatrix}x_{n-1} & y_{n-1}\\ x_n & y_n\end{vmatrix}+\begin{vmatrix}x_n & y_n\\ x_1 & y_1\end{vmatrix}\right)\right|$$

$$=\left|\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\dots\dots\dots\right.$$

$$\left.+(x_ny_1-x_1y_n)]\right|$$

Alternate method

We can also take the help of the **'Stair method'**, after locating all points on XY plane.

Area of polygon $=\frac{1}{2}$ 

$$=\left|\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\dots\dots\dots\right.$$

$$\left.+(x_ny_1-x_1y_n)]\right|$$

NOTES:

(a) By analysing the area expression of triangle and quadrilateral, we can conclude that relative positions of these points are very crucial to get correct area using this formula. Hence, if we mark these points arbitrarily, then we might not get correct area. So **it is must to locate these points first and then mark $A_1, A_2 \dots\dots\dots A_n$.**

(b) When point $A_1, A_2 \dots\dots\dots A_n$ are taken in anti-clockwise order, the area comes out to be positive otherwise for clockwise order it is negative. However, we consider area as positive quantity for calculation so we take absolute value of area.

ILLUSTRATION 29

Show that the area of the triangle with vertices $[(a+1)(a+2), (a+2)]$, $[(a+2)(a+3), (a+3)]$ and $[(a+3)(a+4), (a+4)]$ is independent of a.

SOLUTION

The area of the given triangle is

$$\Delta=\frac{1}{2}\begin{vmatrix}(a+1)(a+2) & (a+2) & 1\\ (a+2)(a+3) & (a+3) & 1\\ (a+3)(a+4) & (a+4) & 1\end{vmatrix}$$

(We are using elementary row transformation concept from determinant chapter)

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get,

$$\Delta=\frac{1}{2}\begin{vmatrix}(a+1)(a+2) & a+2 & 1\\ 2(a+2) & 1 & 0\\ 2(a+3) & 1 & 0\end{vmatrix}=|-1|=1, \text{which is}$$

independent of a.

ILLUSTRATION 30

Find the area of the pentagon, whose vertices taken in order are (4, 3), (–5, 6), (–7, –2), (0, –7) and (3, –6).

SOLUTION

Let $A = (4, 3)$, $B = (-5, 6)$, $C = (-7, -2)$, $D = (0, -7)$, $E = (3, -6)$ be the vertices of the pentagon.

Area of ΔABC

$$=\frac{1}{2}\,|4\,(6+2)+(-5)\,(-2-3)+(-7)\,(3-6)|$$

$$=\frac{1}{2}\,|32+25+21|=39 \text{ sq. units.}$$

Area of ΔACD

$$=\frac{1}{2}\,|4\,(-2+7)+(-7)\,(-7-3)+0\,(3+6)|$$

$$=\frac{1}{2}\,|20+70+0|=45 \text{ sq. units.}$$

Area of ΔADE

$$= \frac{1}{2}\,|\,4\,(-7+6)+0\,(-6-3)+3\,(3+7)\,|$$

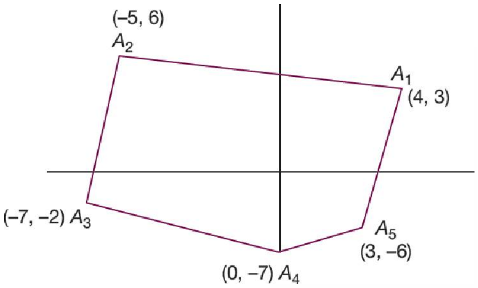
$$= \frac{1}{2}\,|\,-4+0+30\,|\; = 13 \text{ sq. units.}$$

$\therefore$ Area of pentagon $ABCDE = 39 + 45 + 13$

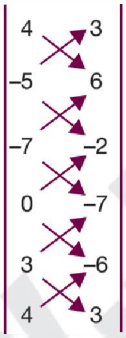
$= 97$ sq. units

Alternate

Before using direct formula, first we have to locate points on argand plane first and then use stair method for calculating area.



Area of pentagon $= \dfrac{1}{2}$



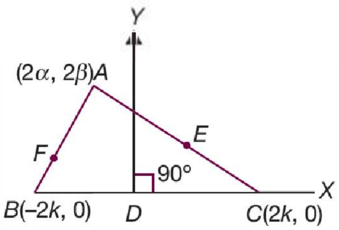
$$= \frac{1}{2}\,(24 + 15 + 10 + 42 + 49 + 0 + 21 + 9 + 24)$$

$$= \frac{1}{2}\,(194) = 97 \text{ sq. units}$$

ILLUSTRATION 31

Prove that the area of a triangle is four times the area of the triangle formed by joining the midpoints of its sides.

SOLUTION



Let ABC be a triangle.

Let D be the middle point of BC and $BC = 4k$.

We take D as the origin and DC as x-axis.

Then $B = (-2k, 0)$, $C = (2k, 0)$

Let $A = (2\alpha, 2\beta)$

Let E and F be the middle points of AC and AB, respectively.

Then $D = (0, 0)$, $E = (\alpha + k, \beta)$, $F = (\alpha - k, \beta)$

$$\text{Now } \Delta ABC = \left|\frac{1}{2}[2\alpha(0-0) - 2k(0-2\beta) + 2k(2\beta - 0)]\right|$$

$$= \frac{1}{2}\,|8k\beta| = 4\,|k\beta| \qquad \ldots(1)$$

Area of ΔDEF

$$= \left|\frac{1}{2}[0(\beta-\beta) + (\alpha+k)(\beta-0) + (\alpha-k)(0-\beta)]\right|$$

$$= \left|\frac{1}{2}[(\alpha+k)\beta - (\alpha-k)\beta]\right| = |k\beta| \qquad \ldots(2)$$

From (1) and (2), it follows that area of $\Delta ABC = 4$ Area of ΔDEF. Hence proved

ILLUSTRATION 32

The point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio k:1. Find the values of k, for which the area of ΔABC, where B (1, 5), C (7, 2) is 2 sq. units.

SOLUTION

Coordinates of A, dividing the join of $P = (-5, 1)$ and $Q = (3, 5)$ in the ratio k:1 are given by $\left(\dfrac{(3k-5)}{k+1}, \dfrac{(5k+1)}{k+1}\right)$

Also, the area of the Δ ABC is given by

$$\Delta = \left|\frac{1}{2}\sum x_1(y_2 - y_3)\right|$$

$$= \frac{1}{2}\left|\left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right]\right|$$

$$= \left|\frac{1}{2}\left\{\frac{3k-5}{k+1}(3) + \left(2 - \frac{5k+1}{k+1}\right) + 7\left(\frac{5k+1}{k+1} - 5\right)\right\}\right| = 2$$

$$\left\{\frac{9k - 15 + 30k + 6}{k+1} - 33\right\} = \pm 4$$

By taking + we get, $k = 23$ and taking – we get, $k = \dfrac{19}{5}$.

ILLUSTRATION 33

If the coordinates of two points A and B are (3, 4) and (5, –2), respectively. Find the coordinates of any point P, if $PA = PB$ and area of $\Delta PAB = 10$.

SOLUTION

Let the coordinates of P be (x, y). Then,

$$PA = PB \Rightarrow PA^2 = PB^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$$

$$\Rightarrow \quad x - 3y - 1 = 0 \qquad \ldots(1)$$

Now, area of $\Delta PAB = 10$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10 \Rightarrow 6x + 2y - 26 = \pm 20$$

$$\Rightarrow \quad 6x + 2y - 46 = 0 \text{ or } 6x + 2y - 6 = 0$$

$$\Rightarrow \quad 3x + y - 23 = 0 \text{ or } 3x + y - 3 = 0 \qquad \ldots(2)$$

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get, $x = 7, y = 2$.

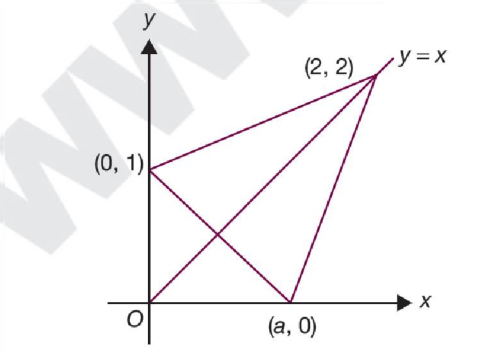
Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get, $x = 1$, $y = 0$.

Thus, the coordinates of P are (7, 2) or (1, 0).

ILLUSTRATION 34

A triangle has two of its vertices at (0, 1) and (2, 2) in the Cartesian plane. Its third vertex lies on the x-axis. If the area of the triangle is 2 sq. units, then find the sum of the possible abscissae of the third vertex.

SOLUTION



$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ a & 0 & 1 \end{vmatrix} = 2; \quad \therefore \begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ a & -1 & 1 \end{vmatrix} = 4$$

$$\Rightarrow \quad |a + 2| = 4; a = 2 \text{ or } a = -6$$

Sum of the abscissa = –4

ILLUSTRATION 35

Find the area of the quadrilateral $ABCD$ with vertices $A(-2, 0)$, $B(0, -4)$, $C(4, -2)$ and $D(2, 2)$.

SOLUTION

$$AB = BC = CD = DA = \sqrt{20}$$

It is a square of side $\sqrt{20}$ hence Area = 20

ILLUSTRATION 36

If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.

SOLUTION

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, x_3)$ be a triangle and $x_1, x_2, x_3, y_1, y_2, y_3$ be integers.

$\therefore BC^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2$ is a positive integer.

If the triangle is equilateral, then $AB = BC = CA = a$(say) and $\angle A = \angle B = \angle C = 60°$.

$\therefore$ Area of the triangle $= \left(\dfrac{\sqrt{3}}{4}\right)a^2$, which is irrational.

$\because a^2$ is a positive integer.

Now, the area of the triangle in terms of the coordinates

$$= \left(\frac{1}{2}\right)[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)],$$

which is a rational number.

This contradicts that the area is an irrational number, if the triangle is equilateral.

ILLUSTRATION 37

Given a ΔABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_2, y_3)$. Let there exists a point $P(a, b)$ such that $6a = 2x_1 + x_2 + 3x_3$, $6b = 2y_1 + y_2 + 3y_3$

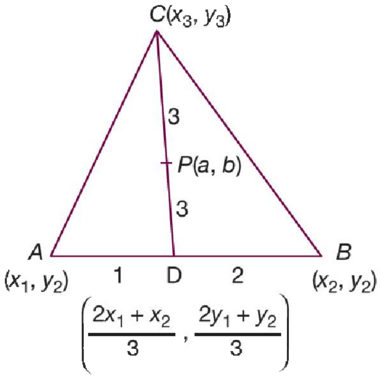
Show that:

(i) Area of ΔPBC must be less than the area of ΔABC.

(ii) P lies inside the ΔABC.

SOLUTION

We are trying to trace given point $P = \left(\dfrac{2x_1 + x_2 + 3x_3}{6}, \dfrac{2y_1 + y_2 + 3y_3}{6}\right)$ from points ABC



Point D divides AB in the ratio 1:2.
P divides CD in the ratio 1:1.
Hence, P lies inside the triangle.
$\therefore$ Area of $\Delta PBC <$ Area of ΔABC

ILLUSTRATION 38

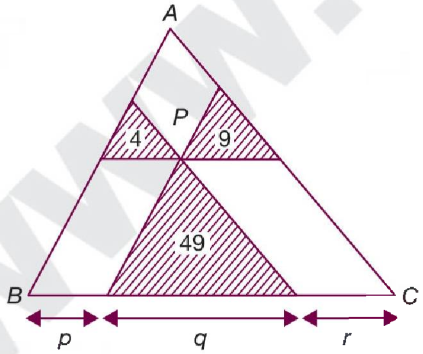
P is a point inside the ΔABC. Lines are drawn through P, parallel to the sides of the triangle. The three resulting triangles with the vertex at P have areas 4, 9 and 49 sq. units. Then find the area of ΔABC.

SOLUTION

From result of similar triangles, we get

$$\frac{49}{\Delta} = \frac{q^2}{a^2} \text{ or } \frac{7}{\sqrt{\Delta}} = \frac{q}{a} \quad \ldots(1)$$

Similarly, $\dfrac{2}{\sqrt{\Delta}} = \dfrac{p}{a}$ …(2)



and $\dfrac{3}{\sqrt{\Delta}} = \dfrac{r}{a}$ …(3)

Side $BC = a$

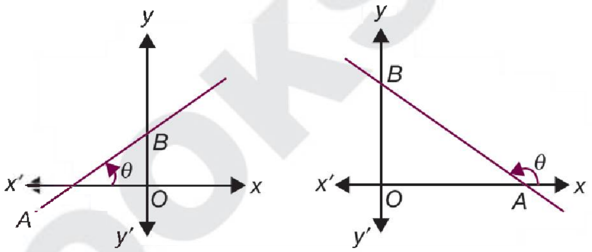
$\therefore$ (1) + (2) + (3) gives $\dfrac{12}{\sqrt{\Delta}} = \dfrac{p+q+r}{a} = 1$

$\Rightarrow$ $\Delta = 144$

SLOPE OF A LINE

Slope of a line means the **inclination of the line with the positive direction of x-axis**. Thus, the trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called **slope or gradient** of a line denoted by (m).

Let the line AB make an angle θ with the x-axis. Then the gradient of line $AB = m = \tan\theta$.



NOTES:

(i) Slope of x-axis or a line parallel to x-axis is zero ($m = \tan 0° = 0$).

(ii) Slope of y-axis or a line parallel to y-axis is not defined ($m = \tan 90°$ (not defined)).

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points, then the slope between A and B is defined as,

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta : 0 \le \theta < \pi;\ \theta \ne \frac{\pi}{2}$$

Where, θ is the angle of inclination of the line joining A and B with the positive direction of x-axis.

Slope of line $ax + by + c = 0$ is $\left(-\dfrac{a}{b}\right)$

Two lines with slopes m_1 and m_2 are:

(i) Parallel, if $m_1 = m_2$.

(ii) Perpendicular, if $m_1 m_2 = -1$.

ILLUSTRATION 39

Show that the line joining the points (2, –3) and (–5, 1) is:

(i) Parallel to the line joining (7, –1) and (0, 3).
(ii) Perpendicular to the line joining (4, 5) and (0, –2).

SOLUTION

(i) Let $A = (2, -3)$, $B = (-5, 1)$, $C = (7, -1)$, $D = (0, 3)$

$$m_1 = \text{Slope of } AB = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-3-1}{2+5} = -\frac{4}{7}$$

$$m_2 = \text{Slope of } CD = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-1-3}{7-0} = -\frac{4}{7}$$

Since $m_1 = m_2$, $AB \,||\, CD$.

(ii) Let $A = (2, -3)$, $B = (-5, 1)$

$E = (4, 5)$, $F = (0, -2)$

$$m_1 = \text{Slope of } AB = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-3-1}{2+5} = -\frac{4}{7}$$

$$m_2 = \text{Slope of } EF = \frac{y_1 - y_2}{x_1 - x_2} = \frac{5+2}{4-0} = -\frac{7}{4}$$

$$\text{Since, } m_1 m_2 = \left(-\frac{4}{7}\right)\left(\frac{4}{7}\right) = -1$$

Hence, $AB \perp EF$.

ILLUSTRATION 40

Consider a triangle, whose vertices are $A(-2, 1)$, $B(1, 3)$ and $C(3x, 2x - 3)$, where x is a real number. Then show that:

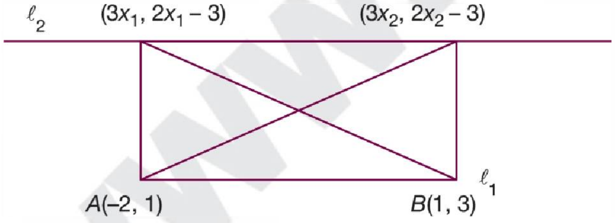
(i) The vertex C of the ΔABC always moves on a line parallel to the base AB.

(ii) The area of the ΔABC is independent of x.

SOLUTION

(i) $m_{AB} = \dfrac{3-1}{1+2} = \dfrac{2}{3}$

For moving C, let there are two positions C_1 and C_2



$$m_{\ell_1} = \frac{2(x_2 - x_1)}{3(x_2 - x_1)} = \frac{2}{3};$$

(ii) $A = \dfrac{1}{2}\begin{vmatrix} -2 & 1 & 1 \\ 1 & 3 & 1 \\ 3x & (2x-3) & 1 \end{vmatrix}$

$$= \frac{1}{2}(3x\,(1-3) - (2x-3)\,(-2-1) - 7)$$

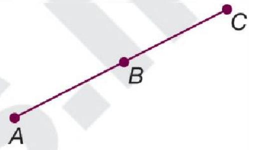
$$= -8$$

CONDITION FOR COLLINEARITY OF THREE POINTS

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear. Then,

Method-I

Area of $\Delta ABC = 0$



$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Method-II

As these points are collinear, they will not follow triangular inequality. Hence,

$$AC = AB + BC.$$

Method-III

As these points are collinear, their slopes are equal.

Hence, $m_{AB} = m_{BC} = m_{CA}$

$$\Rightarrow \quad \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3} = \frac{y_3 - y_1}{x_3 - x_1}$$

ILLUSTRATION 41

Consider the points $P(2, -4)$, $Q(4 - 2)$ and $R = (7, 1)$. Show that the points P, Q, R are collinear.

SOLUTION

$$PQ = 2\sqrt{2};\ QR = 3\sqrt{2};\ PR = 5\sqrt{2} \Rightarrow PQ + QR = PR$$

Alternate method

$$\text{Area formed by } \Delta ABC = \begin{vmatrix} 2 & -4 \\ 4 & -2 \\ 7 & 1 \\ 2 & -4 \end{vmatrix}$$

$$= -4 + 16 + 4 + 14 - 28 - 2 = 0$$

Hence, they are collinear.

ILLUSTRATION 42

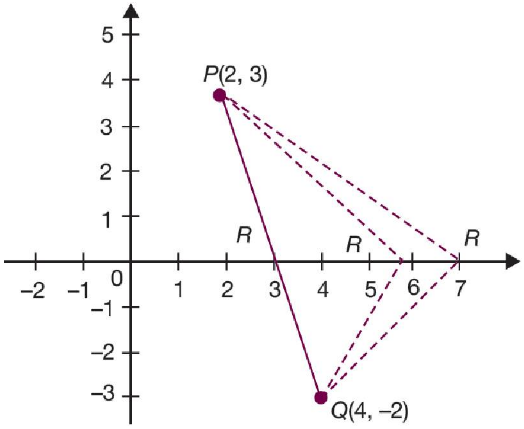
Given three points $P(2, 3)$, $Q(4, -2)$ and $R(\alpha, 0)$.

(i) Find the value of α, if $PR + RQ$ is minimum.

(ii) Find the value of α, if $|PR - RQ|$ is maximum.

SOLUTION

(i) Point $R(\alpha, 0)$ lies on x-axis.



Now, from triangular inequality, $PR + RQ \geq PQ$. Thus, minimum value of $PR + RQ$ is PQ, which occurs when points P, Q and R are collinear.

$$\Rightarrow \left|\begin{array}{cc} 2 & 3 \\ 4 & -2 \\ \alpha & 0 \\ 2 & 3 \end{array}\right| = 0$$

$$\Rightarrow -4 - 12 + 2\alpha + 3\alpha = 0$$

$$\Rightarrow \alpha = \frac{16}{5}$$

(ii)

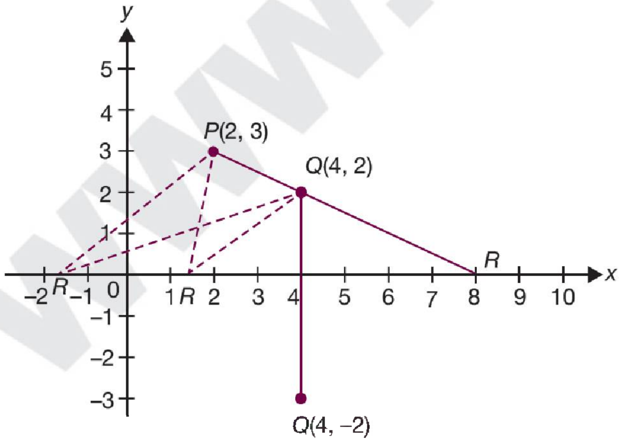


Image of Q in x-axis is $Q'(4, 2)$.

Now, $RQ = RQ'$

$\therefore \quad |PR - RQ| = |PR - RQ'|$

From triangular inequality,

$|PR - RQ'| < PQ'$.

$\therefore \quad |PR - RQ'|_{\max} = PQ'$, when P, Q' and R are collinear.

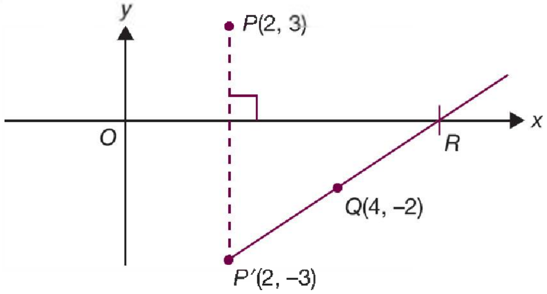
$$\Rightarrow \left|\begin{array}{cc} 2 & 3 \\ 4 & 2 \\ \alpha & 0 \\ 2 & 3 \end{array}\right| = 0$$

$$\Rightarrow 4 - 12 - 2\alpha + 3\alpha = 0$$

$$\Rightarrow \alpha = 8 \quad \therefore \text{ point } R(8, 0)$$

Alternate method

We can also take the image of point P with respect to x-axis as P'.



$PR = P'R$

$|PR - RQ| = |P'R - RQ| \leq P'Q$

$$P', Q, R \text{ are collinear } \left|\begin{array}{cc} 2 & -3 \\ 4 & -2 \\ \alpha & 0 \\ 2 & -3 \end{array}\right| = 0$$

$$\Rightarrow \quad -4 + 12 + 2\alpha - 3\alpha = 0$$

$$\Rightarrow \quad \alpha = 8 \qquad \therefore \text{ point } R(8, 0)$$

Test Your Concepts-III

(Solutions on page H.5)

1. Using distance formula, show that the points (1, 5), (2, 4) and (3, 3) are collinear.
2. Find the maximum value of $y=\sqrt{(x-3)^2+(x^2-2)^2}-\sqrt{x^2+(x^2-1)^2}$.
3. Use distance formula to show that the points ($\text{cosec}^2\theta$, 0), (0, $\sec^2\theta$) and (1, 1) are collinear.
4. $A(1, 1)$ and $B(2, -3)$ are two points and P is a point on AB produced such that $AP = 3AB$. Find the coordinates of P.
5. The coordinates of A, B, C, D are (6, 3), (-3, 5), (4, –2) and (x, $3x$), respectively. If $\dfrac{\Delta DBC}{\Delta ABC}=\dfrac{1}{2}$, find x.
6. Find the coordinates of the points, which trisect the line segment joining the points (2, 3) and (6, 5).
7. Prove that the quadrilateral whose vertices are $A(-2, 5)$, $B(4, -1)$, $C(9, 1)$ and $D(3, 7)$ is a parallelogram. If E divides AC in the ratio 2:1, prove that D, E and the midpoint F of BC are collinear.
8. In what ratio does the x-axis divides the line segment joining the points (2, –3) and (5, 6).
9. If $(t, t-2)$, $(t+3, t)$ and $(t+2, t+2)$ are the vertices of a triangle, show that its area is independent of t.
10. Show that the points $(a, 0)$, $(0, b)$ and (1, 1) are collinear, if $\dfrac{1}{a}+\dfrac{1}{b}=1$.

Answer Keys

2. $\sqrt{10}$ **4.** (4, –11) **5.** $x=\dfrac{11}{8}, x=-\dfrac{3}{8}$ **6.** $\left(\dfrac{10}{3},\dfrac{11}{3}\right),\left(\dfrac{14}{3},\dfrac{13}{3}\right)$

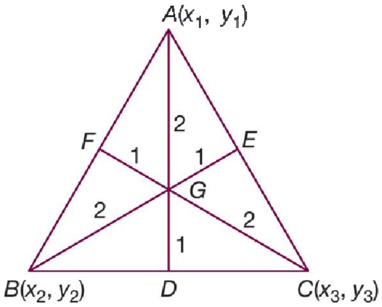
8. internally in the ratio 1 : 2

COORDINATES OF SOME PARTICULAR POINTS

The coordinates of some particular points can be understood by considering a ΔABX with vertices—$A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Centroid (G)

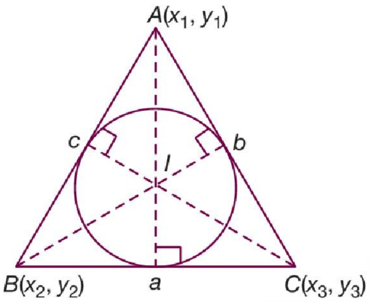
- The centroid is the point of intersection of the medians (line joining the midpoint of sides and opposite vertices). Therefore, centroid divides the median in the ratio 2:1.
  (**Median**: line joining vertex to midpoint of opposite side)
- Centroid is the point which lies inside the triangle and divides the area of triangle in three equal parts on joining it to all three vertices.
- Area of ΔGAB = Area of ΔGBC = Area of ΔGCA $=\dfrac{1}{3}$ Area of ΔABC.



- Three medians divide ΔABC in 6 equal areas.
- Area of ΔGBD = Area of ΔGDC = Area of ΔGCE = Area of ΔGEA = Area of ΔGAF = Area of ΔGFB $=\dfrac{1}{6}$ Area of ΔABC.
- Area of triangle formed by midpoints of sides $\Delta DEF=\dfrac{1}{4}$ Area of ΔABC.
- Coordinates of G can be calculated using, $\left(\dfrac{x_1+x_2+x_3}{3},\dfrac{y_1+y_2+y_3}{3}\right)$

Incentre (*I*)

- The incentre of the triangle is the point of intersection of internal bisector of the angles.
- It is a centre of a circle touching all three sides of a triangle.
- A point lying inside the triangle and equidistant from all sides.



Incentre can be calculated using, $r = \frac{\Delta}{s}$ where Δ is area of triangle ABC

PROOF:

$$\text{Area of } \Delta ABC = \text{Area of } (\Delta IAB + \Delta IBC + ICA)$$

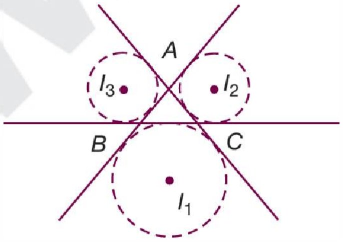
$$\Delta = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br$$

$$\Delta = \left(\frac{a+b+c}{2}\right)r = sr \Rightarrow \boxed{r = \frac{\Delta}{s}}$$

Excentres (I_1, I_2, I_3)

For Excentre (I_1)

- The centre of the circle which touches the side BC and the extended portions of sides AB and AC is called the excentre of ΔABC with respect to the vertex A.

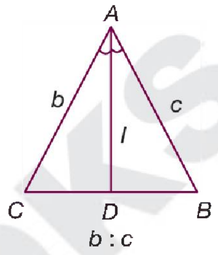


- The excentre I_1 of the triangle is the point of concurrency of internal bisector of the angle A and exterior angle bisector of angle B and angle C.
- Excentres lie outside the triangle and equidistant from all sides.

NOTE:

(i) Angle bisector divides the opposite sides in the ratio of remaining sides.
It means point D divides CB in b:c ratio internally.

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$

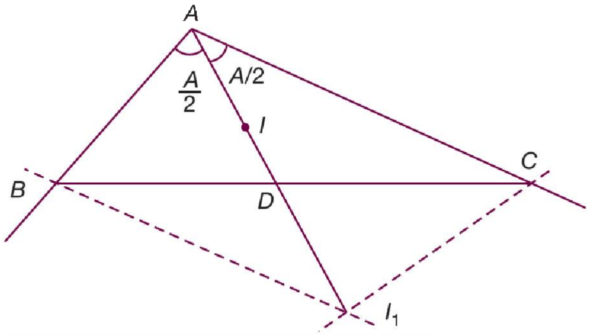


(ii) Incentre (I) divides the angle bisectors AD in the ratio $(b + c)$:a.

(iii) Excentre (I_1) divides the angle bisectors AD in the ratio $(b + c)$:a.

(iv) Incentre (I) and Excentre (I_1) are harmonic conjugate of each other with respect to angle bisector AD. Hence AI, AD, AI_1 are in harmonic progression.

Assume a, b, c as the sides of ΔABC.



Coordinates of Incentre,

$$I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Coordinates of Excentre,

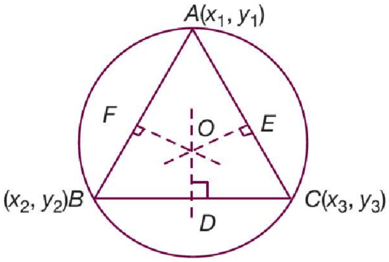
$$I_1\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

$$I_2\left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right),$$

$$I_3\left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

Circumcentre

- It is the point of concurrency of perpendicular bisectors of the sides of the triangle.
- It is also the centre of a circumcircle passing through the vertices of the triangle.
- A point which is equidistant from all three vertices of triangle.



If O is the circumcentre of any ΔABC, then

$$OA^2 = OB^2 = OC^2.$$

- For equilateral triangle, centroid is circumcentre.
- For right angled triangle, midpoint of hypotenuse is circumcentre.
- For scalene triangle, we need to find out circumcentre.

Method-I

If $O\ (x, y)$ be circumcentre of ΔABC, where $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$OA^2 = OB^2 = OC^2$$

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2 = (x - x_3)^2 + (y - y_3)^2$$ on solving, it gives (x, y).

Method-II

If D, E and F are midpoints of BC, AC and AB respectively, then

$$\left.\begin{array}{l}\text{slope of } BC \times \text{Slope of } OD = -1\\ \text{slope of } CA \times \text{Slope of } OE = -1\\ \text{slope of } AB \times \text{Slope of } OF = -1\end{array}\right\}\begin{array}{l}\text{Solving any two}\\ \text{we get } x \text{ and } y\end{array}$$

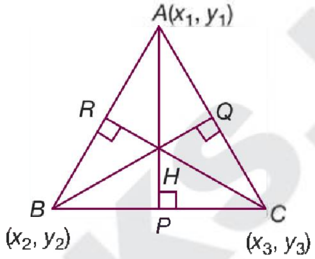
Method-III

Circumcentre is given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Orthocentre (H)

- It is the point of concurrency of the altitudes (perpendicular drawn from the vertices of the triangle on the opposite sides).



- For equilateral triangle, centroid is orthocentre.
- For right angled triangle, vertex which is right angled is orthocentre.
- For scalene triangle, we need to find out orthocentre.

Method-I

When vertices and the angles of the triangle are given, then orthocentre is given by,

$$H\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Method-II

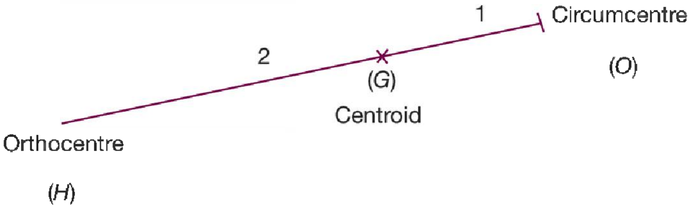
When $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given, we can get slope of sides AB, BC and CA.

$$\left.\begin{array}{l}\text{slope of } BC \times \text{Slope of } HA = -1\\ \text{slope of } CA \times \text{Slope of } HB = -1\\ \text{slope of } AB \times \text{Slope of } HC = -1\end{array}\right\}\begin{array}{l}\text{Now we can get}\\ \text{slopes of altitudes}\\ HA, HB, HC\end{array}$$

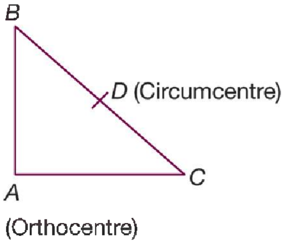
Now write any two altitude's equation and solve them to get orthocentre.

Relation Among Centres

- In case of equilateral triangle centroid, incentre, circumcentre and orthocentre of the triangle are same point.
- Orthocentre H, centroid G and circumcentre C of a triangle are collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2:1.



- If a triangle is right angle, then its circumcentre is the midpoint of hypotenuse and orthocentre is the point where right angle is formed, where D is circumcentre and vertex A is orthocentre.



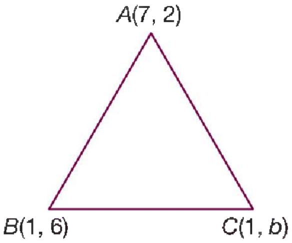
- In an isosceles triangle centroid, orthocentre, incentre, circumcentre are collinear.
- Image of orthocentre with respect to any side lies on circumcircle.
- Centroid and incentre always lie inside the triangle.

Type of Triangle	**Location of Circumcentre**	**Location of Orthocentre**
Acute angled triangle	Inside	Inside
Right angled triangle	On the Δ (Midpoint of hypotenuse)	On the Δ (At 90° vertex)
Obtuse angled triangle	Outside	Outside

ILLUSTRATION 43

If the two vertices of a triangle are (7, 2) and (1, 6) and its centroid is (4, 6), then the coordinate of the third vertex are (a, b), then find the value of $(a + b)$.

SOLUTION



Now $\dfrac{7+1+a}{3} = 4$ and $\dfrac{2+6+b}{3} = 6$

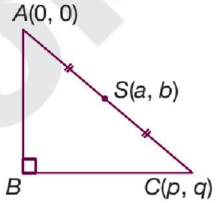
$a = 4$ and $b = 10 \Rightarrow a + b = 14$

ILLUSTRATION 44

The orthocentre of the ΔABC is 'B' and the circumcentre is 'S' (a, b). If A is the origin, then find the vertex C.

SOLUTION

It is given that orthocentre of triangle ABC is B, it means it is right angled at vertex B and circumcentre is mid point of hypotenuse.



Let the coordinates of C be (p, q); $\dfrac{p+0}{2} = a$ and $\dfrac{q+2}{2} = b$.

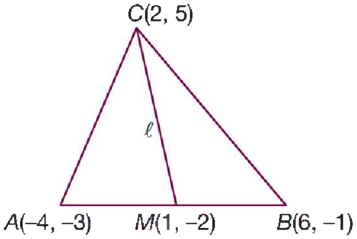
$p = 2a$ and $q = 2b$; $C(2a, 2b)$

ILLUSTRATION 45

Let ΔABC have vertices $A(-4, -3)$, $B(6, -1)$ and $C(2, 5)$. Find the length of the median from C to AB.

SOLUTION

Midpoint of AB is $M(1, -2)$



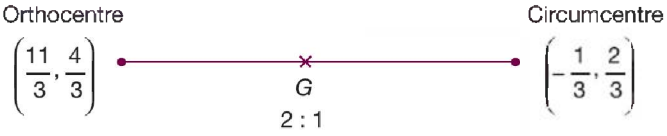
Length of median $\ell = \sqrt{(2-1)^2 + 7^2} = \sqrt{50}$

ILLUSTRATION 46

If in ΔABC, A (1, 10), circumcentre $\left(-\dfrac{1}{3}, \dfrac{2}{3}\right)$ and orthocentre $\left(\dfrac{11}{3}, \dfrac{4}{3}\right)$, then find the coordinates of midpoint of side opposite to A.

www.EEBOOKS.IN

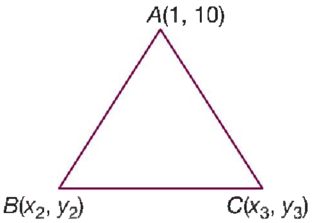
SOLUTION


Orthocentre
$\left(\frac{11}{3}, \frac{4}{3}\right)$
G
2 : 1
Circumcentre
$\left(-\frac{1}{3}, \frac{2}{3}\right)$

O, *G*, *C* are collinear. Get *G*.

$$G\left(\frac{2\left(-\frac{1}{3}\right)+\frac{11}{3}}{2+1}, \frac{2\left(\frac{2}{3}\right)+\frac{4}{3}}{2+1}\right)$$

$$G\left(1, \frac{8}{9}\right) = \left(\frac{1+x_2+x_3}{3}, \frac{10+y_2+y_3}{3}\right)$$



$$x_2 + x_3 = 2$$

$$y_2 + y_3 = -\frac{22}{3}$$

$$\text{Midpoint of } BC \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right) \equiv \left(1, -\frac{11}{3}\right)$$

ILLUSTRATION 47

Let H be the orthocentre of ΔABC, where $A(0, 0)$, $B(8, 0)$ and $C\,(4, 4\sqrt{3})$. If G_1, G_2, G_3, G_4, ….. are centroids of ΔAHB, ΔAG_1B, ΔAG_2B, ….., respectively, then find $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \text{Ar}(\Delta AG_iB)$.

SOLUTION

$\because$ ΔABC is equilateral.

$\therefore$ $H \to$ orthocentre or centroid

$$H = \left(4, \frac{4}{\sqrt{3}}\right); G_1 = \left(4, \frac{4}{3\sqrt{3}}\right); G_2 = \left(4, \frac{4}{9\sqrt{3}}\right)$$

$$\vdots \qquad \vdots$$

$$\vdots \qquad \vdots$$

$$G_n = \left(4, \frac{4}{3^n\sqrt{3}}\right)$$

$\therefore$ $\lim\limits_{n\to\infty}(A_1 + A_2 + \ldots A_n)$. (All triangles are isosceles)

$$= \frac{1}{2}\cdot 8\left[\frac{4}{3\sqrt{3}} + \frac{4}{9\sqrt{3}} + \ldots\ldots \frac{4}{3^n\sqrt{3}}\right]$$

$$= 4\cdot\frac{4}{3\sqrt{3}}\lim_{n\to\infty}\left[1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\frac{1}{3^{n-1}}\right]$$

$$= \frac{16}{3\sqrt{3}}\cdot\frac{1}{1-\frac{1}{3}} = \frac{8}{\sqrt{3}}$$

ILLUSTRATION 48

Let the points A, B and C are represented by $(a\cos\theta_i, a\sin\theta_i)$ $i = 1, 2, 3$ and $\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1) = \frac{-3}{2}$ then prove that orthocentre of ΔABC is at the origin.

SOLUTION

$$2\cos(\theta_1 - \theta_2) + 2\cos(\theta_2 - \theta_3) + 2\cos(\theta_3 - \theta_1) + 1 + 1 + 1 = 0$$

(using $\cos^2\theta + \sin^2\theta = 1$ and $\cos\,(A - B) = \cos A\cos B + \sin A\sin B$)

$$\begin{aligned}&(2\cos\theta_1\cos\theta_2 + 2\cos\theta_2\cos\theta_3\\&\quad + 2\cos\theta_3\cos\theta_1 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3)\\&\quad + (\sin^2\theta_1\,\sin^2\theta_2 + \sin^2\theta_3 + 2\sin\theta_1\sin\theta_2\\&\quad + 2\sin\theta_2\sin\theta_3 + 2\sin\theta_3\sin\theta_1) = 0\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad &(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)^2\\&\quad + (\sin\theta_1 + \sin\theta_2 + \sin\theta_3)^2 = 0\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad &\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0 \text{ and}\\&\quad \sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0\end{aligned}$$

Centroid $G(0, 0)$

As $OA = OB = OC$ it means origin is circumcentre also.

$\because$ Centroid and circumcentre of ΔABC is at origin.

$\therefore$ ABC is equilateral.

$\therefore$ Orthocentre of ABC is also origin.

Test Your Concepts-IV

(Solutions on page H.6)

1. If $A(1, 5)$, $B(-2, 1)$ and $C(4, 1)$ be the vertices of ΔABC and if the internal bisector of $\angle A$ meets BC at D, then find AD.
2. Find the centroid of the ΔABC, whose vertices are $A(9, 2)$, $B(1, 10)$ and $C(-7, -6)$. Also find the centroid of the triangle formed by joining these midpoints. Do the two triangles have same centroid?
3. Find the coordinates of the centre of the circle inscribed in the triangle, whose vertices are $(7, -36)$, $(7, 20)$ and $(-8, 0)$.
4. If P, Q and R divide the sides BC, CA and AB of ΔABC in the same ratio, then prove that the centroid of the triangles ABC and PQR coincide.
5. The coordinates of the centroid of a triangle and those of two of its vertices are $\left(\frac{2}{3}, 2\right)$, $(2, 3)$, $(-1, 2)$, respectively. Find the area of the triangle.
6. If G is the centroid of triangle with vertices $A(a, 0)$, $B(-a, 0)$ and $C(b, c)$, then find $\frac{AB^2 + BC^2 + CA^2}{GA^2 + GB^2 + GC^2}$

Answer Keys

1. 4 units
2. (1, 2); Yes.
3. $(0, -1)$
5. 9/2
6. 3

LOCUS OF POINT

It is the path or the curve traced by a moving point satisfying the given condition.

The equation to the locus of a point is the algebraic relation, which is satisfied by the coordinates of every point on the locus of the point.

To find the locus of a point:

I. Assumes locus point say (h, k).
II. Write the given condition mentioned in question involving (h, k).
III. Eliminate the variable(s), if any.
IV. Replace $h \to x$ and $k \to y$.

Now relation between h and k is the locus of the point.

NOTE:
A point has two variables, hence in locus problems exactly one condition/logic should be given in question.

ILLUSTRATION 49

The extremities of a diagonal of a rectangle are (0, 0) and (4, 4). Then find the locus of the extremities of the other diagonal.

SOLUTION

Using Pythagoras, $AB^2 + BC^2 = AC^2$

$$h^2 + k^2 + (h-4)^2 + (k-4)^2 = 16 + 16$$

$$\Rightarrow \quad 2h^2 + 2k^2 + 32 - 8h - 8k = 32$$

$$\Rightarrow \quad h^2 + k^2 - 4h - 4k = 0$$

Locus of (h, k)

$h = x, k = y$

$$x^2 + y^2 - 4x - 4y = 0$$

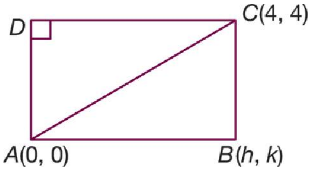
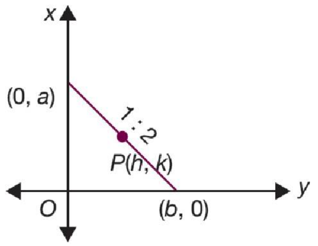


ILLUSTRATION 50

The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod, which divides it in the ratio 2:1.

SOLUTION

Suppose the two perpendicular lines are $x = 0$ and $y = 0$ and the intercepts a and b are cut respectively on the two lines. Then the two points on these lines are $(0, a)$ and $(b, 0)$. The point P has coordinates given by,



$$h = \frac{b+2.0}{2+1},\quad k = \frac{2.a+1.0}{2+1} \Rightarrow b = 3h \text{ and } a = \frac{3k}{2}$$

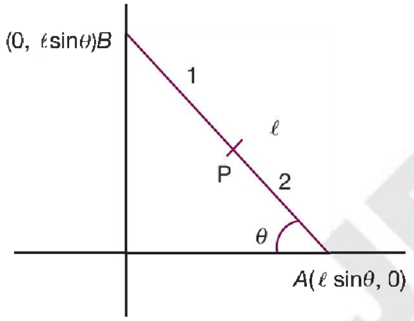
Also, $l^2 = a^2 + b^2$

$$l^2 = \left(\frac{3k}{2}\right)^2 + (3h)^2$$

Thus, the required locus is,

$$x^2 + \frac{y^2}{4} = \frac{l^2}{9}$$

Alternative solution



$$P(h, k) = \left(\frac{\ell\cos\theta + 2(0)}{1+2}, \frac{1(0) + 2(\ell\sin\theta)}{1+2}\right)$$

$$h = \frac{\ell\cos\theta}{3};\ k = \frac{2\ell\sin\theta}{3}$$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{9h^2}{\ell^2} + \frac{9k^2}{4\ell^2} = 1$$

$$\text{locus } h \to x;\ k \to y$$

$$x^2 + \frac{y^2}{4} = \frac{\ell^2}{9}$$

ILLUSTRATION 51

Find the locus of point $P(x, y)$, which is equidistant from the points $A(a+b, b-a)$ and $B(a-b, a+b)$.

SOLUTION

We have $PA = PB$, i.e., $(PA)^2 = (PB)^2$.

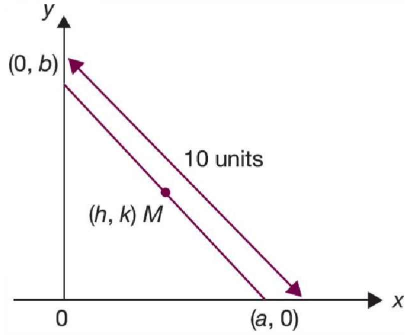
$$\Rightarrow \quad [x-(a+b)]^2 + [y-(b-a)]^2$$
$$= [x-(a-b)]^2 + [y-(a+b)]^2$$
$$\Rightarrow \quad [(x-a)-b]^2 + [(y-b)+a]^2$$
$$= [(x-a)+b]^2 + [(y-b)-a]^2$$
$$\Rightarrow \quad [(x-a)+b]^2 - [(x-a)-b]^2$$
$$= [(y-b)+a]^2 - [(y-b)-a]^2$$
$$\Rightarrow \quad 4b\,(x-a) = 4a(y-b) \Rightarrow bx = ay$$

ILLUSTRATION 52

A stick of length 10 units rests against the floor and the wall of a room. If the stick begins to slide on the floor, then find the locus of its middle point.

SOLUTION

Given $(a^2 + b^2) = 100$; $P(h, k)$ be the middle point. Then,



$$h = \frac{a}{2}, k = \frac{b}{2} \Rightarrow a = 2h, b = 2k$$

$$a^2 + b^2 = 100 \Rightarrow 4h^2 + 4k^2 = 100$$

$$\Rightarrow \quad h^2 + k^2 = 25$$

$\therefore$ The locus is $x^2 + y^2 = 25$.

ILLUSTRATION 53

ABC is a variable triangle with the fixed vertex $C(1, 2)$ and A, B having the coordinates $(\cos t, \sin t)$, $(\sin t, -\cos t)$ respectively, where t is a parameter. Find the locus of the centroid of the ΔABC.

SOLUTION

Let $G(\alpha, \beta)$ be the centroid in any position. Then

$$G(\alpha, \beta) = \left(\frac{1+\cos t+\sin t}{3}, \frac{2+\sin t-\cos t}{3}\right)$$

$$\therefore \quad \alpha = \frac{1+\cos t+\sin t}{3}, \beta = \frac{2+\sin t-\cos t}{3}$$

or $\quad 3\alpha - 1 = \cos t + \sin t \qquad \ldots(1)$

$\quad 3\beta - 2 = \sin t - \cos t \qquad \ldots(2)$

Squaring and adding, (1) and (2) equation, we get,

$$(3\alpha-1)^2 + (3\beta-2)^2 = (\cos t+\sin t)^2 + (\sin t-\cos t)^2$$
$$= 2(\cos^2 t + \sin^2 t) = 2$$

$\therefore$ The equation of the locus of the centroid is

$$(3x-1)^2 + (3y-2)^2 = 2$$

or $\quad 9(x^2+y^2) - 6x - 12y + 3 = 0$

$\therefore \quad 3(x^2+y^2) - 2x - 4y + 1 = 0.$

Test Your Concepts-V

(Solutions on page H.7)

1. Find the equation of the locus of a point such that the sum of its distances from (0, 2) and (0, –2) is 6.

2. If the coordinates of a variable point *P* be ($a\cos\theta$, $b\sin\theta$), where θ is a variable quantity, then find the locus of *P*.

3. If *O* be the origin and *A* be a point on the locus $y^2 = 8x$, then find the locus of the middle point of *OA*.

4. Find the equation of the locus of a point *P*, if the sum of squares of distances of the point *P* from the axes is p^2.

5. If the coordinates of a variable point *P* be ($\cos\theta + \sin\theta$, $\sin\theta - \cos\theta$), where θ is a variable quantity, then find the locus of *P*.

6. *A* and *B* are two given points, whose coordinates are (–5, 3) and (2, 4), respectively. A point *P* moves in such a manner that *PA*:*PB* = 3:2. Find the equation to the locus traced out by *P*.

7. If *A*(1, 1) and *B*(–2, 3) are two fixed points, find the locus of a point *P* so that the area of ΔPAB is 9 units.

8. Find the locus of a point such that the line segment having end points (2, 0) and (–2, 0) subtend a right angle at that point.

9. *A*(2, 3) is a fixed point and *Q*($3\cos\theta$, $2\sin\theta$) a variable point. If *P* divides *AQ* internally in the ratio 3:1, then find the locus of *P*.

Answer Keys

1. $\frac{x^2}{5} + \frac{y^2}{9} = 1$

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

3. $y^2 = 4x$

4. $x^2 + y^2 = p^2$

5. $x^2 + y^2 = 2$

6. $5x^2 + 5y^2 - 76x - 48y + 44 = 0$

7. $2x + 3y + 13 = 0$; $2x + 3y - 23 = 0$

8. $x^2 + y^2 = 4$

9. $\left(\frac{4x-2}{9}\right)^2 + \left(\frac{4y-3}{6}\right)^2 = 1$

SOLVED PROBLEMS

PROBLEM 1

If two opposite vertices of a square are (2, 6) and (0, –2), then find the coordinates of the other vertices.

SOLUTION

Let $ABCD$ be the square and $A = (2, 6)$, $C = (0, -2)$
Let $B = (x, y)$
$ABCD$ is a square.

$$\therefore \quad AB = BC \qquad \ldots(1)$$

$$AB^2 + BC^2 = AC^2 \qquad \ldots(2)$$

From (1), $\sqrt{(x-2)^2+(y-6)^2} = \sqrt{(x-0)^2+(y+2)^2}$

or $x^2 + 4 - 4x + y^2 + 36 - 12y = x^2 + y^2 + 4 + 4y$

or $4x + 16y = 36 \Rightarrow x + 4y = 9 \qquad \ldots(3)$

From (2), $[(x-2)^2 + (y-6)^2] + [(x-0)^2 + (y+2)^2]$

$$= (0-2)^2 + (-2-6)^2$$

$$\Rightarrow \quad x^2 + 4 - 4x + y^2 + 36 - 12y + x^2 + y^2 + 4 + 4y = 4 + 64$$

$$\Rightarrow \quad 2x^2 + 2y^2 - 4x - 8y = 24$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 4y = 12$$

$$\Rightarrow \quad (9-4y)^2 + y^2 - 2(9-4y) - 4y = 12$$

[From (3), $x = 9 - 4y$]

$$\Rightarrow \quad 81 + 16y^2 - 72y + y^2 - 18 + 8y - 4y - 12 = 0$$

$$\Rightarrow \quad 17y^2 - 68y + 51 = 0$$

$$\Rightarrow \quad y^2 - 4y + 3 = 0$$

$$\Rightarrow \quad y = 1, 3$$

From (3), $y = 1 \Rightarrow x = 9 - 4\,(1) = 5$

and $y = 3 \Rightarrow x = 9 - 4\,(3) = -3$

$\therefore \quad (x, y) = (5, 1), (-3, 3).$

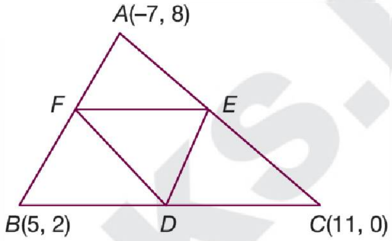
One of these are the coordinates of B and the other, those of D. Hence, the other vertices of the square are (5, 1) and (–3, 2).

PROBLEM 2

The vertices of a ΔABC are $A(-7, 8)$, $B(5, 2)$ and $C(11, 0)$. If D, E, F are the midpoints of the sides BC, CA and AB, respectively, then show that $\Delta ABC = 4\ \Delta DEF$.

SOLUTION

Given, $A = (-7, 8)$, $B = (5, 2)$ and $C = (11, 0)$
D is the midpoint of BC.



$$\therefore \quad D = \left(\frac{5+11}{2}, \frac{2+0}{2}\right) \text{ or } D = (8, 1)$$

E is the midpoint of CA.

$$\therefore \quad E = \left(\frac{11+(-7)}{2}, \frac{0+8}{2}\right) \text{ i.e., } E = (2, 4)$$

F is the midpoint of AB.

$$\therefore \quad F = \left(\frac{-7+5}{2}, \frac{8+2}{2}\right) \text{ or } F = (-1, 5)$$

Now, $\Delta ABC = \frac{1}{2}\,|-7(2-0) + 5(0-8) + 11(8-2)|$

$$= \frac{1}{2}\,|12| = 6 \text{ sq. units}$$

and $\Delta DEF = \frac{1}{2}\,|8(4-5) + 2(5-1) + (-1)(1-4)|$

$$= \frac{1}{2}\,|3| = \frac{3}{2} \text{ sq. units}$$

$$\therefore \quad 4\ \Delta DEF = 4 \times \frac{3}{2} = 6 \text{ sq. units}$$

$$\therefore \quad \Delta DEF = 4\ \Delta ABC = 4\Delta DEF$$

Hence proved

PROBLEM 3

If Δ_1 is the area of the triangle with vertices (0, 0), (a tan α, b cot α), (a sin α, b cos α), Δ_2 is the area of the triangle with vertices (a, b), (a sec^2 α, b cosec^2 α), (a + a sin$^2\alpha$, b + b cos^2 α) and Δ_3 is the area of the triangle with vertices (0, 0), (a tan α, – b cot α), (a sin α, b cos α), Show that there is no value of α, for which Δ_1, Δ_2 and Δ_3 are in G.P.

SOLUTION

We have,

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\tan\alpha & b\cot\alpha & 1 \\ a\sin\alpha & b\cos\alpha & 1 \end{vmatrix} = \frac{1}{2}ab\ |\sin\alpha - \cos\alpha|$$

$$\text{and } \Delta_2 = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ a\sec^2\alpha & b\,\text{cosec}^2\alpha & 1 \\ a + a\sin^2\alpha & b + b\cos^2\alpha & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - aC_3$ and $C_2 \to C_2 - bC_3$, we get,

$$\Delta_2 = \frac{1}{2}ab\begin{vmatrix} 0 & 0 & 1 \\ \tan^2\alpha & \cot^2\alpha & 1 \\ \sin^2\alpha & \cos^2\alpha & 1 \end{vmatrix}$$

$$= \frac{1}{2}ab(\sin^2\alpha - \cos^2\alpha)$$

$$\text{and } \Delta_3 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\tan\alpha & -b\cot\alpha & 1 \\ a\sin\alpha & b\cos\alpha & 1 \end{vmatrix} = \frac{1}{2}ab\,|\sin\alpha + \cos\alpha|$$

So that, $\Delta_1\,\Delta_3 = \frac{1}{2}\,ab\,\Delta_2$

Now Δ_1, Δ_2 and Δ_3 are in G.P. if,

$$\Delta_1\,\Delta_3 = \Delta_2^2 \Rightarrow \frac{1}{2}\,ab\Delta_2 = \Delta_2^2 \Rightarrow \Delta_2 = \frac{1}{2}\,ab$$

$$\Delta_2 = \frac{1}{2}ab\,(\sin^2\alpha - \cos^2\alpha) = \frac{1}{2}ab$$

$$\Rightarrow \quad \sin^2\alpha - \cos^2\alpha = 1$$

That is, $\alpha = (2m+1)\frac{\pi}{2}$, $m \in I$. However, for this value of α, the vertices of the given triangles are not defined. Hence Δ_1, Δ_2 and Δ_3 cannot be in G.P. for any value of α.

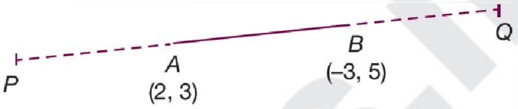
PROBLEM 4

The line segment joining $A(2, 3)$ and $B(-3, 5)$ is extended though each end by a length equal to its original length. Find the coordinates of the new ends.

SOLUTION

P divides AB in the ratio 1:2 externally.

$$\therefore\ P\left(\frac{-3-4}{1-2}, \frac{5-6}{1-2}\right) = P(7, 1)$$



Q divides AB in the ratio 2:1 externally.

$$\therefore \quad Q\left(\frac{-6-2}{2-1}, \frac{10-3}{2-1}\right) = Q(-8, 7)$$

PROBLEM 5

Vertices of a triangle are $A(x_1, x_1\tan\alpha_1)$, $B(x_2, x_2\tan\alpha_2)$ and $C(x_3, x_3\tan\alpha_3)$. If the circumcentre coincides with the origin and the orthocentre $H(\bar{x}, \bar{y})$, then prove that

$$\bar{y}(\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3) = \bar{x}(\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3).$$

SOLUTION

Here, the circumcentre $O(0, 0)$. So

$$OA = OB = OC.$$

$$\therefore \quad x_1^2 + x_1^2\tan^2\alpha_1 = x_2^2 + x_2^2\tan^2\alpha_2 = x_3^2 + x_3^2\tan^2\alpha_3$$

$$\text{or} \quad x_1^2\sec^2\alpha_1 = x_2^2\sec^2\alpha_2 = x_3^2\sec^2\alpha_3$$

$$\text{or} \quad \frac{x_1}{\cos\alpha_1} = \frac{x_2}{\cos\alpha_2} = \frac{x_3}{\cos\alpha_3} = k\ (\text{suppose})$$

$\therefore$ The vertices of the triangle become

$A(k\cos\alpha_1, k\sin\alpha_1)$, $B(k\cos\alpha_2, k\sin\alpha_2)$, $C(k\cos\alpha_3, k\sin\alpha_3)$

$\therefore$ The centroid

$$G\left(k\frac{\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3}{3}, k\frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{3}\right)$$

We know that the orthocentre $H(\bar{x}, \bar{y})$, centroid G and circumcentre $O(0,0)$ are collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2:1.

So, $G\left(\frac{\bar{x}}{3}, \frac{\bar{y}}{3}\right)$

$$= \left(k\frac{\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3}{3}, k\frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{3}\right)$$

$$\therefore \quad \frac{\bar{y}}{\bar{x}} = \frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3}$$

or $\quad \bar{y}(\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3)$

$$= \bar{x}\,(\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3)$$

PROBLEM 6

Three vertices of a ΔABC are $A(2, 1)$, $B(7, 1)$ and $C(3, 4)$. Images of this triangle are taken in x-axis, y-axis and in the line $y = x$. If G_1, G_2 and G_3 are the centroids of the three image triangles, then find the area of $\Delta G_1G_2G_3$.

SOLUTION

Centroid of the given triangle = (4, 2)
So, centroid of the image triangle is itself the image of the original centroid.

$$\therefore \quad G_1 = (4, -2),\ G_2 = (-4, 2),\ G_3 = (2, 4)$$

$$\therefore \quad \text{Area of } \Delta G_1G_2G_3 \text{ is } \frac{1}{2}\left|\begin{vmatrix} 4 & -2 \\ -4 & 2 \\ 2 & 4 \\ 4 & -2 \end{vmatrix}\right| = 20 \text{ sq. units}$$

PROBLEM 7

Prove that the centroid of any triangle is the same as the centroid of the triangle formed by joining the midpoints of its sides.

SOLUTION

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of any ΔABC and G be its centroid.

$$\text{Then, } G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Let D, E and F be the midpoints of the sides BC, CA and AB, respectively. Then

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right),\ E = \left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$$

$$\text{and } F = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Let H be the centroid of the ΔDEF, then

$$H = \left(\frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_1}{2} + \frac{x_1 + x_2}{2}}{3}, \frac{\frac{y_3 + y_2}{2} + \frac{y_3 + y_1}{2} + \frac{y_1 + y_2}{2}}{3}\right)$$

$$\text{or } H = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Clearly, G and H are the same point.

Hence Proved

PROBLEM 8

If $A(5, 2)$, $B(10, 12)$ and $P(x, y)$ is such that $\frac{AP}{PB} = \frac{3}{2}$, then find the internal bisector of $\angle APB$, which always passes through a fixed point.

SOLUTION

Let the internal bisector of $\angle APB$ meets the side AB at C.

$$\text{We know that } \frac{AC}{BC} = \frac{AP}{PB}$$

$$\therefore \quad \text{Coordinates of } C \text{ are } \left(\frac{3(10) + 2(5)}{3 + 2}, \frac{3(12) + 2(2)}{3 + 2}\right)$$

or (8, 8)

Hence, the internal bisector of $\angle APB$ always passes through the point (8, 8).

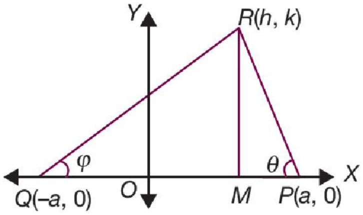
PROBLEM 9

Two points $P(a, 0)$ and $Q(-a, 0)$ are given. R is a variable point on one side of the line PQ, such that $\angle RPQ - \angle RQP$ is a 2α. Find the locus of R.

SOLUTION

Let R (h, k) be the variable point, then $\angle RPQ = \theta$ and $\angle RQP = \phi$, so that $\theta - \phi = 2\alpha$. Let $RM \perp PQ$, so that $RM = k$, $MP = a - h$ and $MQ = a + h$.

$$\text{Then } \tan\theta = \frac{RM}{MP} = \frac{k}{a - h};\ \tan\phi = \frac{RM}{MQ} = \frac{k}{a + h}$$



Therefore, from $2\alpha = \theta - \phi$, we have,

$$\tan 2\alpha = \tan(\theta - \phi)$$

$$= \frac{\tan\theta - \tan\phi}{1 + \tan\theta\,\tan\phi} = \frac{k(a+h) - k(a-h)}{a^2 - h^2 + k^2} = \frac{2hk}{a^2 - h^2 + k^2}$$

$$\Rightarrow \quad a^2 - h^2 + k^2 = 2hk \cot 2\alpha$$

Hence, the locus of R (h, k) is $x^2 - y^2 + 2xy \cot 2\alpha - a^2 = 0$.

PROBLEM 10

If the coordinates of a variable point P be $\left(t + \frac{1}{t}, t - \frac{1}{t}\right)$, where t is a variable quantity, then find the locus of P.

SOLUTION

$$x = t + \frac{1}{t};\; y = t - \frac{1}{t}$$

$$x^2 = t^2 + \frac{1}{t^2} + 2;\; y^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad x^2 - y^2 = 4 \qquad \text{(required locus)}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. For triangle ABC where $A(-1, 11)$, $B(-9, -8)$, and $C(15, -2)$ mark correct option

(A) circumcenter, orthocentre, incenter, and centroid of the triangle are collinear

(B) circumcenter, orthocentre, incenter, and centroid of the triangle are coincident at same point

(C) circumcenter, orthocentre and centroid collinear but incentre is not collinear with them

(D) we cannot comment

2. Without changing the direction of coordinate axis, to which point should origin be transferred, so that the equation $x^2 + y^2 - 4x + 6y - 7 = 0$ is changed to an equation with contains on term of first degree?

(A) (3, 2) (B) (2, –3)

(C) (– 2, 3) (D) None of these

3. To remove xy term from the second degree equation $5x^2 + 8xy + 5y^2 + 3x + 2y + 5 = 0$, the coordinate axis are rotated through an angle θ, then θ equals

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{3\pi}{8}$ (D) $\frac{\pi}{8}$

4. If a, b and c are the pth, qth and rth terms, respectively of an harmonic progression. Then points (bc, p), (ca, q), and (ab, r) are forming:

(A) collinear

(B) non-collinear with area pqr

(C) non-collinear with area abc

(D) None of these

5. In ΔABC, P is any point inside a triangle such that the area of ΔBPC, ΔAPC and ΔAPB are equal. If the line AP cut BC at M, area of ΔPMC is 5 sq. units then area of ΔABC is

(A) 20 sq. units (B) 25 sq. units

(C) 30 sq. units (D) None of these

6. If in a ΔABC, $A(2, 20)$, circumcentre $(-1, 2)$ and orthocentre $(1, 4)$, then the coordinates of midpoint of side opposite to vertex A is

(A) $\left(0, \frac{1}{2}\right)$ (B) (1, 2)

(C) $\left(-\frac{3}{2}, -6\right)$ (D) $\left(\frac{1}{3}, \frac{8}{3}\right)$

7. In a triangle with $BC = 3$ and $AC = 4$ and circle with AB as diameter passes through the centroid of a triangle, then AB is

(A) $\sqrt{5}$ (B) 5

(C) $\sqrt{3}$ (D) None of these

8. The sides of a triangle have length 11, 15 and k, where k is an integer. Then the number of values of k, for which the triangle is obtuse, is

(A) 5 (B) 12

(C) 13 (D) 17

9. A circle is inscribed in a Δ formed by (0, 0) (2, 0) and $\left(1, \sqrt{3}\right)$. Its area is

(A) $\frac{\pi\sqrt{3}}{2}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi\sqrt{2}}{2}$ (D) $\frac{\pi}{3}$

10. If one side of a rhombus has endpoint (4, 5) and (1, 1), then the maximum areas of the rhombus is

(A) 5 (B) 25

(C) 12.5 (D) $\frac{61}{2}$

11. Let ABC be an equilateral triangle with $AB = 6$ and vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \text{ equals}$$

(A) 192 (B) 243

(C) 486 (D) 972

12. The square $ABCD$, where $A(0, 0)$, $B(2, 0)$, $C(2, 2)$ and $D(0, 2)$ undergoes the following transformations successively.

(i) $f_1(x, y) = (y, x)$

(ii) $f_2(x, y) = (x + 3y, y)$

(iii) $f_3(x, y) = \left(\frac{x - y}{2}, \frac{x + y}{2}\right)$

Then the final figure is

(A) a square (B) a rectangle

(C) a parallelogram (D) a rhombus

13. If (0, 0), $(a, 2)$ and $(2, b)$ form the vertices of an equilateral triangle, where a and b not lie between 0 and 2, then the value of $4(a + b) - ab$ equals

(A) 4
(B) –4
(C) 2
(D) 8

14. M is the midpoint of side AB of equilateral ΔABC. P is a point on BC such that $AP + PM$ is minimum. If $AB = 20$, then $AP + PM$ is

(A) $10\sqrt{7}$
(B) $10\sqrt{3}$
(C) $10\sqrt{5}$
(D) 10

15. Let S be the set of all values of α, such that the points $(\alpha, 6)$, $(-5, 0)$ and $(5, 0)$ form an isosceles triangle. Then the value of $\sum_{\alpha \in S} \alpha^2$ is

(A) 356
(B) 18
(C) 178
(D) 338

16. If the orthocentre and circumcentre of a triangle are (0, 0) and (3, 6), respectively then the centroid of the triangle is

(A) (1, 2)
(B) (2, 4)
(C) $\left(\frac{2}{3}, \frac{4}{3}\right)$
(D) $\left(\frac{1}{3}, \frac{2}{3}\right)$

17. Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of ΔOPQ. The point R inside the ΔOPQ is such that the triangles OPR, PQR and OQR are of equal area. The point S is such that $OS = PS = QS$. Then find RS.

(A) $\frac{13}{16}$
(B) $\frac{11}{12}$
(C) $\frac{13}{24}$
(D) $\frac{11}{24}$

18. $A(3x_1, 3y_1)$, $B(3x_2, 3y_2)$ and $C(3x_3, 3y_3)$ are the vertices of a triangle with orthocenter H at $(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$ then find $\angle ABC$.

(A) $\frac{\pi}{2}$
(B) $\frac{\pi}{3}$
(C) $\frac{\pi}{6}$
(D) $\frac{\pi}{4}$

19. The area of the triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2), where a, x_1, x_2 are in G.P. with common ratio 'r' and b, y_1, y_2 are in G.P. with common ratio 's' is:

(A) $|ab(r - 1)(s - 1)(s - r)|$
(B) $\frac{1}{2}|ab(r + 1)(s + 1)(s - r)|$
(C) $\frac{1}{2}|ab(s - 1)(r - 1)(s - r)|$
(D) $\frac{1}{2}abrs$

20. A straight rod of length $3l$ units slides with its ends A and B always on the x and y axes, respectively, then the locus of centroid of ΔOAB is

(A) $x^2 + y^2 = 3l^2$
(B) $x^2 + y^2 = l^2$
(C) $x^2 + y^2 = 4l^2$
(D) $x^2 + y^2 = 2l^2$

21. By translation of axes, the equation $xy - x + 2y - 6 = 0$ changed as $XY = c$ then find c.

(A) 4
(B) 5
(C) 6
(D) 7

22. In a ΔABC, the coordinates of B are (0, 0) $AB = 2$, $\angle ABC = \frac{\pi}{3}$ and the midpoint of BC is (2, 0). The centroid of triangle is:

(A) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
(B) $\left(\frac{5}{3}, \frac{1}{\sqrt{3}}\right)$
(C) $\left(\frac{4 + \sqrt{3}}{3}, \frac{1}{\sqrt{3}}\right)$
(D) $\left(\frac{4 - \sqrt{3}}{3}, \frac{1}{\sqrt{3}}\right)$

23. In a ΔABC, E and F are points on AC and AB, respectively. The lines BE and CF intersect at P. If area $(BPF) = 5$, area $(PFAE) = 22$ and area $(CPE) = 8$, then the area (BPC) is

(A) 22
(B) 16
(C) 10
(D) not uniquely decidable

24. The perimeter of a parallelogram is 40. All the sides of the parallelogram are natural numbers and at least one of its diagonals is a natural number. The number of non-congruent parallelograms so formed is

(A) 10
(B) 30
(C) 60
(D) 100

25. The polar coordinates equivalent to $(-3, \sqrt{3})$ are

(A) $\left(2\sqrt{3}, \frac{\pi}{6}\right)$
(B) $\left(-2\sqrt{3}, \frac{5\pi}{6}\right)$
(C) $\left(2\sqrt{3}, \frac{7\pi}{6}\right)$
(D) $\left(2\sqrt{3}, \frac{5\pi}{6}\right)$

26. If the point $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) internally, then

(A) $t < 0$
(B) $0 < t < 1$
(C) $t > 1$
(D) $t = 1$

27. Coordinates of the vertices of a ΔABC are (12, 8), (–2, 6) and (6, 0), then the correct statement is

(A) triangle is right but not isosceles.
(B) triangle is isosceles but not right.
(C) triangle is obtuse.
(D) the product of the abscissa of the centroid, orthocenter and circumcentre is 160.

28. Three circles of radii 3, 4 and 5 touch each other externally. Tangents are drawn at the point of contact, which meet at 'p'. Find the distance between the point of contact and p.

(A) $\sqrt{5}$ (B) $12\sqrt{5}$
(C) $6\sqrt{5}$ (D) $15\sqrt{5}$

29. If the area of triangle formed by the points $(2a, b)$ $(a + b, 2b + a)$ and $(2b, 2a)$ be λ , then the area of the triangle whose vertices are $(a + b, a - b)$, $(3b - a, b + 3a)$ and $(3a - b, 3b - a)$ will be

(A) $\frac{3}{2}\lambda$ (B) 3λ
(C) 4λ (D) None of these

30. If the origin is shifted to (–2, 3), then the transformed equation of curve $x^2 + 2y - 3 = 0$ with respect to (0, 0) is

(A) $x^2 - 4x + 2y + 4 = 0$ (B) $x^2 - 4x + 2y - 5 = 0$
(C) $x^2 + 4x + 2y - 5 = 0$ (D) None of these

31. The distance between the circumcentre and orthocentre of the triangle whose vertices are (0, 0), (6, 8) and (–4, 3) is

(A) $\frac{125}{8}$ units (B) $\frac{\sqrt{5}}{2}$ units
(C) $\frac{5\sqrt{5}}{2}$ units (D) $5\sqrt{5}$ units

32. If all the 3 vertices of an isosceles right angle triangle be integral points and the length of base is also an integer, then which of the point is never a rational point. (A point $P(x, y)$ is integer point, if both x and y are integers and point is rational, if both x and y are rational)

(A) centroid (B) incentre
(C) circumcentre (D) orthocentre

33. If $A(1, p^2)$, $B(0, 1)$ and $C(p, 0)$ are the coordinates of three points, then the value of p for which the area of the ΔABC is minimum is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{-1}{\sqrt{3}}$
(C) $\frac{1}{\sqrt{3}}$ or $\frac{-1}{\sqrt{3}}$ (D) None

34. If $P(1, 0)$, $Q(-1, 0)$, and $R(2, 0)$ are three given points, then the locus of the point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is

(A) a straight line parallel to the x-axis
(B) a circle passing through the origin
(C) a circle with centre at the origin
(D) a straight line parallel to the y-axis

35. Let $O(0, 0)$, $A(2, 0)$ and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let 'R' be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le \min\{d(P, OB), d(P, AB)\}$, where d denotes the distance from the point to the corresponding line, then the area of the region R is

(A) $\sqrt{2} + 1$ sq. units (B) $\sqrt{2} - 1$ sq. units
(C) $2 + \sqrt{3}$ sq. units (D) $2 - \sqrt{3}$ sq. units

36. Vertices of a variable triangle are (3, 4), $(5\cos\theta, 5\sin\theta)$ and $(5\sin\theta, -5\cos\theta)$, where $\theta \in R$. Locus of its orthocentre is

(A) $(x + y - 1)^2 + (x - y - 7)^2 = 100$
(B) $(x + y - 7)^2 + (x - y - 1)^2 = 100$
(C) $(x + y - 7)^2 + (x + y - 1)^2 = 100$
(D) $(x + y - 7)^2 + (x - y + 1)^2 = 100$

37. $OPQR$ is a square and M and N are the midpoints of the sides PQ and QR, respectively then the ratio of the areas of the square and the triangle OMN is

(A) 4:1 (B) 2:1
(C) 8:3 (D) 4:3

38. The extremities of a diagonal of a rectangle are (0, 0) and (4, 4). The locus of the extremities of the other diagonal is equal to

(A) $x^2 + y^2 - 4x - 4y = 0$
(B) $x^2 + y^2 - 4x + 4y = 0$
(C) $x^2 + y^2 - 4x - 4y + 4 = 0$
(D) $x^2 + y^2 + 4x - 4y = 0$

39. The number of rational points on the line joining $(\sqrt{5}, 3)$ and $(3, \sqrt{3})$ is

(A) 0 (B) 1
(C) 2 (D) infinite

40. The Cartesian coordinates of point having polar coordinates $\left(-2, \frac{2\pi}{3}\right)$ will be

(A) $(1, \sqrt{3})$ (B) $(\sqrt{3}, 1)$
(C) $(1, -\sqrt{3})$ (D) $(-1, \sqrt{3})$

41. Let P be the point (–3, 0) and Q be a moving point $(0, 3t)$. Let PQ be trisected at R so that R is nearer to Q. RN is drawn perpendicular to PQ meeting the x-axis at N. The locus of the midpoint of RN is

(A) $(x + 3)^2 - 3y = 0$ (B) $(y + 3)^2 - 3x = 0$
(C) $x^2 - y = 1$ (D) $y^2 - x = 1$

42. If in an isosceles triangle with base 'a', vertical angle 20° and lateral side each of length 'b' is given then value of $a^3 + b^3$ equals

(A) $3ab$ (B) $3ab^2$
(C) $3a^2b$ (D) 3

43. The base of a triangle lies along $x = a$ and is of length $2a$. The area of triangle is a^2 if the vertex lies along

(A) $x = -a$ (B) $x = a$

(C) $x = \frac{a}{2}$ (D) $x = 2a$

44. Vertex A of ΔABC moves in such a way that $\tan B + \tan C$ = constant, then locus of the orthocentre of ΔABC is (BC is fixed)

(A) straight line (B) parabola

(C) circle (D) ellipse

45. If x_1, x_2 and x_3 as well as y_1, y_2 and y_3 are in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2), (x_3, y_3)

(A) lie on a straight line

(B) lie on an ellipse

(C) lie on a circle

(D) are vertices of a triangle

46. The number of integral points (integral points means both the coordinates should be integer) exactly in the interior of the triangle with vertices (0, 0), (0, 21) and (21, 0) is

(A) 133 (B) 190

(C) 233 (D) 105

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If (–6, –4), (3, 5) and (–2, 1) are the vertices of a parallelogram, then the remaining vertex can be

(A) (0, – 1) (B) (7, 10)

(C) (–1, 0) (D) (–11, –8)

2. If $(a\cos\theta_1, a\sin\theta_1)$, $(a\cos\theta_2, a\sin\theta_2)$ and $(a\cos\theta_3, a\sin\theta_3)$ represent the vertices of an equilateral triangle inscribed in a circle, then

(A) $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

(B) $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$

(C) $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = 0$

(D) $\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = 0$

3. If the points $A(0, 0)$, $B(\cos\alpha, \sin\alpha)$ and $C(\cos\beta, \sin\beta)$ are the vertices of a right angled triangle, then

(A) $\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{\sqrt{2}}$ (B) $\cos\left(\frac{\alpha-\beta}{2}\right) = -\frac{1}{\sqrt{2}}$

(C) $\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{\sqrt{2}}$ (D) $\sin\left(\frac{\alpha-\beta}{2}\right) = -\frac{1}{\sqrt{2}}$

4. If all the vertices of a triangle have integral coordinates, then the triangle maybe

(A) right angled (B) equilateral

(C) isosceles (D) None of these

5. In an acute ΔABC, if the coordinates of orthocentre H are $(4, b)$ of centroid G are $(b, 2b - 8)$, and of circumcentre C are $(-4, 8)$, then b cannot be

(A) 4 (B) 8

(C) 12 (D) –12

6. The base of a triangle lies along $x = a$ and is of length $2a$. The area of triangle is a^2 if the vertex lies along

(A) $x = -a$ (B) $x = 0$

(C) $x = \frac{a}{2}$ (D) $x = 2a$

7. For a ΔABC, $P(-2, 3)$ is the circumcentre and $Q(1, 5)$ is the nine point centre then

(A) the centroid of ΔABC is $\left(0, \frac{13}{3}\right)$.

(B) the distance between the orthocentre and centroid of ΔABC is $\frac{4}{3}\sqrt{13}$ units.

(C) the distance between orthocentre and circumcentre of ΔABC is $2\sqrt{13}$ units.

(D) orthocentre of ΔABC is (4, 6).

8. Let S_1, S_2, S_3 .... be squares such that for each $n > 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm then for which of the following values of 'n' is the area of S_n less than 1 cm^2?

(A) 7 (B) 8

(C) 9 (D) 10

9. If $A\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$, $B\left(\frac{-3}{\sqrt{2}}, \sqrt{2}\right)$, $C\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}\right)$ and $D(3\cos\theta, 2\sin\theta)$ are four points and if the area of quadrilateral $ABCD$ is maximum $\left(\text{where } \theta \in \left(\frac{3\pi}{2}, 2\pi\right)\right)$, then

(A) maximum area is 10 sq. units.

(B) $\theta = \frac{7\pi}{4}$

(C) $\theta = 2\pi - \sin^{-1}\left(\frac{3}{\sqrt{85}}\right)$

(D) maximum area is 12 sq. units.

10. In ΔOAB where O is origin, coordinate of A are rational number. $\angle AOB = \theta$, coordinate of B always irrational then θ can take the value

(A) 30°
(B) 15°
(C) 60°
(D) 45°

11. Three vertices of a triangle are $A(4, 3)$, $B(1, -1)$ and $C(7, k)$. Value(s) of k for which centroid, orthocentre, incentre and circumcentre of the ΔABC lie on the same straight line is/are

(A) 7
(B) –1
(C) $\frac{-19}{8}$
(D) 8

12. If $(-4, 0)$ and $(1, -1)$ are two vertices of a triangles of are 4 sq. units, then its third vertex lies on

(A) $y = x$
(B) $5x + y + 12 = 0$
(C) $x + 5y - 4 = 0$
(D) $x + 5y + 12 = 0$

13. Let $O\ (0, 0)$, $A(0,4)$, $B(6, 0)$. The P be a moving point such that the area of triangle POA is two time the area of triangle POB, The locus of P will be a straight line whose equation can be

(A) $x + 3y = 0$
(B) $x + 2y = 0$
(C) $2x - 3y = 0$
(D) $3y - x = 0$

14. If the point $P(x, y)$ is equidistant from the points $A(a + b, b - a)$, and $B(a - b, a + by)$, then

(A) $ax = by$
(B) $bx = ay$
(C) $x^2 - y^2 = 2(ax + by)$
(D) P can be (a, b)

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Let $ABCD$ is a square with sides of unit length. Points E and F are taken on sides AB and AD, respectively so that $AE = AF$. Let P be a point inside the square $ABCD$.

1. The maximum possible area of quadrilateral $CDFE$ is

(A) $\frac{1}{8}$
(B) $\frac{1}{4}$
(C) $\frac{5}{8}$
(D) $\frac{3}{8}$

2. The value of $(PA)^2 - (PB)^2 + (PC)^2$ is equal to

(A) 3
(B) 2
(C) 1
(D) 0

3. Let a line passing through point A divides the square $ABCD$ into two parts so that the area of one portion is double the other, then the length of portion of line inside the square is

(A) $\frac{\sqrt{10}}{3}$
(B) $\frac{\sqrt{13}}{3}$
(C) $\frac{\sqrt{11}}{3}$
(D) $\frac{2}{\sqrt{3}}$

Comprehension 2

Let $A(0, \beta)$, $B(-2, 0)$ and $C(1, 1)$ be the vertices of a triangle.

4. Angle A of the ΔABC will be obtuse, if β lies in

(A) $(-1, 2)$
(B) $\left(2, \frac{5}{2}\right)$
(C) $\left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right)$
(D) None of these

5. All the values of β, for which angle A of triangle ABC is largest lie in the interval

(A) $(-1, 2)$
(B) $\left(-2, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right)$
(C) $\left(-2, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \sqrt{6}\right)$
(D) None of these

Comprehension 3

Let ABC be a triangle, in which AD is the angular bisector of $\angle A$. Then D divides BC in the ratio of sides containing the angle, i.e., $AB:AC$. If the length of BC is x, then $BD = \left(\frac{AB}{AB + AC}\right)x$ and $DC = \left(\frac{AC}{AB + AC}\right)x$ and

(i) If $PA = PB$, then P lies on perpendicular bisector of the line joining points A and B.
(ii) If P is equidistant from two non-parallel lines $\Rightarrow P$ lies on angular bisector of given two lines.

For a square $ABCD$ having vertices at $A(0, 0)$, $B(2, 0)$, $C(0, 2)$ and $D(2, 2)$.

6. Let $d(P, AB)$ represent the distance of point P from side AB. Then the area of region R consisting of all points P inside the square satisfying $d\ (P, AB) < \min\ \{d(P, BC), d(P, CD), d(P, DA)\}$ is

(A) $\frac{1}{2}$ sq. units
(B) 1 sq. unit
(C) 2 sq. units
(D) 4 sq. units

7. Let the centre of the square is F. Then the area of region R consisting of all points P inside the square satisfying $PF \le \min\{PA, PB, PC, PD\}$ is

(A) $\frac{1}{2}$ sq. units (B) 1 sq. unit

(C) 2 sq. units (D) $\frac{1}{4}$ sq. units

8. The area of the region R consisting of all points P inside the square such that the distance of P from the nearest diagonal is less than distance of P from any of the sides is

(A) $2\sqrt{2}$ sq. units (B) $2(2\sqrt{2})$ sq. units

(C) $4\left(2-\sqrt{2}\right)$ sq. units (D) $\frac{2-\sqrt{2}}{2}$ sq. units

Comprehension 4

Consider a triangle PQR with coordinates of its vertices as $P(-8, 5)$, $Q(-15, -19)$, and $R(1, -7)$.

9. The distance between the orthocentre and circumcentre of the ΔPQR is

(A) $\frac{25}{2}$ (B) $\frac{29}{2}$

(C) $\frac{37}{2}$ (D) $\frac{51}{2}$

10. Radius of nine point circle is

(A) 25 (B) $\frac{25}{2}$

(C) $\frac{25}{4}$ (D) $\frac{25}{8}$

Comprehension 5

$A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are three vertices of a ΔABC. $lx + my + n = 0$ is an equation of the line L.

11. If L intersects the sides BC, CA and AB of the ΔABC at P, Q and R, respectively, then

$$\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} \text{ is equal to}$$

(A) -1 (B) $-\frac{1}{2}$

(C) $\frac{1}{2}$ (D) 1

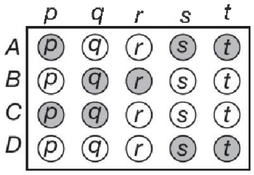
12. If P divides BC in the ratio 2:1 and Q divides CA in the ratio 1:3, then R divides AB in the ratio (P, Q and R are the points as in previous problem)

(A) 2:3 internally (B) 2:3 externally

(C) 3:2 internally (D) 3:2 externally

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:



	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is	(p) $\left(1, \frac{1}{\sqrt{3}}\right)$
(B) The incentre of the triangle with vertices $\left(1, \sqrt{3}\right)$, (0, 0) and (2, 0) is	(q) (0, 0)
(C) Let $O(0,0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of the ΔOPQ. The point R inside the ΔOPQ is such that the triangles OPR, PQR, OQR are of equal area. Then the coordinates of R are	(r) (2, 3)
(D) If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then (a, b) equals	(s) $\left(3, \frac{4}{3}\right)$

(*Continued*)

2. If $A(2a, 4a)$ and $B(2a, 6a)$ are two vertices of a ΔABC and the vertex C is given by,

COLUMN-I	COLUMN-II
(A) $(4a, 5a)$	(p) Equilateral
(B) $((2+\sqrt{3})a, 5a)$	(q) Right angled
(C) $(6a, 4a)$	(r) Obtuse angled
(D) $(a, 3a)$	(s) Isosceles

3. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If (3, 2), (–3, 2) and (0, h) are the vertices of an equilateral triangle and $h < 0$, then h equals	(p) $2 - 3\sqrt{3}$
(B) If $A(1, 1)$, $B(4, 5)$ and $C(6, 13)$, then $\cos A$ equals	(q) 1
(C) If G is the centroid of ΔABC and $BC = 3$, $CA = 4$ and $AB = 5$, then BG equals	(r) $\frac{63}{65}$
(D) Number of points which are equidistant from vertices $A(1, 3)$, $B(-3, 5)$ and $C(5, -1)$ are	(s) $\frac{\sqrt{52}}{3}$
	(t) $1 - 3\sqrt{3}$

4. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If $A(3, -2)$ and $B(0, 2)$ P moves on $y - 1 = 0$, then the least value of $PA + PB$ is	(p) 4
(B) Locus of orthocentre of triangle with vertices (0, 1) $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$ is $(x+y-1)^2 + (x-y+1)^2 = k$, then k is	(q) 5
(C) $A(0, 0)$, $B(2, 0)$, P moves such that $PA = 2PB$. Then the number of positions of P such that $\Delta PAB = 1$ unit is	(r) 2
(D) Number of points in the locus of $(x^2 - y^2)^2 + (x^2 + y^2 - 1)^2 = 0$ is	(s) 1

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Consider two fixed points A and B. The vertex 'C' of ΔABC moves such that $\sin(A + B) = 2 \sin A \sin B$. If the locus of '$C$' intersects a circle, then find the maximum number of points of intersection.

2. If locus of the point $(t^2 - t + 1, t^2 + t + 1)$, where t is real parameter is $x = \left(\frac{y-x}{2}\right)^2 - \left(\frac{y-x}{2}\right) + k$, then find k.

3. A ΔABC has vertices $A(5, 1)$, $B(-1, -7)$ and $C(1, 4)$, respectively. L be the line mirror passing through C and parallel to AB and a light ray emanating from point A goes along the direction of the internal bisector of the angle A, which meets the mirror and BC at E and D, respectively. If the sum of the areas of ΔACE and ΔABC is $(10 + a)$ sq. units, then find the value of a.

4. If the line segment joining $P(1, 3)$ and $Q(5, 7)$ subtend a right angle at $R(x, y)$ and area of triangle $PQR = 2$, then find the maximum number of such points R in the xy-plane.

5. If the sum of the squares of the sides of triangle is 16, then the sum of the squares of the medians is $3k$. Then find k.

6. Point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio k:1. Then find the integral value of k, for which the area of ΔABC, where B is $(1, 5)$ and C is $(7, -2)$ is equal to 2 units in magnitude.

7. The distance between the circumcentre and the orthocentre of the triangle, whose vertices are $(0, 0)$, $(6, 8)$, and $(-4, 3)$ is L. Then find the value of $\frac{2}{\sqrt{5}}L$.

8. Let $A(0, 1)$, $B(1, 1)$, $C(1, -1)$ and $D(-1, 0)$ be four points. If P is any other point, then $PA + PB + PC \geq d$, when $[d]$ is ______, where [.] represents greatest integer.

9. If the area of the triangle formed by the points $(2a, b)$ $(a + b, 2b + a)$ and $(2b, 2a)$ is 2 sq. units, then find the area of the triangle whose vertices are $(a + b, a - b)$, $(3b - a, b + 3a)$ and $(3a - b, 3b - a)$.

10. The centroid of the triangle formed by (a, b), (b, c) and (c, a) is the origin and $a^3 + b^3 + c^3 = kabc$, then find the value of k.

11. When $(0, 0)$ is shifted to $(3, -3)$, then the points $P(5, 5)$, $Q(-2, 4)$ and $R(7, -7)$ are changed as A, B and C and if the area of ΔABC is k, then find $\frac{86}{k}$.

12. Consider a ΔABC with $BC = 3$. Choose a point D on BC such that $BD = 2$. Find the value of $AB^2 + 2AC^2 - 3AD^2$.

13. A ΔABC is given with $A(1, -2)$, $B(-4, 3)$ and $C(3, 2)$. Find the sum of all the possible coordinates of 'P' in the plane of the ΔABC such that the area of $\Delta PAB = \Delta PBC = \Delta PAC$.

14. Consider a 2×2 graph paper. If the number of isosceles right angled triangles whose vertices are integer points of the graph and whose sides forming a right angle are parallel to coordinate axes is 4λ, then find λ.

15. A point P is selected inside an equilateral triangle. If the sum of lengths of the perpendicular dropped on to sides from P is 2004, then find $\frac{\text{length of altitude of triangle}}{2004}$.

16. Two medians drawn from the acute angles of a right angled triangle intersect at an angle $\frac{\pi}{6}$. If the length of the hypotenuse of the triangle is 3 units, then the area of the triangle (in sq. units) is $\sqrt{K}$, then find K.

17. Find the number of triangles with points $(8, 9)$ $(8, 16)$ and $(20, 25)$ as the feet of perpendiculars drawn from the vertices on the opposite side.

18. In ΔABC, $B(0, 0)$, $AB = 2$, $\angle ABC = \frac{\pi}{3}$ and midpoint of BC has coordinates $(2, 0)$. Then find $k_1 + k_2$, if the centroid of the triangle is $\left(\frac{k_1}{3}, \frac{k_2}{\sqrt{3}}\right)$.

ARCHIVE: JEE MAIN

1. **[JEE MAIN 2020: Jan Attempt]**

Let $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertices of a triangle ABC. If P is a point inside the triangle ABC such that the triangles APC, APB and BPC have equal areas, then the length of the line segment PQ, where Q is the point $\left(-\frac{7}{6}, -\frac{1}{3}\right)$, is ______ .

2. **[JEE Main 2019]**

Let $O(0, 0)$ and $A(0, 1)$ be two fixed points. Then the locus of a point P such that the perimeter of ΔAOP is 4 is:

(A) $8x^2 + 9y^2 - 9y = 18$ (B) $9x^2 + 8y^2 - 8y = 16$

(C) $9x^2 - 8y^2 + 8y = 16$ (D) $8x^2 - 8y^2 + 9y = 18$

3. **[JEE Main 2019]**

Two vertices of a triangle are $(0, 2)$ and $(4, 3)$. If its orthocentre is at the origin, then its third vertex lies in which quadrant?

(A) Third (B) Fourth

(C) Second (D) First

4. **[JEE Main 2019]**

Let S be the set of all triangles in the xy-plane, each having one vertex at the origin and the other two vertices lie on coordinate axes with integral coordinates. If each triangle in S has area 50 sq. units, then the number of elements in the set S is:

(A) 32 (B) 36

(C) 9 (D) 18

5. **[JEE Main 2019]**
A triangle has a vertex at (1, 2) and the midpoints of the two sides through it are (–1, 1) and (2, 3). Then the centroid of this triangle is:

(A) $\left(\frac{1}{3}, 2\right)$ (B) $\left(\frac{1}{3}, \frac{5}{3}\right)$

(C) $\left(1, \frac{7}{3}\right)$ (D) $\left(\frac{1}{3}, 1\right)$

6. **[JEE Main 2018]**
A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is:

(A) $2x + 3y = xy$ (B) $3x + 2y = xy$
(C) $3x + 2y = 6xy$ (D) $3x + 2y = 6$

7. **[JEE Main 2018]**
Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$, respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter is:

(A) $2\sqrt{10}$ (B) $3\sqrt{\frac{5}{2}}$

(C) $\frac{3\sqrt{5}}{2}$ (D) $\sqrt{10}$

8. **[JEE Main 2017]**
Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point:

(A) $\left(2, \frac{1}{2}\right)$ (B) $\left(2, -\frac{1}{2}\right)$

(C) $\left(1, \frac{3}{4}\right)$ (D) $\left(1, -\frac{3}{4}\right)$

9. **[AIEEE 2012]**
If the line $2x + y = k$ passes through the point, which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3:2, then k equals:

(A) $\frac{11}{5}$ (B) $\frac{29}{5}$

(C) 5 (D) 6

ARCHIVE: JEE ADVANCED

1. **[JEE Advanced 2007]**
Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of the ΔOPQ. The point R inside the ΔOPQ is such that the triangles OPR, PQR and OQR are of equal area. The coordinates of R are:

(A) $\left(\frac{4}{3}, 3\right)$ (B) $\left(3, \frac{2}{3}\right)$

(C) $\left(3, \frac{4}{3}\right)$ (D) $\left(\frac{4}{3}, \frac{2}{3}\right)$

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A **2.** B **3.** B **4.** A **5.** C **6.** C **7.** A **8.** C **9.** D **10.** B
11. D **12.** C **13.** A **14.** A **15.** A **16.** B **17.** D **18.** B **19.** C **20.** B
21. A **22.** B **23.** C **24.** D **25.** D **26.** B **27.** D **28.** A **29.** C **30.** C
31. C **32.** B **33.** D **34.** D **35.** D **36.** D **37.** C **38.** A **39.** A **40.** C
41. D **42.** B **43.** D **44.** A **45.** A **46.** B

Multiple Correct Choice Type Questions

1. B, C, D **2.** A, B **3.** A, C, D **4.** A, C **5.** A, B, C, D
6. B, D **7.** A, B, C **8.** B, C, D **9.** B, D **10.** A, B, C
11. B, C **12.** C, D **13.** A, D **14.** B, D

Linked Comprehension Type Questions

1. C **2.** D **3.** B **4.** C **5.** C **6.** B **7.** C **8.** C **9.** A **10.** C
11. A **12.** D

Matrix Match/Column Match Type Questions

1. A → (q) B → (p) C → (s) D → (r)
2. A → (s) B → (p, s) C → (q) D → (r)
3. A → (p) B → (r) C → (s) D → (q)
4. A → (q) B → (p) C → (p) D → (p)

Integer/Numerical Answer Type Questions

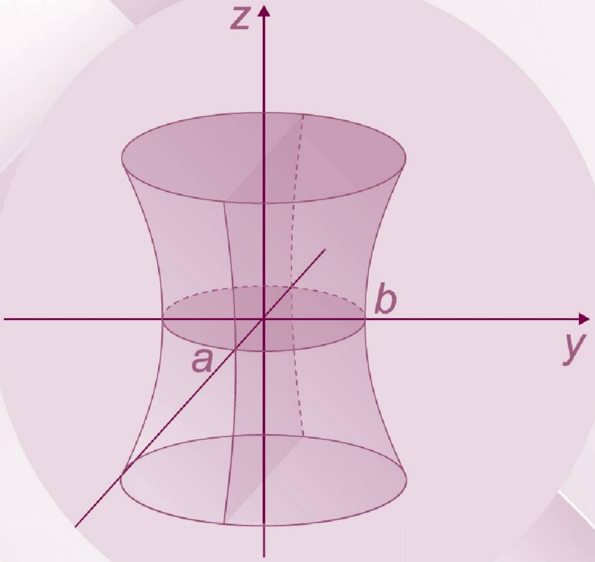
1. 2 **2.** 1 **3.** 9 **4.** 4 **5.** 4
6. 7 **7.** 5 **8.** 4 **9.** 8 **10.** 3
11. 2 **12.** 6 **13.** 1 **14.** 5 **15.** 1
16. 3 **17.** 4 **18.** 6

ARCHIVE: JEE MAIN

1. 5 **2.** B **3.** C **4.** B **5.** A **6.** B **7.** B **8.** A **9.** D

ARCHIVE: JEE ADVANCED

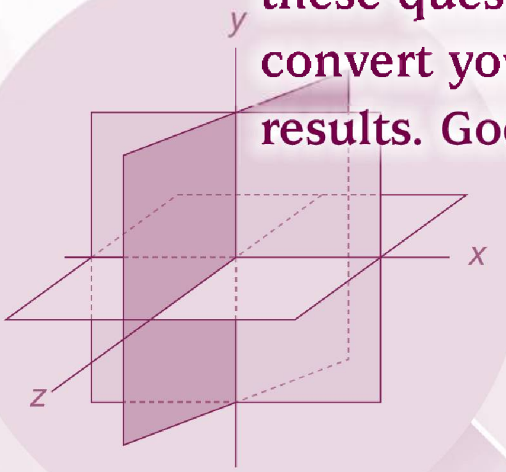
1. C

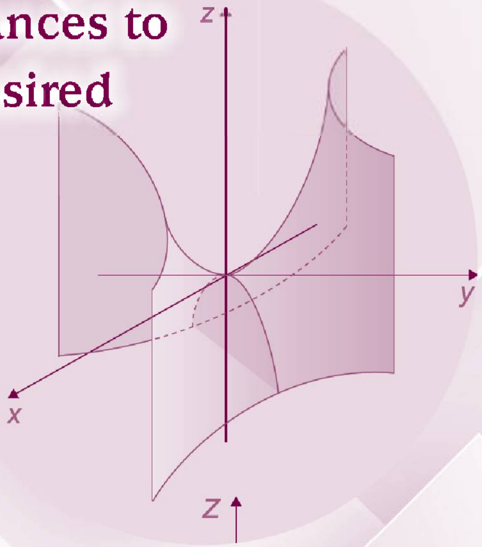


From the Author's Desk

Dear learners,

Straight line chapter consists of essential fundamentals of co-ordinate geometry. Let me count a few topics—various forms of straight line, their interconversion, parallel and perpendicular lines, concurrent lines, family of straight line, angle bisectors of lines, and so on. These are all concept building topics. Parametric form of straight line, locus problems and homogenization of a curve are tricky concepts. More you practice these questions, more your chances to convert your hard work into desired results. Good luck!!





CHAPTER 2 Straight Line

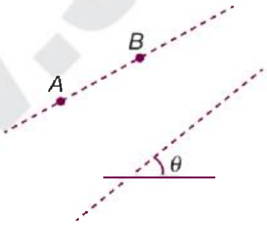
Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Various forms of straight line	(e) Family of lines
(b) Angle between two lines	(f) Concurrent lines
(c) Image/foot of perpendicular from a point	(f) Pair of straight lines
(d) Angle bisectors of lines	(h) Homogenization of curve

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

DEFINITION OF A STRAIGHT LINE



A straight line is a path of extension of shortest distance between two points or locus which is traced by a point travelling in a constant direction.

A relation between x and y, which is satisfied by the co-ordinates of every point lying on a straight line is called the **equation of a straight line**.

Every Ist degree equation in x and y, i.e., $ax + by + c = 0$ represents a line, where a, b and c are constants.

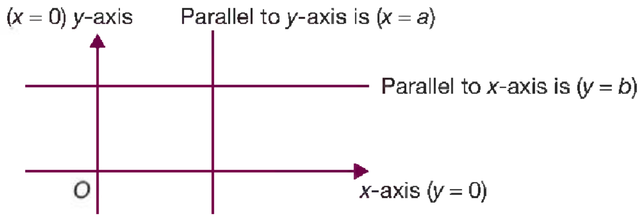
EQUATION OF STRAIGHT LINE PARALLEL TO AXES

(i) Equation of x-axis: $y = 0$

(ii) Equation of a line parallel to x-axis at a distance of $b : y = b$

(iii) Equation of y-axis: $x = 0$

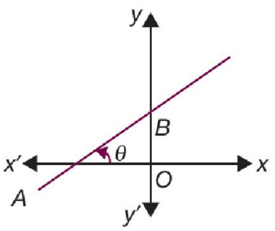
(iv) Equation of a line parallel to y-axis and at a distance of a: $x = a$

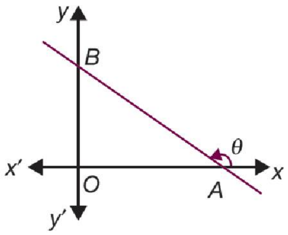


NOTE:
The combined equation of the co-ordinate axes is $xy = 0$.

SLOPE OF A LINE

Slope of a line means the inclination of the line with the positive direction of x-axis. Thus, the trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called the **slope or gradient** of a line denoted by (m). Let the line AB make an angle θ with the x-axis, then the gradient of line $AB = m = \tan\theta$.





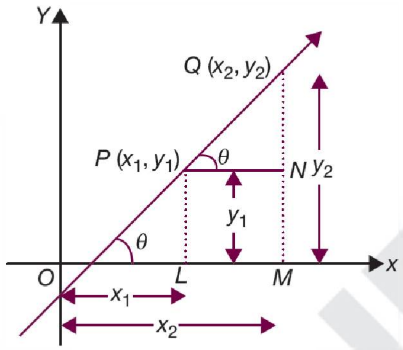
NOTES:

(i) Slope of x-axis or a line parallel to x-axis is zero.

(ii) Slope of y-axis or a line parallel to y-axis is not defined.

Slope of a Line in Term of Co-ordinates of any Two Points on it

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line making an angle θ with the positive direction of x-axis.



$$\text{In } \Delta PQN,\ \tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus, the slope of a line is

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

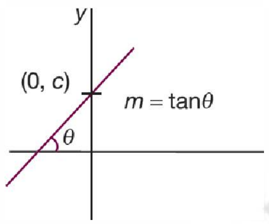
DIFFERENT FORMS OF THE EQUATION OF A STRAIGHT LINE

1. **General form**

   We know that a 1st degree equation in x and y, which is $ax + by + c = 0$ always represents a straight line. This form is known as the general form of a straight line.

2. **Slope-Intercept form**

   The equation of a line with slope m and making an intercept c on y-axis is $y = mx + c$.

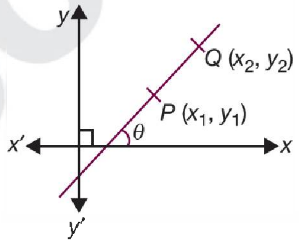


   If the line passes through the origin, then $c = 0$. Thus, the equation of a line with slope m and passing through the origin is $y = mx$.

3. **Point and slope form**

   Equation of a straight line passing through point (x_1, y_1) and having slope m is given by $y - y_1 = m(x - x_1)$.

4. **Two points form**



   Let m be the slope of the line passing through the points (x_1, y_1) and (x_2, y_2) then,

   $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

   So, the equation of the line is

   $$y - y_1 = m\,(x - x_1) \qquad \ldots(1)$$

   Putting the value of m in equation (1), we get,

   $$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

   This is the required equation of the line in two points. We can also write equation of required line using point Q

   $$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

5. **Determinant form**

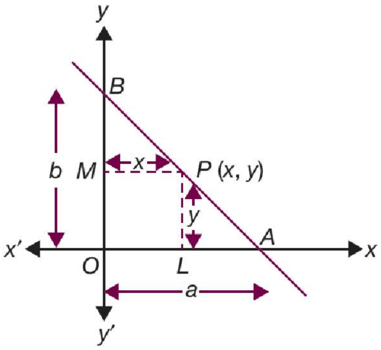
   Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two given points, then the equation of the line passing through these points is given by $(y - y_1) = \dfrac{y_2 - y_1}{x_2 - x_1}\ (x - x_1)$.

   This can also be written in determinant form as

   $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

6. **Intercept form**

Let AB be the line which cuts off the intercepts $OA = a$ and $OB = b$ on the x and y-axis respectively



let $P(x, y)$ be any point on the line.

Area of ΔOAB = Area of ΔOPA + Area of ΔOPB

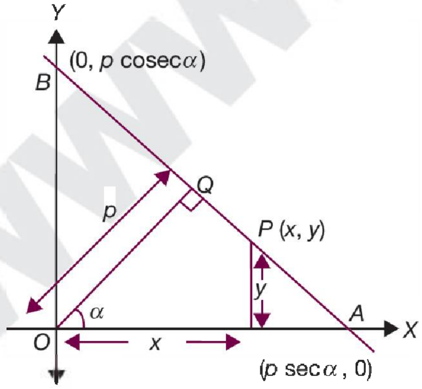
$$\Rightarrow \frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$$

$$\therefore \quad \frac{\boldsymbol{x}}{\boldsymbol{a}} + \frac{\boldsymbol{y}}{\boldsymbol{b}} = \mathbf{1} \quad (a \neq 0,\ b \neq 0)$$

This is the equation of a line in the intercept form.

7. **Normal form**

Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be P and $\angle XOQ = \alpha$. By using intercept form of line we get $\dfrac{x}{p\sec\alpha} + \dfrac{y}{p\operatorname{cosec}\alpha} = 1$



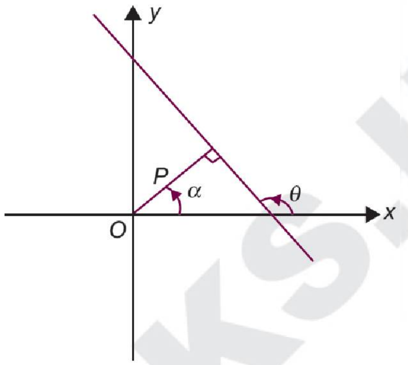
$\Rightarrow$ $\boldsymbol{x}\ \mathbf{cos}\boldsymbol{\alpha} + \boldsymbol{y}\ \mathbf{sin}\boldsymbol{\alpha} = \boldsymbol{p}$ $(p > 0)$ and $\alpha \in [0, 2\pi)$, which is the required equation of line.

Here, θ be the indication angle of straight line made with $+x$-axis.

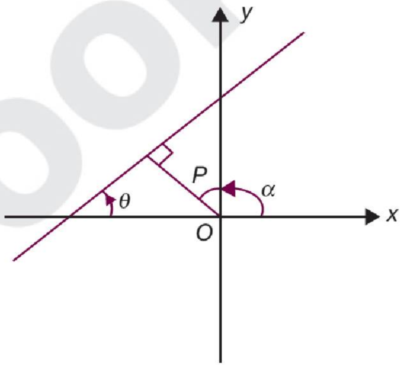
α is angle made by perpendicular OQ with $+x$-axis.

A straight line can exist in following cases, where we can understand α and θ.

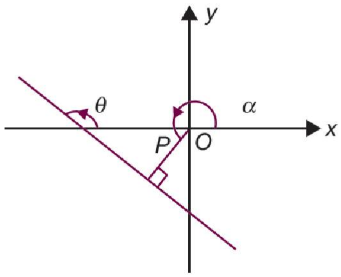
(i)



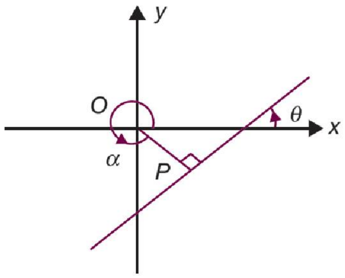
(ii)



(iii)



(iv)



Conversion of the General Equation of Line to Different Forms

1. **Slope-Intercept form**

To reduce the equation $Ax + By + C = 0$ to the form $y = mx + c$.

The given equation is $Ax + By + C = 0$.

$$\Rightarrow\ y=\frac{-A}{B}x-\frac{C}{B}\qquad (B\neq 0)$$

Slope of line $(m)=-\frac{A}{B}$,

y-intercept of line $(c)=-\frac{C}{B}$

2. **Intercept form**

   To reduce the equation $Ax + By + C = 0$ to the form $\frac{x}{a}+\frac{y}{b}=1$.

$$Ax+By=-C$$

$$\Rightarrow\ \frac{x}{-C/A}+\frac{y}{-C/B}=1$$

Intercept on x-axis $=-\frac{C}{A}$

Intercept on y-axis $=-\frac{C}{B}$

NOTE:

(i) To get the intercept of a line on x-axis, put $y=0$, value of x gives the intercept on x-axis.

(ii) Similarly, to get the intercept of a line on y-axis, put $x=0$, value of y gives the intercept on y-axis.

(iii) Intercept can be negative, positive or zero.

3. **Normal form**

   To reduce the equation $Ax + By + C = 0$ to the form $x\cos\alpha + y\sin\alpha = p$.

NOTE:

In normal form, $x\cos\alpha + y\sin\alpha = p$, where p is taken as positive and $\alpha\in[0, 2\pi)$.

The given equation is $Ax + By + C = 0$ or $Ax + By = -C$.

CASE-1:

When C < 0, then the normal form is

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}},$$

where, $\cos\alpha=\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha=\frac{B}{\sqrt{A^2+B^2}}$,

$$p=-\frac{C}{\sqrt{A^2+B^2}}$$

CASE-2:

When C > 0, then write the equation as $-Ax-By=C$.

$$\frac{-A}{\sqrt{A^2+B^2}}x+\frac{-B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}},$$

where, $\cos\alpha=\frac{-A}{\sqrt{A^2+B^2}}$, $\sin\alpha=\frac{-B}{\sqrt{A^2+B^2}}$,

$$p=\frac{C}{\sqrt{A^2+B^2}}$$

ILLUSTRATION 1

Reduce the line $2x - 3y + 5 = 0$ in slope-intercept, intercept and normal form.

SOLUTION

Slope-Intercept Form

$$y=\frac{2x}{3}+\frac{5}{3},\quad \tan\theta=m=\frac{2}{3},\quad c=\frac{5}{3}$$

Intercept Form

$$\frac{x}{\left(-\frac{5}{2}\right)}+\frac{y}{\left(\frac{5}{3}\right)}=1$$

Hence, $a=-\frac{5}{2}$ and $b=\frac{5}{3}$

Normal Form

$$-2x+3y=5$$

$$\Rightarrow\ -\frac{2x}{\sqrt{13}}+\frac{3y}{\sqrt{13}}=\frac{5}{\sqrt{13}}$$

Where, $\sin\alpha=\frac{3}{\sqrt{13}}$, $\cos\alpha=\frac{-2}{\sqrt{13}}$, $p=\frac{5}{\sqrt{13}}$

ILLUSTRATION 2

Find the equation of the straight line which makes an angle of 15° with the positive direction of x-axis and which cuts an intercept of length 4 units on the negative direction of y-axis.

SOLUTION

Here, $m=$ slope of the line $=\tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$

and $c=$ intercept on y-axis $=-4$

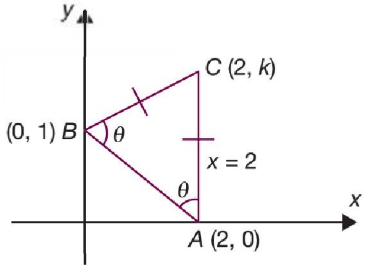
$\therefore$ Equation of the required line will be

$$y=\left(2-\sqrt{3}\right)x-4\ \Rightarrow\ \left(2-\sqrt{3}\right)x-y-4=0$$

ILLUSTRATION 3

The extremities of the base of an isosceles triangle ABC are the points $A(2, 0)$ and $B(0, 1)$. If the equation of the side AC is $x = 2$, then find the slope of the side BC.

SOLUTION



$$AC = BC$$

$$\therefore \quad \sqrt{k^2} = \sqrt{4+(k-1)^2}$$

$$k^2 = 4 + k^2 - 2k + 1$$

$$\therefore \quad k = \frac{5}{2}$$

$$\therefore \quad \text{Slope of } BC = \frac{\frac{5}{2}-1}{2} = \frac{5-2}{4} = \frac{3}{4}$$

ILLUSTRATION 4

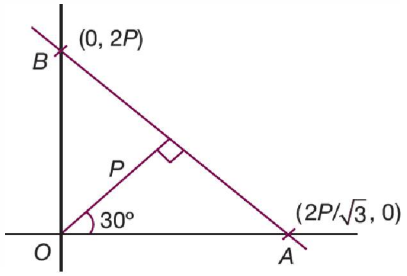
Find the equation of the straight line on which the perpendicular from the origin makes an angle 30° with x-axis and which forms a triangle of area $\frac{50}{\sqrt{3}}$ sq. units with the co-ordinate axes.

SOLUTION

Let p be the length of the perpendicular drawn from the origin on the required straight line. Given that the line make an angle $\alpha = 30°$ with x-axis.

$\therefore$ The equation of the line is $x \cos 30° + y \sin 30° = p$.

$$\Rightarrow \quad x\left(\frac{\sqrt{3}}{2}\right)+\frac{y}{2} = p \qquad \ldots(1)$$



If the line (1) meets the co-ordinates axes at A and B respectively, then putting $y = 0$ in (1), we get,

$x = \frac{2p}{\sqrt{3}} = OA$ and putting $x = 0$ in (1), we get,

$y = 2p = OB$. (We can also observe it from intercept form)

Now, the area of right angled $\Delta OAB = \frac{1}{2} OA \cdot OB$

$= \frac{50}{\sqrt{3}}$ (Given)

$$\Rightarrow \quad \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}}$$

$$\Rightarrow \quad p^2 = 25$$

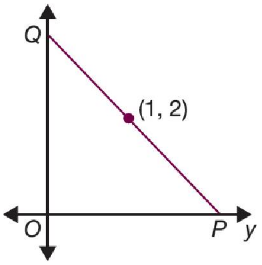
$$\therefore \quad p = \pm 5 \text{ (but } p > 0)$$

Hence, from (1), the equation of the required line is $x\sqrt{3} + y = 10$.

ILLUSTRATION 5

A line is drawn through the point (1, 2) to meet the positive co-ordinate axes at P and Q such that it forms a ΔOPQ, where O is the origin. If the area of the ΔOPQ is least, then find the slope of the line PQ.

SOLUTION



Equation of line PQ

$$(y - 2) = m(x - 1)$$

$$\therefore \quad OP = 1 - \frac{2}{m}$$

$$OQ = 2 - m$$

$$\text{Area of } \Delta POQ = \frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(1-\frac{2}{m}\right)(2-m)$$

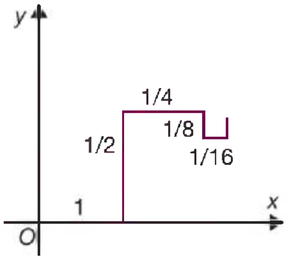
$$= \frac{1}{2}\left(2-m-\frac{4}{m}+2\right) = \frac{1}{2}\left(4-\left[m+\frac{4}{m}\right]\right)$$

$$= \frac{1}{2}\left(\left[\sqrt{-m}-\frac{2}{\sqrt{-m}}\right]^2+8\right)$$

Which has least value, when $m = -2$.

ILLUSTRATION 6

A particle begins at the origin and moves successively in the following manner as shown in the Figure. 1 unit to the right, $\frac{1}{2}$ unit up, $\frac{1}{4}$ unit to the right, $\frac{1}{8}$ unit down, $\frac{1}{16}$ unit to the right, etc. The length of each move is half the length of the previous move and the movement continues in the 'flip flop' manner indefinitely. Find the point, at which the 'flip flop' converges.



SOLUTION

x-co-ordinates: $1+\frac{1}{4}+\frac{1}{16}+\ldots.\infty=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$

y-co-ordinates: $\frac{1}{2}-\frac{1}{8}+\frac{1}{32}\ldots.-\infty=\frac{\frac{1}{2}}{1+\frac{1}{4}}=\frac{2}{5}$

Hence, $\left(\frac{4}{3},\frac{2}{5}\right)$

ILLUSTRATION 7

Each number of the family of parabolas $y=ax^2+2x+3$ has a maximum or a minimum point depending upon the value of a. Find the equation to the locus of the maxima or minima for all the possible values of 'a'.

SOLUTION

Maxima/or minima occurs at $\left(-\frac{1}{a},3-\frac{1}{a}\right)$ $=\left(-\frac{B}{2A},-\frac{D}{4A}\right)$ (vertex)

$(f(x))=ax^2+bx+c$ has a maxima or minima, if $x=-\frac{b}{2a}$)

$\therefore\quad h=-\frac{1}{a}$ and $k=3-\frac{1}{a}$

Eliminating 'a' we get, $k=3+h$, locus is $y=x+3$.

ILLUSTRATION 8

A line drawn from the origin touches the curve $y^2=x^2-x+1$ in the I$^{\text{st}}$ quadrant. If the line is inclined at an acute angle θ with the positive x-axis, then find $\tan\theta$.

SOLUTION

Let the line is $y=mx$, where $\tan\theta=m$ and $0<\theta<\frac{\pi}{2}$

$\therefore\quad m>0\qquad y^2=x^2-x+1$

$\Rightarrow\quad m^2x^2=x^2-x+1$

$\Rightarrow\quad (m^2-1)x^2+x-1=0$

$\therefore\quad D=0$; As it touches given curve, both roots should be equal

$\Rightarrow\quad 1+4(m^2-1)=0\Rightarrow 4m^2-3=0$

$\Rightarrow\quad m^2=\frac{3}{4}\Rightarrow m=\frac{\sqrt{3}}{2}=\tan\theta$

ILLUSTRATION 9

A line with gradient 2 intersects the line with gradient 6 at the point (40, 30). Find the distance between the x-intercepts of these lines.

SOLUTION

Let C_1 and C_2 be the x-intercepts of the line with slopes 2 and 6, respectively.

$$y-0=2(x-C_1)$$

$\Rightarrow\quad y=2x-2C_1 \qquad \ldots(1)$

Similarly,

$$y=6x-6C_2 \qquad \ldots(2)$$

Both (1) and (2) satisfy $x=40$ and $y=30$.

$\therefore\quad 30=80-2C_1\Rightarrow C_1=25$

And $30=240-6C_2\Rightarrow C_2=35$

Hence, $C_2-C_1=10$ units

ILLUSTRATION 10

Start with an equilateral triangle of side length 1 unit. Construct a 2nd triangle by connecting the midpoints of the sides of the 1st triangle. Construct a 3rd triangle by connecting the midpoints of the edges of the 2nd triangle. Continue this process indefinitely and then find the sum of the areas of all the triangles.

SOLUTION

New side = $\frac{1}{2}$(old side of Δ)

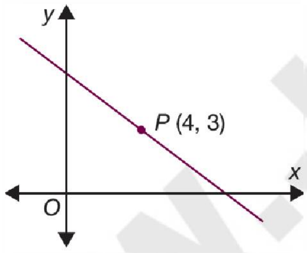
Sum of the area of all the triangles

$$=\frac{\sqrt{3}}{4}\left(1^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^2+\ldots\infty\right)=\frac{\sqrt{3}}{4}\left(\frac{1}{1-\frac{1}{4}}\right)=\frac{1}{\sqrt{3}}$$

ILLUSTRATION 11

Find the number of straight line(s) passing through the point $P(4, 3)$, whose x-intercept is a prime number and whose y-intercept is a positive integer.

SOLUTION



(If $m > 0$, then either x or y intercept must be negative)

Let $(y - 3) = m(x - 4)$

Now, x-intercept = $4-\frac{3}{m}$ and y-intercept = $3 - 4m$

For x-intercept to be positive prime and y-intercept to be positive integer, the possible values of m are

$m = -1; x = 7; y = 7$

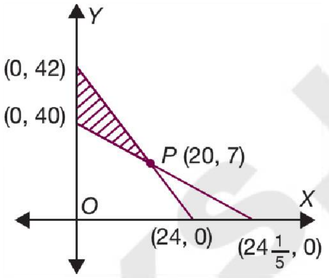
$m = -3; x = 5; y = 15$

ILLUSTRATION 12

Find the area of the triangular region in the Ist quadrant bounded on the left by the y-axis, bounded below the line $7x + 4y = 168$ and bounded above by the line $5x + 3y = 121$.

SOLUTION

From the figure, it is clear that, the lines intersect at $P(20, 7)$.



$\therefore$ Area of shaded region

$$=\frac{1}{2}\left(42-40\frac{1}{3}\right)20$$

$$=\frac{1}{2}\left(\frac{5}{3}\right)20=\frac{50}{3}\text{ sq. units}$$

ILLUSTRATION 13

Find the number of points which are equidistant from the lines $3x + 4y = 0$, $5x - 12y = 0$ and $y - 15 = 0$.

SOLUTION

As these lines are forming a triangle, there are 4 required points.

$\therefore$ There is an incentre (inside the triangle) and 3 excentres (outside the triangle).

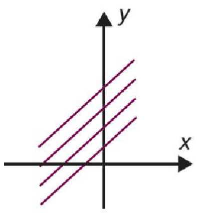
ILLUSTRATION 14

If m and b are real numbers and $mb > 0$, then find the quadrant in which the intercept of the line $y = mx + b$ between the co-ordinate axes may lie.

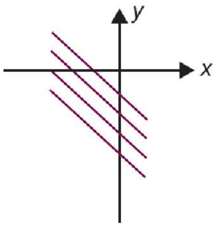
SOLUTION

$mb > 0 \Rightarrow m > 0$ and $b > 0$ or $m < 0$ and $b < 0$.

Hence, the possible lines are as shown in the figures.



$m > 0$ and $b > 0$



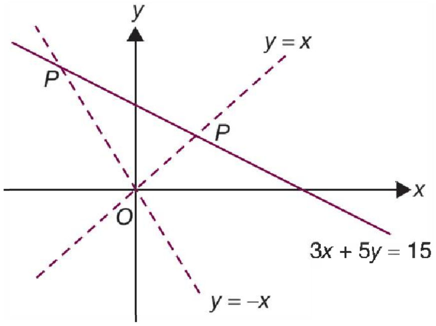
$m < 0$ and $b < 0$

Segment of straight line between the co-ordinate axes may lie in IInd or IIIrd quadrants.

ILLUSTRATION 15

Point 'P' lies on the line $\lambda = \{(x, y) \mid 3x + 5y = 15\}$. If 'P' is also equidistant from the co-ordinate axes, then locate the point P.

SOLUTION



Note that P can lie either on $y = x$ or $y = -x$.

CASE-1:

Let $P(x, y)$ lies on the line $x = y$.

$$3x + 5x = 15$$

$$\Rightarrow \quad x = \frac{15}{8}$$

Hence, $P\left(\frac{15}{8}, \frac{15}{8}\right)$ lies in I$^{\text{st}}$ quadrant.

CASE-2:

Let P lies on $y = -x$ and the line, we get $x = \frac{-15}{2}$.

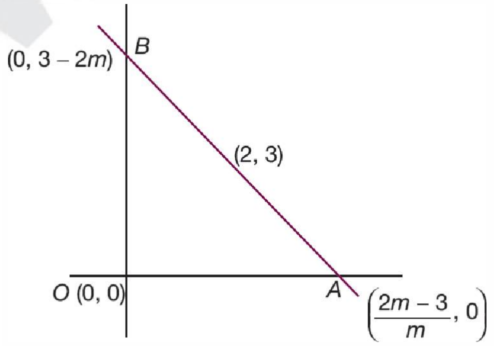
Hence, $P = \left(\frac{-15}{2}, \frac{15}{2}\right)$ lies in II$^{\text{nd}}$ quadrant.

$\therefore$ P may lie on I$^{\text{st}}$ or II$^{\text{nd}}$ quadrants.

ILLUSTRATION 16

Find the number of possible straight lines, passing through (2, 3) and forming a triangle with co-ordinate axes, whose area is 12 sq. units.

SOLUTION



Equation of any line through (2, 3) is

$$y - 3 = m(x - 2)$$

$$\Rightarrow \quad y = mx - 2m + 3$$

With the help of the figure, area of $\Delta OAB = \pm 12$ sq. units.

$$\Rightarrow \quad \frac{1}{2}\left(\frac{2m-3}{m}\right)(3 - 2m) = \pm 12$$

Taking positive sign we get, $(2m + 3)^2 = 0$

This gives one value of m, i.e., $-\frac{3}{2}$

Taking negative sign we get, $4m^2 - 36m + 9 = 0 \; (D > 0)$

Quadratic in m gives two values of m.

Hence, 3 straight lines are possible.

INTERSECTION OF TWO LINES

Let the equations of two lines be

$$a_1x + b_1y + c_1 = 0 \qquad \ldots(1)$$

$$a_2x + b_2y + c_2 = 0 \qquad \ldots(2)$$

Let these two lines intersect at a point $P(x_1, y_1)$. Then (x_1, y_1) satisfies each of the given equations, $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$.

Solving these two equations,

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Hence, the co-ordinates of the point of intersection are $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

ILLUSTRATION 17

For a, b and $c > 0$, if the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$, then find their relation.

SOLUTION

By solving the given lines for their point of intersection, we get the point of intersection as $\left(\frac{-c}{(a+b)}, \frac{-c}{(a+b)}\right)$.

Its distance from (1, 1) is $\sqrt{\left(1 + \frac{c}{a+b}\right)^2 + \left(1 + \frac{c}{a+b}\right)^2}$

$< 2\sqrt{2}$ (Given)

$$\Rightarrow \quad \sqrt{2}\left|\frac{a+b+c}{a+b}\right| < 2\sqrt{2}$$

$$\Rightarrow \quad (a+b+c)^2 < 4(a+b)^2 \text{ or } (a+b+c)^2 - (2a+2b)^2 < 0$$

$$\Rightarrow \quad (c-a-b)(c+3a+3b) < 0$$

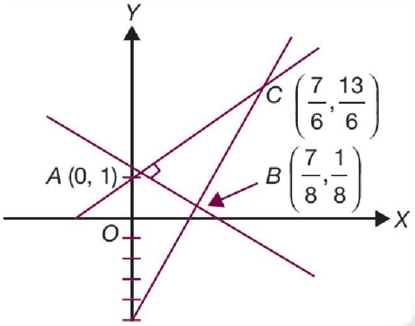
$$(\because\ a, b, c > 0,\ 3a+3b+c > 0)$$

$$\therefore \quad (c-a-b) < 0 \text{ or } (a+b-c) > 0 \text{ or } a+b > c$$

ILLUSTRATION 18

Find the orthocentre and circumcentre of the triangle whose sides are $x + y = 1$, $x - y + 1 = 0$, $7x - y = 6$.

SOLUTION



Y
C $\left(\frac{7}{6}, \frac{13}{6}\right)$
A (0, 1)
B $\left(\frac{7}{8}, \frac{1}{8}\right)$
O
X

Given lines of ΔABC are

$AB: x + y - 1 = 0$

$BC: 7x - y - 6 = 0$

$CA: x - y + 1 = 0$

Solve and get vertices as

$$A = (0, 1),\ B = \left(\frac{7}{8}, \frac{1}{8}\right),\ C = \left(\frac{7}{6}, \frac{13}{6}\right)$$

$$\angle BAC = 90^\circ$$

We get orthocentre as $(0, 1)$ = vertex A

$$\text{Circumcentre} = \text{Midpoint of } B \text{ and } C = \left(\frac{49}{48}, \frac{55}{48}\right)$$

ILLUSTRATION 19

Find the coordinates of the centroid, circumcentre and orthocentre of the triangle formed by the lines $3x - 2y = 6$, $3x + 4y + 12 = 0$ and $3x - 8y + 12 = 0$.

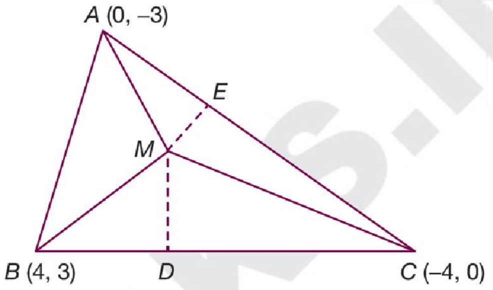
SOLUTION

To find the centroid

Solving the given equations, we get, $A\ (0, -3)$, $B\ (4, 3)$ and $C\ (-4, 0)$.

We know that centroid $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$$= \left(\frac{0+4-4}{3}, \frac{-3+3+0}{3}\right) = (0, 0)$$



To find the circumcentre

Let $M\ (\alpha, \beta)$ be the circumcentre.

Then $MA = MB = MC$

$$\therefore \quad (\alpha - 0)^2 + (\beta + 3)^2 = (\alpha - 4)^2 + (\beta - 3)^2$$

$$= (\alpha + 4)^2 + (\beta - 0)^2$$

$$\alpha^2 + \beta^2 + 6\beta + 9 = \alpha^2 + \beta^2 - 8\alpha - 6\beta + 25$$

$$= \alpha^2 + \beta^2 + 8\alpha + 16$$

$$\Rightarrow \quad 6\beta + 9 = -8\alpha - 6\beta + 25 = 8\alpha + 16$$

$$\therefore \quad 6\beta + 9 = -8\alpha - 6\beta + 25 \text{ and } 6\beta + 9 = 8\alpha + 16$$

$$\Rightarrow \quad 2\alpha + 3\beta - 4 = 0 \text{ and } 8\alpha - 6\beta + 7 = 0$$

Solving them, we get $\alpha = \frac{1}{12}, \beta = \frac{23}{18}$

$$\therefore \quad \text{Circumcentre} = \left(\frac{1}{12}, \frac{23}{18}\right)$$

To find the orthocentre

To find the orthocentre H, we know that the line joining orthocentre and circumcentre is divided by centroid in the ratio 2:1.

Let $H\ (\alpha, \beta)$ be the orthocentre.

$$\Rightarrow \quad 0 = \frac{\alpha + (2)\left(\frac{1}{12}\right)}{3} \Rightarrow \alpha = -\frac{1}{6};$$

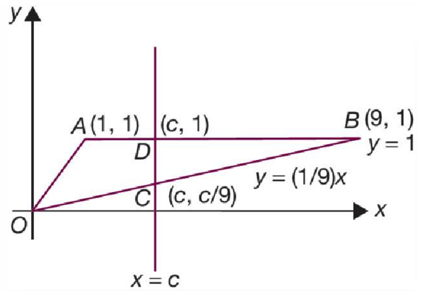
$$0 = \frac{\beta + (2)\left(\frac{23}{18}\right)}{3} \Rightarrow \beta = -\frac{23}{9}$$

$$\therefore \text{Orthocentre } H = \left(-\frac{1}{6}, -\frac{23}{9}\right)$$

ILLUSTRATION 20

The line $x = c$ cuts the triangle with the corners (0, 0), (1, 1) and (9, 1) into two regions. For the area of the two regions to be the equal, find c.

SOLUTION



Area of $\Delta BCD = \dfrac{1}{2}$ area of ΔOAB

$$2\cdot\frac{1}{2}\cdot\begin{vmatrix} 9 & 1 & 1 \\ c & 1 & 1 \\ c & \frac{c}{9} & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} 9 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \pm 4$$

We get $c^2 - 18c + 117 = 0$; no real c

or $\quad c^2 - 18c + 45 = 0 \Rightarrow c = 3$ or $c = 15$

But $c \neq 15$ so correct value of c is 3.

Test Your Concepts-I

(Solutions on page H.23)

1. Find the equation of the line joining the points (–1, 3) and (4, –2).
2. Find the equation of a line with slope –1 and cutting off an intercept of 4 units on negative direction of y-axis.
3. Find the equation of the diagonal, through the origin of the quadrilateral formed by the lines $x = 0$, $y = 0$, $x + y = 1$ and $6x + y = 3$.
4. A line intersects the straight lines $5x - y - 4 = 0$ and $3x - 4y - 4 = 0$ at A and B, respectively. If a point $P(1, 5)$ on the line AB is such that $AP{:}PB = 2{:}1$ (internally), then find the point A.
5. Find the equation of the straight line, which passes through the point (1, 2) and makes an angle θ with the positive direction of the x-axis, where $\cos\theta = -\dfrac{1}{3}$.
6. The ends of the rod of length l moves on two mutually perpendicular lines, find the locus of the point on the rod, which divides it in the ratio $m_1 : m_2$.
7. Find the ratio in which the line joining the points $A(1, 2)$ and $B(-3, 4)$ is divided by the line $x + y - 5 = 0$.
8. Find the equation of the line, which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.
9. $ABCD$ is a variable rectangle having its sides parallel to fixed directions (say m). The vertices B and D lie on $x = a$ and $x = -a$ and A lies on the line $y = 0$. Find the locus of C.
10. Find the quadratic equation whose roots are the a and b intercepts of the line passing through (1, 1) and making a triangle of area A with the co-ordinate axes.
11. Find the locus of the point of intersection of the lines $x\cos\alpha + y\sin\alpha = a$, and $x\sin\alpha - y\cos\alpha = b$, where α is a variable.
12. Find the equation of the straight line, which passes through the point (–5, 4) and is such that the portion of it between the axes is divided by the given point in the ratio 1:2.
13. A variable line through the point $\left(\dfrac{6}{5}, \dfrac{6}{5}\right)$ cuts the co-ordinate axes at the points A and B. If the point P divides AB internally in the ratio 2:1, show that the locus of P is $5xy = 2(2x + y)$.
14. Find the locus of the circumcentre of a triangle, whose two sides are along the co-ordinate axes and third side passes through the point of intersection of the lines $ax + by + c = 0$ and $lx + my + n = 0$.

Answer Keys

1. $x+y-2=0$

2. $x+y+4=0$

3. $3x-2y=0$

4. $A\left(\frac{75}{17},\frac{307}{17}\right)$

5. $\sqrt{8}x+y-\sqrt{8}-2=0$

6. $m_1^2x^2+m_2^2y^2=\left(\frac{m_1m_2\ell}{m_1+m_2}\right)^2$

7. 1 : 2 externally

8. $x+y=7$ and $4x+3y=24$

9. $(1-m^2)x+my=(m^2+1)a$

10. $x^2-2Ax+2A=0$

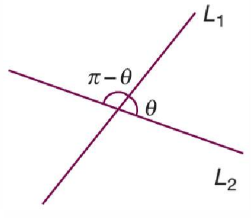
11. $x^2+y^2=a^2+b^2$

12. $-\frac{2x}{15}+\frac{y}{12}=1$

14. $2xy(ma-bl)+x(an-lc)+y(mc-bn)=0$

ANGLE BETWEEN TWO LINES

If θ be the acute angle between the two lines $y=m_1x+c_1$ and $y=m_2x+c_2$, then $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$



NOTES:

(i) There are two angles formed between the two lines, i.e., θ and π-θ.

(ii) For parallel or coincident lines, $\theta=0$

$\Rightarrow\ m_1=m_2$

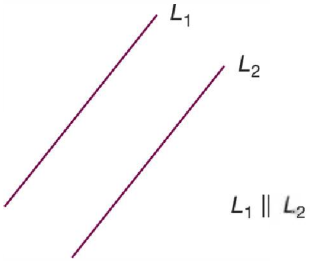
(iii) For perpendicular lines, $\theta=90°$

$\Rightarrow\ m_1m_2=-1$

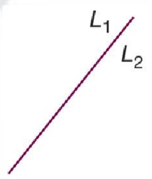
Nature of Straight Lines

Lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are

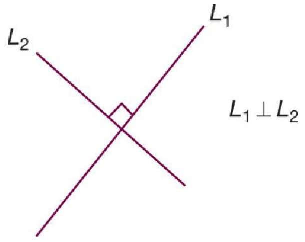
(i) **Parallel**, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$



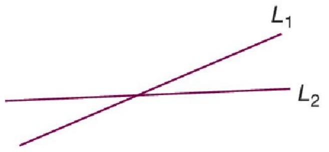
(ii) **Coincident**, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$



(iii) **Perpendicular**, if $a_1a_2+b_1b_2=0$



(iv) **Intersecting at one point**, if $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$



Equation of Parallel Straight Lines

Equation of a line parallel to a line $ax+by+c=0$ is $ax+by+\lambda=0$, where λ is a real parameter.

Equation of Perpendicular Straight Line

Equation of a line perpendicular to a line $ax+by+c=0$ is $bx-ay+k=0$, where k is a real parameter.

Straight Line Making a Given Angle with a Line

Equations of lines passing through a point (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as, $y - y_1 = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$

Computation of Internal Angles of Triangle

Let equation of sides of a triangle are given as $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$.
Get the slopes of straight lines and arrange them in decreasing order.

Let m_1, m_2 and m_3 are the slopes of 3 lines $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$, where $m_1 > m_2 > m_3$, then the interior angles of the ΔABC found by these lines are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1m_2},\ \tan B = \frac{m_2 - m_3}{1 + m_2m_3} \text{ and}$$

$$\tan C = \frac{m_3 - m_1}{1 + m_3m_1}$$

COROLLARY:

If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the sides of a triangle, then the area of the triangle is given by,

$$\Delta = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2 and C_3 are the cofactors of elements c_1, c_2, c_3 in the above determinant.
(We learn about the cofactors, in the determinant chapter in Algebra)

ILLUSTRATION 21

If the angle between the lines is $\dfrac{\pi}{4}$ and slope of one of the lines is $\dfrac{1}{2}$, then find the slope of the other line.

SOLUTION

$m_1 = \dfrac{1}{2}$, $m_2 = m$ and the angle between the lines is $\dfrac{\pi}{4}$

$$\tan 45° = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right| = 1$$

$\Rightarrow \quad 2m - 1 = \pm(m + 2)$

By taking positive sign, we get $m = 3$

By taking negative sign, we get $m = -\dfrac{1}{3}$; so we get slopes as $3, -\dfrac{1}{3}$

ILLUSTRATION 22

If the line L, which is perpendicular to the line with equation $y = 3x - 5$, contains the point $(1, 4)$, then find the x-intercept of L.

SOLUTION

Slope of given line = 3; slope of line $L = \dfrac{-1}{3}$

Required line L: $(y - 4) = \dfrac{-1}{3}(x - 1)$

Putting $y = 0$, we get $x = 13$.

ILLUSTRATION 23

The perpendicular bisector of the line segment joining $A(1, 4)$ and $B(t, 3)$ has y-intercept as -4, then find the possible value of t.

SOLUTION

Slope of $AB = \dfrac{4-3}{1-t} = \dfrac{1}{1-t}$

Slope of bisector $= t - 1$

Midpoint $= \left(\dfrac{t+1}{2}, \dfrac{7}{2}\right)$

$\therefore$ Equation of bisector is $\left(y - \dfrac{7}{2}\right) = (t-1)\left(x - \left(\dfrac{t+1}{2}\right)\right)$

By putting $x = 0$ and $y = -4$.

$\Rightarrow \quad -\dfrac{15}{2} = -\dfrac{(t^2 - 1)}{2} \Rightarrow t^2 = 16$

$\Rightarrow \quad t = \pm 4$

ILLUSTRATION 24

Find the equation of the straight line, which passes through the origin and making angle 60° with the line $x + \sqrt{3}\,y + 3\sqrt{3} = 0$.

SOLUTION

The given line is $x + \sqrt{3}y + 3\sqrt{3} = 0$.

$\Rightarrow \quad y = \left(-\dfrac{1}{\sqrt{3}}\right)x - 3 \quad \therefore$ Slope of given line $= -\dfrac{1}{\sqrt{3}}$.

Let slope of the required line be m. Also, between these lines is given to be 60°.

$$\Rightarrow \quad \tan 60^\circ = \left|\frac{m-\left(-1/\sqrt{3}\right)}{1+m\left(-1/\sqrt{3}\right)}\right|$$

$$\Rightarrow \quad \sqrt{3} = \left|\frac{\sqrt{3}m+1}{\sqrt{3}-m}\right| \Rightarrow \frac{\sqrt{3}m+1}{\sqrt{3}-m} = \pm\sqrt{3} \qquad \ldots(1)$$

Take positive sign

$$\frac{\sqrt{3}m+1}{\sqrt{3}-m} = \sqrt{3}$$

$$\Rightarrow \quad \sqrt{3}\,m + 1 = 3 - \sqrt{3}\,m \Rightarrow m = \frac{1}{\sqrt{3}}$$

Using $y = mx + c$, the equation of the required line is $y = \frac{1}{\sqrt{3}}x + 0$, i.e., $x - \sqrt{3}\,y = 0$. ($\because$ This passes through origin, so $c = 0$)

From (1), take negative sign

$$\frac{\sqrt{3}m+1}{\sqrt{3}-m} = -\sqrt{3} \Rightarrow \sqrt{3}\,m + 1 = -3 + \sqrt{3}\,m$$

$\Rightarrow \quad m$ is not defined.

$\therefore$ The slope of the required line is not defined. Thus, the required line is a vertical line. This line is to pass through the origin.

$\therefore$ The equation of the required line is $x = 0$.

ILLUSTRATION 25

If $x + y = 0$, $4x + 5y = 0$ and $2x + y = 8$ are the sides of a ΔABC, then show that the line $x - 2y = 0$ passes through orthocentre of ΔABC.

SOLUTION

$x - 2y = 0$ is perpendicular to $2x + y = 8$ and also passes through origin which is the vertex opposite to third side. Hence, $x - 2y = 0$ is an altitude.

$\therefore$ Orthocentre lies on altitude.

ILLUSTRATION 26

Find the orthocentre of the triangle formed by the lines $x - 7y + 6 = 0$, $2x - 5y - 6 = 0$ and $7x + y - 8 = 0$.

SOLUTION

As the lines $x - 7y + 6 = 0$ and $7x + y - 8 = 0$ are perpendicular, the triangle is right angled.

$\therefore$ Hence, the orthocentre is the point of intersection of the lines $x - 7y + 6 = 0$ and $7x + y - 8 = 0$.

$\therefore$ Orthocentre is $H(1, 1)$.

ILLUSTRATION 27

Find the equations of the straight lines passing through the point (2, 3) and inclined at $\frac{\pi}{4}$ radians to the line $2x + 3y = 5$.

SOLUTION

Let the line $2x + 3y = 5$ makes angle θ with the positive x-axis. Then, $\tan\theta = -\frac{2}{3}$

$$\left|\frac{m-(-2/3)}{1+m(-2/3)}\right| = \tan 45^\circ = 1$$

$$\Rightarrow \quad 3m + 2 = \pm(3 - 2m)$$

On solving, we get, $m = \frac{1}{5}$ and -5

$\therefore$ The equations of the required lines are

$$y - 3 = \frac{1}{5}(x - 2) \Rightarrow x - 5y + 13 = 0 \text{ and}$$

$$y - 3 = -5(x - 2) \Rightarrow 5x + y - 13 = 0$$

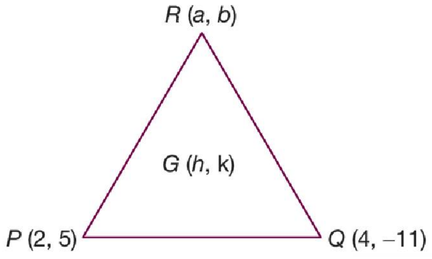
ILLUSTRATION 28

If the vertices P and Q of a ΔPQR are given by (2, 5) and (4, –11) respectively and the point R moves along the line N: $9x + 7y + 4 = 0$, then show that the locus of the centroid of the ΔPQR is a straight line parallel to the line N.

SOLUTION

$R(a, b)$ lies on the line $9x + 7y + 4 = 0$.

$$\Rightarrow \quad R\left(a, \frac{-(4+9a)}{7}\right), \text{ centroid of } \Delta PQR = (h, k)$$



$$h = \left(\frac{2+4+a}{3}\right) = \frac{6+a}{3} \qquad \ldots(1)$$

$$k = \frac{5 - 11 - \frac{(4+9a)}{7}}{3} = \frac{-46-9a}{7\times 3} \qquad \ldots(2)$$

From (1) and (2), we get, $3h - 6 = \dfrac{(-21k - 46)}{9}$

$\Rightarrow \quad 27h + 21k - 54 + 46 = 0$

$\therefore$ Locus is $9x + 7y - \dfrac{8}{3} = 0$ which is a straight line | | to N.

ILLUSTRATION 29

Find the equation to the sides of an isosceles right angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

SOLUTION

We need to find out equations to the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -\dfrac{3}{4}$.

$$\Rightarrow \quad \tan 45° = \pm\left(\frac{m - m_1}{1 + m_1 m_2}\right), \text{ i.e., } 1 = \pm\left(\frac{m + \frac{3}{4}}{1 - \frac{3}{4}m}\right)$$

On solving, we get,

$$m_A = \frac{1}{7} \quad \text{and} \quad m_B = -7$$

Hence, the required equations of the two lines are $y - 2 = mA\,(x - 2)$ and $y - 2 = mB\,(x - 2)$.

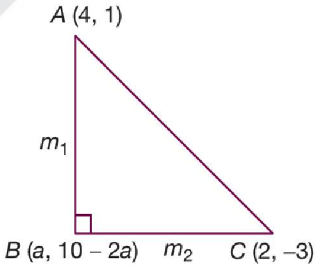
$\Rightarrow \quad 7y - x - 12 = 0$ and $7x + y = 16$

ILLUSTRATION 30

The vertex of a right angle of a right angled triangle lies on the straight line $2x + y - 10 = 0$ and the two other vertices, at points (2, –3) and (4, 1), then find the area of triangle.

SOLUTION

Let slope of AB is m_1 and BC is m_2



$$m_1 m_2 = -1; \ \frac{9 - 2a}{a - 4} \times \frac{13 - 2a}{a - 2} = -1$$

$$117 - 26a - 18a + 4a^2 = -(a^2 - 6a + 8)$$

$$5a^2 - 50a + 125 = 0 \Rightarrow 5(a - 5)^2 = 0$$

$\Rightarrow \quad a = 5$ so B is (5, 0)

$\therefore \quad \text{Area} = \dfrac{1}{2} AB \times AC = \dfrac{1}{2}\sqrt{2} \times 3\sqrt{2} = 3$ sq. units.

ILLUSTRATION 31

If a line in the xy plane is parallel to the line $2x + 7y = 3$ and passes through the point (3, – 1), then find the y-intercept of the line.

SOLUTION

An equation of the parallel line is $2x + 7y = c$.

Since (3, –1) is on the line, $c = (2)(3) + 7(-1) = -1$

Thus, the equation is $2x + 7y = -1$. When $x = 0$, we get the y-intercept as $y = \dfrac{-1}{7}$

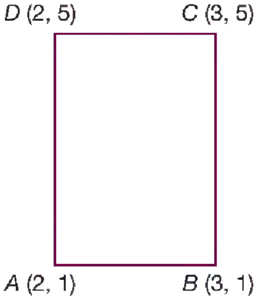
ILLUSTRATION 32

If the equations to a pair of opposite sides of a parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$, then find the equations to its diagonals.

SOLUTION

The sides of parallelogram are $x = 2$, $x = 3$, $y = 1$, $y = 5$.

$\therefore$ Vertices of the parallelogram are as shown in the figure.



(This parallelogram is a rectangle)

The equation of diagonal AC is

$$\frac{y - 1}{5 - 1} = \frac{x - 2}{3 - 2} \Rightarrow y = 4x - 7$$

The equation of diagonal BD is

$$\frac{x - 2}{3 - 2} = \frac{y - 5}{1 - 5} \Rightarrow 4x + y = 13$$

ILLUSTRATION 33

Find the line passing through point $\left(-1, \frac{\pi}{2}\right)$ and perpendicular to straight line $\sqrt{3}\ \sin\theta + 2\cos\theta = \frac{4}{r}$.

SOLUTION

Perpendicular to $\sqrt{3}\sin\theta + 2\cos\theta = \frac{4}{r}$ is

$$\sqrt{3}\sin\left(\frac{\pi}{2}+\theta\right) + 2\cos\left(\frac{\pi}{2}+\theta\right) = \frac{k}{r}$$

It is passing through $\left(-1, \frac{\pi}{2}\right)$

$$\therefore \quad \sqrt{3}\sin\pi + 2\cos\pi = \frac{k}{-1}$$

$$\Rightarrow \quad k = 2$$

$$\therefore \quad \sqrt{3}\cos\theta - 2\sin\theta = \frac{2}{r}$$

$$\Rightarrow \quad \sqrt{3}\ r\cos\theta - 2r\sin\theta = 2$$

Alternate

Point $A\left(-1, \frac{\pi}{2}\right) = (0, -1)$

Given line, $\sqrt{3}r\ \sin\theta + 2r\cos\theta = 4$

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\Rightarrow \quad \sqrt{3}\ y + 2x = 4$$

$$m = -\frac{2}{\sqrt{3}}$$

Required line is $y + 1 = \frac{\sqrt{3}}{2}(x - 0)$

$$\Rightarrow \quad 2y + 2 = \sqrt{3}x$$

$$\Rightarrow \quad \sqrt{3}r\cos\theta - 2r\sin\theta = 2$$

ILLUSTRATION 34

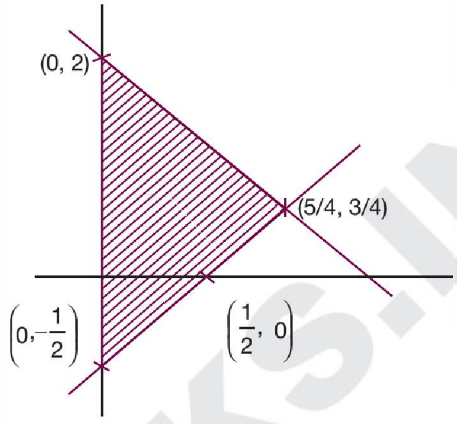
Find the area of the triangle formed by y-axis, the straight line L passing through the points (1, 1) and (2, 0) and the straight line perpendicular to the line L and passing through the points $\left(\frac{1}{2}, 0\right)$.

SOLUTION

The line L is $x + y = 2$ ...(1)

The line perpendicular to L and passing through $\left(\frac{1}{2}, 0\right)$ is

$$x - y = \lambda \text{ or } x - y = \frac{1}{2} \qquad \text{...(2)}$$



At y-axis, $x = 0$ ...(3)

Solving them in pairs, we get the points

$$(0, 2), \left(0, -\frac{1}{2}\right) \text{ and } \left(\frac{5}{4}, \frac{3}{4}\right)$$

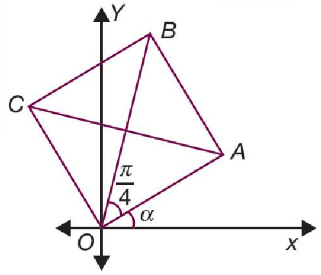
$\therefore$ The required area is $\frac{1}{2} \cdot \frac{5}{4}\left(2 + \frac{1}{2}\right) = \frac{25}{16}$ sq. units.

ILLUSTRATION 35

A square of side a lies above the x-axis and has one vertex at the origin. If the side passing through the origin makes an angle $\alpha\left(0 < \alpha < \frac{\pi}{4}\right)$ with the positive direction of x-axis, then find the equation of its diagonal not passing through the origin.

SOLUTION

The co-ordinates of A are $(a\cos\alpha, a\sin\alpha)$. The equation of OB is $y = \tan\left(\frac{\pi}{4}+\alpha\right)x$



$$\Rightarrow \quad CA \perp OB$$

$$\therefore \quad \text{Slope of } CA = -\cot\left(\frac{\pi}{4}+\alpha\right)$$

The equation of CA is

$$y - a\sin\alpha = -\cot\left(\frac{\pi}{4}+\alpha\right)(x - a\cos\alpha)$$

$$\Rightarrow \quad y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a$$

ILLUSTRATION 36

A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$. Find the equations of the other two sides.

SOLUTION

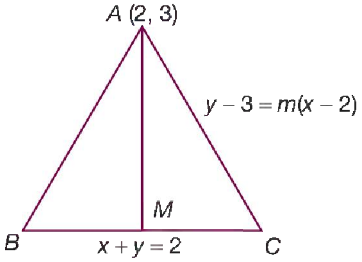
Other sides pass through the vertex (2, 3).

∴ The equations of other sides will be of the form

$$y - 3 = m(x - 2) \quad \ldots(1)$$

The equation of the third side is

$$x + y = 2 \quad \ldots(2)$$



Being equilateral triangle, the angle between these two lines is 60°.

$$\therefore \quad \tan 60^\circ = \left|\frac{m-(-1)}{1+m(-1)}\right| \text{(By angle formula)}$$

$$\Rightarrow \quad \sqrt{3} = \left|\frac{m+1}{1-m}\right|$$

$$\therefore \quad \frac{m+1}{1-m} = \pm\sqrt{3}$$

$$\therefore \quad m + 1 = \sqrt{3}(1 - m) \text{ or } m + 1 = \sqrt{3}(m - 1)$$

$$\Rightarrow \quad (1 + \sqrt{3})m = \sqrt{3} - 1 \text{ or } (1 - \sqrt{3})m = -\sqrt{3} - 1$$

$$\Rightarrow \quad m = \frac{\sqrt{3}-1}{\sqrt{3}+1}, \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}-1)^2}{2}, \frac{(\sqrt{3}+1)^2}{2}$$

$$m = 2 - \sqrt{3} \text{ or } 2 + \sqrt{3}$$

∴ The equations of the other two sides are

$$y - 3 = (2 - \sqrt{3})(x - 2) \text{ and } y - 3 = (2 + \sqrt{3})(x - 2).$$

ILLUSTRATION 37

If the lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line, then find p.

SOLUTION

Given lines are perpendicular to a common line then the lines must be parallel.

∴ Slopes are equal.

So, $p(p^2 + 1) = -(p^2 + 1) \Rightarrow p^3 + p + p^2 + 1 = 0$

$\Rightarrow (p^2 + 1)(p + 1) = 0$

$\Rightarrow \quad p = -1$

Test Your Concepts-II

(Solutions on page H.25)

1. Find the equation of a straight line, which cuts off an intercept of length 3 units on y-axis and is parallel to the line joining the points (3, –2) and (1, 4).
2. Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$.
3. Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line $2x - 3y = 7$.
4. Consider the ΔOAB, where 'O' is the origin. If $B(3, 4)$ and orthocentre of the triangle is $P(1, 4)$, find the co-ordinates of A.
5. Find the equation to the straight line, which bisects and is perpendicular to the straight line joining the points $A(a, b)$ and $B(a', b')$.
6. If k be a non-zero real number, then comment on nature of the two lines with equations $y - 3x + k = 0$ and $6x - 2y = k$.
7. The straight line $y = x - 2$ rotates about a point, where it cuts the x-axis and becomes perpendicular to the straight line $ax + by + c = 0$, then find its equation.

Answer Keys

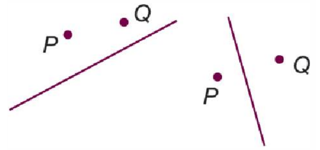
1. $3x + y - 3 = 0$ **2.** $2x - 3y + 6 = 0$ **3.** $2x - 3y + 12 = 0$ **4.** $\left(0, \frac{19}{4}\right)$

5. $2(b' - b)y + 2(a' - a)x = (b'^2 - b^2 + a'^2 - a^2)$ **6.** Parallel **7.** $ay - bx + 2b = 0$

POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE

Straight line is not a closed curve so we cannot tell position of one point but we can talk about relative position of two points.

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.



$(ax_1 + by_1 + c)(ax_2 + by_2 + c) > 0$	Both points are on same side of the line
$(ax_1 + by_1 + c)(ax_2 + by_2 + c) < 0$	Both points are on opposite side of the line

ILLUSTRATION 38

Find the range of θ in the interval $(0, \pi)$ such that the points $(3, 5)$ and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y - 1 = 0$.

SOLUTION

Required condition is

$$(3 + 5 - 1)(\sin\theta + \cos\theta - 1) > 0$$

$$\Rightarrow \quad \sin\theta + \cos\theta > 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta > \frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin\left(\frac{\pi}{4} + \theta\right) > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{\pi}{4} < \frac{\pi}{4} + \theta < \frac{3\pi}{4}$$

$$\Rightarrow \quad 0 < \theta < \frac{\pi}{2}$$

ILLUSTRATION 39

If the point $P\left(\frac{3a}{2}, 1\right)$ lies between the two different lines $x + y = a$ and $x + y = 2a$, then find the least positive integral value of a.

SOLUTION

Point $\left(\frac{3a}{2}, 1\right)$ lies between two different lines, $x + y = a$ and $x + y = 2a$.

$$\therefore \left(\frac{3}{2}a + 1 - a\right)\left(\frac{3}{2}a + 1 - 2a\right) < 0 \text{ or } (a + 2)(2 - a) < 0$$

or $a \in (-\infty, -2) \cup (2, \infty)$

ILLUSTRATION 40

Let u, v and w satisfy the equations $uvw = -6$, $uv + vw + wu = -5$ and $u + v + w = 2$, where $u > v > w$, then find 'a' for which the points $P(u, -w)$ and $Q(v, a^2)$ lie on the same side of the line $4x - y + 5 = 0$.

SOLUTION

The equation whose roots are u, v and w is $x^3 - (u + v + w)x^2 + (uv + vw + wu)x - uvw = 0$.

$$\Rightarrow \quad x^3 - 2x^2 - 5x + 6 = 0$$

$$\therefore \quad x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$$

$$= (x - 1)(x - 3)(x + 2)$$

$$\therefore \quad x = 1, 3, -2$$

$$\therefore \quad u = 3, v = 1, w = -2 \text{ (As } u > v > w)$$

$$\Rightarrow \quad P(u, -w) = (3, 2) \text{ and } Q(v, a^2) = (1, a^2)$$

Now the given line is $4x - y + 5 = 0$

$$\therefore \quad (4(3) - 2 + 5)(4(1) - a^2 + 5) > 0$$

$$\Rightarrow \quad 15(9 - a^2) > 0$$

$$\Rightarrow \quad a^2 - 9 < 0$$

$$\therefore \quad a \in (-3, 3)$$

ILLUSTRATION 41

Given the lines $y + 2x = 3$ and $y + 2x = 5$ cut the axes at A, B and C, D respectively. Find the area of quadrilateral $ABDC$ and locate the point $(2, 3)$.

SOLUTION

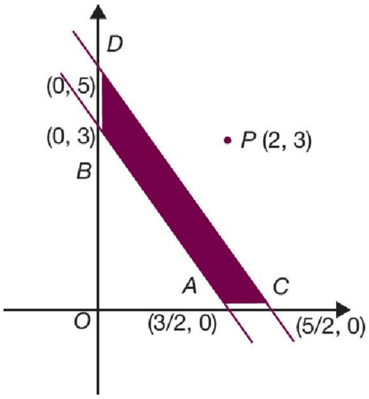
$$2x + y - 3 = 0$$

$$2x + y - 5 = 0$$

For $P(2, 3)$

$$(4 + 3 - 3)(4 + 3 - 5) > 0$$

$\therefore$ P lies outside the quadrilateral.



Area of quadrilateral

$$= \text{Area of } \Delta OCD - \text{Area of } \Delta OAB$$

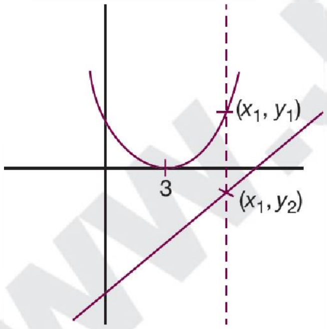
$$= \frac{1}{2}\left(\frac{5}{2}\right)5 - \frac{1}{2}\left(\frac{3}{2}\right)3$$

$$= \frac{25-9}{4} = 4$$

ILLUSTRATION 42

Find the least positive integral value of b for which the point $(2b+3, b^2)$ lies above the line $3x - 4y - a(a-2) = 0$ $\forall a \in R$.

SOLUTION



$(2b+3, b^2)$ lies on $(x-3)^2 = 4y$

(x_1, y_2) lies on line

$\therefore \quad 3x_1 - 4y_2 - a(a-2) = 0 \Rightarrow 4y_2 = 3x_1 - a(a-2)$

Now, $y_2 < y_1 \Rightarrow \dfrac{3x_1 - a(a-2)}{4} < y_1$

Put $\quad x_1 = 2b+3$ and $y_1 = b^2$

$\Rightarrow \quad 3(2b+3) - a(a-2) < 4b^2$

$\Rightarrow \quad a^2 - 2a + 4b^2 - 6b - 9 > 0 \ \forall a \in R$

$\because \quad D < 0 \Rightarrow B^2 - 4AC < 0$

$\therefore \quad 4 - 4(4b^2 - 6b - 9) < 0$

$\Rightarrow \quad 1 - 4b^2 + 6b + 9 < 0$

$\Rightarrow \quad 4b^2 - 6b - 10 > 0$

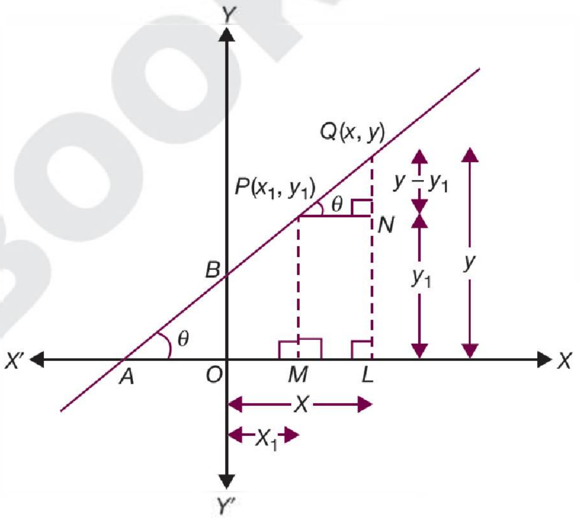
$\Rightarrow \quad 2b^2 - 3b - 5 > 0$

$\Rightarrow \quad (2b-5)(b+1) > 0$

$\Rightarrow \quad b \in (-\infty, -1) \cup \left(\frac{5}{2}, \infty\right)$

Hence, the least positive integral value 'b' is 3.

PARAMETRIC FORM OF STRAIGHT LINE



Let the given line meets x-axis at A and y-axis at B and passes through the point $P(x_1, y_1)$. Let $Q(x, y)$ be any point on the line at a distance r from $P(x_1, y_1)$. Then

$$PN = ML = OL - OM = x - x_1$$

and $\quad QN = QL - NL = y - y_1$

From ΔPQN, $\cos\theta = \dfrac{PN}{PQ} = \dfrac{x - x_1}{r}$

$$\Rightarrow \quad \cos\theta = \frac{x - x_1}{r} \qquad \ldots(1)$$

Again $\sin\theta = \dfrac{QN}{PQ}$

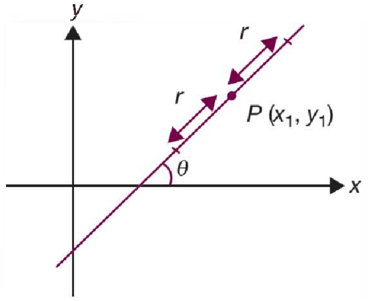
$$\Rightarrow \quad \sin\theta = \frac{y - y_1}{r} \qquad \ldots(2)$$

$\therefore$ From (1) and (2) we get,

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the equation of the line in the parametric form.

Working Rule



(i) The equation of the line is $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\pm r$, where $r \in (0, \infty)$

$\therefore\ x-x_1=\pm r\cos\theta$ and $y-y_1=\pm r\sin\theta$

$\Rightarrow\ x=x_1\pm r\cos\theta$ and $y=y_1\pm r\sin\theta$

Thus, the co-ordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$, where r is variable and θ is fixed.

(ii) If θ is **variable** and x_1, y_1 and r are **fixed**, then the same equation will represent a **circle** whose centre is (x_1, y_1) and radius is r.

ILLUSTRATION 43

A line inclined at an angle of 135° to be the x-axis passes through the point $A(3, 4)$. There are two points P and Q on the line such that $AP = AQ = 2\sqrt{2}$. Find the points P and Q.

SOLUTION

$$\frac{x-3}{\cos\theta}=\frac{y-4}{\sin\theta}=\pm 2\sqrt{2}$$

$$\Rightarrow\quad x=3\pm 2\sqrt{2}\cos\theta \text{ and } y=4\pm 2\sqrt{2}\sin\theta,\ \theta=135^\circ$$

We get the points (1, 6) and (5, 2), either of them as P or Q.

ILLUSTRATION 44

Find the equation of the straight line, which passes through the point (3, 2) and whose gradient is $\frac{3}{4}$.

Find the co-ordinates of the points on the lines that are 5 units away from the point (3, 2).

SOLUTION

Let the required line make an angle θ with the positive direction x-axis and pass through $P(3, 2)$.

$$\text{Slope } m=\frac{3}{4}$$

$$\tan\theta=\frac{3}{4}\ \left(\because\ \cos\theta=\frac{4}{5} \text{ and } \sin\theta=\frac{3}{5}\right)$$

Now, the parametric equation of the required line will be $\frac{x-3}{\frac{4}{5}}=\frac{y-2}{\frac{3}{5}}$ $\left(\text{by formula, } \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}\right)$.

Now, the co-ordinates of the points which are at a distance of 5 units from P are $(3 \pm 5\cos\theta, 2 \pm 5\sin\theta)$ or $(3 \pm 4), (2 \pm 3)$ or (7, 5) and (– 1, – 1).

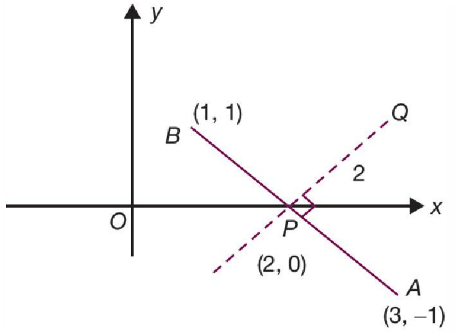
ILLUSTRATION 45

The midpoint of the line segment joining (3, –1) and (1, 1) is shifted by two units (in the sense of increasing y) perpendicular to the line segment. Find the co-ordinates of the point in the new position.

SOLUTION

Let P be the midpoint of the line segment joining $A(3, -1)$ and $B(1, 1)$.

$$\text{Then } P\left(\frac{3+1}{2}, \frac{-1+1}{2}\right)=(2, 0).$$



Let P be shifted to Q, where $PQ = 2$ and y-co-ordinate of Q is greater than that of P (from question).

$$\text{Now, slope of } AB=\frac{1-(-1)}{1-3}=\frac{2}{-2}=-1$$

$\therefore$ Slope of $PQ = 1 = \tan\theta$; $\theta = 45^\circ$

Co-ordinates of Q by distance formula $= (2 \pm 2\cos\theta, 0 \pm 2\sin\theta)$

$$= \left(2 \pm 2\cdot\frac{1}{\sqrt{2}}, 0 \pm 2\cdot\frac{1}{\sqrt{2}}\right) = (2 \pm \sqrt{2}, \pm \sqrt{2})$$

As y-co-ordinate of Q is greater than that of P.

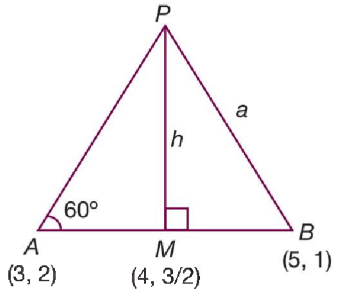
$Q = (2 + \sqrt{2}, \sqrt{2})$, which is the required point.

ILLUSTRATION 46

Let $A(3, 2)$ and $B(5, 1)$. ABP is an equilateral triangle constructed on the side of AB remote from the origin, then find the orthocentre of ΔABP.

SOLUTION

Let mid point of AB is M and PM is $\perp AB$.



$$m_{AB} = \frac{-1}{2}\ ;\ m_{PM} = 2 = \tan\theta$$

$$h = \frac{a\sqrt{3}}{2};\ \sin\theta = \frac{2}{\sqrt{5}},\ \cos\theta = \frac{1}{\sqrt{5}}$$

Now, $a = \sqrt{5} = AB;\ h = \dfrac{\sqrt{15}}{2}$

Equation of line PM: $\dfrac{x-4}{\cos\theta} = \dfrac{y-3/2}{\sin\theta} = h = \pm\dfrac{\sqrt{15}}{2}$

It is given that P is away from origin. So we take h as positive

$$P: x = 4 + \frac{\sqrt{3}}{2},\ y = \frac{3}{2} + \sqrt{3}$$

So, orthocentre or centroid is $\left(\dfrac{\Sigma x_i}{3}, \dfrac{\Sigma y_i}{3}\right)$

$$= \left(4 + \frac{\sqrt{3}}{6}, \frac{3}{2} + \frac{\sqrt{3}}{3}\right)$$

ILLUSTRATION 47

Find the equation of the line passing through the point $P(1, 2)$ and cutting the lines $x + y - 4 = 0$ and $2x - y = 1$ at A and B, respectively such that the harmonic mean of PA and PB is 10.

SOLUTION

Let the equation of the line passing through $P(1, 2)$ be

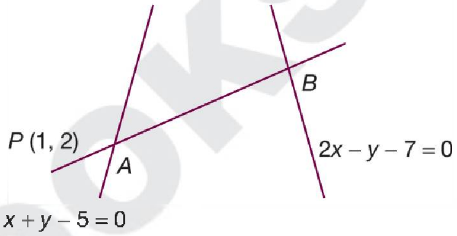
$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r \quad \text{...(1)}$$

Let $PA = r_1$, $PB = r_2$

Then,

$$A = (1 + r_1\cos\theta, 2 + r_1\sin\theta)$$

$$B = (1 + r_2\cos\theta, 2 + r_2\sin\theta)$$



As A is on the line $x + y - 4 = 0$,

$$1 + r_1\cos\theta + 2 + r_1\sin\theta - 4 = 0$$

$$\Rightarrow r_1(\cos\theta + \sin\theta) = 1$$

$$\therefore \frac{1}{r_1} = \cos\theta + \sin\theta \quad \text{...(2)}$$

As B is on the line $2x - y - 1 = 0$

$$\Rightarrow 2(1 + r_2\cos\theta) - (2 + r_2\sin\theta) - 1 = 0$$

$$\Rightarrow r_2(2\cos\theta - \sin\theta) = 1$$

$$\therefore \frac{1}{r_2} = 2\cos\theta - \sin\theta \quad \text{...(3)}$$

Now, H.M. of PA and PB is 10.

$$\therefore 10 = \frac{2PA\cdot PB}{PA + PB} = \frac{2r_1r_2}{r_1 + r_2}$$

$$\Rightarrow \frac{r_1 + r_2}{r_1r_2} = \frac{1}{5}$$

$$\therefore \frac{1}{r_2} + \frac{1}{r_1} = \frac{1}{5}$$

$$\Rightarrow \cos\theta + \sin\theta + 2\cos\theta - \sin\theta = \frac{1}{5} \text{ (From (2) and (3))}$$

$$\Rightarrow \cos\theta = \frac{1}{15} \Rightarrow \tan\theta = 4\sqrt{14}$$

Required line: $y - 2 = 4\sqrt{14}(x - 1)$

ILLUSTRATION 48

A variable line L is drawn through $O(0, 0)$ to meet the lines L_1: $2x + 3y = 5$ and L_2: $2x + 3y = 10$ at points P and Q, respectively. A point R is taken on L such that, $2OP \cdot OQ = OR \cdot OP + OR \cdot OQ$. Find locus of R.

SOLUTION

Let the line L be $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta} = r$

Let $OP = r_1$, then point $P(r_1 \cos q_1, r_1 \sin \theta)$

$OQ = r_2$, then point $Q(r_2 \cos \theta, r_2 \sin \theta)$

Point P and Q lie on lines L_1 and L_2 respectively

Then $OP = r_1 = \dfrac{5}{2\cos\theta + 3\sin\theta}$ and

$OQ = r_2 = \dfrac{10}{2\cos\theta + 3\sin\theta}$

Let, $OR = r$

Then, according to the condition,

$$2OP \times OQ = OR(OP + OQ)$$

$$\Rightarrow \quad 2r_1r_2 = r(r_1 + r_2)$$

$$\Rightarrow \quad r = \frac{2r_1r_2}{r_1 + r_2}$$

$$\Rightarrow \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{r}$$

After putting value of r_1 and r_2

$$20 = 6r \cos \theta + 9r \sin \theta$$

Point $R(x, y) = (r \cos \theta, r \sin \theta)$

$\therefore$ Locus is $6x + 9y = 20$.

ILLUSTRATION 49

A variable line $L = 0$ is drawn through $O(0, 0)$ to meet the lines L_1: $x + 2y - 3 = 0$ and L_2: $x + 2y + 4 = 0$ at points M and N, respectively. A point P is taken on $L = 0$ such that $\dfrac{1}{OP^2} = \dfrac{1}{OM^2} + \dfrac{1}{ON^2}$, then find the locus of P.

SOLUTION

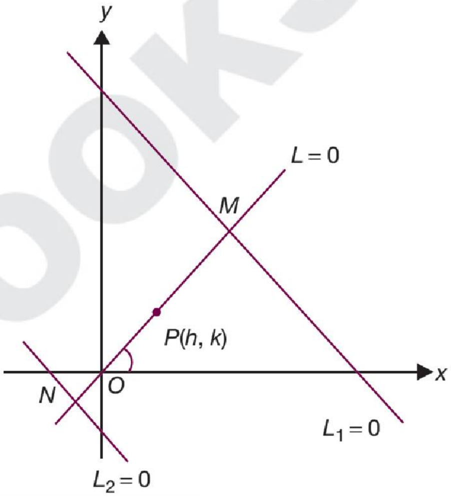
Let the parametric equation of the variable line is

$$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r$$

$\Rightarrow$ $x = r \cos \theta; y = r \sin \theta$

$\therefore$ Putting $(x = r \cos \theta; y = r \sin \theta)$ in $L_1 = 0$, we get,

$$\frac{1}{OM} = \frac{(\cos\theta + 2\sin\theta)}{3} \quad \ldots(1)$$



Similarly, putting the general point in $L_2 = 0$, we get,

$$\frac{1}{ON} = \frac{-(\cos\theta + 2\sin\theta)}{4} \quad \ldots(2)$$

Let, $P(h, k)$ and $OP = r$

$\Rightarrow$ $r \cos \theta = h, r \sin \theta = k$

$$\therefore \quad \frac{1}{OP^2} = \frac{1}{OM^2} + \frac{1}{ON^2}$$

$$\Rightarrow \quad \frac{1}{r^2} = \frac{(\cos\theta + 2\sin\theta)^2}{9} + \frac{(\cos\theta + 2\sin\theta)^2}{16}$$

$$\Rightarrow \quad 144 = 16(r \cos \theta + 2r \sin \theta)^2 + 9(r \cos \theta + 2r \sin \theta)^2$$

$$\Rightarrow \quad 144 = 16(h + 2k)^2 + 9(h + 2k)^2$$

$\therefore$ Locus of $P(h, k)$ is $(x + 2y)^2 = \dfrac{144}{25}$

$$\Rightarrow \quad x + 2y = \pm\frac{12}{5}$$

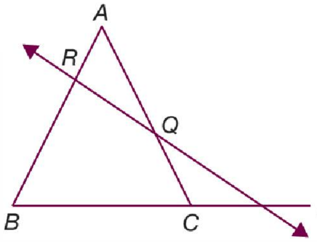
ILLUSTRATION 50

A line L intersects the sides BC, CA and AB of a ΔABC at P, Q and R, respectively. Show that $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1.$

SOLUTION

Let A (x_1, y_1), B(x_2, y_2) and C (x_3, y_3) be the vertices of ΔABC and let $\ell x + my + n = 0$ be the equation of the line L. If P divides BC in the ratio λ:1, then the co-ordinates of P are $\left(\frac{\lambda x_3+x_2}{\lambda+1},\frac{\lambda y_3+y_2}{\lambda+1}\right)$.

Also, as P lies on L, we have,



$$\ell\left(\frac{\lambda x_3+x_2}{\lambda+1}\right)+m\left(\frac{\lambda y_3+y_2}{\lambda+1}\right)+n=0$$

$$\Rightarrow \quad -\frac{\ell x_2+my_2+n}{\ell x_3+my_3+n}=\lambda=\frac{BP}{PC} \qquad \ldots(1)$$

Similarly, we obtain,

$$\frac{CQ}{QA}=-\frac{\ell x_3+my_3+n}{\ell x_1+my_1+n} \qquad \ldots(2)$$

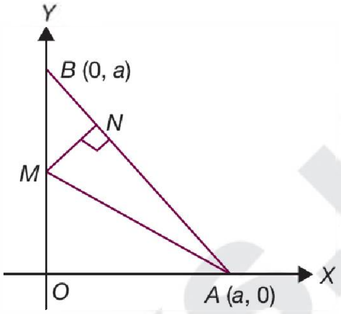
and $$\frac{AR}{RB}=-\frac{\ell x_1+my_1+n}{\ell x_2+my_2+n} \qquad \ldots(3)$$

Multiplying (1), (2) and (3), we get the required result as $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$

ILLUSTRATION 51

A triangle ΔAMN is inscribed in the ΔOAB, where $A(a, 0)$, $B(0, a)$ and O being the origin, with right angle at N. M and N lie respectively on OB and AB. If the area of the ΔAMN is $\frac{3}{8}$ th of the area of the ΔOAB, then find $\frac{AN}{BN}$.

SOLUTION



Let $\frac{AN}{BN}=\lambda$

$A(a, 0)$ and $B(0, a)$

Then the co-ordinates of N are $\left(\frac{a}{1+\lambda},\frac{\lambda a}{1+\lambda}\right)$

Now, equation of MN perpendicular to AB is $y-\frac{\lambda a}{1+\lambda}=x-\frac{a}{1+\lambda}$ or $x-y=\frac{1-\lambda}{1+\lambda}a$

So, the co-ordinates of M are $\left(0,\frac{\lambda-1}{\lambda+1}a\right)$.

$\therefore$ The area of the ΔAMN is $\frac{1}{2}\begin{vmatrix} a & 0 & 1\\ \frac{a}{1+\lambda} & \frac{\lambda a}{1+\lambda} & 1\\ 0 & \frac{(\lambda-1)a}{\lambda+1} & 1\end{vmatrix}$

$$=\frac{1}{2}\left|\left[a\left(\frac{-a}{\lambda+1}\right)+\frac{1-\lambda}{(1+\lambda)^2}a^2\right]\right|=\frac{\lambda a^2}{(1+\lambda)^2}$$

Also, the area of the $\Delta OAB=\frac{a^2}{2}$

So, according to the given condition, $\frac{\lambda a^2}{(1+\lambda)^2}=\frac{3}{8}\cdot\frac{1}{2}a^2$

$\Rightarrow \quad 3\lambda^2-10\lambda+3=0$

$\Rightarrow \quad \lambda=3$ or $\lambda=\frac{1}{3}$

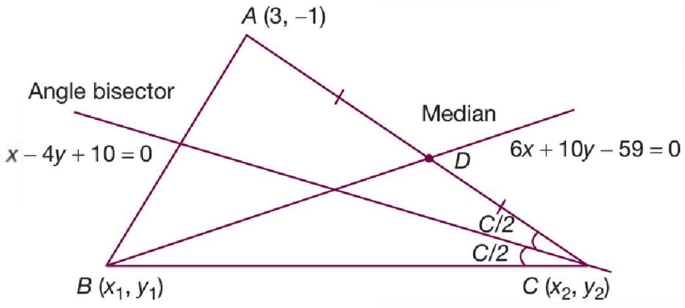
For $\lambda=\frac{1}{3}$, M lies outside the segment OB and hence acceptable λ is 3.

It means the required value of $\frac{AN}{BN}=3$

ILLUSTRATION 52

Find equations of the sides of the triangle having (3, –1) as a vertex, $x - 4y + 10 = 0$ and $6x + 10y - 59 = 0$ being the equations of an angle bisector and a median respectively drawn from different vertices.

SOLUTION



Let the vertices of the triangle be

$A(3, -1)$, $B(x_1, y_1)$ and $C(x_2, y_2)$.

Equation of the median through B be $6x + 10y - 59 = 0$ and the equation of the angle bisector from C be $x - 4y + 10 = 0$.

$$\Rightarrow \quad x_2 - 4y_2 + 10 = 0 \qquad \ldots(1)$$

Also, $D\left(\dfrac{x_2+3}{2}, \dfrac{y_2-1}{2}\right)$, the midpoint of AC lies on the median through B, i.e., $6x + 10y - 59 = 0$.

$$\Rightarrow \quad 6\left(\frac{x_2+3}{2}\right) + 10\left(\frac{y_2-1}{2}\right) - 59 = 0$$

$$\Rightarrow \quad 3x_2 + 5y_2 - 55 = 0 \qquad \ldots(2)$$

Solving (1) and (2) we get, $x_2 = 10$, $y_2 = 5$, i.e., the co-ordinates of C are (10, 5) and thus the equation of AC is $6x - 7y = 25$.

Let the slope of BC be m, then since, BC and AC are equally inclined to the angle bisector $x - 4y + 10 = 0$, we have,

$$\frac{\frac{1}{4} - m}{1 + \frac{1}{4}m} = \frac{\frac{6}{7} - \frac{1}{4}}{1 + \frac{6}{7} \times \frac{1}{4}} \Rightarrow \frac{1 - 4m}{4 + m} = \frac{17}{34} \Rightarrow m = -\frac{2}{9}$$

So that equation of BC is $y - 5 = -\dfrac{2}{9}(x - 10)$

$$\Rightarrow \quad 2x + 9y = 65$$

Since, B lies on BC and the median $6x + 10y - 59 = 0$ passes through B.

Solving these two we get, $B\left(\dfrac{-7}{2}, 8\right)$

Hence, the equation of AB is $y + 1 = \dfrac{8+1}{-7/2 - 3}(x - 3)$

$$\Rightarrow \quad 18x + 13y = 41$$

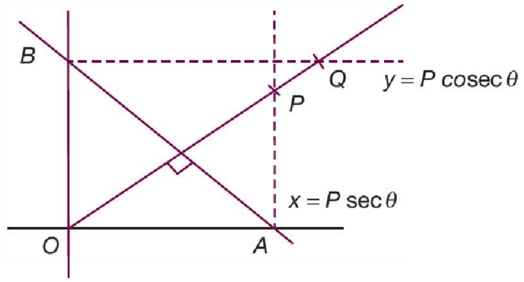
ILLUSTRATION 53

The line $x \cos \theta + y \sin \theta = p$ meets the axes of co-ordinates at A and B, respectively. Through A and B, lines are drawn parallel to the axes so as to meet the perpendicular drawn from origin to the given line in P and Q, respectively. Show that $|PQ| = \dfrac{4p|\cos 2\theta|}{\sin^2 2\theta}$.

SOLUTION

We get $A\left(\dfrac{p}{\cos\theta}, 0\right)$, $B\left(0, \dfrac{p}{\sin\theta}\right)$

Lines through A and B parallel to axes are $x = \dfrac{p}{\cos\theta}$, $y = \dfrac{p}{\sin\theta}$



These meet the line through the origin and perpendicular to given line, i.e., $x \sin\theta - y\cos\theta = 0$ in P and Q.

$$\therefore \quad P\left(\frac{p}{\cos\theta}, \frac{p\sin\theta}{\cos^2\theta}\right), Q\left(\frac{p\cos\theta}{\sin^2\theta}, \frac{p}{\sin\theta}\right)$$

$\therefore \quad PQ^2$ (by distance formula)

$$= p^2\cos^2 2\theta\left(\frac{1}{\sin^4\theta\cos^2\theta} + \frac{1}{\cos^4\theta\sin^2\theta}\right)$$

$$= 16p^2\cos^2 2\theta \frac{1}{(2\sin\theta\cos\theta)^4}$$

$$\therefore \quad PQ = \frac{4p|\cos 2\theta|}{\sin^2 2\theta}$$

ILLUSTRATION 54

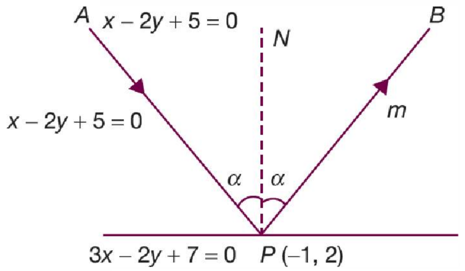
A ray of light is sent along the line $x - 2y + 5 = 0$ and upon reaching the line $3x - 2y + 7 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

SOLUTION

A ray of light is sent along the line (say AP), whose equation is L_1: $x - 2y + 5 = 0$.
Equation of mirror line,

$$L_2: 3x - 2y + 7 = 0.$$

We get, $P(-1, 2)$ as their point of intersection.



Thus, if AP is the incident ray and L_2 is the surface then the reflected ray will be along the line PB, such that $\angle APN = \angle BPN = \alpha$ (say).

Where, PN is normal to the surface (straight line) given by (2)

Now, slope of $L_1 = \dfrac{1}{2}$ and slope of $L_2 = \dfrac{3}{2}$

$\therefore$ Slope of PN normal to $L_2 = \dfrac{-2}{3}$

Let, m be the slope of PB.

$\therefore$ $\tan \angle APN = \tan \angle BPN$

$$\Rightarrow \frac{1/2-(-2/3)}{1+(1/2)(-2/3)} = \pm \frac{m-(-2/3)}{1+m(-2/3)}$$

$$\Rightarrow 7(3 - 2m) = \pm 4(3m + 2)$$

$$\Rightarrow m = \frac{1}{2} \text{ or } \frac{29}{2}$$

However, $\dfrac{1}{2}$ is the slope of the line $AP(L_1)$.

$\therefore$ Slope of reflected ray (line PB) $= \dfrac{29}{2}$

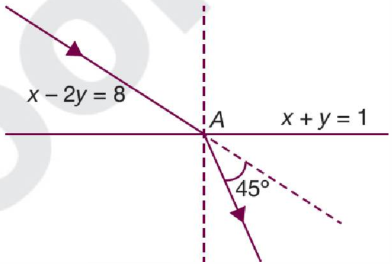
$\therefore$ Equation of line PB (reflected ray), which passes through $P(-1, 2)$ is $y - 2 = \left(\dfrac{29}{2}\right)(x + 1)$.

$\therefore$ The required equation is $29x - 2y + 33 = 0$.

ILLUSTRATION 55

A ray of light is sent along the line $x - 2y = 8$. After refracting across the line $x + y = 1$, it enters the opposite side after turning by 45° away from the line $x + y = 1$. Find the equation of the line along which the refracted ray travels.

SOLUTION



The point of incidence A is the intersection of lines $x - 2y = 8$ and $x + y = 1$.

Solving $x - 2y = 8$ and $x + y = 1$

We get the point $A\left(\dfrac{10}{3}, \dfrac{-7}{3}\right)$

Let the refracted ray have the slope m.

$$\text{Then, } \tan 45° = \left|\frac{m-\frac{1}{2}}{1+m\cdot\frac{1}{2}}\right| = \left|\frac{2m-1}{m+2}\right| = 1$$

$$\Rightarrow \quad \frac{2m-1}{m+2} = \pm 1$$

On solving, we get, $m = 3$ and $-\dfrac{1}{3}$

As we can see that refracted ray should have more angle with line $x + y = 1$ is required reflected ray.

Let the angle between $x + y = 1$ and the line through $A\left(\dfrac{10}{3}, \dfrac{-7}{3}\right)$ with the slope 3 be ϕ. Then

$$\tan \phi = \left|\frac{3-(-1)}{1+3(-1)}\right| = 2$$

If the angle between $x + y = 1$ and the line through $A\left(\frac{10}{3}, \frac{-7}{3}\right)$ with the slope $\frac{-1}{3}$ be ψ, then

$$\tan \psi = \left|\frac{\frac{-1}{3} - (-1)}{1 + \frac{-1}{3}(-1)}\right| = \frac{1}{2}$$

$\therefore \quad \tan \phi > \tan \psi;\ \phi > \psi$

$\therefore$ The slope of the refracted ray = 3

$\therefore$ The equation of the refracted ray is $y + \frac{7}{3} = 3\left(x - \frac{10}{3}\right)$.

ILLUSTRATION 56

A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC, whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation to the line AC, so that $AB = AC$.

SOLUTION

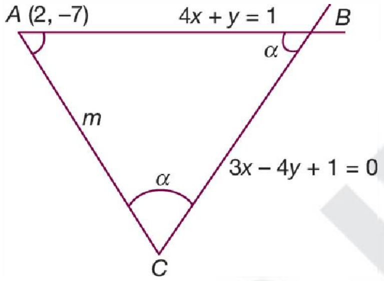
Equation of line AB is $4x + y = 1$

Its slope $= -4$

Slope of line BC, $3x - 4y + 1 = 0$ is $\frac{3}{4}$

If α is $\angle ABC$, then

$$\tan \alpha = \left|\frac{-4 - 3/4}{1 + (-4)(3/4)}\right| = \frac{19}{8}$$



Given that, $AB = AC$

$\therefore \quad \angle ACB = \angle ABC = \alpha$

If slope of AC is m, then

$$\tan \alpha = \frac{19}{8} = \pm \frac{m - 3/4}{1 + m.(3/4)}$$

$\Rightarrow \quad 19(4 + 3m) = \pm 8(4m - 3)$

$\Rightarrow \quad m = -4$ or $\frac{-52}{89}$

However, -4 is the slope of line AB.

Slope of $AC = m = \frac{-52}{89}$

Hence, the equation of line AC which passes through $A(2, -7)$ and have slope $m = \frac{-52}{89}$ is $y + 7 = \frac{-52}{89}(x - 2)$

Hence, the required equation is $52x + 89y + 519 = 0$.

Test Your Concepts-III

(Solutions on page H.25)

1. Find α, if the point (α, α^2) lies inside the triangle having sides along the lines $2x + 3y = 1$, $x + 2y - 3 = 0$ and $6y = 5x - 1$.
2. Show that the points (1, 4) and (0, –3) lie on the opposite sides of the line $x + 3y + 7 = 0$.
3. Let $P(\sin\theta, \cos\theta)$ $(0 \le \theta \le 2\pi)$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\sqrt{\frac{3}{2}}, 0\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. Find θ, if P lies inside the ΔOAB.
4. Represent the straight line $y = x + 2$ in the parametric form.
5. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of $PA.PB$ (where, $P[\sqrt{3}, 0]$).
6. Find the equation of the line through the point A (2, 3) and making an angle of 45° with the x-axis. Also, determine the length of intercept on it between A and the line $x + y + 1 = 0$.
7. A line joining two points $A(2, 0)$ and $B(3, 1)$ is rotated about A in the anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C, what will be the co-ordinates of C in the new position?

8. Find the equation of the line passing through (2, 3) and making intercepts of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$.

9. Find the co-ordinates of two points, which are $3\sqrt{2}$ distance from the point (1, 3) and lie on a straight line passing from this point and inclined at $\frac{2\pi}{3}$ angle to x-axis.

10. Find the co-ordinates of those points on the line $3x + 2y = 5$, which are equidistant from the lines $4x + 3y = 7$ and $2y = 5$.

Answer Keys

1. $\alpha \in \left(\frac{1}{2}, 1\right) \cup \left(-\frac{3}{2}, -1\right)$

3. $0 < \theta < \frac{\pi}{12}$ or $\frac{5\pi}{12} < \theta < \frac{\pi}{2}$

4. $\frac{y-2}{\frac{1}{\sqrt{2}}} = \frac{x}{\frac{1}{\sqrt{2}}} = r$

5. $\frac{4\left(2+\sqrt{3}\right)}{3}$

6. $3\sqrt{2}$

7. $\left(2+\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$

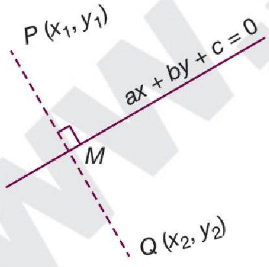
8. $x = 2$ and $5(y-3) = 3(x-2)$

9. $\left(1+\frac{3}{\sqrt{2}}, 3-\frac{3\sqrt{3}}{\sqrt{2}}\right)$ and $\left(1+\frac{3}{\sqrt{2}}, 3-\frac{3\sqrt{3}}{\sqrt{2}}\right)$

10. $x = \frac{1}{16}, y = \frac{77}{32}$ and $x = -\frac{1}{14}, y = \frac{73}{28}$

LENGTH OF PERPENDICULAR, FOOT OF PERPENDICULAR AND IMAGE OF A POINT RELATIVE TO A STRAIGHT LINE

Working Rule



(i) Length of perpendicular from a point $P(x_1, y_1)$ on the line

$$ax + by + c = 0 \text{ is } PM = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

(ii) Foot of perpendicular $M(x, y)$, drawn from the point $P(x_1, y_1)$ to line $ax + by + c = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

(iii) Image of the point $P(x_1, y_1)$ with respect to the line (mirror) $ax + by + c = 0$ is $Q(x, y)$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

PROOF:

The line PQ joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is perpendicular to the line $ax + by + c = 0$.

$$\therefore \quad \frac{y_1 - y_2}{x_1 - x_2}\left(-\frac{a}{b}\right) = -1$$

$$\Rightarrow \quad \frac{x_1 - x_2}{a} = \frac{y_1 - y_2}{b}$$

$$\Rightarrow \quad \frac{x_1 - x_2}{a} = \frac{y_1 - y_2}{b} = \frac{a(x_1 - x_2) + b(y_1 - y_2)}{a^2 + b^2}$$

(By ratio and proportion)

$$= \frac{(ax_1 + by_1 + c) - (ax_2 + by_2 + c)}{a^2 + b^2} \qquad \ldots(1)$$

The midpoint $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ of PQ is on the line $ax + by + c = 0$.

$$a\cdot\frac{x_1+x_2}{2}+b\cdot\frac{y_1+y_2}{2}+c=0$$

$$\Rightarrow \quad a(x_1+x_2)+b(y_1+y_2)+2c=0$$

$$\Rightarrow \quad (ax_1+by_1+c)+(ax_2+by_2+c)=0$$

Using this result in relation (1), we get,

$$\frac{x_1-x_2}{a}=\frac{y_1-y_2}{b}=\frac{2(ax_1+by_1+c)}{a^2+b^2}$$

$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

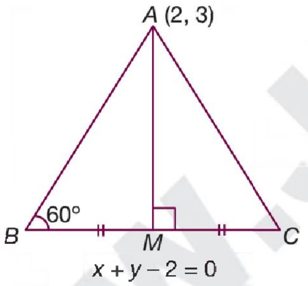
NOTE:

The length of the perpendicular from the origin on the line $ax+by+c=0$ is $p=\frac{|c|}{\sqrt{a^2+b^2}}$

ILLUSTRATION 57

If one of the vertices of an equilateral triangle is (2, 3) and the equation of its opposite side is $x+y-2=0$, then find the area of triangle.

SOLUTION



$$AM=\frac{|2+3-2|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}$$

$$AB\sin 60^\circ=AM$$

$$AB=\frac{3}{\sqrt{2}}\cdot\frac{2}{\sqrt{3}}=\sqrt{6}$$

$$\therefore \text{ Area of equilateral triangle}=\frac{\sqrt{3}}{4}(\sqrt{6})^2=\frac{6\sqrt{3}}{4}$$

$$=\frac{3\sqrt{3}}{2}\text{ sq. units.}$$

ILLUSTRATION 58

A line with positive rational slope, passes through the point $M(6, 0)$ and it is at a distance of 5 from $N(1, 3)$, then find the slope of line.

SOLUTION

Let the required line be $(y-0)=m(x-6)$, $m>0$

$$\Rightarrow \quad mx-y-6m=0$$

As distance of above line from $N(1, 3)$ is 5, then

$$\left|\frac{m-3-6m}{\sqrt{1+m^2}}\right|=5$$

$$\Rightarrow \quad (3+5m)^2=25(1+m^2)$$

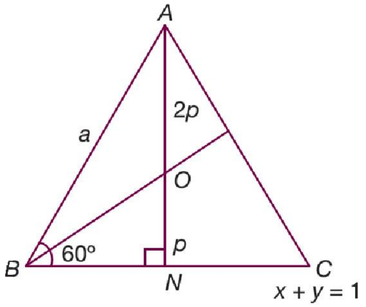
$$\Rightarrow \quad 9+30m=25$$

$$\Rightarrow \quad m=\frac{16}{30}=\frac{8}{15}$$

ILLUSTRATION 59

If an equilateral triangle has its centroid at the origin and one of its sides is along the line $x+y=1$, then find the side of the triangle.

SOLUTION



$$ON=\frac{|0+0-1|}{\sqrt{1+1}}=\left|\frac{1}{\sqrt{2}}\right|=p;\ AN=3p=\frac{3}{\sqrt{2}}$$

Now $a\sin 60^\circ=AN$; let a be side of triangle

$$a\sin 60^\circ=3P=\frac{3}{\sqrt{2}}\Rightarrow a\frac{\sqrt{3}}{2}=\frac{3}{\sqrt{2}}\Rightarrow a=\sqrt{6}$$

ILLUSTRATION 60

If the image of the $M(\lambda, \lambda^2)$ in the line $x+y=\lambda^2$ is $(0, 2)$, then find λ.

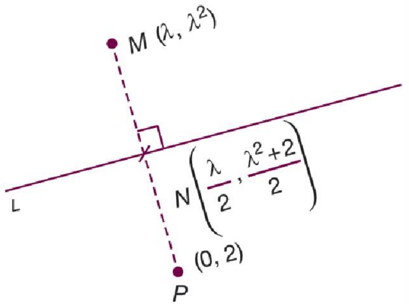
SOLUTION

N is mid point of M and its image P

N lies on the given line.

$$\text{Now, } \frac{\lambda}{2}+\frac{\lambda^2+2}{2}=\lambda^2\Rightarrow \lambda^2-\lambda-2=0 \quad \ldots(1)$$

MP is perpendicular to the given line.



$$\left(\frac{\lambda^2-2}{\lambda-0}\right)\times(-1)=-1\Rightarrow\lambda^2-\lambda-2=0 \qquad \ldots(2)$$

As (1) and (2) must be satisfied simultaneously, on solving, we get $\lambda = -1, 2$.

ILLUSTRATION 61

The line L has intercepts a and b on the co-ordinate axes. Keeping the origin as fixed, the co-ordinate axes are rotated through a fixed angle. The line L has now intercepts p and q on the rotated axes, then show that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$

SOLUTION

The equation of the line L is $\frac{x}{a}+\frac{y}{b}=1$. After the rotation of the axes, the line L has intercepts p and q on the new axes. In this system, equation of the line is

$$\frac{X}{p}+\frac{Y}{q}=1$$

$\because$ The origin and the line, both are fixed and the distance between them remains the same

$$\Rightarrow\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right|=\left|\frac{-1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}\right|\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$$

ILLUSTRATION 62

Prove that the product of the perpendiculars from the points $\left(\left[\pm\sqrt{a^2-b^2}\right],0\right)$ to the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ is b^2.

SOLUTION

Equation of the given lines is x (b cos θ) + y (a sin θ) – $ab = 0$.

$$\therefore \quad p_1=\left|\frac{\sqrt{(a^2-b^2)}(b\cos\theta)-ab}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}}\right|,$$

$$p_2=\left|\frac{-\sqrt{(a^2-b^2)}(b\cos\theta)-ab}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}}\right|$$

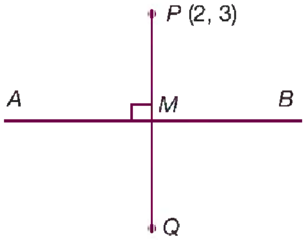
$$\therefore \quad p_1p_2=\left|\frac{(a^2-b^2)b^2\cos^2\theta-a^2b^2}{b^2\cos^2\theta+a^2\sin^2\theta}\right|$$

$$=b^2\frac{\left(a^2-a^2\cos^2\theta+b^2\cos^2\theta\right)}{b^2\cos^2\theta+a^2\sin^2\theta}=b^2$$

ILLUSTRATION 63

Find the foot of the perpendicular, drawn from the point (2, 3) to the line $3x - 4y + 5 = 0$. Also, find the image of (2, 3) in the given line.

SOLUTION



Let, AB: $3x - 4y + 5 = 0$

$P(2, 3)$ and $PM \perp AB$

Foot of normal from P (2, 3) to $3x - 4y + 5 = 0$ is given by $\frac{x-2}{3}=\frac{y-3}{-4}=\frac{-(3\times2-4\times3+5)}{9+16}=\frac{1}{25}$

$$\Rightarrow \quad x=\frac{53}{25}, y=\frac{71}{25} \quad \therefore\left(\frac{53}{25},\frac{71}{25}\right)$$

Image of $P(2, 3)$ on $3x - 4y + 5 = 0$ is given by $\frac{x-2}{3}=\frac{y-3}{-4}=-2\frac{(3\times2-4\times3+5)}{9+16}=\frac{2}{25}$

$$\Rightarrow \quad x=\frac{56}{25}, y=\frac{67}{25} \quad \therefore\left(\frac{56}{25},\frac{67}{25}\right)$$

ILLUSTRATION 64

The vertices of a ΔOBC are O (0, 0), B (–3, –1) and C (–1, –3). Find the equation of the line parallel to BC and intersecting the sides OB and OC and whose perpendicular distance from the point (0, 0) is $\frac{1}{2}$.

SOLUTION

Here, $B(-3, -1)$ and $C(-1, -3)$.

Slope of $BC = \frac{-3-(-1)}{-1-(-3)} = -1$.

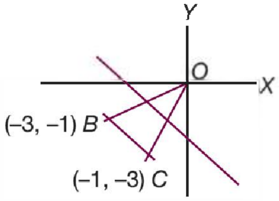
Hence, any line parallel to BC will have its slope $= -1$

Hence, its equation is $x + y - c = 0$.

Its distance from origin is $\frac{1}{2}$

$$\Rightarrow \quad \frac{c}{\sqrt{1+1}} = \pm\frac{1}{2}$$

$$\Rightarrow \quad c = \pm\frac{\sqrt{2}}{2}$$



$\therefore$ The required equation is $x + y \pm \frac{\sqrt{2}}{2} = 0$.

Now, the sides OB and OC are in IIIrd quadrant. This line meets both OB and OC and hence it will also be in IIIrd quadrant. So, the intercepts on the axes should be negative. Therefore we should choose positive sign, hence the required line is $\sqrt{2}\,x + \sqrt{2}\,y + 1 = 0$.

ILLUSTRATION 65

If $x\cos\alpha + y\sin\alpha = p$, where $p = \frac{-\sin^2\alpha}{\cos\alpha}$ be a straight line. Prove that the perpendiculars, p_1, p_2 and p_3 on this line from the points $(m^2, 2m)$, $(mm', m + m')$ and $(m'^2, 2m')$, respectively are in G.P.

SOLUTION

The given line is $x\cos\alpha + y\sin\alpha = p$.

Length of perpendiculars from $(m^2, 2m)$, $(mm', m+m')$ $(m'^2, 2m')$ are p_1, p_2 and p_3, respectively.

$$p_1 = \frac{m^2\cos\alpha + 2m\sin\alpha - p}{\sqrt{(\cos^2\alpha + \sin^2\alpha)}}$$

$$= m^2\cos\alpha + 2m\sin\alpha + \frac{\sin^2\alpha}{\cos\alpha} = \frac{(m\cos\alpha + \sin\alpha)^2}{\cos\alpha}$$

Similarly, $p_2 = \frac{mm'\cos\alpha + (m+m')\sin\alpha - p}{\sqrt{(\cos^2\alpha + \sin^2\alpha)}}$

$$= mm'\cos\alpha + (m + m')\sin\alpha + \sin^2\alpha/\cos\alpha$$

$$= \left(\frac{mm'\cos^2\alpha + [m+m']\sin\alpha\cos\alpha + \sin^2\alpha}{\cos\alpha}\right)$$

$$= \left(\frac{[m\cos\alpha + \sin\alpha][m'\cos\alpha + \sin\alpha]}{\cos\alpha}\right)$$

$$p_3 = \left(\frac{(m'\cos\alpha + \sin\alpha)^2}{\cos\alpha}\right), \text{ similar as in } p_1$$

$$\therefore \quad p_1p_3 = \frac{(m\cos\alpha + \sin\alpha)^2(m'\cos\alpha + \sin\alpha)^2}{\cos^2\alpha} = p_2^2$$

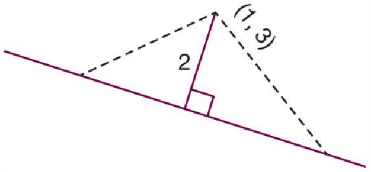
Hence, p_1, p_2 and p_3 are in G.P.

ILLUSTRATION 66

Find the number of points on the line $3x + 4y = 5$, which are at a distance of $\sec^2\theta + 2\,\mathrm{cosec}^2\theta$, $\theta \in R$ from the point (1, 3).

SOLUTION

The perpendicular distance of (1, 3) from the line $3x + 4y = 5$ is $\left|\frac{3+12-5}{\sqrt{9+16}}\right| = 2$ units.



While, $\sec^2\theta + 2\,\mathrm{cosec}^2\theta$

$$= 3 + (\tan\theta - \sqrt{2}\cot\theta)^2 + 2\sqrt{2}$$

$$\geq 3 + 2\sqrt{2} > 2$$

Evidently, there will be two such points on the line.

ILLUSTRATION 67

p_1 and p_2 are the distances of a variable point from the two lines $2x + y - 1 = 0$ and $x + 2y + 1 = 0$ satisfying the condition that $p_1^2 = p_2^2 + 3$. Prove that the product of the distances of the same point from the lines $x + y + 2 = 0$ and $x - y - 3 = 0$ varies as its distance from the line $x - 3y - 1 = 0$.

SOLUTION

$$p_1^2 - p_2^2 = 3$$

$$\Rightarrow \quad \left(\frac{2h+k-1}{\sqrt{5}}\right)^2 - \left(\frac{h+2k+1}{\sqrt{5}}\right)^2 = 3$$

$$\Rightarrow \quad (h + k)(h - k - 2) = 5$$

$$\Rightarrow \quad h^2 - k^2 = 5 + 2h + 2k \qquad \ldots(1)$$

Now, the product of perpendicular from (h, k) to other two lines is $d_1d_2 = \frac{h+k+2}{\sqrt{2}} \cdot \frac{h-k-3}{\sqrt{2}}$

$$= \frac{1}{2}\left((h^2-k^2)+(2h-2k)-3(h+k)-6\right)$$

$$= \frac{1}{2}(5 + 2h + 2k + 2h - 2k - 3h - 3k - 6)$$

$$= \frac{h-3k-1}{2} \qquad \ldots(2)$$

Again, if p be distance of (h, k) from $x - 3y - 1 = 0$,

then, $p = \frac{h-3k-1}{\sqrt{10}} \qquad \ldots(3)$

Taking ratio of (2) and (3)

$$\therefore \quad \frac{d_1d_2}{p} = \frac{\sqrt{10}}{2} = \text{Constant}$$

Hence, $d_1 \cdot d_2$ varies as p. Hence proved

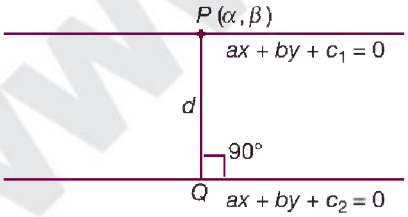
DISTANCE BETWEEN TWO PARALLEL LINES

The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

NOTE:

The coefficients of x and y in both equations should be the same.

PROOF:



Let $P(\alpha, \beta)$ be a point on Ist line,

$$\therefore \quad a\alpha + b\beta + c_1 = 0$$

We draw $PQ \perp$ IInd line

$d = PQ$ = Length of perpendicular from P to second line $= \left|\frac{a\alpha + b\beta + c_2}{\sqrt{a^2+b^2}}\right|$

Now, distance between parallel lines

$$= \left|\frac{-c_1 + c_2}{\sqrt{a^2+b^2}}\right| \qquad (\because a\alpha + b\beta = -c_1)$$

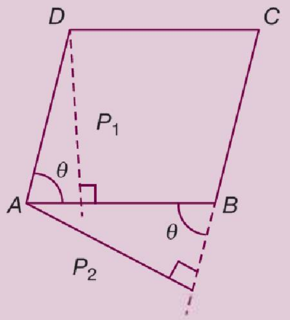
$$= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$$

Area of Parallelogram

(i) The area of the parallelogram $= \frac{p_1p_2}{\sin\theta}$,

where p_1 and p_2 are the distances between two pairs of opposite sides and θ is the angle between any two adjacent sides.

PROOF:



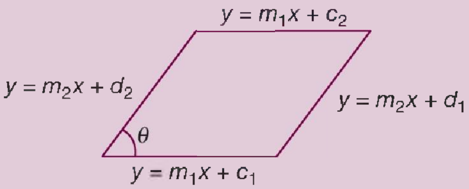
Area of parallelogram = Base(height)

$$= (P_2 \operatorname{cosec} \theta)\, P_1 = \frac{P_1P_2}{\sin\theta}$$

(ii) Area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$

Note: Equation of sides of parallelogram must be in $y = mx + c$ form

PROOF:



$$P_1 = \left|\frac{c_1-c_2}{\sqrt{1+m_1^2}}\right|;\ p_2 = \frac{|d_1-d_2|}{\sqrt{1+m_2^2}}\ \ \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\sin\theta=\left|\frac{m_1-m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}\right|$$

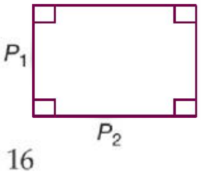
$$\therefore\quad \text{Area of the parallelogram}=\frac{p_1p_2}{\sin\theta}$$

$$=\left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$$

ILLUSTRATION 68

If $x+y+1=0$, $x+y+4=0$, $x-y+1=0$ and $2x-2y+\lambda=0$ are the sides of a rectangle whose area is 12 sq. units, then find the sum of all the possible values of λ.

SOLUTION



We have $P_1P_2=12$

$$\Rightarrow\quad \left(\frac{3}{\sqrt{2}}\right)\left(\frac{\left|\frac{\lambda}{2}-1\right|}{\sqrt{2}}\right)=12\Rightarrow|\lambda-2|=16$$

$$\Rightarrow\quad \lambda=18,-14$$

Now, the sum of possible $\lambda=18-14=4$.

ILLUSTRATION 69

Three lines $x+2y+3=0$, $x+2y-7=0$ and $2x-y-4=0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

SOLUTION

Distance between the two parallel lines is $\frac{|7+3|}{\sqrt{5}}=2\sqrt{5}$. The equations of sides A and C are of the form $2x-y+k=0$.

$\because$ Distance between sides A and B = Distance between sides B and C

Hence, $\frac{|k-(-4)|}{\sqrt{5}}=2\sqrt{5}$

$$\Rightarrow\quad \frac{k+4}{\sqrt{5}}=\pm 2\sqrt{5}$$

$$\Rightarrow\quad k=6,-14$$

Hence, the 4th sides of the two squares are

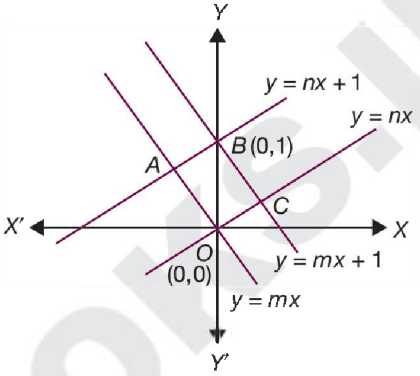
$$2x-y+6=0 \text{ and } 2x-y-14=0.$$

ILLUSTRATION 70

Find the area of the parallelogram formed by the lines $y=mx$, $y=nx$, $y=mx+1$ and $y=nx+1$.

SOLUTION

$y=mx$ and $y=nx$ are the lines through (0, 0), $y=mx+1$ and $y=nx+1$ are the lines parallel to $y=mx$ and $y=nx$, respectively with y-intercept 1.



The vertices are $O(0,0)$ and $A\left(\frac{1}{(m-n)},\frac{m}{(m-n)}\right)$.

The area of parallelogram is given by $2\times Ar(\Delta OAB)$.

$$=2\times\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ \frac{1}{m-n} & \frac{m}{m-n} & 1\end{vmatrix}=\frac{1}{|m-n|}$$

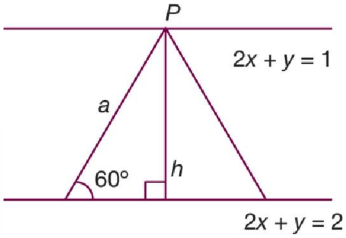
Alternate

Using the direct formula of area of parallelogram = $\left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|=\frac{1}{|m-n|}$

ILLUSTRATION 71

P is a point on the line $y+2x=1$ and Q and R are two points on the line $3y+6x=6$ such that the ΔPQR is an equilateral triangle, find the area of this triangle.

SOLUTION



The distance between the given parallel lines (h) is

$$\left|\frac{2-1}{\sqrt{5}}\right|=\frac{1}{\sqrt{5}}\qquad (\because a\sin 60^\circ=h)$$

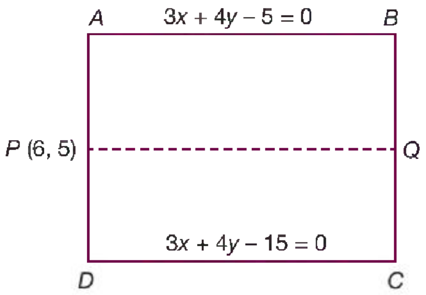
$\therefore$ Length of the side of the triangle is $= \dfrac{2h}{\sqrt{3}}$

$$\text{Area of triangle} = \frac{\sqrt{3}}{4} \times \frac{4h^2}{3} = \frac{h^2}{\sqrt{3}} = \frac{1}{5\sqrt{3}} \text{ sq. units}$$

ILLUSTRATION 72

The equations of two sides of a square are $3x + 4y - 5 = 0$ and $3x + 4y - 15 = 0$. The line along the third side has a point (6, 5) on it. Find the equations of remaining sides of the square.

SOLUTION



Let us consider a square $ABCD$ and given sides as AB and DC and the point P (6, 5) be on the side AD.

Clearly, $3x + 4y - 5 = 0$ and $3x + 4y - 15 = 0$ are two parallel lines, their 'm' being equal to $\dfrac{-3}{4}$ now slope of AD and BC are $\dfrac{4}{3}$

Equation of 3rd side (AD) $y - 5 = \dfrac{4}{3}(x - 6)$

$\Rightarrow \quad 4x - 3y = 9$

Now, 4th side BC is parallel to AD is $4x - 3y = k$

Side of square = Distance between parallel sides AB and CD $\left|\dfrac{-5+15}{\sqrt{3^2+4^2}}\right| = 2$

Now, the distance between AD and BC = $\left|\dfrac{k-9}{\sqrt{3^2+4^2}}\right| = 2 \Rightarrow |k-9| = 10$

$k = 19, -1$

$\therefore$ The equation of the remaining side BC is $4x - 3y = -1$ or $4x - 3y = 19$.

NOTE:

As it has not been given whether P lies between A and D or not, use of $PQ = AD = PA + PD$ may lead to a wrong result.

Test Your Concepts-IV

(Solutions on page H.27)

1. A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.
2. Prove that no line can be drawn through the point (4, –5) so that its distance from (–2, 3) will be equal to 12.
3. Two points A and B move on the positive direction of x-axis and y-axis respectively, such that $OA + OB = K$. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.
4. Find all points on $x + y = 4$, those lie at a unit distance from the line $4x + 3y - 10 = 0$.
5. Two sides of a square lie on the lines $x + y = 1$ and $x + y + 2 = 0$. What is its area?
6. Drawn from the origin are two mutually perpendicular lines forming an isosceles triangle together with the straight line $2x + y = a$, then find the area of this triangle.
7. In the co-ordinate plane, the point $P(a, 0)$ has a distance 2 units from the line $y = 2x$. If $a > 0$, then find a.
8. The algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line is zero, then find the line that passes through a fixed point.
9. The co-ordinates of the vertices A, B and C of a triangle are (6, 3), (–3, 5) and (4, –2), respectively

and P is any point (x, y). Show that the ratio of the areas of the ΔPBC and ΔABC is $|x + y - 2|:7$.

10. A straight line through the origin 'O' meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at the points P and Q, respectively. Find the ratio in which the point 'O' divides the segment PQ.

11. Consider the lines

$$L_1: \frac{x}{3} + \frac{y}{4} = 1,\ L_2: \frac{x}{4} + \frac{y}{3} = 1,\ L_3: \frac{x}{3} + \frac{y}{4} = 2 \text{ and}$$

$L_4: \frac{x}{4} + \frac{y}{3} = 2$, then confirm whether the quadrilateral formed by these lines is a rhombus or a square or a rectangle or a parallelogram.

Answer Keys

1. $2y - 29x + 31 = 0$

4. $(-7, 11)$ and $(3, 1)$

5. $\frac{9}{2}$

6. $\frac{a^2}{5}$

7. $\sqrt{5}$

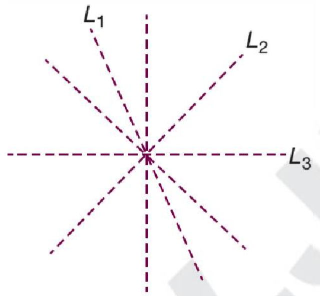
8. $(1, 1)$

10. 3: 4

11. The lines form a rhombus

CONCURRENCY OF LINES

More than two non-parallel straight lines intersecting at one point are known as concurrent lines.



Concurrency Test

(a) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e., co-ordinates of the point satisfy the equation of the line), then the three lines are concurrent. Otherwise, they are not concurrent.

(b) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

ILLUSTRATION 73

For what values of λ, the following three lines are concurrent.

$$x + y = 1,\ \lambda x + 2y = 3,\ \lambda^2 x + 4y + 9 = 0$$

SOLUTION

The three lines will be concurrent if

$$\begin{vmatrix} 1 & 1 & -1 \\ \lambda & 2 & -3 \\ \lambda^2 & 4 & 9 \end{vmatrix} = 0$$

$\Rightarrow$ $(18 + 12) - (9\lambda + 3\lambda^2) - (4\lambda - 2\lambda^2) = 0$

$\Rightarrow$ $\lambda^2 + 13\lambda - 30 = 0 \Rightarrow (\lambda - 2)(\lambda + 15) = 0$

$\therefore$ $\lambda = 2, -15$

ILLUSTRATION 74

If the straight lines, $ax + amy + 1 = 0$, $bx + (m + 1)\,by + 1 = 0$ and $cx + (m + 2)cy + 1 = 0$, $m \neq 0$ are concurrent, then show that a, b and c are in harmonic progression.

SOLUTION

$$\begin{vmatrix} a & am & 1 \\ b & b(m+1) & 1 \\ c & c(m+2) & 1 \end{vmatrix} = 0$$ (Apply $C_2 \to C_2 - mC_1$ or we can also open it directly)

$$\begin{vmatrix} a & 0 & 1 \\ b & b & 1 \\ c & 2c & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad a(b - 2c) + (2bc - bc) = 0$

$\Rightarrow \quad ab - 2ac + bc = 0$

$\Rightarrow \quad b(a + c) = 2ac$

$\Rightarrow \quad b = \dfrac{2ac}{a+c}$

$\therefore \quad a, b$ and c are in H.P.

ILLUSTRATION 75

Find the values of k for which the lines $kx + 2y + 2 = 0$, $2x + ky + 3 = 0$ and $3x + 3y + k = 0$ are concurrent.

SOLUTION

Three non-parallel lines are concurrent, if $\Delta = 0$.

$$\begin{vmatrix} k & 2 & 2 \\ 2 & k & 3 \\ 3 & 3 & k \end{vmatrix} = 0$$

$\Rightarrow \quad (k - 2)(k + 5)(k - 3) = 0$

$\Rightarrow \quad k = 2, 3, -5$

However, for $k = 2$, lines are parallel.
Hence, the required k is 3, –5.

ILLUSTRATION 76

Find the set of real values of k for which the lines $x + 3y + 1 = 0$, $kx + 2y - 2 = 0$ and $2x - y + 3 = 0$ form a triangle.

SOLUTION

Lines form a triangle.

$\therefore \quad x + 3y + 1 = 0$ is not parallel to $kx + 2y - 2 = 0$

$\therefore \quad \dfrac{-k}{2} \neq \dfrac{-1}{3} \Rightarrow k \neq \dfrac{2}{3}$

Also, line $2x - y + 3 = 0$ is not parallel to $kx + 2y - 2 = 0$

$\therefore \quad \dfrac{-k}{2} \neq 2 \Rightarrow k \neq -4$

For non-concurrency, $\begin{vmatrix} 1 & 3 & 1 \\ k & 2 & -2 \\ 2 & -1 & 3 \end{vmatrix} \neq 0$

$\Rightarrow \quad -10k \neq 12$

$\Rightarrow \quad k \neq -\dfrac{6}{5}$

Hence, $k \in R - \left\{-4, \dfrac{-6}{5}, \dfrac{2}{3}\right\}$

ILLUSTRATION 77

If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b and c being distinct and different from 1) are concurrent, then prove that $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 1$.

SOLUTION

The given lines are concurrent.

$$\therefore \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

[We are using elementary row/column transformations from determinant chapter (Algebra)]

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get,

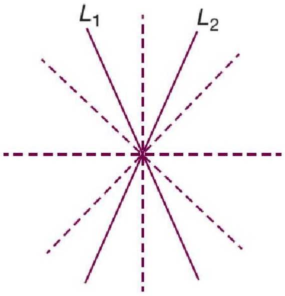
$$\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

$\Rightarrow \quad a(b - 1)(c - 1) - (b - 1)(1 - a) - (c - 1)(1 - a) = 0$

$\Rightarrow \quad \dfrac{a}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 0$

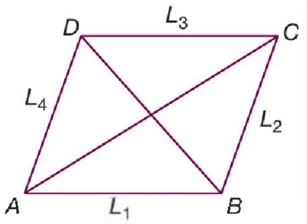
$\Rightarrow \quad \left(\dfrac{a}{1-a} + 1\right) + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 1 \Rightarrow \dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 1$

FAMILY OF STRAIGHT LINES



If L_1: $a_1x + b_1y + c_1 = 0$ and L_2: $a_2x + b_2y + c_2 = 0$ are two given lines, then family of lines through the point of intersection of the two given lines is $L_1 + \lambda L_2 = 0$ $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, where λ is an arbitrary real number.

Parallelogram



If $L_1 = ax + by + c$, $L_2 = a'x + b'y + d$, $L_3 = ax + by + c'$, $L_4 = a'x + b'y + d'$ then, $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ form a parallelogram.

$L_2L_3 - L_1L_4 = 0$ represents equation of the diagonal BD.

$L_1L_2 - L_3L_4 = 0$ represents equation of the diagonal AC.

PROOF:

Since $L_2L_3 - L_1L_4 = 0$ is the Ist degree equation in x and y it is a straight line. Secondly, point B satisfies the equation because the co-ordinates of B satisfy $L_2 = 0$ and $L_1 = 0$, so B lies on it and point D lies on line L_3 and L_4, it is also satisfied by point D. Hence it is equation of diagonal BD.

Similarly line $L_1L_2 - L_3L_4 = 0$ represents the diagonal AC.

Alternate approach:

The diagonal AC is also given by $L_1 + \lambda L_4 = 0$ and $L_2 + \mu L_3 = 0$, if the two equations are identical for some λ and μ.

(For getting the values of λ and μ, compare the coefficients of x, y and the constant terms).

One Parameter Family of Straight Line

If a linear expression L_1 contains an unknown coefficient, then the line $L_1 = 0$ cannot be a fixed line. Rather, it represents a family of straight lines known as one parameter family of straight lines, for example, family of lines parallel to the x-axis, i.e., $y = C$ and family of straight lines passing through the origin, i.e., $y = mx$.

If a variable straight line passes through a fixed point, we have two methods to find the fixed point.

Method-I

We should try to write a family of straight lines as $L_1 + \lambda L_2 = 0$, where L_1 and L_2 are two fixed lines and λ is a parameter then each member of it will pass through a fixed point, given by point of intersection of $L_1 = 0$ and $L_2 = 0$.

Method-II

Let the family of straight lines be of the form $ax + by + c = 0$ where a, b and c are variable parameters satisfying the condition $al + bm + cn = 0$ where l, m and n are

given and $n \neq 0$.

Rewriting the condition as $a\left(\frac{l}{n}\right) + b\left(\frac{m}{n}\right) + c = 0$ and

comparing with the given family of straight lines, we find that each member of it passes through the fixed point $\left(\frac{l}{n}, \frac{m}{n}\right)$.

ILLUSTRATION 78

Find the equation of the straight line passing through the point (1, 2) and through the point of intersection of the lines $x + 2y = 3$ and $2x - 3y = 4$.

SOLUTION

Equation of any straight line passing through the intersection of the lines $x + 2y = 3$ and $2x - 3y - 4 = 0$ is

$$(x + 2y - 3) + \lambda(2x - 3y - 4) = 0. \qquad \ldots(1)$$

Since it passes through the point (1, 2)

$$\therefore \quad (1 + 4 - 3) + \lambda(2 - 6 - 4) = 0$$

i.e., $\lambda = \frac{1}{4}$

Now, substituting this value of λ in (1) we get, $6x + 5y - 16 = 0$ as the equation of the required line.

Alternate

Point of intersection of given lines $P\left(\frac{17}{7}, \frac{2}{7}\right)$ and given point (1, 2)

Required straight line $y - 2 = \dfrac{2 - \frac{2}{7}}{1 - \frac{17}{7}}(x - 1)$

$$= -\frac{6}{5}(x - 1)$$

$$\Rightarrow \quad 6x + 5y = 16$$

ILLUSTRATION 79

If a, b and c are in A.P., then prove that the straight lines $ax + by + c = 0$ will always pass through the point.

SOLUTION

As a, b and c are in A.P., then $2b = a + c$
$a - 2b + c = 0$ we can rewrite it as $a(1) + b(-2) + c = 0$ and the given line is $ax + by + c = 0$.
Clearly, we can observe the fixed point as (1, –2).

Alternate

$$2b = a + c \text{ or } c = 2b - a$$

$$\therefore \quad ax + by + (2b - a) = 0$$

$$\Rightarrow \quad a(x - 1) + b(y + 2) = 0$$

$$\Rightarrow \quad (x - 1) + \lambda(y + 2) = 0$$

$$\Rightarrow \quad L_1 + \lambda L_2 = 0$$

It passes through the intersection of $x - 1 = 0$ and $y + 2 = 0$, i.e., the point (1, –2).

ILLUSTRATION 80

Let a, b and $c \in R$ and satisfying $(a + b)^2 + 4c^2 - 4ac - 4bc = 0$, then the variable line $ax + by + c = 0$ passes through a fixed point. Find the point.

SOLUTION

We have $(a + b)^2 + 4c^2 - 4ac - 4bc = 0$

$$(a + b)^2 + (2c)^2 - 4c(a + b) = 0$$

$$\Rightarrow \quad (a + b - 2c)^2 = 0$$

$$\Rightarrow \quad a + b - 2c = 0$$

$$\Rightarrow \quad a\left(-\frac{1}{2}\right) + b\left(-\frac{1}{2}\right) + c = 0 \text{ and}$$

Given line $ax + by + c = 0$

Hence, the fixed point is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.

ILLUSTRATION 81

Family of lines represented by the equation $(\cos\theta + \sin\theta)x + (\cos\theta - \sin\theta)y - 3(3\cos\theta + \sin\theta) = 0$ passes through a fixed point M for all real values of θ, then find reflection of M in the line $x - y = 0$.

SOLUTION

Given equation is $\cos\theta(x + y - 9) + \sin\theta(x - y - 3) = 0$

$\Rightarrow \quad (x + y - 9) + \tan\theta\,(x - y - 3) = 0$, which is of the form $L_1 + \lambda L_2 = 0$

Where, L_1: $x + y - 9 = 0$ and L_2: $x - y - 3 = 0$
Hence, the lines will always pass through the point of intersection of $L_1 = 0$ and $L_2 = 0$, i.e., $M(6, 3)$.
Reflection of $M(6, 3)$ w.r.t the line $x - y = 0$ is $M'(3, 6)$.

Alternate

For objective, put $\theta = 0$ and $\frac{\pi}{2}$, we get $x + y = 9$ and $x - y = 3$.

$$\therefore \quad x = 6, y = 3$$

$$\Rightarrow \quad M(6, 3)$$

$\therefore$ Reflection of $M(6, 3)$ w.r.t the line $x - y = 0$ is $M'(3, 6)$.

ILLUSTRATION 82

Prove that each member of the family of straight lines $(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

SOLUTION

The given family of straight lines can be rewritten as

$$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$$

or $\quad (4x - 7y + 2) + \tan\theta\,(3x + 2y + 1) = 0$

which is of the form $L_1 + \lambda L_2 = 0$
Hence, each member of it will pass through a fixed point, which is the intersection of $4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$, i.e., $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

ILLUSTRATION 83

The line $(k+1)^2x + ky - 2k^2 - 2 = 0$ passes through a point regardless of the value k. Find the line with slope 2 passing through that point?

SOLUTION

$$(k+1)^2x + ky = 2k^2 + 2$$

$$(k^2+2k+1)x + ky = 2k^2 + 2$$

$$k^2(x-2) + k(2x+y) + (x-2) = 0$$

$$(k^2+1)(x-2) + k(2x+y) = 0$$

$$(2x+y) + \left(\frac{k+1}{k}\right)(x-2) = 0$$

Hence, the line passes through fixed point $x = 2$ and $y = -4$.

$\therefore$ Required line $y + 4 = 2(x-2)$.

ILLUSTRATION 84

Show that the straight line given by $x(a+2b) + y(a+3b) = a + b$ for different values of a and b pass through a fixed point.

SOLUTION

The given equation can be put in the form

$$a(x+y-1) + b(2x+3y-1) = 0$$

or $\quad (x+y-1) + \lambda(2x+3y-1) = 0$

Where, $\lambda = \dfrac{b}{a}$

Above is of the form $P + \lambda Q = 0$ and as such it represent a line through the intersection of the line $P = 0$ and $Q = 0$,

i.e., $\quad x + y - 1 = 0$ and $2x + 3y - 1 = 0$.

Solving the above, we get the point $(2, -1)$, which is fixed.

ILLUSTRATION 85

The algebraic sum of distances of the lines $ax + by + 2 = 0$ from $(1, 2)$, $(2, 1)$ and $(3, 5)$ is zero and the lines $bx - ay + 4 = 0$ and $3x + 4y + 5 = 0$ cuts the co-ordinate axes at concyclic points, then find.

(i) $a + b$

(ii) Area of the triangle formed by the line $ax + by + 2 = 0$ with co-ordinates axes

SOLUTION

(i) $P_1 + P_2 + P_3 = 0$

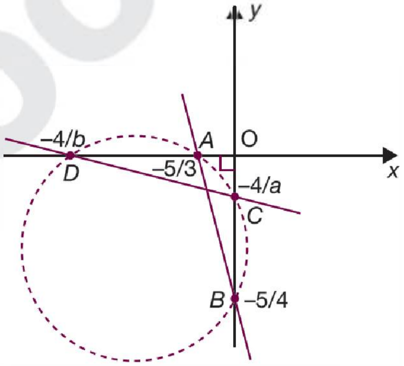
$$\frac{a+2b+2+2a+b+2+3a+5b+2}{\sqrt{a^2+b^2}} = 0$$

$\Rightarrow \quad 6a + 8b + 6 = 0 \quad \ldots(1)$

(2) $a + \left(\frac{8}{3}\right)b + 2 = 0$ compare it with $ax + by + 2 = 0$

Hence, the line $ax + by + 2 = 0$ always passes through the point $\left(2, \frac{8}{3}\right)$

Now, $bx - ay + 4 = 0$ and $3x + 4y + 5 = 0$ meets axis in concyclic points.



$ABCD$ are concyclic then $OA \cdot OD = OC \cdot OB$

$$\left(\frac{-5}{3}\right)\left(\frac{-4}{b}\right) = \left(\frac{4}{a}\right)\left(\frac{-5}{4}\right)$$

$\Rightarrow \quad 4a + 3b = 0 \quad \ldots(2)$

Solving (1) and (2), we get $a = \dfrac{9}{7}$, $b = \dfrac{-12}{7}$ and $a + b = -\dfrac{3}{7}$

(ii) Given line $ax + by + 2 = 0$

Rewriting the equation as $\dfrac{x}{\left(-\frac{2}{a}\right)} + \dfrac{y}{\left(-\frac{2}{b}\right)} = 1$

Area with co-ordinate axes $= \dfrac{1}{2}\left(-\dfrac{2}{a}\right)\left(-\dfrac{2}{b}\right)$

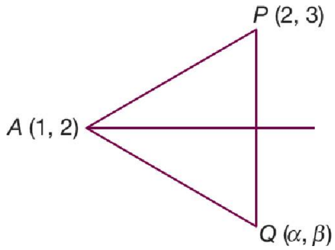
$$= \frac{2}{\frac{9}{7}\cdot\frac{12}{7}} = \frac{49}{54}$$

ILLUSTRATION 86

Find locus of the image of the point $(2, 3)$ in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$.

SOLUTION

Lines $(2x - 3y + 4) + k(x - 2y + 3) = 0$ pass through the point of intersection of $2x - 3y + 4 = 0$ and $x - 2y + 3 = 0$, which is $A(1, 2)$.



Let image of $P(2, 3)$ in any line through A be $Q(\alpha, \beta)$.

$\therefore \quad AP = AQ$

$\therefore \quad (\alpha - 1)^2 + (\beta - 2)^2 = (2 - 1)^2 + (3 - 2)^2 = 2$

$\therefore \quad \alpha^2 + \beta^2 - 2\alpha - 4\beta + 3 = 0;\ \alpha = x,\ \beta = y$

or $\quad x^2 + y^2 - 2x - 4y + 3 = 0$, which is a circle.

ILLUSTRATION 87

Find the equations of the lines through the point of intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and whose distance from the origin is $\sqrt{5}$.

SOLUTION

The required line by $L_1 + \lambda L_2 = 0$ is

$$(x - 3y + 1) + \lambda(2x + 5y - 9) = 0.$$

$$\Rightarrow \quad x(1 + 2\lambda) + y(5\lambda - 3) + (1 - 9\lambda) = 0 \qquad \ldots(1)$$

Its distance from origin is given to be $\sqrt{5}$.

$$\therefore \quad \frac{|1-9\lambda|}{\left([1+2\lambda]^2+[5\lambda-3]^2\right)^{1/2}} = \sqrt{5} \text{ (Square it)}$$

$$\therefore \quad (1 - 9\lambda)^2 = 5(1 + 4\lambda^2 + 4\lambda + 25\lambda^2 - 30\lambda + 9)$$

$$\Rightarrow \quad 64\lambda^2 - 112\lambda + 49 = 0$$

$$\Rightarrow \quad (8\lambda - 7)^2 = 0$$

$$\therefore \quad \lambda = \frac{7}{8}$$

Putting for λ in (1) and simplifying we get the required line as $2x + y - 5 = 0$.

ILLUSTRATION 88

Given the family of lines $a(2x + y + 4) + b(x - 2y - 3) = 0$. Prove that among the lines of the family, there exists only one line, whose distance from the point $(2, -3)$ is $\sqrt{10}$ and write the equation of this line.

SOLUTION

Put $\frac{b}{a} = \lambda$, then family of line becomes as lines

$$2x + y + 4 + \lambda(x - 2y - 3) = 0.$$

$$(2 + \lambda)x + (1 - 2\lambda)\,y + 4 - 3\lambda = 0$$

Its distance from $(2, -3)$ to line is given to be $\sqrt{10}$

$$\therefore \quad \left|\frac{4+2\lambda-3+6\lambda+4-3\lambda}{\left([2+\lambda]^2+[1-2\lambda]^2\right)^{1/2}}\right| = \sqrt{10}$$

$$\Rightarrow \quad |\lambda + 1| = \sqrt{2(\lambda^2+1)}$$

$$\Rightarrow \quad (\lambda - 1)^2 = 0$$

$$\therefore \quad \lambda = 1,\ 3x - y + 1 = 0$$

As there is only one value of λ, there will be only one line.

ILLUSTRATION 89

If the algebraic sum of the perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point. Also show that this fixed point is centroid of n sided polygon formed by these given points.

SOLUTION

Let n given points be (x_i, y_i), where $i = 1, 2, \ldots n$ and the variable straight line is $ax + by + c = 0$.

Given that $\sum_{i=1}^{n}\left(\frac{ax_i+by_i+c}{\sqrt{a^2+b^2}}\right) = 0$

$$\Rightarrow \quad a\sum x_i + b\sum y_i + cn = 0$$

$\Rightarrow \quad a\frac{\sum x_i}{n} + b\frac{\sum y_i}{n} + c = 0$ compare it with variable line $ax + by + c = 0$.

Hence, the variable straight line always passes through the fixed point $G\left(\frac{\sum x_i}{n}, \frac{\sum y_i}{n}\right)$, which is the centroid of n sided polygon formed by these given points.

ILLUSTRATION 90

Show that the sum of the reciprocals of the intercepts made on the co-ordinate axes by any line not passing through the origin and through the point of intersection of the lines $2x + 3y = 6$ and $3x + 2y = 6$ is constant.

SOLUTION

Equation of any line through the points of intersection of the given lines is $2x + 3y - 6 + k(3x + 2y - 6) = 0$.

$$\Rightarrow \quad (2 + 3k)x + (3 + 2k)y - 6(k + 1) = 0$$

$$\Rightarrow \quad \frac{x}{\frac{6(k+1)}{2+3k}} + \frac{y}{\frac{6(k+1)}{3+2k}} = 1, \text{ when } k \neq -1$$

In this case, sum of the reciprocals of the intercepts made by this line on the co-ordinate axis is equal to

$$\frac{2+3k+3+2k}{6(k+1)} = \frac{5(k+1)}{6(k+1)} = \frac{5}{6}$$

However, for $k = -1$, the line becomes $x = y$, which passes through the origin.

However, this special case is already not been considered.

ILLUSTRATION 91

(i) If $\frac{x}{c} + \frac{y}{d} = 1$ is a line through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, then find their relation.

(ii) If lengths of the perpendiculars drawn from the origin to these lines are equal, then find their relation.

SOLUTION

(i) Use the condition of concurrency for three lines.

$$\begin{vmatrix} \frac{1}{c} & \frac{1}{d} & 1 \\ \frac{1}{a} & \frac{1}{b} & 1 \\ \frac{1}{b} & \frac{1}{a} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{bc} + \frac{1}{a^2} + \frac{1}{bd} - \frac{1}{b^2} - \frac{1}{ac} - \frac{1}{ad} = 0$$

(ii) $P_1 = P_2 = P_3$ (using length of perpendicular formula)

$$\frac{1}{\sqrt{\frac{1}{c^2} + \frac{1}{d^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{b^2} + \frac{1}{a^2}}}$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2} + \frac{1}{d^2}$$

Test Your Concepts-V

(Solutions on page H.29)

1. If the lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$ $(p \neq 0)$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then find $a^2 + b^2$.
2. If the lines $x = a + m$, $y = -2$ and $y = mx$ are concurrent, then find the least value of $|a|$.
3. Find the area of the rhombus enclosed by the lines $ax \pm by \pm c = 0$.
4. If the straight lines of the family $x(a + b) + y(a - b) = 2a$ (a and b being parameters) are concurrent at point P, then find P.
5. Find the point of concurrency of the altitudes drawn from the vertices $(at_1t_2, a(t_1+t_2))$, $(at_2t_3, a(t_2+t_3))$ and $(at_3t_1, a(t_3+t_1))$ respectively of a ΔABC.
6. Let ABC be a given isosceles triangle with $AB = AC$. Sides AB and AC are extended upto E and F respectively, such that $BE.CF = AB^2$. Prove that the line EF always passes through a fixed point.

Answer Keys

1. 2

2. $|a| \geq 2\sqrt{2}$

3. $\frac{2c^2}{ab}$

4. (1, 1)

5. $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$

ANGLE BISECTORS OF TWO INTERSECTING STRAIGHT LINES

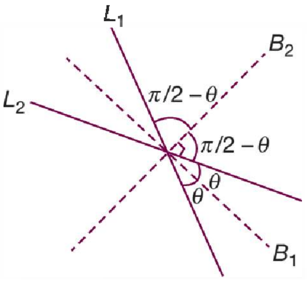
(i) Straight lines which pass through the point of intersection of given straight lines and bisect the angle containing them

- Let there are two straight lines L_1 and L_2 and there are two angles between given lines 2θ and $\pi - 2\theta$. Angle bisectors B_1 and B_2 making equal angles with given lines L_1 and L_2.

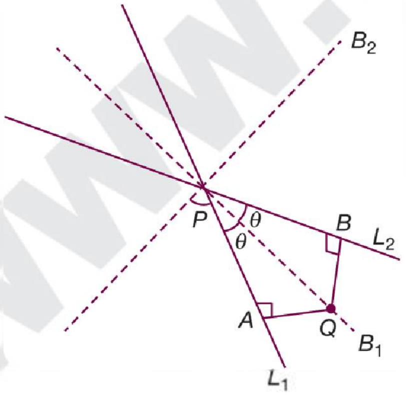
$$\text{Angle between } B_1 \text{ and } B_2 = \frac{\pi}{2} - \theta + \theta$$

$$= \frac{\pi}{2}$$

- Both angle bisectors are always perpendicular i.e., $B_1 \perp B_2$ always



(ii) Locus of a point which is at equal distance from two given lines, is the angle bisectors of given straight lines



PROOF:

From any angle bisector, let B_1, two perpendiculars QA and QB are drawn to given lines.

$$\Delta QPA \cong \Delta QPB \text{ (Congruent)}$$

$$\therefore \quad QA = QB$$

Equations of the angle bisectors of the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are given by

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}} \qquad (\text{Where, } ab' \neq a'b)$$

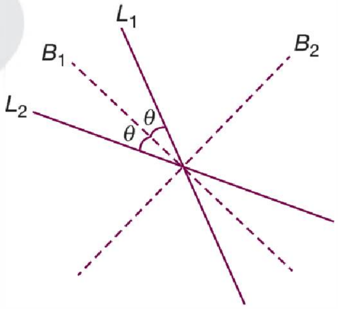
Identify Acute Angle Bisector and Obtuse Angle Bisector

Ist Approach

If θ be the angle between one of the lines and one of the bisectors, find $\tan\theta$.

If $|\tan \theta| < 1$, then $2\theta < 90°$, then selected angle bisector is the acute angle bisector.

If $|\tan \theta| > 1$, then $2\theta > 90°$, then selected angle bisector is the obtuse angle bisector.

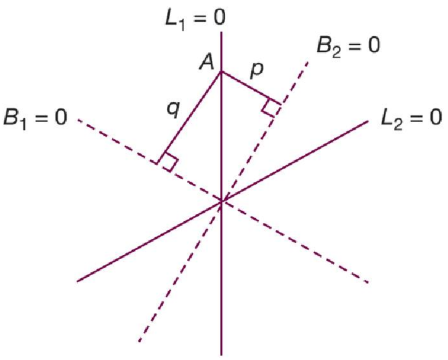


IInd Approach

Another way of identifying an acute and obtuse angle bisector is as follows.

Let $L_1 = 0$ and $L_2 = 0$ are the given lines and $B_1 = 0$ and $B_2 = 0$ are the angle bisectors of lines $L_1 = 0$ and $L_2 = 0$. Take a point A on any one of the lines (here let $L_1 = 0$) and drop a perpendicular on $B_1 = 0$ and $B_2 = 0$ as shown.

Now compare length of perpendicular q and p to angle bisectors B_1 and B_2 respectively.



$\lvert p\rvert < \lvert q\rvert$	Obtuse angle bisector B_1 Acute angle bisector B_2
$\lvert p\rvert > \lvert q\rvert$	Obtuse angle bisector B_2 Acute angle bisector B_1

NOTE:

If $\lvert p\rvert = \lvert q\rvert$, then lines L_1 and L_2 are perpendicular.

IIIrd Approach

Compute $aa' + bb'$ and follow the table.

Condition	Sign taken in angle bisector equation	Selected angle bisector
$aa' + bb' > 0$	Positive sign $\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$	Obtuse angle bisector
	Negative sign $\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$	Acute angle bisector
$aa' + bb' < 0$	Positive sign $\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$	Acute angle bisector
	Negative sign $\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$	Obtuse angle bisector

Remark(s)

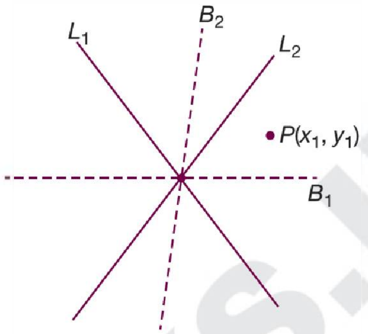
Equation of straight lines passing through $P(x_0, y_0)$ and equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the angle bisectors of these two lines and passing through the point $P(x_0, y_0)$.

Identify Angle Bisector Containing Point (X_1, Y_1)

Working Rule

To discriminate between angle bisectors containing the point (x_1, y_1)

Equation of straight lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$



Compute E (let) $= (ax_1 + by_1 + c)(a'x_1 + b'y_1 + c')$ to identify the angle bisector, which contains the point (x_1, y_1).

If $E > 0$, then by taking positive sign in angle bisector equation, we get the required angle bisector containing the point (x_1, y_1).

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

If $E < 0$, then by taking negative sign in angle bisector equation, we get the required angle bisector containing the point (x_1, y_1).

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

Angle Bisector Containing Origin

Given straight lines are $ax + by + c = 0$ and $a'x + b'y + c' = 0$.

Make the constant term positive (let c and c' are positive), then

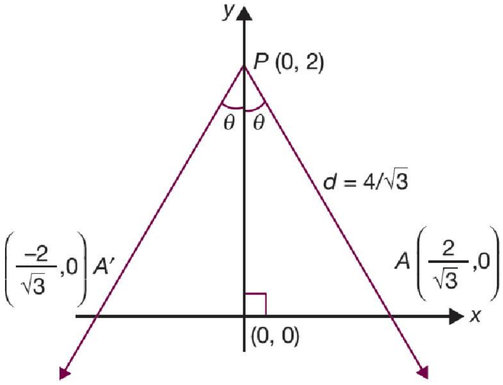
$\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing the origin.

$\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin.

ILLUSTRATION 92

If A is a point on either of two rays $y + \sqrt{3}\,\lvert x\rvert = 2$ at a distance of $\frac{4}{\sqrt{3}}$ units from their point of intersection, then find foot of perpendicular from A on the bisector of the angle between.

SOLUTION


y
P (0, 2)
θ
θ
d = 4/√3
$\left(\frac{-2}{\sqrt{3}}, 0\right)$ A′
A $\left(\frac{2}{\sqrt{3}}, 0\right)$
x
(0, 0)

$$y + \sqrt{3}\, x = 2; \qquad \text{for } x > 0$$

$$y - \sqrt{3}\, x = 2; \qquad \text{for } x < 0$$

Given rays $y + \sqrt{3}|x| = 2$

$$\Rightarrow \qquad \sqrt{3}|x| = 2 - y \geq 0$$

$$y \leq 2$$

Point of intersection of rays $P(0, 2)$

$$AP = AP' = \frac{4}{\sqrt{3}}$$

Angle bisectors of rays = y-axis

Required foot of perpendicular on angles bisector of rays = Origin (0, 0)

ILLUSTRATION 93

Find the equation of the line, which bisects the obtuse angle between the lines $3x + 4y + 4 = 0$ and $4x + 3y - 2 = 0$.

SOLUTION

Given lines, $L_1: 3x + 4y + 4 = 0$ and $L_2: 4x + 3y - 2 = 0$.

∴ The equations of the angle bisectors of given lines is

$$\frac{3x + 4y + 4}{\sqrt{9+16}} = \pm \frac{4x + 3y - 2}{\sqrt{16+9}}$$

$$\Rightarrow \qquad 3x + 4y + 4 = \pm(4x + 3y - 2)$$

Taking positive sign we get,

$$B_1: x - y - 6 = 0$$

Taking negative sign we get,

$$B_2: 7x + 7y + 2 = 0$$

Now, let θ be the angle between the line L_1 and the bisector B_1 whose slopes are

$$m_1 = \frac{-3}{4} \text{ and } m_2 = 1 \text{ respectively, then}$$

$$\tan\theta = \left| \frac{-\frac{3}{4} - 1}{1 - \frac{3}{4}} \right| = 7$$

$$\Rightarrow \qquad \theta > 45^\circ \Rightarrow 2\theta > 90^\circ$$

Hence, the bisector B_1 is the obtuse angle bisector and B_2 is the acute angle bisector.

Alternate I

Given lines,

$$L_1: 3x + 4y + 4 = 0$$

$$L_2: 4x + 3y - 2 = 0$$

Angle bisector equations,

$$B_1: x - y - 6 = 0$$

$$B_2: 7x + 7y + 2 = 0$$

Let us assume a point $A(0, -1)$ on L_1

Now, length of the perpendicular of angle bisector B_1 is $p = \left| \frac{0 + 1 - 6}{\sqrt{1+1}} \right| = \frac{5}{\sqrt{2}}$

Length of perpendicular of angle bisector B_2 is $q = \left| \frac{0 - 7 + 2}{7\sqrt{1+1}} \right| = \frac{5}{7\sqrt{2}}$

Clearly, $p > q$ which means, the angle bisector B_1 is an obtuse angle bisector and B_2 is an acute angle bisector.

Alternate II

Given lines are $L_1: 3x + 4y + 4 = 0$ and $L_2: 4x + 3y - 2 = 0$.

Now $aa' + bb' = (3)(4) + (4)(3) = 24 > 0$

It means, by taking positive sign in angle bisector equation, we get the obtuse angle bisector $B_1: x - y - 6 = 0$ and acute angle bisector $B_2: 7x + 7y + 2 = 0$.

ILLUSTRATION 94

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the following.

(i) Bisector of the obtuse angle bisector and acute angle bisector.

(ii) Bisector of the angle which contains (0, 0).

(iii) Bisector of the angle which contains (1, 2).

SOLUTION

(i) Given lines,

L_1: $4x + 3y - 6 = 0$

L_2: $5x + 12y + 9 = 0$

Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$\Rightarrow 9x - 7y - 41 = 0$ and $7x + 9y - 3 = 0$

We get angle bisector

B_1: $9x - 7y - 41 = 0$

B_2: $7x + 9y - 3 = 0$

If θ is the acute angle between the line L_1: $4x + 3y - 6 = 0$ and the bisector

B_1: $9x - 7y - 41 = 0$, then

$$\tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

The bisector of the obtuse angle is B_1: $9x - 7y - 41 = 0$

The bisector of the acute angle is B_2: $7x + 9y - 3 = 0$

Alternate

Given lines,

L_1: $4x + 3y - 6 = 0$

L_2: $5x + 12y + 9 = 0$

$aa' + bb' = (4)(5) + (3)(12) = 56 > 0$

By taking positive sign, we get the bisector of the obtuse angle as B_1: $9x - 7y - 41 = 0$

By taking negative sign, we get the bisector of the acute angle as B_2: $7x + 9y - 3 = 0$

(i) Given lines,

L_1: $4x + 3y - 6 = 0$ and L_2: $5x + 12y + 9 = 0$

For origin (0, 0)

As $(0 + 0 - 6)(0 + 0 + 9) < 0$, then we get the angle bisector containing the origin, by taking negative sign in angle bisector equation

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$\Rightarrow 7x + 9y - 3 = 0$ (B_2)

(ii) Given lines,

L_1: $4x + 3y - 6 = 0$ and L_2: $5x + 12y + 9 = 0$

For point (1, 2)

As $(4 + 6 - 6)(5 + 24 + 9) > 0$, then we get angle bisector containing the (1, 2), by taking positive sign in angle bisector equation

$$\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13} \Rightarrow 9x-7y-41=0\ (B_1)$$

ILLUSTRATION 95

If $px + qy + r + \lambda(qx - py + r') = 0$ and $px + qy + r - \lambda(qx - py + r') = 0$ are two given lines, then determine the equations of their bisectors.

SOLUTION

The required bisectors are

$$\frac{P+\lambda Q}{\sqrt{\{(p+\lambda q)^2+(q-p\lambda)^2\}}} = \pm\frac{P-\lambda Q}{\sqrt{\{(p-\lambda q)^2+(q+p\lambda)^2\}}}$$

The denominator on both sides cancel as each is $\sqrt{\{(p^2+q^2)(1+\lambda)^2\}}$

Hence, the bisectors are $P + \lambda Q = \pm (P - \lambda Q)$.

$\Rightarrow$ $P = 0$ or $Q = 0$

$\therefore$ $px + qy + r = 0$ or $qx - py + r' = 0$

Clearly, these lines are perpendicular as we know that the bisectors of two given lines are always perpendicular.

PAIR OF STRAIGHT LINES

An algebraic equation in two variables x and y is said to be a homogeneous equation of second degree, if the sum of indices of x and y in each term is 2. Thus, the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight line passing through origin under some condition on coefficients a, b and h.

We can write $ax^2 + 2hxy + by^2 = 0$

as $$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \qquad \ldots(1)$$

which is a quadratic equation in $\frac{y}{x}$

Thus, $$\frac{y}{x} = \frac{-2h \pm \sqrt{4h^2-4ab}}{2b}$$

$$\Rightarrow \quad y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin

(a) if $h^2 > ab$ then Lines are real and distinct

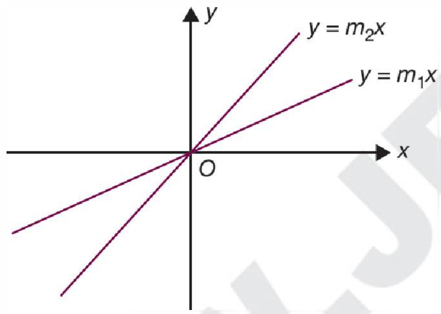
(b) if $h^2 = ab$ then Lines are coincident

(c) if $h^2 < ab$ then No real straight lines but only origin satisfies this equation

If $y = m_1x$ and $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$

$\Rightarrow \quad bm^2 + 2hm + a = 0$, where m_1 and m_2 are roots of this equation

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$



If θ is the acute angle between the pair of straight lines represented by, $ax^2 + 2hxy + by^2 = 0$, then, we know, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$$\Rightarrow \quad \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

Remark(s)

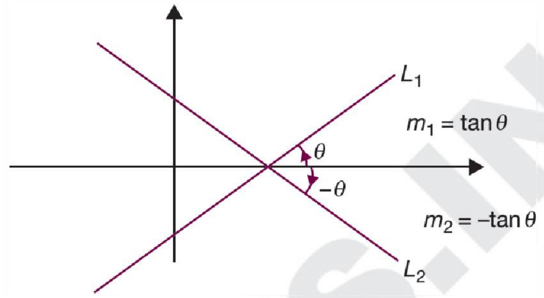
The condition that these lines are:

(a) At right angles to each other is $a + b = 0$, i.e., coefficient of x^2 + coefficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $h = 0$, i.e., coefficient of $xy = 0$.

PROOF:



Both straight lines are equally inclined to the x-axis. Here, $m_1 + m_2 = \tan\theta - \tan\theta = 0$

$$\Rightarrow \quad \frac{-2h}{b} = 0 \Rightarrow h = 0$$

$\therefore$ Coefficient of $xy = 0$

NOTE:

A homogeneous equation of degree n may represent n straight lines passing through the origin.

Angle Bisectors of Homogeneous Straight Lines

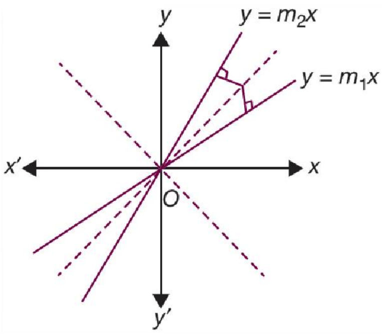
Angle bisectors equation of $ax^2 + 2hxy + by^2 = 0$ is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

PROOF:

Let $y - m_1, x = 0$ and $y - m_2x = 0$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

Since, bisector of the angles between two lines is the locus of a point which is equidistant from the two given lines.

Then, equation of angle bisector is



$$\frac{y-m_1x}{\sqrt{1+m_1^2}}=\pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$$

The joint equation of the two bisectors is

$$\left(\frac{y-m_1x}{\sqrt{1+m_1^2}}+\frac{y-m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y-m_1x}{\sqrt{1+m_1^2}}-\frac{y-m_2x}{\sqrt{1+m_2^2}}\right)=0$$

On simplification we get,

$$(m_1+m_2)(x^2-y^2)=2xy(1-m_1m_2)$$

$$\Rightarrow \quad \frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

Hence proved

NOTE:

Coefficient of x^2 + coefficient of $y^2 = 0$. Thus, the bisectors of the angle between the lines are perpendicular to each other.

ILLUSTRATION 96

Prove that the equations to the straight lines passing through the origin and making an angle α with the straight line $y + x = 0$ are given by $x^2 + 2xy \sec 2\alpha + y^2 = 0$.

SOLUTION

Let $y = mx$ be the equation of the straight line passing through the origin and making angle α with $y + x = 0$. Now, slope of line $y + x = 0$ is -1.

$$\therefore \quad \tan\alpha=\pm\left(\frac{m+1}{1-m}\right) \text{ or } \tan^2\alpha=\frac{(m+1)^2}{(m-1)^2} \text{ (squaring)}$$

$$\Rightarrow \quad m^2(1-\tan^2\alpha)+2m(1+\tan^2\alpha)+(1-\tan^2\alpha)=0$$

$$\Rightarrow \quad m^2+2m\left(\frac{1+\tan^2\alpha}{1-\tan^2\alpha}\right)+1=0$$

$$\Rightarrow \quad m^2+2m\sec 2\alpha+1=0, \text{ where } m=\frac{y}{x}$$

$$\Rightarrow \quad \frac{y^2}{x^2}+2\left(\frac{y}{x}\right)\sec 2\alpha+1=0$$

$$\Rightarrow \quad y^2+2xy\sec 2\alpha+x^2=0$$

ILLUSTRATION 97

Prove that the pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

SOLUTION

If the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$, then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

The lines perpendicular to them and passing through origin will be $y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$

Their combined equation is $(m_1y + x)(m_2y + x) = 0$.

$$\Rightarrow \quad m_1m_2y^2+xy(m_1+m_2)+x^2=0$$

$$\Rightarrow \quad \frac{a}{b}y^2+xy\left(-\frac{2h}{b}\right)+x^2=0$$

$$\Rightarrow \quad bx^2-2hxy+ay^2=0$$

TRICK:

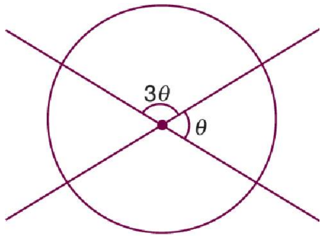
Interchange the coefficients of x^2 and y^2 and change the sign of the term containing xy.

ILLUSTRATION 98

If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along the diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector, then find the relation.

SOLUTION

According to the question, we have the following figure.



$$3\theta+\theta=180°$$

$$\therefore \quad \theta=45°$$

Now, for $ax^2 + 2(a + b)xy + by^2 = 0$

Angle between the lines is

$$\tan 45° = \frac{2\sqrt{(a+b)^2 - ab}}{|a+b|} = 1$$

$\Rightarrow \quad (a+b)^2 = 4(a^2 + b^2 + ab)$

$\Rightarrow \quad 3a^2 + 3b^2 + 2ab = 0$

ILLUSTRATION 99

If the line $y = mx$ bisects the angle between the lines $ax^2 + 2h\,xy + by^2 = 0$, then find the quadratic equation whose root is m.

SOLUTION

Equation of the angle bisector $\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$

$$\Rightarrow \quad \frac{1 - \left(\frac{y}{x}\right)^2}{a-b} = \frac{\frac{y}{x}}{h}$$

Now, put $y = mx$

$\Rightarrow \quad (1 - m^2)h = (a-b)m$

$\Rightarrow \quad m^2h + (a-b)m - h = 0$

ILLUSTRATION 100

If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then find m.

SOLUTION

The equation of the bisectors of the lines $xy = 0$ are $y = \pm x$

Putting $y = \pm x$ in $my^2 + (1 - m^2)xy - mx^2 = 0$, we get $(1 - m^2)x^2 = 0$

$\Rightarrow \quad m = \pm 1$

GENERAL EQUATION OF SECOND DEGREE CURVE

An equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where a, b, c, f, g, and h are constants is said to be the general equation of second degree in x and y.

Conditions for General Equation to Represent Pair of Straight Lines

The necessary and sufficient conditions for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent pair of straight lines are as follows.

(i) $h^2 - ab \geq 0$;

(ii) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

PROOF:

Let us rewrite the general two degree equation as quadratic in y.

$$by^2 + 2y(hx + f) + ax^2 + 2gx + c = 0$$

$$y = \frac{-2(hx+f) \pm \sqrt{4(hx+f)^2 - 4b(ax^2 + 2gx + c)}}{2b}$$

$$by = -(hx + f) \pm \sqrt{(h^2 - ab)x^2 + 2x(hf - gb) + f^2 - bc}$$

It is a quadratic in x.

For this to represent pair of straight lines, in square root, it should be a perfect square.

$D = 0$

$\Rightarrow \quad B^2 - 4AC = 0$

$\Rightarrow \quad 4(hf - gb)^2 - 4(h^2 - ab)(f^2 - bc) = 0$

$\Rightarrow \quad h^2f^2 + g^2b^2 - 2hfgb - (h^2f^2 - h^2bc - abf^2 + ab^2c) = 0$

$\Rightarrow \quad b(g^2b - 2hgf + ch^2 + af^2 - abc) = 0$

$\Rightarrow \quad abc + 2hgf - af^2 - bg^2 - ch^2 = 0$ this term is represented as Δ.

Point of Intersection of Pair of Straight Lines

$F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is given by (by making partial derivatives with respect to x and y as zero)

$$\frac{\delta F(x,y)}{\delta x} = 0 \Rightarrow 2ax + 2hy + 2g = 0 \qquad \ldots(1)$$

$$\frac{\delta F(x,y)}{\delta y} = 0 \Rightarrow 2hx + 2by + 2f = 0 \qquad \ldots(2)$$

Solving (1) and (2)

Gives the point of intersection of the lines $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$.

Angle Between Lines

Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be the lines represented by general equation of 2nd degree then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (y - m_1x - c_1)(y - m_2x - c_2) = 0$

$$m_1m_2x^2 - (m_1 + m_2)xy + y^2 + \alpha x + \beta y + c_1c_2 = 0$$

Comparing coefficient, we get,

$$\frac{m_1m_2}{a} = \frac{m_1 + m_2}{-2h} = \frac{1}{b}$$

$$\Rightarrow \quad m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

Let θ be the angle between the lines.

$$\text{Thus, } \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

NOTES:

(i) The lines are parallel or coincident, if $h^2 = ab$.

(ii) The lines are perpendicular, if $a + b = 0$.

$\Rightarrow$ Coefficient of x^2 + coefficient of $y^2 = 0$

Equation of Bisector of Angles Between the Lines

Let $P(x_1, y_1)$ be the point of intersection of the lines and (x, y) be the moving point of the shifting of origin. Then the equation of bisector is given by

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h}$$

Alternate

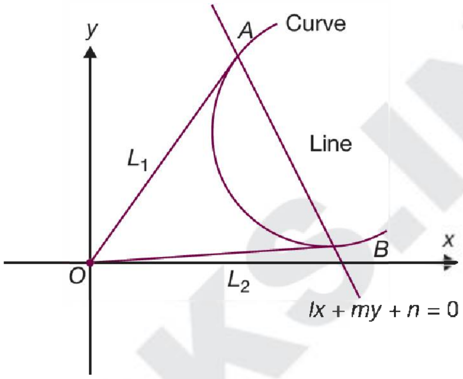
First, we can get separate straight lines equations from given pair of straight lines by forming quadratic equation in x or y.

As we get separate straight line equations, we can use angle bisector equation formula.

HOMOGENISATION OF GENERAL EQUATION

Let the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the line $lx + my + n = 0$ are given. Given line cuts curve at two distinct point A and B.

Now, the straight lines joining the origin to the point of intersection of the curve and the line is obtained by homogenising the equation of curve with the help of given straight line.



As $\quad lx + my + n = 0 \Rightarrow \dfrac{lx + my}{-n} = 1$

Equation of curve can be written as

$$ax^2 + 2hxy + by^2 + 2gx.1 + 2fy.1 + c.1^2 = 0$$

Then, the combined equation of OA and OB is given by

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right)$$

$$+ c\left(\frac{lx + my}{-n}\right)^2 = 0$$

$$\Rightarrow \quad Ax^2 + 2Hxy + By^2 = 0$$

It represents the pair of lines from origin to point of intersection of curve and the given lines (OA and OB).

ILLUSTRATION 101

Discuss the nature of the lines represented by the following equations:

(i) $2x^2 - 5xy + 2y^2 + 4x - 5y + 2 = 0$

(ii) $2x^2 + 2xy + y^2 + 6x + 2y + 5 = 0$

(iii) $x^2 + 2xy + y^2 + 12 = 0$

SOLUTION

It is easy to see that in each case, $\Delta = 0$.

In (i), $h^2 - ab > 0$,

$$\Delta = \begin{vmatrix} 2 & \frac{-5}{2} & 2 \\ \frac{-5}{2} & 2 & \frac{-5}{2} \\ 2 & \frac{-5}{2} & 2 \end{vmatrix} = 0$$

Hence (i) represents a pair of intersecting lines.

In (ii), $h^2 - ab < 0$, hence (ii) represents a pair of imaginary lines.

In (iii), $h^2 - ab = 0$ and $f^2 - bc < 0$, hence (iii) represents a pair of imaginary lines.

ILLUSTRATION 102

For what value of λ does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represent a pair of straight lines. Find their equations and the angle between them.

SOLUTION

By comparing the given equation, with the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we get,

$$a = 12, h = -5, b = 2, g = \frac{11}{2}, f = \frac{-5}{2}, c = \lambda$$

If the given equation represents a pair of straight lines, then $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow \quad (12)(2)(\lambda) + 2\frac{-5}{2}\cdot\frac{11}{2}\cdot(-5) - 12\cdot\left(\frac{-5}{2}\right)^2$$

$$-2\cdot\left(\frac{11}{2}\right)^2 - \lambda\cdot(-5)^2 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Also, $h^2 - ab = 25 - 24 > 0$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{25-24}}{12+2} = \frac{1}{7}$$

Angle between lines are $\tan^{-1}\left(\frac{1}{7}\right)$ or $\pi - \tan^{-1}\left(\frac{1}{7}\right)$

∴ The given equation will represent a pair of straight line, if $\lambda = 2$.

The given equation

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \qquad \ldots(1)$$

∵ $12x^2 - 10xy + 2y^2 = 2(3x - y)(2x - y)$, the factors of (1) can be taken as, $2(3x - y + a)(2x - y + b)$.

So that,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2$$

$$= 2(3x - y + a)(2x - y + b)$$

$$\Rightarrow \quad 12x^2 - 10xy + 2y^2 + 2(3b + 2a)x - 2(a + b)y + 2ab$$

Comparing, $2a + 3b = \frac{11}{2}, a + b = \frac{5}{2}, ab = 1$

Solving, we get $a = 2$ and $b = \frac{1}{2}$ Thus, the factors of (1) are $2(3x - y + 2)\left(2x - y + \frac{1}{2}\right) = 0$

$$\Rightarrow \quad (3x - y + 2)(4x - 2y + 1) = 0$$

Alternate

$$2y^2 - 5y(1 + 2x) + 12x^2 + 11x + 2 = 0$$

$$y = \frac{5(1+2x) \pm \sqrt{25(1+2x)^2 - 8(12x^2 + 11x + 2)}}{4}$$

$$4y = 5 + 10x \pm (2x + 3)$$

On solving, we get the two straight lines as $3x - y + 2 = 0$ and $4x - 2y + 1 = 0$.

ILLUSTRATION 103

If the equation $ax^2 - 6xy + y^2 + 2gx + 2fy + c = 0$ represents a pair of lines, whose slopes are m and m^2, then find the sum of all the possible values of a.

SOLUTION

The given lines will be parallel to lines $ax^2 - 6xy + y^2 = 0$. So,

$$\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + a = 0; \ \frac{y}{x} = m$$

$m^2 - 6m + a = 0$ has roots m and m^2

$$\therefore \quad m + m^2 = 6 \Rightarrow m^2 + m - 6 = 0$$

$$\Rightarrow \quad (m + 3)(m - 2) = 0$$

$$m = -3 \text{ or } 2 \text{ and } m^3 = a$$

$$a = -27 \text{ or } 8$$

Hence, the sum of all the possible values of $a = -27 + 8 = -19$.

ILLUSTRATION 104

If the equation $9x^2 + 2hxy + 4y^2 + 6x + 2fy - 3 = 0$ represents two parallel lines, then find h and f.

SOLUTION

The given equation represents a pair of parallel lines.

$\therefore$ We have, $h^2 = ab$

$\Rightarrow$ $h = \pm 6$

Condition for pair of lines $\begin{vmatrix} 9 & h & 3 \\ h & 4 & f \\ 3 & f & -3 \end{vmatrix} = 0$

$\Rightarrow$ $9f^2 - 6hf + 36 = 0$

$\Rightarrow$ $3f^2 \pm 12f + 12 = 0$

$\Rightarrow$ $f \pm 4f + 4 = 0$

$\Rightarrow$ $f = \pm 2$

ILLUSTRATION 105

If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y-axis, then find the condition.

SOLUTION

Given equation of pair of straight lines is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \ldots(1)$$

These lines intersect on y-axis at $(0, y_1)$.

Thus, from the equation (1), $by_1^2 + 2fy_1 + c = 0$.

Above equation have equal roots.

$\therefore$ $D = 0$

$\therefore$ $4f^2 - 4bc = 0$

$\Rightarrow$ $f^2 - bc = 0$

Now, equation (1) represents a pair of straight lines.

So, $abc + 2fgh = af^2 + ch^2 + bg^2$

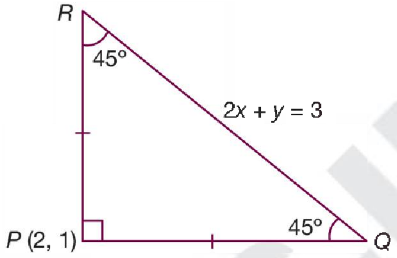
$\therefore$ $abc + 2fgh = abc + bg^2 + ch^2$

$2fgh = bg^2 + ch^2$ and $f^2 = bc$

ILLUSTRATION 106

Let PQR be a right angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, then find the equation representing the pair of lines PQ and PR.

SOLUTION



Let m be the slope of PQ. Then,

$$\tan 45° = \left|\frac{m-(-2)}{1+m(-2)}\right| = \left|\frac{m+2}{1-2m}\right|$$

$\therefore$ $m + 2 = 1 - 2m$ or $-1 + 2m = m + 2$

$\therefore$ $m = \frac{-1}{3}$ or $m = 3$

Hence, the equation of PQ is $y - 1 = \frac{-1}{3}(x - 2)$

$\Rightarrow$ $x + 3y - 5 = 0$

The equation of PR is $y - 1 = 3(x - 2)$.

$\Rightarrow$ $3x - y - 5 = 0$

Hence, the combined equation of PQ and PR is $(x + 3y - 5)(3x - y - 5) = 0$.

$\Rightarrow$ $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

ILLUSTRATION 107

Let $S = \{(x, y) \mid x^2 + 2xy + y^2 - 3x - 3y + 2 = 0\}$, then comment on locus of S.

SOLUTION

$x^2 + 2xy + y^2 - 3(x + y) + 2 = 0$

or $(x + y)^2 - 3(x + y) + 2 = 0$

$\Rightarrow$ $(x + y - 2)(x + y - 1) = 0$

$\therefore$ There are 2 parallel lines, which are non-coincident.

ILLUSTRATION 108

Show that the four lines given by $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ lie along the sides of a square.

SOLUTION

$$\begin{aligned} 12x^2 + 7xy - 12y^2 &= 12x^2 + 16xy - 9xy - 12y^2 \\ &= 4x(3x + 4y) - 3y(3x + 4y) \\ &= (3x + 4y)(4x - 3y) \end{aligned}$$

$\therefore$ The pair of lines given by the 1st equation has their separate equations $3x + 4y = 0$ and $4x - 3y = 0$, which are at right angles.

Again, $12x^2 + 7xy - 12y^2 - x + 7y - 1$

$$= (3x + 4y)(4x - 3y) - x + 7y - 1$$

(Factorised first three terms)

$$= (3x + 4y - 1)(4x - 3y + 1), \text{ by observation}$$

$\therefore$ The pair of lines given by the 2nd equation has their separate equations $3x + 4y - 1 = 0$ and $4x - 3y + 1 = 0$, which are at right angles.

Thus, the 4 lines are $3x + 4y = 0$, $4x - 3y = 0$, $3x + 4y - 1 = 0$ and $4x - 3y + 1 = 0$ of which the 1st and the 3rd are parallel while the second and the fourth are also parallel. Moreover, the first and the second are at right angles. Hence, these lines form a rectangle.

The distance between one set of parallel sides $3x + 4y = 0$ and $3x + 4y - 1 = 0$ is given by

$$d_1 = \left|\frac{-1}{\sqrt{3^2 + 4^2}}\right| = \frac{1}{5} \text{ units}$$

The distance between other two parallel sides $4x - 3y = 0$ and $4x - 3y + 1 = 0$ is given by

$$d_2 = \left|\frac{1}{\sqrt{3^2 + 4^2}}\right| = \frac{1}{5} \text{ units} = d_1$$

Hence, these lines form a square.

ILLUSTRATION 109

If the straight lines joining the origin and the points of intersection of the curve $5x^2 + 12xy - 6y^2 + 4x - 2y + 3 = 0$ and $x + ky - 1 = 0$ are equally inclined to the $+x$-axis, then find the value of k.

SOLUTION

Required lines we get homogenisation of curve with the given straight line. $(x + ky = 1)$

$$5x^2 + 12xy - 6y^2 + (4x - 2y)(1) + 3(1)^2 = 0$$

$$\Rightarrow \quad 5x^2 + 12xy - 6y^2 + (4x - 2y)(x + ky) + 3(x + ky)^2 = 0$$

$$\Rightarrow \quad 5x^2 + 12xy - 6y^2 + 4x^2 + (4k - 2)xy - 2ky^2 + 3x^2 + 3k^2y^2 + 6kxy = 0$$

$$\Rightarrow \quad 12x^2 + (3k^2 - 2k - 6)y^2 + 10xy(k + 1) = 0$$

As it is given that required lines are equally inclined with x-axis

Put coefficient of $xy = 0$

$$\therefore \quad k = -1$$

ILLUSTRATION 110

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight line, then prove that the product of the perpendiculars drawn from the origin to the lines is $\dfrac{c}{\sqrt{(a-b)^2 + 4h^2}}$.

SOLUTION

Let $ax^2 + 12hxy + by^2 + 2gx + 2fy + c$

$$= (\ell_1x + m_1y + n_1)(\ell_2x + m_2y + n_2).$$

Then the separate lines are $\ell_1x + m_1y + n_1 = 0$ and $\ell_2x + m_2y + n_2 = 0$.

Where, $\ell_1\ell_2 = a_1$, $m_1m_2 = b$, $n_1n_2 = c$, $\ell_1m_2 + \ell_2m_1 = 2h$, $m_1n_2 + m_2n_1 = 2f$ and $n_1\ell_2 + n_2\ell_1 = 2g$.

Lengths of perpendiculars from (0, 0) to the lines are respectively

$$\frac{n_1}{\sqrt{\ell_1^2 + m_1^2}} \text{ and } \frac{n_2}{\sqrt{\ell_2^2 + m_2^2}}$$

$\therefore$ The required product of perpendiculars drawn from the origin

$$= \frac{n_1}{\sqrt{\ell_1^2 + m_1^2}} \cdot \frac{n_2}{\sqrt{\ell_2^2 + m_2^2}}$$

$$= \frac{n_1n_2}{\sqrt{\ell_1^2\ell_2^2 + m_1^2m_2^2 + \ell_1^2m_2^2 + \ell_2^2m_1^2}}$$

$$= \frac{c}{\sqrt{a^2 + b^2 + (\ell_1m_2 + \ell_2m_1)^2 - 2\ell_1\ell_2 \cdot m_1m_2}}$$

$$= \frac{c}{\sqrt{a^2 + b^2 + (2h)^2 - 2ab}}$$

$$= \frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

ILLUSTRATION 111

Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\dfrac{2\sqrt{2}}{3}$.

SOLUTION

Equation of the given curve is $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ and equation of the given straight line is $y - 3x = 2$.

$$\therefore \quad \frac{y-3x}{2} = 1$$

Making equation (1) as the homogeneous equation of the 2nd degree in x any y with the help of (1), we have,

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y-3x}{2}\right) + 8y\left(\frac{y-3x}{2}\right)$$

$$-11\left(\frac{y-3x}{2}\right)^2 = 0$$

$$\Rightarrow \quad x^2 + 2xy + 3y^2 + \frac{1}{2}(4xy + 8y^2 - 12x^2 - 24\,xy)$$

$$-\frac{11}{4}(y^2 - 6xy + 9x^2) = 0$$

$$\Rightarrow 4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy)$$

$$-11\,(y^2 - 6xy + 9x^2) = 0$$

$$\Rightarrow \quad -119x^2 + 34xy + 17y^2 = 0 \text{ or } 119x^2 - 34xy - 17y^2 = 0$$

$$\Rightarrow \quad 7x^2 - 2xy - y^2 = 0$$

This is the equation of the lines joining the origin to the points of intersection of (1) and (2).

Comparing equation (3) with the equation $ax^2 + 2hxy + by^2 = 0$, we have $a = 7$, $b = -1$ and $2h = -2$, i.e., $h = -1$

If θ be the acute angle between pair of lines (3), then

$$\tan\,\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{1+7}}{7-1}\right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$$(\therefore\ \theta = \tan^{-1}\frac{2\sqrt{2}}{3})$$

Test Your Concepts-VI

(Solutions on page H.30)

1. If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq = -1$.
2. The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at the origin, then find the value of p.
3. If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then find λ.
4. A straight line through $P(-2, -3)$ cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line, if $PQ.PR = 20$.
5. Show that the equation $6x^2 - 5xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.
6. Find the equations to the pair of lines through the origin, which are perpendicular to the lines represented by $2x^2 - 7xy + 2y^2 = 0$.
7. Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the co-ordinates of their point of intersection and also the angle between them.
8. Show that the straight lines $\left(A^2 - 3B^2\right)x^2 - 8ABxy + \left(B^2 - 3A^2\right)y^2 = 0$ form with the line $Ax + By + C = 0$ an equilateral triangle of area $\dfrac{C^2}{\sqrt{3}\left(A^2 + B^2\right)}$.
9. Find the condition on a and b, such that the portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + by = 0$ subtends a right angle at the origin.

Answer Keys

2. $\dfrac{-9 \pm \sqrt{33}}{8}$ **3.** $\lambda = 2$ **4.** $3x - y + 3 = 0$ **5.** $y - 3x = 0$; $y - 2x = 0$

6. $y + 3x = 0$ and $2y + x = 0$ **7.** $(1, -2)$ **9.** $2a + b + b^2 = 0$

SOLVED PROBLEMS

PROBLEM 1

Two fixed points A and B are taken on the co-ordinate axes such that $OA = a$ and $OB = b$. Two variable points A' and B' are taken on the same axes such that $OA' + OB' = OA + OB$. Find the locus of the point of intersection of AB' and $A'B$.

SOLUTION

Let $A(a, 0), B(0, b), A'(a', 0), B'(0, b')$.

Equation of $A'B$ is $\frac{x}{a'} + \frac{y}{b} = 1$ …(1)

and equation of AB' is $\frac{x}{a} + \frac{y}{b'} = 1$ …(2)

Subtracting (1) from (2), we get,

$$x\left(\frac{1}{a} - \frac{1}{a'}\right) + y\left(\frac{1}{b'} - \frac{1}{b}\right) = 0$$

$$\Rightarrow \quad \frac{x(a'-a)}{aa'} + \frac{y(b-b')}{bb'} = 0 \qquad (\text{Using } a' - a = b - b')$$

$$\Rightarrow \quad \frac{x}{a(b-b'+a)} + \frac{y}{bb'} = 0 \Rightarrow b' = \frac{a(a+b)y}{ay-bx} \qquad \ldots(3)$$

From (2), $b'x + ay = ab'$

$$\Rightarrow \quad b' = \frac{ay}{a-x} \qquad \ldots(4)$$

Equating (3) and (4), we get, $x + y = a + b$, which is the required locus.

PROBLEM 2

If (x_r, y_r), where $r = 1, 2$ and 3 be the co-ordinates of the points A, B and C respectively, then prove that the equation of the internal bisector of angle A, is $b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, where $b = AC$ and $c = AB$.

SOLUTION

Let, AD be the internal bisector of $\angle BAC$.

Then, $\frac{BC}{DC} = \frac{AB}{AC} = \frac{c}{b}$.

Thus, D divides BC internally in the ratio $c{:}b$, so that $D\left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$. Now, the equation of line AD is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{bx_2 + cx_3}{b+c} & \frac{by_2 + cy_3}{b+c} & 1 \end{vmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ bx_2 + cx_3 & by_2 + cy_3 & b+c \end{vmatrix} = 0 \quad \text{(Using properties of determinant)}$$

$$\Rightarrow \quad b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \text{Hence Proved}$$

PROBLEM 3

Given n straight lines and a fixed point O, a straight line is drawn through O meeting lines in the points $R_1, R_2, R_3, \ldots, R_n$ and on it, a point R is taken such that $\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \ldots + \frac{1}{OR_n}$. Show that the locus of R is a straight line.

SOLUTION

Let the equations of the given lines be $a_ix + b_iy + c_i = 0$, $i = 1, 2, \ldots, n$ and the point O be the origin $(0, 0)$.

Then equation of the line through O can be written as $\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r$, where θ is the angle made by the line with the positive direction of x-axis and r is the distance of any point on the line from the origin O.

Let $r, r_1, r_2, \ldots r_n$ be the distances of the points, $R, R_1, R_2, \ldots R_n$ from O.

$\Rightarrow \quad OR = r$ and $OR_i = r_i$

Then the co-ordinates of R are $(r\cos\theta, r\sin\theta)$ and of R_i are $(r_i\cos\theta, r_i\sin\theta)$, $i = 1, 2, \ldots, n$.

Since, R_i lies on $a_ix + b_iy + c_i = 0$

$$\Rightarrow \quad a_i r_i \cos\theta + b_i r_i \sin\theta + c_i = 0$$

For $i = 1, 2, \ldots, n$

$$\Rightarrow \quad -\frac{a_i}{c_i}\cos\theta - \frac{b_i}{c_i}\sin\theta = \frac{1}{r_i}, \; i = 1, 2, \ldots, n.$$

$$\Rightarrow \quad -\left[\sum_{i=1}^{n}\frac{a_i}{c_i}\right]\cos\theta - \left[\sum_{i=1}^{n}\frac{b_i}{c_i}\right]\sin\theta = \sum_{i=1}^{n}\frac{1}{r_i}$$

(Given $\sum_{i=1}^{n}\frac{1}{r_i} = \frac{n}{r}$)

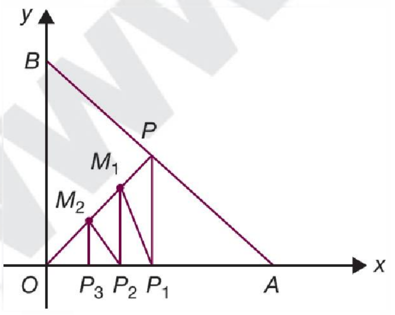
$$\Rightarrow \quad -\left[\sum_{i=1}^{n}\frac{a_i}{c_i}\right]\cos\theta - \left[\sum_{i=1}^{n}\frac{b_i}{c_i}\right]\sin\theta = \frac{n}{r}$$

$$\Rightarrow \quad \left(\sum_{i=1}^{n}\frac{a_i}{c_i}\right) r\cos\theta + \left(\sum_{i=1}^{n}\frac{b_i}{c_i}\right) r\sin\theta + n = 0$$

Hence, the locus of R is $\left(\sum_{i=1}^{n}\frac{a_i}{c_i}\right)x + \left(\sum_{i=1}^{n}\frac{b_i}{c_i}\right)y + n = 0$, which is a straight line.

PROBLEM 4

The line $x + y = 1$ meets x-axis at A and y-axis at B. P is the midpoint of AB (Figure). P_1 is the foot of the perpendicular from P to OA, M_1 is that of P_1 from OP, P_2 is that of M_1 from OA, M_2 is that of P_2 from OP, P_3 is that of M_2 from OA and so on. If P_n denotes the n^{th} foot of the perpendicular on OA from M_{n-1}, then find OP_n.



SOLUTION

Let $x + y = 1$ meets x-axis at $A(1, 0)$ and y-axis at $B(0, 1)$.

The co-ordinates of P are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and PP_1 is perpendicular to OA.

$$\Rightarrow \quad OP_1 = P_1P = \frac{1}{2}$$

Equation of line OP is $y = x$.

We have,

$$OM_{n-1}^2 = OP_n^2 + P_nM_{n-1}^2 = 2OP_n^2 = 2p_n^{\,2}$$

(say $OP_n = p_n$)

Also, $OP_{n-1}^2 = OM^2_{n-1} + P_{n-1}M^2_{n-1} = 2OM^2_{n-1} = 4p^2_n$

$$\Rightarrow \quad p_{n-1}^2 = 4p_n^2 \;\Rightarrow\; p_n^2 = \frac{1}{4}p_{n-1}^2$$

$$\Rightarrow \quad p_n = \frac{1}{2}p_{n-1}$$

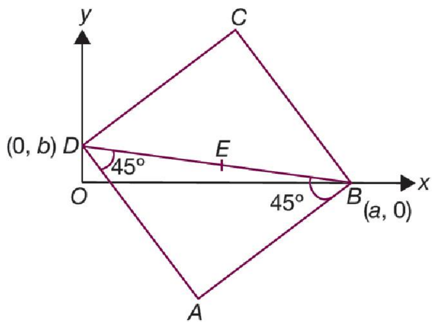
$$\therefore \quad OP_n = P_n = \frac{1}{2}p_{n-1} = \frac{1}{2^2}p_{n-2} = \ldots\ldots\ldots = \frac{1}{2^{n-1}}\,p_1 = \frac{1}{2^n}$$

PROBLEM 5

One diagonal of a square is the portion of the line $\frac{x}{a} + \frac{y}{b} = 1$, which is intercepted between the axes. Find the co-ordinates of other two vertices of the square. Also prove that if two opposite vertices of a square move on two perpendicular lines, the other two vertices also move on two perpendicular lines.

SOLUTION

The slope of the given lines is $\frac{-b}{a}$ and the two vertices are clearly on the diagonal BD of the square $ABCD$. If m be the slope of the line inclined at an angle of 45° to BD, then



$$\tan(\pm 45°) = \frac{m + \frac{b}{a}}{1 - m\frac{b}{a}} = \pm 1$$

$$\therefore \quad m = \frac{a-b}{a+b} \text{ or } -\frac{(a+b)}{(a-b)}$$

$$\therefore \quad AB \text{ is } y - 0 = \frac{a-b}{a+b}(x-a)$$

$$AD \text{ is } y - b = -\frac{a+b}{a-b}(x-0)$$

Solving these by subtracting, the point A is $\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$, C is obtained by using the fact that midpoint of AC and BD is same.

$$\therefore \quad C \text{ is } \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$$

Now for 2nd part, the opposite vertices B and D move on two perpendicular lines x-axis and y-axis. Now, the point $A\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$ lies on $y = -x$ and point C $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ lies on $y = x$. Both these lines are clearly perpendicular.

PROBLEM 6

The circumcentre of a triangle having vertices $A(a, a\tan\alpha)$, $B(b, b\tan\beta)$, $C(c, c\tan\gamma)$ is at the origin, where $\alpha + \beta + \gamma = \pi$. Show that the orthocentre lies on the line

$$\left(4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)x - 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cdot y = y$$

SOLUTION

$\because$ Circumcentre O is the origin, $OA = OB = OC = r$

$$a^2 + a^2\tan^2\alpha = b^2 + b^2\tan^2\beta = c^2 + c^2\tan^2\gamma = r^2$$

$$a\sec\alpha = b\sec\beta = c\sec\gamma = r$$

$$a\sec\alpha = r \Rightarrow a = r\cos\alpha$$

Similarly, $b = r\cos\beta$, $c = r\cos\gamma$

$\therefore \quad A(r\cos\alpha, r\sin\alpha)$

Similarly, $B(r\cos\beta, r\sin\beta)$ and $C(r\cos\gamma, r\sin\gamma)$

$$\therefore \quad \text{Centroid } G\left(\frac{r(\cos\alpha + \cos\beta + \cos\gamma)}{3}, \frac{r(\sin\alpha + \sin\beta + \sin\gamma)}{3}\right)$$

Circumcentre $O(0, 0)$ and orthocentre $H(h, k)$ and centroid G are collinear.

$\therefore \quad$ Slope of OG = Slope of OH

$$\Rightarrow \quad \frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma} = \frac{k}{h}$$

$\therefore$ Point (h, k) will be upon $x(\sin\alpha + \sin\beta + \sin\gamma) - y(\cos\alpha + \cos\beta + \cos\gamma) = 0$.

$$\Rightarrow \quad x\left(2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\right)$$

$$-y\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\sin^2\frac{\gamma}{2}\right) = 0$$

$\because \quad \alpha + \beta + \gamma = \pi$

$$\Rightarrow \quad x\left(2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\frac{\gamma}{2}\right)$$

$$-y\left(1 + 2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} - 2\sin^2\frac{\gamma}{2}\right) = 0$$

$$\Rightarrow \quad x\left(2\cos\frac{\gamma}{2}\left[\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\right]\right)$$

$$-y\left(1 + 2\sin\frac{\gamma}{2}\left[\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right]\right) = 0$$

$$\Rightarrow \quad x\left(4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)$$

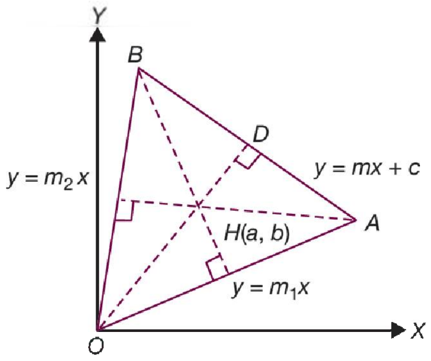
$$-y\left(1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right) = 0$$

Hence, the result because $\alpha + \beta + \gamma = \pi$.

PROBLEM 7

A triangle has the lines $y = m_1x$ and $y = m_2x$ for two of its sides, with m_1 and m_2 being the roots of the equation $bx^2 + 2hx + a = 0$. If $H(a, b)$ is the orthocentre of the triangle, then show that the equation of the third side is $(a + b)(ax + by) = ab(a + b - 2h)$.

SOLUTION



The given lines $y = m_1x$ and $y = m_2x$ intersect at the origin $O(0, 0)$. Thus, one vertex of the triangle is at the origin O.

∴ Let OAB be the triangle and OA and OB be the lines $y = m_1x$ and $y = m_2x$, respectively.

Let the equation of the 3rd side AB be $y = mx + c$.

On solving, we get $A\left(\frac{c}{m_1 - m}, \frac{cm_1}{m_1 - m}\right)$, $B\left(\frac{c}{m_2 - m}, \frac{cm_2}{m_2 - m}\right)$

Given that, $H\ (a, b)$ is the orthocentre of the ΔOAB.

∴ OH is ⊥ to AB

$$\Rightarrow \left(\frac{b}{a}\right) \times m = -1 \Rightarrow m = \frac{-a}{b} \qquad \ldots(1)$$

Now, the equation of line through A and perpendicular to OB is

$$AH: y - \frac{cm_1}{m_1 - m} = -\frac{1}{m_2}\left(x - \frac{c}{m_1 - m}\right)$$

$$\Rightarrow \quad y = -\frac{x}{m_2} + \frac{c(m_1m_2 + 1)}{m_2(m_1 - m)} \qquad \ldots(2)$$

Similarly, equation of the line through B and perpendicular to OA is

$$y = -\frac{x}{m_1} + \frac{c(m_1m_2 + 1)}{m_1(m_2 - m)} \qquad \ldots(3)$$

The point of intersection of (ii) and (iii) is the orthocentre $H\ (a, b)$.

∴ Subtracting (3) from (2), we get,

$$x = a = \frac{-cm(m_1m_2 + 1)}{(m_1 - m)(m_2 - m)}$$

$$\text{or} \quad c = \frac{-[m_1m_2 - m(m_1 + m_2) + m^2]a}{m(m_1m_2 + 1)} \qquad \ldots(4)$$

∵ m_1 and m_2 are the roots of the equation $bx^2 + 2hx + a = 0$.

$$\therefore \quad m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

∴ From (4), we have

$$c = \frac{-\left(a/b + 2hm/b + m^2\right)a}{m(a/b + 1)} = \frac{-\left(a + 2hm + bm^2\right)a}{m(a + b)}$$

∴ The equation of 3rd side AB is $y = mx + c$.

$$y = mx - \frac{(a + 2hm + bm^2)a}{m_1(a + b)} \quad \left(m = \frac{-a}{b}\right)$$

$$\Rightarrow \quad y = -\frac{a}{b}x - \frac{(a - 2ha/b + b.a^2/b^2)a}{(-a/b)(a + b)}$$

$$\Rightarrow \quad (ax + by)(a + b) = ab\ (a + b - 2h)$$

PROBLEM 8

Find the values of non-negative real numbers h_1, h_2, h_3, k_1, k_2, k_3 such that the algebraic sum of the perpendiculars drawn from the points $(2, k_1)$ $(3, k_2)$ $(7, k_3)$, $(h_1, 4)$ $(h_2, 5)$ $(h_3, -3)$, on a variable line passing through $(2, 1)$ is zero.

SOLUTION

Let the equation of the variable line be $ax + by + c = 0$. It is given that

$$\sum_{i=1}^{6} \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{6}\left(ax_i + by_i + c\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{6} ax_i + \sum_{i=1}^{6} by_i + \sum_{i=1}^{6} c = 0$$

$$\Rightarrow \quad \sum_{i=1}^{6} ax_i + \sum_{i=1}^{6} by_i + 6c = 0$$

$$\Rightarrow \quad a\left(\frac{\Sigma x_i}{6}\right) + b\left(\frac{\Sigma y_i}{6}\right) + c = 0$$

Hence, the fixed point must be $\left(\frac{\Sigma x_i}{6}, \frac{\Sigma y_i}{6}\right)$.

However, the fixed point is (2, 1) so that, $\frac{2 + 3 + 7 + h_1 + h_2 + h_3}{6} = 2$

$$\Rightarrow \quad h_1 + h_2 + h_3 = 0$$

$$\Rightarrow \quad h_1 = 0,\ h_2 = 0,\ h_3 = 0$$

(As h_1, h_2 and h_3 are non-negative)

Similarly, $\frac{k_1 + k_2 + k_3 + 4 + 5 - 3}{6} = 1$

$$\Rightarrow \quad k_1 = k_2 = k_3 = 0$$

Alternate

Let the variable line be $y - 1 = m(x - 2)$ or $mx - y + 1 - 2m = 0$.

$\because$ The algebraic sum of the perpendicular from the given points is zero, i.e.,

$$\frac{m(2+3+7+h_1+h_2+h_3-12)+(6-k_1-k_2-k_3-4-5+3)}{\sqrt{1+m^2}}$$

for all m

For all $m \Rightarrow m(h_1+h_2=h_3)-(k_1+k_2+k_3)=0$

$\Rightarrow \quad h_1+h_2=h_3=0$ and $k_1+k_2+k_3=0$

$\Rightarrow \quad h_1=0=h_2=h_3$ and $k_1=0=k_2=k_3$

PROBLEM 9

Find the equation of the straight line passing through $(-2, -7)$ and having intercept of length 3 units between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.

SOLUTION

Distance AC between the two given parallel lines

$$= \frac{|c_1-c_2|}{\sqrt{a^2+b^2}} = \frac{12-3}{\sqrt{16+9}} = \frac{9}{5}.$$

Let, AB be the intercept of length 3 units.

$$\Rightarrow \quad BC = \frac{12}{5}$$

If θ is the angle between BC and AB, then $\tan\theta = \frac{9}{12}$

$$= \frac{3}{4}.$$

Slope of the parallel lines $= -\frac{4}{3} = m_2$

If m_1 is the slope of the required line, then

$$\tan\theta = \pm\frac{m_1-m_2}{1+m_1m_2}$$

$$\Rightarrow \quad \frac{3}{4} = \pm\frac{m_1+\frac{4}{3}}{1-\frac{4}{3}m_1}$$

i.e., $m_1+\frac{4}{3}=\frac{3}{4}\left(1-\frac{4}{3}m_1\right)$ and $m_1+\frac{4}{3}=-\frac{3}{4}\left(1-\frac{4}{3}m_1\right)$

The slopes are

(i) $m_1 = \frac{-7}{24}$ and (ii) $m_1 = \infty$ (The line is parallel to the y-axis).

The required equations of the lines are $7x + 24y + 182 = 0$ and $x + 2 = 0$.

Alternate

Equation of the line, through $P(-2, -7)$ and making an angle θ with the x-axis is $\frac{x+2}{\cos\theta} = \frac{y+7}{\sin\theta} = r$.

If this line intersects the given lines at A and B with $AB = 3$, then the points A and B are

$$A\left(-2+r_1\cos\theta, -7+r_1\sin\theta\right)$$

and $B\left(-2+(r_1+3)\cos\theta, -7+(r_1+3)\sin\theta\right)$

$\because$ A and B lie on the lines $4x + 3y = 3$ and $4x + 3y = 12$, we have

$$4r_1\cos\theta+3r_1\sin\theta=32$$

and $\quad 4r_1\cos\theta+3r_1\sin\theta+12\cos\theta+9\sin\theta=41.$

So that, $12\cos\theta+9\sin\theta=9$ or $4\cos\theta+3\sin\theta=3$.

By solving this equation we find that, $\theta=\frac{\pi}{2}$ and $\tan\theta=-\frac{7}{24}$.

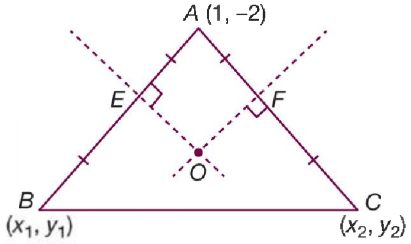
Hence, the required lines are $x + 2 = 0$ and $y+7=-\frac{7}{24}(x+2)$, i.e., $7x + 24y + 182 = 0$.

PROBLEM 10

The equations of the perpendicular bisectors of the sides AB and AC of a ΔABC are respectively $x - y + 5 = 0$ and $x + 2y = 0$. If the co-ordinates of A are $(1, -2)$, then find the equation of BC.

SOLUTION

Let E and F are mid point of AB and AC and O is circumcentre.



From the figure, $E\left(\frac{x_1+1}{2}, \frac{y_1-2}{2}\right)$ $F\left(\frac{x_2+1}{2}, \frac{y_2-2}{2}\right)$

∵ E and F lie on OE and OF respectively, we get

$$x_1-y_1+13=0 \quad \ldots(1)$$

$$x_2+2y_2-3=0 \quad \ldots(2)$$

Also, slope of $AB = -1$ and slope of AC is 2, so that

$$x_1+y_1+1=0 \quad \ldots(3)$$

$$2x_2-y_2-4=0 \quad \ldots(4)$$

Solving these equations, we get the co-ordinates of B and C as

$$B(-7, 6) \text{ and } C\left(\frac{11}{5}, \frac{2}{5}\right)$$

∴ Equation of BC is $14x + 23y - 40 = 0$.

Alternate Method

Image of A w.r.to perpendicular side bisector of AB is = point B

$$\frac{x-1}{1}=\frac{y+2}{-1}=\frac{-2(1+2+5)}{1+1}$$

Point $B(-7, 6)$

Similarly image of A w.r.to perpendicular side bisector of AC is = point C

$$\frac{x-1}{1}=\frac{y+2}{2}=\frac{-2(1-4)}{1+4}$$

Point $C\left(\frac{11}{5}, \frac{2}{5}\right)$

Now equation of BC

$$14x+23y-40=0$$

PROBLEM 11

A variable straight line drawn through the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$, meets the co-ordinate axes in A and B. Show that the locus of the midpoint of AB is the curve $2xy(a+b)=ab(x+y)$.

SOLUTION

The equation of the variable line through the point of intersection of the given lines is given by

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)+k\left(\frac{x}{b}+\frac{y}{a}-1\right)=0.$$

$$\Rightarrow \left(\frac{1}{a}+\frac{k}{b}\right)x+\left(\frac{1}{b}+\frac{k}{a}\right)y=(1+k)$$

$$\Rightarrow (ak+b)x+(bk+a)y=ab(1+k)$$

∴ The points A and B are $\left(\frac{ab(1+k)}{ak+b}, 0\right)$ and $\left(0, \frac{ab(1+k)}{bk+a}\right)$.

Let $P(x_1, y_1)$ be the midpoint of AB.

Then

$$x_1=\frac{ab(1+k)}{2(ak+b)} \text{ and } y_1=\frac{ab(1+k)}{2(bk+a)} \quad \ldots(1)$$

Now adding their reciprocal,

$$\frac{1}{x_1}=\frac{2(ak+b)}{ab(1+k)}, \frac{1}{y_1}=\frac{2(bk+a)}{ab(1+k)}$$

$$\Rightarrow \frac{1}{x_1}+\frac{1}{y_1}=\frac{2(a+b)(1+k)}{ab(1+k)} \Rightarrow \frac{x_1+y_1}{x_1y_1}=\frac{2(a+b)}{ab}$$

(This eliminates parameter k).

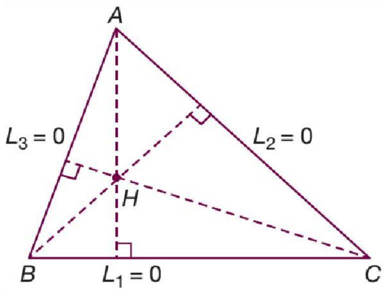
Hence, the locus of $P(x_1, y_1)$ is $2xy(a+b)=ab(x+y)$.

PROBLEM 12

The three sides of a triangle are $L_r \equiv x\cos\theta_r + y\sin\theta_r - p_r = 0$; $r = 1, 2, 3$. Show that the orthocentre of the triangle is given by $L_1\cos(\theta_2-\theta_3)=L_2\cos(\theta_3-\theta_1)=L_3\cos(\theta_1-\theta_2)$.

SOLUTION

Let the equations of BC, CA and AB be respectively $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$.



Any line passing through A is the intersection of the lines $L_2 = 0$, $L_3 = 0$, which is

$$L_2 + \lambda L_3 = 0 \qquad \ldots(1)$$

or $\quad x(\cos\theta_2 + \lambda\cos\theta_3) + y(\sin\theta_2 + \lambda\sin\theta_3)$

$$- (p_2 + \lambda p_3) = 0$$

'm' of this line $= -\dfrac{\cos\theta_2 + \lambda\cos\theta_3}{\sin\theta_2 + \lambda\sin\theta_3}$

It is perpendicular to BC.

$$\therefore \quad -\frac{\cos\theta_2 + \lambda\cos\theta_3}{\sin\theta_2 + \lambda\sin\theta_3}\cdot\left(-\frac{\cos\theta_1}{\sin\theta_1}\right) = -1$$

$$\left(\because\ m_{BC} = -\frac{\cos\theta_1}{\sin\theta_1}\right)$$

$$\Rightarrow \quad \cos\theta_1(\cos\theta_2 + \lambda\cos\theta_3) + \sin\theta_1(\sin\theta_2 + \lambda\sin\theta_3) = 0$$

$$\Rightarrow \quad \cos(\theta_1 - \theta_2) + \lambda\cos(\theta_3 - \theta_1) = 0$$

$$\therefore \quad \lambda = -\frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_3 - \theta_1)}$$

Putting in (1), the equation of the altitude through A becomes,

$$L_2 - \frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_3 - \theta_1)}L_3 = 0$$

$$\Rightarrow \quad L_2\cos(\theta_3 - \theta_1) = L_3\cos(\theta_1 - \theta_2) \qquad \ldots(2)$$

Similarly, the equation of the altitudes through B and C are respectively

$$L_3\cos(\theta_1 - \theta_2) = L_1\cos(\theta_2 - \theta_3) \qquad \ldots(3)$$

and $\quad L_1\cos(\theta_2 - \theta_3) = L_2\cos(\theta_3 - \theta_1) \qquad \ldots(4)$

The orthocentre being the intersection of altitudes, satisfies (2), (3) and (4). So, the orthocentre is given by

$$L_1\cos(\theta_2 - \theta_3) = L_2\cos(\theta_3 - \theta_1) = L_3\cos(\theta_1 - \theta_2).$$

PROBLEM 13

Prove that the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent two parallel straight lines, if $g^2 - ac \geq 0$ and ($[h = \sqrt{ab}, g\sqrt{b} = f\sqrt{a}]$ or $[h = -\sqrt{ab}, g\sqrt{b} = -f\sqrt{a}]$). Also prove that, $\dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f}$

Also prove that the distance between them is $2\sqrt{\dfrac{g^2 - ac}{a(a+b)}}$

SOLUTION

The given equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

For parallel lines, $h^2 = ab$ $\left(\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b} = 0\right)$

CASE-1:

$$(h = \sqrt{ab})$$

$$abc + 2fg\sqrt{ab} - af^2 - bg^2 - c\cdot ab$$

$$= -(f\sqrt{a} - g\sqrt{b})^2 = 0, \text{ if } g\sqrt{b} = f\sqrt{a}$$

Thus, one set of conditions is

$$h = \sqrt{ab},\ g\sqrt{b} = f\sqrt{a} \qquad \ldots(1)$$

CASE-2:

$$(h = -\sqrt{ab})$$

$$abc - 2fg\cdot\sqrt{ab} - af^2 - bg^2 - c\cdot ab$$

$$= -(f\sqrt{a} + g\sqrt{b})^2 = 0, \text{ if } g\sqrt{b} = -f\sqrt{a}$$

Thus, the 2nd set of condition is

$$h = -\sqrt{ab},\ g\sqrt{b} = -f\sqrt{a} \qquad \ldots(2)$$

Now from (1),

$$\frac{a}{h} = \frac{a}{\sqrt{ab}} = \sqrt{\frac{a}{b}},\ \frac{h}{b} = \frac{\sqrt{ab}}{b} = \sqrt{\frac{a}{b}} \text{ and } \frac{g}{f} = \sqrt{\frac{a}{b}}$$

$$\therefore \quad \frac{a}{h} = \frac{h}{b} = \frac{g}{f}$$

Also from (2), we have in the similar manner that

$$\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$$

Thus in either case, we have,

$$\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$$

The distance between the parallel lines

Using $h = \sqrt{ab}$ in $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have,

$$ax^2 + 2\sqrt{ab}\cdot xy + by^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow \quad (x\sqrt{a} + y\sqrt{b})^2 + 2gx + 2fy + c = 0 \qquad ...(3)$$

Let, $(x\sqrt{a} + y\sqrt{b} + p)(x\sqrt{a} + y\sqrt{b} + q) = 0$

$$\Rightarrow \quad ax^2 + 2\sqrt{ab}\cdot xy + by^2 + (p+q)\sqrt{a}\cdot x$$

$$+ (p+q)\sqrt{b}\cdot y + pq = 0 \qquad ...(4)$$

Comparing (3) and (4), we get

$$p + q = \frac{2g}{\sqrt{a}} = \frac{2f}{\sqrt{b}},\ pq = c \qquad ...(5)$$

$\therefore$ Distance between the parallel lines $= \dfrac{|p-q|}{\sqrt{(a+b)}}$

$$= \sqrt{\left[\frac{(p+q)^2 - 4pq}{a+b}\right]}$$

$$= \sqrt{\left(\frac{4g^2}{a} - 4c\right)}\Big/\sqrt{(a+b)} = 2\sqrt{\left[\frac{g^2 - ac}{a(a+b)}\right]}$$

$\because$ The distance of two parallel lines is real, so $g^2 - ac \geq 0$.

This distance can also be proved to be equal to $2\sqrt{\left[\dfrac{f^2 - bc}{b(a+b)}\right]}$.

PROBLEM 14

Prove that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = 2hk$ and the curve $(x-k)^2 + (y-h)^2 = c^2$ are at right angles, if $h^2 + k^2 = c^2$.

SOLUTION

Making the equation of the curve homogeneous with the help of that of the line, we get

$$x^2 + y^2 - 2(kx + hy)\left(\frac{hx+ky}{2hk}\right) + (h^2 + k^2 - c^2)\left(\frac{hx+ky}{2hk}\right)^2 = 0$$

$$\text{or} \quad 4h^2k^2x^2 + 4h^2k^2y^2 - 4hk^2x(hx+ky)$$

$$-4h^2ky(hx+ky) + (h^2 + k^2 - c^2)$$

$$(h^2x^2 + k^2y^2 + 2hxy) = 0$$

This is the equation of the pair of lines joining the origin to the points of intersection of the given line and the curve. They will be at right angles,

If coefficient of x^2 + coefficient of y^2 = 0, i.e., $(h^2 + k^2)(h^2 + k^2 - c^2) = 0$

$$\Rightarrow \quad h^2 + k^2 = c^2 \qquad (\because h^2 + k^2 \neq 0)$$

PROBLEM 15

A variable line L passing though the point B (2, 5) intersects the lines $2x^2 - 5xy + 2y^2 = 0$ at P and Q. Find the locus of the point R on L such that the distances BP, BR and BQ are in H.P.

SOLUTION

Let, $BP = r_1$, $BQ = r_2$ and $BR = r$ be the co-ordinates of P, Q and R and they can be taken as $(2 + r_1\cos\theta, 5 + r_1\sin\theta)$, $(2 + r_2\cos\theta, 5 + r_2\sin\theta)$ and $(2 + r\cos\theta, 5 + r\sin\theta)$ respectively.

Now, the lines $2x^2 - 5xy + 2y^2 = 0$ are $y - 2x = 0$ ...(1)

and $2y - x = 0$ ...(2)

Now, if P lies on (1), then $\dfrac{1}{r_1} = 2\cos\theta - \sin\theta$

Q lies on (2), $\dfrac{1}{r_2} = \dfrac{\cos\theta - 2\sin\theta}{8}$

Given r_1, r, r_2 are in $H.P.$ $\Rightarrow \dfrac{2}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$

$$\Rightarrow \quad 17r\cos\theta - 10r\sin\theta = 16$$

$$\Rightarrow \quad 17(2 + r\cos\theta) = 10(5 + r\sin\theta)$$

If R be (h, k) $17h = 10k$

Locus of (h, k) will be $17x = 10y$.

PROBLEM 16

Show that the difference of the tangents of the angles, which the lines $(\tan^2\alpha + \cos^2\alpha)x^2 - 2xy\tan\alpha + \sin^2\alpha\, y^2 = 0$ make with the axis of x is 2.

SOLUTION

The given equation is

$$(\tan^2\alpha + \cos^2\alpha)\,x^2 - 2xy\tan\alpha + \sin^2\alpha\, y^2 = 0 \quad ...(1)$$

$$\therefore \quad a = \tan^2\alpha + \cos^2\alpha,\ h = -\tan\alpha,\ b = \sin^2\alpha$$

If the lines represented by (1) are

$y - m_1x = 0$ and $y - m_2x = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} = \frac{2\tan\alpha}{\sin^2\alpha} = \frac{2}{\sin\alpha\,\cos\alpha}$$

and $$m_1m_2 = \frac{a}{b} = \frac{\tan^2\alpha + \cos^2\alpha}{\sin^2\alpha}$$

Now, if the two lines make angles θ and ϕ respectively to the axis of x, then

$m_1 = \tan\theta$ and $m_2 = \tan\phi$.

$$\therefore \quad (\tan\theta - \tan\phi)^2 = (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$= \frac{4}{\sin^2\alpha\,\cos^2\alpha} - \frac{4(\tan^2\alpha + \cos^2\alpha)}{\sin^2\alpha}$$

$$= \frac{4}{\sin^2\alpha\,\cos^2\alpha}\,(1 - \cos^2\alpha[\tan^2\alpha + \cos^2\alpha])$$

$$= \frac{4}{\sin^2\alpha\,\cos^2\alpha}\,(1 - \sin^2\alpha - \cos^4\alpha)$$

$$= \frac{4}{\sin^2\alpha\,\cos^2\alpha}\,(\cos^2\alpha - \cos^4\alpha)$$

$$= \frac{4\cos^2\alpha(1 - \cos^2\alpha)}{\sin^2\alpha\cos^2\alpha} = 4$$

Hence, $|\tan\theta - \tan\phi| = 2$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The range of α for which the points $(\alpha, 2+\alpha)$ and $\left(\frac{3\alpha}{2}, \alpha^2\right)$ lie on opposite side of the line $2x + 3y = 6$ is:

(A) $(-\infty, -2) \cup (0, 1)$ (B) $(0, 4)$

(C) $\left(\frac{1}{2}, 1\right)$ (D) none of these

2. If the sides of a triangle are $x + y = 2$, $3x + 4y = 7$ and $2x - 5y = 8$, then the line passing through its orthocentre is:

(A) $5x + 2y = 7$ (B) $5x + 2y = 3$

(C) $5x + 2y = 1$ (D) none of these

3. $\log_\lambda\left(x^2 + x\right) > \log_\lambda(2x + 12)$, where the point $(\lambda, 2)$ lies between the lines $x - 2y + 3 = 0$ and $x - 2y + 4 = 0$, then the set of values of x is:

(A) $(3, 4)$ (B) $(-3, 4)$

(C) $(-4, 3)$ (D) none of these

4. λ and μ are integers with $0 < \mu < \lambda$. A is the point (λ, μ) on Cartesian plane. B is the reflection of A in the line $y = x$, C is the reflection of B in y-axis, D is the reflection of C in x-axis, E is the reflection of D in y-axis, then the area of pentagon $ABCDE$ is:

(A) $2\lambda(\lambda + \mu)$ (B) $\lambda(\lambda + 3\mu)$

(C) $\lambda(2\lambda + 3\mu)$ (D) $2\lambda(\lambda + 3\mu)$

5. If $x > 0$, $y > 0$, $2x - 5y = 20$, $3x + my = m$, then $m \in$

(A) $(0, 30)$ (B) $\left(-\frac{15}{2}, 0\right)$

(C) $\left(-\frac{15}{2}, 30\right)$ (D) $R - \left[\frac{-15}{2}, 30\right]$

6. Locus of the foot of perpendicular from $(-2, 3)$ to a variable line with x-intercept 2 is:

(A) $x^2 + y^2 - 3y - 4 = 0$ (B) $y = 1$

(C) $x^2 + y^2 = 4$ (D) $y = \frac{1}{2}$

7. The number of integral values of m, for which the abscissa of the point of intersection of the lines $5x + 3y = 87$ and $y = mx + 1$ is also an integer is:

(A) 2 (B) 3

(C) 4 (D) none of these

8. If the vertices of a triangle are $(4, 3)$, $(5\sin\theta, 5\cos\theta)$, $(5\cos\theta, -5\sin\theta)$, then the locus of the orthocentre is:

(A) a straight line (B) a circle

(C) an ellipse (D) a hyperbola

9. If P, Q, R and S are the points of intersection with the co-ordinate axes of the lines $px + qy = pq$ and $qx + py = pq$. $(p, q > 0)$, then:

(A) P, Q, R and S form a parallelogram

(B) P, Q, R and S form a rhombus

(C) P, Q, R and S are concyclic

(D) none of the these

10. The lines $2x + 3y = 6$, $2x + 3y = 8$ cut the x-axis at A and B. A line 'l' is drawn through the point $(2, 2)$, which meets the x-axis at C in such a way that the abscissa of A, B and C are in G.P., then the equation of the line l is:

(A) $3x - 5y = 16$ (B) $3x + 5y = 16$

(C) $5x + 3y = 16$ (D) $5x - 3y = 16$

11. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals intersect at $(1, 2)$ and the vertex $A(0, a)$, where $a \neq 0$, then 'a' is:

(A) 2 (B) $\frac{-1}{2}$

(C) 4 (D) $\frac{5}{2}$

12. If the line $y = x\sqrt{3}$ cuts the curve $x^3 + y^3 + 3xy + 5x^2 + 3y^2 + 4x + 5y - 1 = 0$ at the points A, B and C, then $OA.OB.OC$ (where O is origin) is:

(A) $\frac{12\sqrt{3} - 4}{13}$ (B) $\frac{3\sqrt{3} + 12}{13}$

(C) $\frac{4\left(3\sqrt{13} - 1\right)}{13}$ (D) $\frac{4\sqrt{3} - 3}{13}$

13. $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines, if:

(A) $h^2=ab,\ f^2>bc$

(B) $h^2=ab,\ hf=gb,\ f^2<bc$

(C) $h^2=ab,\ hf=gb,\ f^2>bc$

(D) none of these

14. A ray of light passing through the point $A(2, 3)$ is reflected at a point B on the line $x + y = 0$ and then passes through $(5, 3)$, then the co-ordinates of B are:

(A) $\left(\frac{1}{3},-\frac{1}{3}\right)$ (B) $\left(\frac{2}{5},-\frac{2}{5}\right)$

(C) $\left(\frac{1}{13},-\frac{1}{13}\right)$ (D) none of these

15. If AD, BE and CF are the altitudes of a ΔABC, whose vertex A is the point $(-4, 5)$. The co-ordinates of the points E and F are $(4, 1)$ and $(-1, -4)$, respectively, then the equation of BC is:

(A) $3x-4y-28=0$ (B) $4x+3y-28=0$

(C) $3x-4y+28=0$ (D) $x+2y+7=0$

16. If the straight lines $x + y - 2 = 0$, $2x - y + 1 = 0$ and $px+qy-r=0$ are concurrent, then the slope of member of family of lines $2px + 3qy + 4r = 0$, which is the farthest from the origin is:

(A) $-\frac{1}{2}$ (B) -2

(C) $-\frac{2}{3}$ (D) $-\frac{3}{10}$

17. A variable line $\frac{x}{a}+\frac{y}{b}=1$ moves in such a way that the harmonic mean of a and b is 8. If c be the least area of triangle made by the line with the co-ordinate axis, then the last digit of c^c is:

(A) 2 (B) 4

(C) 1 (D) 6

18. Let (x, y) be a point in the system of co-ordinate axes not inclined at right angle and (x', y') is the new position of the same w.r.t. another axes having same origin if the transformation equation are $x = px' + qy'$ and $y = p'x' + q'y'$, then $\frac{p^2+p'^2-1}{q^2+q'^2-1}$ equals:

(A) $\frac{pq}{p'q'}$ (B) $\frac{p'q'}{pq}$

(C) $\frac{pp'}{qq'}$ (D) $\frac{qq'}{pp'}$

19. If the equation $2x^2+2hxy+6y^2-4x+5y-6=0$ represents a pair of straight lines, then the length of intercept on the x-axis cut by the lines is:

(A) 2 (B) 4

(C) $\sqrt{7}$ (D) 0

20. If $P(-1, 0)$, $Q(0, 0)$ and $R\left(3,3\sqrt{3}\right)$ be three points, then the equation of the bisector of the angle PQR is:

(A) $\frac{\sqrt{3}}{2}x+y=0$ (B) $x+\sqrt{3}y=0$

(C) $\sqrt{3}x+y=0$ (D) $x+\frac{\sqrt{3}}{2}y=0$

21. In the xy-plane, the number of straight lines, whose x-intercept is a prime number and whose y-intercept is a positive integer that passes through the point $(4, 3)$ is/are:

(A) 1 (B) 2

(C) 3 (D) 4

22. If the reflection of the curve $xy=1$ in the line $y=2x$ is the curve $12x^2+rxy+sy^2+t=0$, then the value of 'r' is:

(A) -7 (B) 25

(C) -175 (D) 90

23. Let the point O lies inside the equilateral triangle ABC. If its distances from the three sides are $4\sqrt{3}$, $5\sqrt{3}$ and $7\sqrt{3}$, respectively, then the length of the side of the triangle is:

(A) 16 (B) $16\sqrt{3}$

(C) 32 (D) $32\sqrt{3}$

24. Consider the two points $A(1, 2)$ and $B(7, 5)$. The line AB is rotated through 45° in anticlockwise direction about the point of trisection of AB, which is nearest to B. The equation of the line in new position is:

(A) $2x-y-6=0$ (B) $3x-y-11=0$

(C) $x-3y-1=0$ (D) $x-y-1=0$

25. If $L_1=0$ and $L_2=0$ are two intersecting lines such that the reflection of L_1 in L_2 and the reflection of L_2 in L_1 coincide, then the angle between the lines is:

(A) 30° (B) 45°

(C) 60° (D) 90°

26. Two fixed points A and B are taken on the rectangular co-ordinate axes such that $OA=a$ and $OB=b$. Two variable points P and Q are taken on the same axes so that $OP + OQ = OA + OB$. The locus of the intersection of AQ and BP is:

(A) $x-y=a$ (B) $x+y=a+b$

(C) $x-y=b-a$ (D) $x-y=b$

27. The lines $ax+by+(a\alpha+b)=0$, $bx+cy+(b\alpha+c)=0$ and $(a\alpha+b)x+(b\alpha+c)y=0$ are concurrent, if:

(A) a, b, c are in AP.

(B) $(x-\alpha)$ is a factor of $ax^2+2bx+c$

(C) $(x-\alpha)$ is a factor of $ax^2+bx+c=0$

(D) a, b, c are in H.P.

28. If $x^2 - 3xy + 2y^2 = 0$ represents two sides of triangle and the orthocentre as (2, 1), then the equation of third side is:

(A) $x + 2y = 5$ (B) $2x + y = 1$

(C) $2x + y = 5$ (D) $2x + y = 6$

29. If $2x + 3y = 7$ makes equal angles with pair of lines $9x^2 + 12xy + ky^2 = 0$, then k equals:

(A) 7 (B) 14

(C) –14 (D) –7

30. If one angular bisector of $a(x + 1)^2 + 2h(x + 1)(y - 3) + b(y - 3)^2 = 0$ is $2x - 3y + 11 = 0$, then the equation of other angular bisector is:

(A) $x + y = 2$ (B) $3x + 2y - 11 = 0$

(C) $3x + 2y = 0$ (D) $3x + 2y - 3 = 0$

31. The equation of the line belonging to the family of the lines given by $3x + 4y + 6 + \lambda\ (x + y + 2) = 0$ (where λ being a parameter) and situated at the greatest distance from the point (2, 3) is:

(A) $4x - 3y + 8 = 0$ (B) $4x + 3y + 8 = 0$

(C) $4x + 3y - 8 = 0$ (D) none of these

32. If the algebraic sum of distances of the line $ax + by + 2 = 0$ from (1, 2), (2, 1) and (3, 5) is zero and the lines $bx - ay + 4 = 0$ and $3x + 4y + 5 = 0$ cut the co-ordinate axes at concyclic points, then:

(A) $a + b = \frac{2}{7}$

(B) area of the triangle formed by the line $ax + by + 2 = 0$ with co-ordinate axes is $\frac{14}{5}$.

(C) line $ax + by + 3 = 0$ always passes through the point (–1, 1)

(D) maximum {a, b} = $\frac{5}{7}$

33. A line passes through (2, 0). The slope of the line, for which its intercept between $y = x - 1$ and $y = -x + 1$ subtends a right angle at the origin, is

(A) $\sqrt{3}$ (B) $-\sqrt{3}$

(C) $\frac{1}{\sqrt{3}}$ (D) $\frac{-1}{2\sqrt{3}}$

34. The complete set of values of 'a' for which the point (a, a^2), $a \in R$ lies inside the triangle formed by the lines $x - y + 2 = 0$, $x + y = 2$ and x-axis is:

(A) (– 2, 2) (B) (–1, 1)

(C) (0, 2) (D) (–2, 0)

35. The values of k, for which the lines $kx + 2y + 2 = 0$, $2x + ky + 3 = 0$ and $3x + 3y + k = 0$ are concurrent are:

(A) {2, 3, 5} (B) {2, 3, – 5}

(C) {3, – 5} (D) {– 5}

36. A straight line passes through the point of intersection $x - 2y - 2 = 0$ and $2x - by - 6 = 0$ and the origin then the complete set of values of b for which the acute angle between this line and $y = 0$ is less than 45°:

(A) $(-\infty, 4) \cup (7, \infty)$

(B) $(-\infty, 5) \cup (7, \infty)$

(C) $(-\infty, 4) \cup (5, 7) \cup (7, \infty)$

(D) $(-\infty, 4) \cup (4, 5) \cup (7, \infty)$

37. Equation of angle bisector of the lines $3x - 4y + 1 = 0$ and $12x + 5y - 3 = 0$ containing the point (1, 2) is:

(A) $3x + 11y - 4 = 0$

(B) $99x - 27y - 2 = 0$

(C) $3x + 11y + 4 = 0$

(D) $99x + 27y - 2 = 0$

38. The line $L_1 = 4x + 3y - 12 = 0$ intersects the x-axis and y-axis at A and B, respectively. A variable line perpendicular to L_1 intersects the x and y-axes at P and Q respectively. The locus of the circumcentre of ΔABQ is:

(A) $6x - 8y + 7 = 0$

(B) $6x + 8y - 25 = 0$

(C) $8x - 6y + 7 = 0$

(D) $14x - 12y + 3 = 0$

39. If the area of the rhombus enclosed by the lines $\ell x \pm my \pm n = 0$ is 2 sq. units, then:

(A) $\ell, 2m, n$ are in G.P. (B) ℓ, n, m are in G.P.

(C) $\ell m = n$ (D) $\ell n = m$

40. In ΔABC, if the orthocentre is (6, 10), circumcentre is (2, 3) and equation of side BC is $2x + y = 17$, then the radius of the circumcircle of ΔABC is:

(A) 4 (B) 5

(C) 2 (D) 3

41. The area of the triangle formed by the line $x + y = 3$ and the angular bisectors of pair of straight lines $x^2 - y^2 + 2y = 1$ is,

(A) 8 sq.units (B) 6 sq.units

(C) 4 sq.units (D) 2 sq.units

42. An equilateral triangle has its centroid at origin and one side is $x + y = 1$. The equations of the other sides are:

(A) $y + 1 = (2 \pm \sqrt{3})(x + 1)$

(B) $y + 1 = (2 \pm \sqrt{3})x$

(C) $y + 1 = (3 \pm \sqrt{3})(x - 1)$

(D) $y \pm 1 = (3 \pm \sqrt{3})(x - 1)$

43. Triangle is formed by the lines $x + y = 0$, $x - y = 0$ and $\ell x + my = 1$. If l and m vary subject to the condition $\ell^2 + m^2 = 1$, then the locus of its circumcentre is:

(A) $(x^2 - y^2)^2 = x^2 + y^2$ (B) $(x^2 + y^2)^2 = (x^2 - y^2)$

(C) $(x^2 + y^2)^2 = 4x^2y^2$ (D) $(x^2 - y^2)^2 = (x^2 + y^2)^2$

44. A piece of cheese is located at (12, 10) in a co-ordinate plane. A mouse is at (4, –2) and it is running up the line $y = -5x + 18$. At the point (a, b), the mouse starts getting farther from the cheese, rather than closer to it. The value of $(a + b)$ is:

(A) 6 (B) 10

(C) 18 (D) 14

45. If h denotes the A.M., k denotes G.M. of the intercepts made on axes by the lines passing through (1, 1), then (h, k) lies on:

(A) $y^2 = 2x$ (B) $y^2 = 4x$

(C) $y = 2x$ (D) $x + y = 2xy$

46. A line has intercepts a, b on axes when the axes are rotated through an angle a, if the lines make equal intercepts on axes, then tan a equals:

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$

(C) $\frac{a}{b}$ (D) $\frac{b}{a}$

47. A point moves in the XY plane such that the sum of its distances from two mutually perpendicular lines is always equal to 3. The area enclosed by the locus of the point is:

(A) 18 sq. units (B) $\frac{9}{2}$ sq. units

(C) 9 sq. units (D) 27 sq. units

48. If the point (a, a) falls between the lines $|x + y| = 2$, then:

(A) $|a| = 2$ (B) $|a| = 1$

(C) $|a| < 1$ (D) $|a| < \frac{1}{2}$

49. A ray travelling along the line $3x - 4y = 5$ after being reflected from a line 'l' travels along the line $5x + 12y = 13$. Then the equation of the line 'l' is:

(A) $x + 8y = 0$ (B) $x - 8y = 0$

(C) $32x + 4y + 65 = 0$ (D) $32x - 4y + 65 = 0$

50. The values of m, for which the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfying the conditions $x > 0$ and $y > 0$ are given by the set:

(A) $\left\{m : m < \frac{-13}{2}\right\}$

(B) $\left\{m : m > \frac{17}{2}\right\}$

(C) $\left\{m : m < \frac{-13}{2} \text{ or } m > \frac{17}{2}\right\}$

(D) $\left\{m : m > 30 \text{ or } m < \frac{-15}{2}\right\}$

51. $A_1, A_2 \ldots\ldots A_n$ are the points on the line $y = x$ lying in the positive quadrant such that $OA_n = nOA_{n-1}$ being the origin. If $OA_1 = 1$ and the co-ordinates of A_n are $(2520\sqrt{2}, 2520\sqrt{2})$, then n:

(A) 5 (B) 6

(C) 7 (D) 8

52. All points inside the triangle formed by $A(1, 3)$, $B(5, 6)$, $C(-1, 2)$ will satisfy:

(A) $2x + 2y \leq 0$ (B) $2x + y + 1 \geq 0$

(C) $2x + 3y - 12 \geq 0$ (D) $-2x + 11 \leq 0$

53. Let $P(1, 1)$, Q (2, 4) and $R(\alpha, \beta)$ be the vertices of the ΔPQR. The point $S(2, 2)$ inside the ΔPQR is such that Area (ΔPQS) = Area (ΔPSR) = Area (ΔRSQ), then (α, β) equals:

(A) (2, 3) (B) $\left(2, \frac{5}{2}\right)$

(C) (3, 1) (D) $\left(\frac{5}{2}, 2\right)$

54. A system of line is given as $y = m_i x + c_i$, where m_i can take any value out of 0, 1, –1 and when m_i is positive, then c_i can be 1 or –1 when m_i equal 0, c_i can be 0 or 1 and when m_i equals – 1, c_i can take 0 or 2. Then the area enclosed by all these straight line is:

(A) $\frac{3}{\sqrt{2}}(\sqrt{2} - 1)$ sq. units (B) $\frac{3}{\sqrt{2}}$ sq. units

(C) $\frac{3}{2}$ sq. units (D) $\frac{3}{4}$ sq. units

55. The joint equation of two altitudes of an equilateral triangle is $\left(\sqrt{3}x - y + 8 - 4\sqrt{3}\right)\left(\sqrt{3}x + y - 12 - 4\sqrt{3}\right) = 0$. The equation of the third altitude is:

(A) $\sqrt{3}x = 2 - 4\sqrt{3}$ (B) $y - 10 = 0$

(C) $\sqrt{3}x = 2 + 4\sqrt{3}$ (D) $y + 10 = 0$

56. A line $x + 2y = 4$ is translated by 3 units away from the origin and then rotated by 30° in clockwise sense about the point, where the shifted line cuts x-axis. If m is the slope of line in new position then find $[m]$, where [.] denotes G.I.F is:

(A) -1 (B) -2
(C) -3 (D) -4

57. The orthocentre of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in:

(A) I^{st} quadrant (B) II^{nd} quadrant
(C) III^{rd} quadrant (D) IV^{th} quadrant

58. If $9x^2 + 2hxy + 4y^2 + 6x + 2fy - 3 = 0$ represents two parallel lines, then:

(A) $h = 6, f = 2$ (B) $h = -6, f = 2$
(C) $h = 6, f = -2$ (D) none of these

59. The point $P(2, 1)$ is shifted through a distance $3\sqrt{2}$ units measured parallel to the line $x + y = 1$ in the direction of decreasing ordinates, to reach at Q. The image of Q with respect to the given line is:

(A) $(3, -4)$ (B) $(-3, 2)$
(C) $(0, -1)$ (D) none of these

60. In a ΔABC, $A(\alpha, \beta), B(2, 3)$ and $C(1, 3)$ and point A lies on the line $y = 2x + 3$, where $\alpha \in I$. Area of ΔABC, Δ is such that $[\Delta] = 5$, then the possible co-ordinates of Δ is where [.] is G.I.F are:

(A) $(2, 3)$ (B) $(5, 13)$
(C) $(-5, -7)$ (D) $(-3, -5)$

61. If the image of $P(a, b)$ on $y = -x$ is Q and the image of Q on the line $y = x$ is R, then the midpoint of PR is:

(A) $(a + b, b + a)$ (B) $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$
(C) $(a - b, b - a)$ (D) $(0, 0)$

62. A straight line L with negative slope passes through the point $(8, 2)$ and cuts the positive co-ordinate axes at points P and Q. As L varies, the absolute minimum value of $OP + OQ$ is (O is origin):

(A) 10 (B) 18
(C) 16 (D) 12

63. Drawn from the origin are two mutually perpendicular lines forming an isosceles triangle together with the straight line $2x + y = a$, then the area of this triangle is:

(A) $\frac{a^2}{2}$ sq. units (B) $\frac{a^2}{3}$ sq. units
(C) $\frac{a^2}{5}$ sq. units (D) none of these

64. If the distance of any point (x, y) from origin is defined as $d(x, y) = |x| + |y|$, then the locus $d(x, y) = 1$ is a:

(A) circle of area π sq. units
(B) square of area 1 sq. unit
(C) square of area 2 sq. units
(D) none of the above

65. The line $3x - 4y + 7 = 0$ is rotated through an angle $\frac{\pi}{4}$ in the clockwise direction about the point $(-1, 1)$. The equation of the line in its new position is:

(A) $7y + x - 6 = 0$ (B) $7y - x - 6 = 0$
(C) $7y + x + 6 = 0$ (D) $7y - x + 6 = 0$

66. The number of integral values of m, for which the x-co-ordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is:

(A) 2 (B) 0
(C) 4 (D) 1

67. For all real values of a and b lines $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ and $mx + 2y + 6 = 0$ are concurrent, then m is equal to:

(A) -2 (B) -3
(C) -4 (D) -5

68. Let $A(a, b)$ and $B(c, d)$ where $c > a > 0$ and $d > b > 0$, then the point C on the x-axis such that $AC + BC$ is the minimum is:

(A) $\left(\frac{bc - ad}{b - d}, 0\right)$ (B) $\left(\frac{ac + bd}{b + d}, 0\right)$
(C) $\left(\frac{ac - bd}{b - a}, 0\right)$ (D) $\left(\frac{ad + bc}{b + d}, 0\right)$

69. If $A(5, 2)$, $B(10, 12)$ and $P(x, y)$ are such that $\frac{AP}{PB} = \frac{3}{2}$, then the internal bisector of $\angle APB$ always passes through:

(A) $(20, 32)$ (B) $(8, 8)$
(C) $(8, -8)$ (D) $(-8, -8)$

70. From a point P, perpendiculars PM and PN are drawn upon x and y axes, respectively. If MN passes through a fixed point (a, b) then the locus of P is:

(A) $\frac{x}{a} + \frac{y}{b} = 1$ (B) $\frac{x}{a} - \frac{y}{b} = 1$
(C) $\frac{a}{x} + \frac{b}{y} = 1$ (D) $\frac{a}{x} - \frac{b}{y} = 1$

71. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). The co-ordinates of the point A are:

(A) $\left(\frac{13}{5}, 0\right)$ (B) $\left(\frac{5}{13}, 0\right)$

(C) $(-7, 0)$ (D) $(7, 0)$

72. The bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y - 1 = 0$ is:

(A) $x + y + 5 = 0$ (B) $x - y + 1 = 0$

(C) $x - y - 5 = 0$ (D) $x - y - 1 = 0$

73. The area of triangle formed by the line $x + y = 3$ and the angle bisectors of the pair of straight lines $x^2 - y^2 + 2y = 1$ is:

(A) 2 (B) 4

(C) 6 (D) 8

74. If P_1, P_2 denote the lengths of the perpendiculars from the origin on the lines $x\sec\alpha + y\operatorname{cosec}\alpha = 2a$ and $x\cos\alpha + y\sin\alpha = a\cos 2\alpha$ respectively, then $\left(\frac{P_1}{P_2} + \frac{P_2}{P_1}\right)^2$ is:

(A) $4\sin^2 4\alpha$ (B) $4\cos^2 4\alpha$

(C) $4\operatorname{cosec}^2 4\alpha$ (D) $4\sec^2 4\alpha$

75. The orthocentre and circumcentre of ΔABC are (1, 2) and (2, 4), respectively. If the base BC has the equation $2x - y - 3 = 0$, then the radius of the circumcircle of ΔABC is:

(A) $\sqrt{\frac{57}{5}}$ (B) $\sqrt{\frac{59}{5}}$

(C) $\sqrt{\frac{61}{5}}$ (D) $\sqrt{\frac{63}{5}}$

76. A straight line passes through the point (2, 2) and cuts the straight lines, jointly given by $3x^2 - y^2 = 0$, at A and B. Given that ΔOAB (O is the origin) is equilateral, what is its area?

(A) $\frac{7}{\sqrt{3}}$ (B) $\frac{5}{\sqrt{3}}$

(C) $\frac{10}{\sqrt{3}}$ (D) $\frac{4}{\sqrt{3}}$

77. The condition that the equation $lx + my + n = 0$ represents the equation of a straight line in the normal form is:

(A) $l^2 + m^2 \neq 0; n > 0$ (B) $l^2 + m^2 \neq 0; n < 0$

(C) $l^2 + m^2 = 1; n < 0$ (D) $l^2 + m^2 = 2;\ n > 0$

78. If $p, x_1, x_2, \ldots$ and $q, y_1, y_2, \ldots$ are in A.P. with common difference a and b respectively, then the centre of mean position of the points $A_i(x_i, y_i)$, $i = 1, 2, 3,$ ----n lies on the line:

(A) $ax - by = aq - bp$ (B) $bx - ay = ap - bq$

(C) $bx - ay = bp - aq$ (D) $ax - by = bq - ap$

79. If the line passing through $P(1,2)$ making an angle 45° with the x-axis in the positive direction meets the pair of lines $x^2 + 4xy + y^2 = 0$ at A and B, then PA. PB equals:

(A) $\frac{13}{3}$ (B) $\frac{13}{6}$

(C) $\frac{11}{6}$ (D) $\frac{11}{3}$

80. If the point $(\cos\theta, \sin\theta)$ does not fall in that angle between the lines $y = |x - 1|$ in which the origin lies, then θ belongs to:

(A) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (B) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(C) $\left(0, \frac{\pi}{2}\right)$ (D) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

81. The locus of $P(x, y)$ such that

$\sqrt{x^2 + y^2 + 8y + 16} - \sqrt{x^2 + y^2 - 6x + 9} = 5$ is:

(A) a hyperbola (B) a circle

(C) a finite line segment (D) an infinite ray

82. The line $x + y = a$, meets the axis of x and y at A and B, respectively. A ΔAMN is inscribed in the ΔOAB, O being the origin with right angle at N. M and N lie respectively on OB and AB. If the area of the ΔAMN is $\frac{3}{8}$ of the area of the ΔOAB, then $\frac{AN}{BN}$ is:

(A) 3 (B) $\frac{1}{3}$

(C) 2 (D) $\frac{1}{2}$

83. Equation of a line which is parallel to the line common to the pair of lines given by $6x^2 - xy - 12y^2 = 0$ and $15x^2 + 14xy - 8y^2 = 0$ and at a distance 7 units from it is:

(A) $3x - 4y = -35$ (B) $5x - 2y = 7$

(C) $3x + 4y = -35$ (D) $2x - 3y = 7$

84. If the internal bisectors of the ΔABC, having the sides $BC = 3$, $CA = 5$ and $AB = 4$ meet the sides BC, CA and AB in D, E and F respectively, then the area of ΔDEF is:

(A) $\frac{10}{3}$ (B) $\frac{5}{7}$

(C) $\frac{10}{7}$ (D) $\frac{7}{10}$

85. If $P(m, n)$ (where m, n are natural numbers) is any point in the interior of the quadrilateral formed by the pair of lines $xy = 0$ and the lines $2x + y - 2 = 0$ and $4x + 5y = 20$, then the possible number of positions of the point p is:

(A) 7 (B) 5

(C) 4 (D) 6

86. If $a^2 + b^2 - c^2 - 2ab = 0$, then the point of concurrency of family of lines $ax + by + c = 0$ lies on the line:

(A) $y = x$ (B) $y = x + 1$

(C) $y = -x$ (D) $3x = y$

87. The curves $y = x^3 - 3x + 2$ and $x + 3y = 2$ intersect at the points (x_1, y_1) (x_2, y_2) and (x_3, y_3), then the point $P(\alpha, \beta)$ lies on the line, where $\alpha = \Sigma x_i$ and $\beta = \Sigma y_i$ is:

(A) $x - 3y = 5$ (B) $x + y - 1 = 0$

(C) $3x - 7 = y$ (D) $2x + y = 2$

88. Number of values of λ such that $\left(\lambda^2 + 1, \lambda\right)$ is the mirror image of $\left(\lambda, \lambda + 1\right)$ w.r.t. $x + 2y - 1 = 0$ is:

(A) 0 (B) 1

(C) 2 (D) 3

89. Triangle whose sides are represented by $2hxy + 2gx + 2fy + c = 0$ and $ax + by + k = 0$, then the product of abscissa and ordinate of orthocentre of triangle is:

(A) $\frac{-fg}{h^2}$ (B) $\frac{-c}{2h}$

(C) $\frac{c}{2h}$ (D) $\frac{fg}{h^2}$

90. The perpendicular bisectors $x + y + 2 = 0$ and $x - y - 1 = 0$ of the sides AB and AC of a ΔABC intersect them at $(-1, -1)$ and $(2, 1)$, respectively. If the midpoint of side BC is P, then the distance of this point from the orthocentre of triangle is:

(A) $\sqrt{85}$ (B) $\sqrt{41}$

(C) $\sqrt{13}$ (D) none of these

91. A rectangular billiard table has vertices at $P(0, 0)$, $Q(0, 6)$, $R(10, 6)$ and $S(10, 0)$. A small billiard ball starts at $M(2, 3)$ and moves in a straight line to the top of the table, bounces to the right side of the table, then comes to rest at $N(7, 1)$. The y-co-ordinate of the point, where it hits the right side is:

(A) $\frac{25}{7}$ (B) $\frac{35}{11}$

(C) $\frac{45}{13}$ (D) $\frac{55}{17}$

92. Point A lies on $y = x$ and point B on $y = mx$ so that the length of AB is 4 units. Then the value of 'm' for which the locus of midpoint of AB represents a circle is:

(A) $m = 0$ (B) $m = -1$

(C) $m = 2$ (D) $m = -2$

93. Let a and b be non-zero real numbers, then the equation $\left(ax^2 + by^2 + c\right)\left(x^2 - 5xy + 6y^2\right) = 0$ represents:

(A) four straight lines, when $c = 0$ and a, b are of the same sign

(B) two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of opposite sign to that of a.

(D) a circle and an ellipse, when a and b are of the same sign and c is of the sign opposite to that of a.

94. The locus of the orthocentre of the triangle formed by the lines $\left(1 + px\right) - py + p\left(1 + p\right) = 0$, $\left(1 + q\right)x - qy + q\left(1 + q\right) = 0$ and $y = 0$, where $p \neq q$ is:

(A) a hyperbola (B) a parabola

(C) an ellipse (D) a straight line

95. Number of straight lines through $(1, 0)$, whose intercept between the lines $y = x - 2$ and $y = -x + 2$ subtend a right angle at the origin is:

(A) 2 (B) 4

(C) 1 (D) 0

96. Equation to the pair of lines through the point $(1, 2)$ and forming a right angled isosceles triangle with $x + y - 4 = 0$ is:

(A) $xy - 2x - y + 2 = 0$

(B) $xy + 2x + y + 2 = 0$

(C) $x^2 + y^2 + 2x + y + 2 = 0$

(D) $x^2 + y^2 + 2xy + 2x + y + 1 = 0$

97. Let I be the incentre and I_1, I_2 and I_3 be the excentres opposite to the vertices A, B and C of a ΔABC. If $I(2,3)$ and the circumcentre of the $\Delta\ I_1I_2I_3$ is $(4,7)$, then the circumcentre of the ΔABC is:

(A) $(3,5)$ (B) $\left(\frac{10}{3},\frac{17}{3}\right)$

(C) $(2,2)$ (D) $\left(\frac{9}{4},\frac{5}{2}\right)$

98. Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of the ΔOPQ. The point R inside the ΔOPQ is such that the triangles OPR, PQR, OQR are of equal area. Then the co-ordinates of R are:

(A) $\left(\frac{4}{3},3\right)$ (B) $\left(3,\frac{2}{3}\right)$

(C) $\left(3,\frac{4}{3}\right)$ (D) $\left(\frac{4}{3},\frac{2}{3}\right)$

99. The equation of line which equally inclined to the axis and passes through common points of family of lines $4acx+y(ab+bc+ca-abc)+abc=0$ where a, b and $c>0$ are in H.P. is:

(A) $y-x=\frac{7}{4}$ (B) $y+x=\frac{7}{4}$

(C) $y-x=\frac{1}{4}$ (D) $y+x=\frac{1}{4}$

100. If 2 consecutive vertices of a regular hexagon $ABCDEF$ are $A(1, 0)$ and $B(2, 0)$, respectively, then the equation of the diagonal CE is:

(A) $\sqrt{3}x+y=4$ (B) $x+\sqrt{3}y=4$

(C) $x+\sqrt{3}y+4=0$ (D) $x-\sqrt{3}y+4=0$

101. If the portion of the line $ax+by-1=0$ intercepted between the lines $ax+y=0$ and $x+by=0$ subtends a right angle at the origin, then:

(A) $a=b$ (B) $a+b=0$

(C) $2a+b=0$ (D) $2a=b$

102. If the lines represented by the equation $3y^2-x^2+2\sqrt{3}x-3=0$ are rotated about $(\sqrt{3},0)$ through an angle 15° one in clockwise direction and other in anticlockwise direction so that they become perpendicular, then the equation of pair of lines in the new position is:

(A) $y^2-x^2+2\sqrt{3}x-3=0$

(B) $y^2-x^2+2\sqrt{3}x+3=0$

(C) $y^2-x^2-2\sqrt{3}x+3=0$

(D) $y^2-x^2=3$

103. The graph $y^2+2xy+40|x|=400$ divides the plane into regions. The area of the bounded region is:

(A) 400 sq. units (B) 800 sq. units

(C) 40 sq. units (D) 80 sq. units

104. Two sides of a rhombus $OABC$ (lying entirely in Ist quadrant or IVth quadrant) of area equal to 2 sq. units are $y=\frac{x}{\sqrt{3}}$, $y=\sqrt{3}x$, then the possible co-ordinates of B is/are (O being origin):

(A) $\left(1+\sqrt{3},1+\sqrt{3}\right)$ (B) $\left(\sqrt{3}-1,\sqrt{3}-1\right)$

(C) $\left(\sqrt{3},\sqrt{3}\right)$ (D) none of these

105. If $4x+3y=13$ and $3x-4y+9=0$ are the equations of two equal sides of an isosceles triangle and (1, 3) is a point on the base, then the equation of the base is:

(A) $7x-y-4=0$ (B) $2x+y-5=0$

(C) $3x-4y+9=0$ (D) $x+7y+3=0$

106. The point $A(2,-1)$ undergoes the following transformations.

(i) Reflection about $x=4$

(ii) Translated in the direction of $y=x$ in increasing ordinates by 2 units

(iii) Reflection about $y=4$ co-ordinates of A in the final position is:

(A) $\left(6+\sqrt{2},8-\sqrt{2}\right)$ (B) $\left(6+\sqrt{2},9-\sqrt{2}\right)$

(C) $\left(6+\sqrt{2},8+\sqrt{2}\right)$ (D) $\left(6\sqrt{2},8\sqrt{2}\right)$

107. If $A\left(\alpha,\frac{1}{\alpha}\right)$, $B\left(\beta,\frac{1}{\beta}\right)$ and $C\left(\gamma,\frac{1}{\gamma}\right)$ be the vertices of a ΔABC, where α and β are the roots of equation $x^2-6p_1x+2=0$ and β and γ are the roots of the equation $x^2-6p_2x+3=0$ and γ and α are the roots of the equation $x^2-6p_3x+6=0$, p_1, p_2, p_3 being positive, then the co-ordinates of the centroid of ΔABC is:

(A) $\left(2,\frac{8}{11}\right)$ (B) $\left(\frac{2}{11},18\right)$

(C) $\left(2,\frac{11}{18}\right)$ (D) $\left(18,\frac{11}{2}\right)$

108. Two vertices of a triangle are (1, 3) and (4, 7). If the orthocentre lies on the line $x + y = 3$, then the locus of the third vertex is:

(A) $x^2 - 2xy + 2y^2 - 3x - 4y + 36 = 0$

(B) $2x^2 - 4xy + 3y^2 - 4x - y + 42 = 0$

(C) $3x^2 + xy - 4y^2 - 2x + 24y - 40 = 0$

(D) $x^2 - 4xy + 3y^2 - 2x - y + 40 = 0$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Consider the points $P(-\sin(\beta-\alpha), -\cos\beta)$, $Q(\cos(\beta-\alpha), \sin\beta)$ and $R(\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$, where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$ then mark the **incorrect** options.

(A) P lies on the line segment RQ
(B) Q lies on the line segment PR
(C) R lies on the line segment QP
(D) P, Q and R are non-collinear

2. If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle, then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ represents:}$$

(A) the altitude through A
(B) the perpendicular bisector of side BC
(C) the median through A
(D) the line joining the centroid with vertex A.

3. Three lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent, if:

(A) $p + q + r = 0$

(B) $p^2 + q^2 + r^2 = pr + rq + pq$

(C) $p^3 + q^3 + r^3 - 3pqr = 0$

(D) none of these

4. The equations of the sides AB, BC and CA of a ΔABC are $2x + y = 0$, $x + py = q$ and $x - y = 3$, respectively and if the centroid is (2, 3), then:

(A) $p + q = 74$
(B) equation of BC is $x + 15y = 59$
(C) equation of median through A is $5x - y - 7 = 0$
(D) equation of attitude through A is $11x - y - 13 = 0$

5. Which of the following line is/are parallel to an angle bisector of the pair of lines $x^2 - y^2 - 4y - 4 = 0$?

(A) $x + 1 = 0$
(B) $y - 2 = 0$
(C) $x + y + 2 = 0$
(D) $x - 1 = 0$

6. A circle 'C' rolls between pair of lines $9x^2 + 24xy + 16y^2 - 25 = 0$ touching both of them. Then its:

(A) area is 4π sq. units
(B) radius is 1 unit
(C) radius is 2 units
(D) area is π sq. units

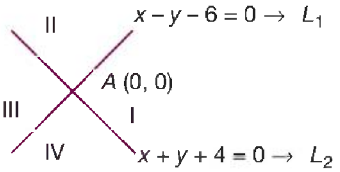
7. The four sides of a quadrilateral are given by the equation $xy(x - 2)(y - 3) = 0$. The equation of the line parallel to $x - 4y = 0$ that divides the quadrilateral in two equal areas is:

(A) $x - 4y - 5 = 0$
(B) $x - 4y + 5 = 0$
(C) $x - 4y - 1 = 0$
(D) $x - 4y + 1 = 0$

8. If the lines $x + y + 4 = 0$, $x - y + 6 = 0$ and $x + ky + k_1 = 0$ do not form a triangle, then the value of k is (if k_1 is a variable):

(A) $k = 1$
(B) $k = -1$
(C) $k = 2$
(D) $k = 3$

9. Two lines $x - y - 6 = 0$ and $x + y + 4 = 0$ divide the xy plane into 4 quadrants as mentioned in diagram. If (0, 0) lies in the Ist quadrant, then:



II
$x - y - 6 = 0 \rightarrow L_1$
$A\ (0, 0)$
III
I
IV
$x + y + 4 = 0 \rightarrow L_2$

(A) $(1,1) \in \text{I}$

(B) $(1,1) \in \text{II}$

(C) $(-3,-3) \in \text{IV}$

(D) $(-3,-3) \in \text{III}$

10. Two straight lines $u = 0$ and $v = 0$ pass through the origin and angle between them is $\tan^{-1}\left(\frac{7}{9}\right)$. If the ratio of the slope of $v = 0$ and $u = 0$ is $\frac{9}{2}$, then their equations are:

(A) $y + 3x = 0$ and $3y + 2x = 0$
(B) $2y + 3x = 0$ and $3y + x = 0$
(C) $2y = 3x$ and $3y = x$
(D) $y = 3x$ and $3y = 2x$

11. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior point and $(b, 1)$ is an exterior point of the triangle, then:

(A) $2 < a < \frac{10}{3}$
(B) $-2 < a < \frac{10}{3}$
(C) $-1 < b < \frac{9}{2}$
(D) $-1 < b < 1$

12. Two sides of a rhombus $OABC$ (lying entirely in Ist quadrant or IIIrd quadrant) of area equal to 2 sq. units are $y = \frac{x}{\sqrt{3}}$, $y = \sqrt{3}\,x$. Then the possible co-ordinates of B is/are (O being origin):

(A) $(1+\sqrt{3}, 1+\sqrt{3})$
(B) $(-1-\sqrt{3}, -1-\sqrt{3})$
(C) $(3+\sqrt{3}, 3+\sqrt{3})$
(D) $(\sqrt{3}-1, \sqrt{3}-1)$

13. The co-ordinates of three points O, A and B are (0, 0), (0, 4) and (6, 0), respectively. If a point P moves so that the area of ΔPOA is always twice the area of ΔPOB, then P lies on:

(A) $x - 3y = 0$
(B) $x + 3y = 0$
(C) $3x + 4y = 0$
(D) $3x - 4y = 0$

14. The equations $(b - c)x + (c - a)y + a - b = 0$, $(b^3 - c^3)x + (c^3 - a^3)y + a^3 - b^3 = 0$ will represent the same line, if:

(A) $b = c$
(B) $c = a$
(C) $a = b$
(D) $a + b + c = 0$

15. The area of a triangle is 5 sq. units. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. The co-ordinates of the third vertex can be:

(A) $\left(\frac{-3}{2}, \frac{3}{2}\right)$
(B) $\left(\frac{3}{4}, \frac{-3}{2}\right)$
(C) $\left(\frac{7}{2}, \frac{13}{2}\right)$
(D) $\left(\frac{-1}{4}, \frac{11}{4}\right)$

16. If the lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then:

(A) $\lambda = 2$
(B) $\lambda = -3$
(C) $\lambda = 4$
(D) $\lambda = -4$

17. Equation of a straight line passing through the point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ are perpendicular to one of them is:

(A) $x + y + 3 = 0$
(B) $x + y - 3 = 0$
(C) $x - 3y - 5 = 0$
(D) $x - 3y + 5 = 0$

18. If one vertex of an equilateral triangle of side a, lies at the origin and the other lies on the line $x - \sqrt{3}y = 0$, the co-ordinates of the third vertex are:

(A) (0, a)
(B) $\left(\frac{\sqrt{3}a}{2}, \frac{-a}{2}\right)$
(C) (0, -a)
(D) $\left(\frac{-\sqrt{3}a}{2}, \frac{a}{2}\right)$

19. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$, are concurrent $(a + b + c \neq 0)$, then:

(A) $a^3 + b^3 + c^3 - 3abc = 0$
(B) $a = b$
(C) $a = b = c$
(D) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

20. If two of the lines represented by $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ bisects the angle between other two, then:

(A) $c + 6a = 0$
(B) $c - 6a = 0$
(C) $b + d = 0$
(D) $b - d = 0$

21. If P is a point on the curve $5x^2 - 8xy + 5y^2 = 4$ and 'O' is the origin, then OP has:

(A) minimum value 1
(B) minimum value $\frac{2}{3}$
(C) maximum value 2
(D) maximum value $\frac{4}{3}$

22. $9x^2 + 2hxy + 4y^2 + 6x + 2fy - 3 = 0$ represents two parallel lines if:

(A) $h = 6, f = 2$
(B) $h = -6, f = -2$
(C) $h = -6, f = 2$
(D) $h = 6, f = -2$

23. If $A(1, 3)$ and $C(7, 5)$ are opposite vertices of a square then the equation of a side through A is:

(A) $x + 2y - 7 = 0$
(B) $x - 2y + 5 = 0$
(C) $2x + y - 5 = 0$
(D) $2x - y + 1 = 0$

24. The quadrilateral formed by the two pairs of lines $6x^2 - 5xy - 6y^2 = 0$ and $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ is a:

(A) square
(B) rhombus
(C) parallelogram
(D) rectangle

25. Equation of a straight line passing through the point (4, 5) and equally inclined to the lines $3x - 4y - 7 = 0$ and $12x - 5y + 6 = 0$ is:

(A) $9x - 7y = 1$
(B) $9x + 7y = 71$
(C) $7x + 9y = 73$
(D) $7x - 9y + 27 = 0$

26. The triangle formed by the lines $x + y = 0$, $3x + y - 4 = 0$ and $x + 3y - 4 = 0$ is:

(A) isosceles (B) scalene

(C) acute angled (D) obtuse angled

27. Let $u = ax + by + a\sqrt[3]{b} = 0, v = bx - ay + b\sqrt[3]{a} = 0, a, b \in R$ be two straight lines. The equation of the bisectors of the angle formed by L_1: $(\tan\theta_1)u - (\tan\theta_2)v = 0$ and L_2: $(\tan\theta_1)u - (\tan\theta_2)v = 0$ for $\theta_1, \theta_2 \in \left(0, \frac{\pi}{2}\right)$ is:

(A) $u = 0$

(B) $(\tan\theta_2)u + (\tan\theta_1)v = 0$

(C) $(\tan\theta_2)u - (\tan\theta_1)v = 0$

(D) $v = 0$

28. Equations of bisectors of angles between intersecting the lines $\frac{x-3}{\cos\theta} = \frac{y+5}{\sin\theta}, \frac{x-3}{\cos\phi} = \frac{y+5}{\sin\phi}$ are $\frac{x-3}{\cos\alpha} = \frac{y+5}{\sin\alpha}$ and $\frac{x-3}{\beta} = \frac{y+5}{\gamma}$, then:

(A) $\alpha = \frac{\theta+\phi}{2}$ (B) $\beta^2 + \gamma^2 = 1$

(C) $\tan\alpha = \frac{-\beta}{\gamma}$ (D) $\tan\alpha = \frac{\beta}{\gamma}$

29. The equation of the line passing through (2, 3) and making an intercept of 2 units between the lines $y + 2x = 5$ and $y + 2x = 3$ is:

(A) $5x - 4y + 2 = 0$ (B) $3x + 4y = 18$

(C) $x = 2$ (D) $y = 3$

30. The equation of the diagonal of the square formed by the pairs of lines $xy + 4x - 3y - 12 = 0$ and $xy - 3x + 4y - 12 = 0$ is:

(A) $x - y = 0$ (B) $x + y + 1 = 0$

(C) $x + y = 0$ (D) $x - y + 1 = 0$

31. Under rotation of axes through θ, $x\cos\alpha + y\sin\alpha = P$ changes to $X\cos\beta + Y\sin\beta = P$, then:

(A) $\cos\beta = \cos(\alpha - \theta)$ (B) $\cos\alpha = \cos(\beta - \theta)$

(C) $\sin\beta = \sin(\alpha - \theta)$ (D) $\sin\alpha = \sin(\beta - \theta)$

32. If (3, 2), (–4, 1) and (–5, 8) are vertices of triangle, then:

(A) orthocentre is (4, 1) (B) orthocentre is (–4, 1)

(C) circumcentre is (–1,5) (D) circumcentre is (3, 2)

33. The point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio k:1. The values of k for which the area of ΔABC, where $B(1, 5)$, $C(7, -2)$ is equal to 2 sq. units are:

(A) 7 (B) 4

(C) $\frac{30}{4}$ (D) $\frac{31}{9}$

34. The angle through which the co-ordinate axes be rotated so that xy-term in the equation $5x^2 + 4\sqrt{3}xy + 9y^2 = 0$ may be missing is:

(A) $\frac{\pi}{6}$ (B) $\frac{-\pi}{6}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$

35. Sides of a rhombus are parallel to the lines $x + y - 1 = 0$ and $7x - y - 5 = 0$. It is given that diagonals of the rhombus intersect at (1, 3) and one vertex 'A' of the rhombus lies on the line $y = 2x$. Then the co-ordinates of the vertex A are:

(A) $\left(\frac{8}{5}, \frac{16}{5}\right)$ (B) $\left(\frac{7}{15}, \frac{14}{15}\right)$

(C) $\left(\frac{6}{5}, \frac{12}{5}\right)$ (D) $\left(\frac{4}{15}, \frac{8}{15}\right)$

36. Equations of the diagonals of a rectangle are $y + 8x - 17 = 0$ and $y - 8x + 7 = 0$. If the area of the rectangle is 8 sq. units, then the equation of the sides of the rectangle is/are:

(A) $x = 1$ (B) $x + y = 1$

(C) $y = 9$ (D) $x - 2y = 3$

37. If the points $(-2, 1), (3, 4)$ and (a^2, a) lie on the same side of the straight line $6x - 7y - 3 = 0$ then a can lie in:

(A) (0, 1) (B) $\left(\frac{6}{5}, \frac{7}{5}\right)$

(C) (2, 4) (D) (–1, 1)

38. If the straight line $3x + 4y = 24$ intersect the axes at A and B and the straight line $4x + 3y = 24$ intersect the axes at C and D then points A, B, C, D lie on:

(A) the circle (B) the parabola

(C) an ellipse (D) the hyperbola

39. If $x^2 + 2hxy + y^2 = 0$ $(h \neq 1)$ represents the equations of the straight lines through the origin which make an angle α with the straight line $y + x = 0$, then:

(A) $\sec 2\alpha = h$ (B) $\cos\alpha = \sqrt{\frac{1+h}{2h}}$

(C) $m_1 + m_2 = -2\sec 2\alpha$ (D) $\cot\alpha = \sqrt{\frac{h+1}{h-1}}$

40. If $6a^2-3b^2-c^2+7ab-ac+4bc=0$, then the family of lines $ax+by+c=0, |a|+|b|\neq 0$ is concurrent at:

(A) $(-2,-3)$ (B) $(3,-1)$

(C) $(2,3)$ (D) $(-3,1)$

41. If one diagonal of a square is the portion of the line $7x+5y=35$ is intercepted by the axes, then the other extremities may be:

(A) (6, 6) (B) (5, 7)

(C) (–1, 1) (D) (7, 5)

42. ABC is an isosceles triangle with vertices $A(2a,0)$ $B(0,a)$ and C lies in the positive quadrant. If the equation of AC is $x=2a$, then the equation of BC is:

(A) $3x-4y+4a=0$

(B) $(\sqrt{5}-1)x-2y+2a=0$

(C) $x-2y+2a=0$

(D) $x+2y-2a=0$

43. A line not parallel to the axes is drawn through the point $P(3,4)$ and inclined at an angle θ to the positive direction of x-axis. If this line meets the lines $x=6$ and $y=8$ at R and S, respectively, then:

(A) $PR=3\sec\theta$ (B) $PS=4\operatorname{cosec}\theta$

(C) $PR+PS=\dfrac{1}{\sin 2\theta}$ (D) $\dfrac{9}{(PR)^2}+\dfrac{16}{(PS)^2}=1$

44. Two lines from the family of lines $(1+2\lambda)x-(1+\lambda)y+1=0$ and the line $x+y=5$ from an equilateral triangle. The equation of the two lines can be:

(A) $y-2=(2+\sqrt{3})(x-1)$

(B) $y-2=(2-\sqrt{3})(x-1)$

(C) $y-2=(3-\sqrt{2})(x-1)$

(D) $y-2=(3-\sqrt{2})(x-2)$

45. Consider a family of lines $(a+3)x-(2a+1)y+(6a-2)=0$, where $a\in R$ then:

(A) the locus of foot of the perpendicular from origin to each line of the family is $x^2+y^2-2x-4y=0$

(B) the equation of the line of the family at the maximum distance from origin is $x+2y-10=0$

(C) the equation of the line of the family at the minimum distance from origin is $2x-y-1=0$

(D) the number of lines of the family at a distance 5 units from origin is exactly two

46. If $A(3,4)$ and $B(-5,-2)$ are the extremities of the base of an isosceles triangle with $\angle C=\tan^{-1}2$, then C can be:

(A) $\left(\dfrac{3\sqrt{5}+1}{2}, -(1+2\sqrt{5})\right)$

(B) $\left(-\dfrac{(3\sqrt{5}+5)}{2}, 3+2\sqrt{5}\right)$

(C) $\left(\dfrac{3\sqrt{5}-1}{2}, 3-2\sqrt{5}\right)$

(D) $\left(-\dfrac{(3\sqrt{5}-5)}{2}, -(1-2\sqrt{5})\right)$

47. The equation of the line passing through the origin and making an intercept of 3 units between the parallel lines $5x+12y=4$ and $5x+12y=40$ is:

(A) $y=0$ (B) $x=0$

(C) $119x-120y=0$ (D) $119x+120y=0$

48. All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy:

(A) $3x+2y>0$ (B) $2x+y-13>0$

(C) $2x-3y-12<0$ (D) $-2x+y>0$

49. If x_1,x_2,x_3 as well as y_1,y_2,y_3 are in G.P. with the same common ratio, then the points (x_1,y_1), (x_2,y_2), (x_3,y_3):

(A) lie on a straight line

(B) lie on an ellipse

(C) lie on a circle

(D) are vertices of a triangle

50. A line $x=k$ intersects the graph of $y=\log_5 x$ and $y=\log_5(x+4)$. The distance between the points of intersection is 0.5 given that $k=a+\sqrt{b}$, where $a,b\in N$, then:

(A) $a=2, b=4$ (B) $a=1, b=5$

(C) $a+b=6$ (D) $a=b=3$

51. Locus of mirror image of $(4,3)$ about the variable line $(2-3m)x+(1+4m)y+2-m=0$ (m is variable) is:

(A) $x^2+y^2+\dfrac{8}{11}x+\dfrac{8}{11}y=0$

(B) $11x^2+11y^2+18x+8y-96=0$

(C) $11x^2+11y^2-18x-8y-96=0$

(D) passing through $\left(-\dfrac{62}{11}, -\dfrac{41}{11}\right)$

52. The diagonals of rhombus $ABCD$ intersect at point (1, 2) and its sides are parallel to the lines $x-\sqrt{3}y+2\sqrt{3}=0$ and $\sqrt{3}x-y+3=0$. If the vertex A lies on X-axis, then possible co-ordinates of vertex C are:

(A) $(-1, 4)$ (B) $(-1, -4)$
(C) $(3, 4)$ (D) none of these

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

A ΔABC with vertex A (1, 2) is given. If the equations of the internal angle bisectors of $\angle B$ and $\angle C$ are $x-y-1=0$ and $y+2x-9=0$, then answer the following.

1. The equation of BC is:
(A) $y=2x-6$ (B) $y=x-4$
(C) $2y=x+6$ (D) $2y=x-4$

2. The equation of altitude from A on BC is:
(A) $2y=x$ (B) $2y+x=7$
(C) $2y+x=5$ (D) $2y-x=3$

3. Area of ΔABC is:
(A) 6 sq. units (B) $\frac{15}{4}$ sq. units
(C) $\frac{9}{4}$ sq. units (D) $\frac{3}{2}$ sq. units

Comprehension 2

$A(4, 0)$, $B(-4, 0)$ are two points, then the locus of P such that,

4. $PA+PB=10$ is:
(A) $\frac{x^2}{25}+\frac{y^2}{16}=1$ (B) $\frac{x^2}{25}+\frac{y^2}{9}=1$
(C) $\frac{x^2}{16}+\frac{y^2}{9}=1$ (D) $\frac{x^2}{9}+\frac{y^2}{25}=1$

5. $PA=PB$ is:
(A) $x=0$ (B) $y=0$
(C) $x-y=0$ (D) $x+y=0$

6. Area of ΔPAB is 4 sq. units, then:
(A) $x+y-1=0$ (B) $x-1=0$
(C) $y-1=0$ (D) $x-y+1=0$

Comprehension 3

ABC is an isosceles triangle with $AB=AC=5$ and $BC=6$. Let P be a point inside the ΔABC such that the distance from P to the base BC equals the Geometric mean of the distance to the sides AB and AC.

7. The locus of the point P is:
(A) a semicircle
(B) a minor arc of a circle
(C) a major arc of circle
(D) a complete circle

8. The minimum distance of the point A from the locus of the point P is:
(A) $\frac{5}{2}$ (B) $\frac{3}{2}$
(C) 2 (D) none of these

9. If the tangents to the locus at B and C intersect at point P, then the area of the ΔPBC is:
(A) 10 (B) 12
(C) 14 (D) 18

Comprehension 4

Let $y=f(x)$ be the given continuous function. Let, (x_1, y_1) be any point on it. The point (x_1, y_1) is moving on the curve corresponding to each position of (x_1, y_1). There is an image point with respect to the straight line $ax+by+c=0$ ($a, b, c \in R$), then the locus of image points is called image of the curve $y=f(x)$ with respect to the given line $ax+by+c=0$. Based on this information, attempt the following questions.

10. Image of the curve $y=1+\frac{1}{x^2}$ w.r.t the line $x+y+2=0$ is:
(A) $(x-2)^2(y+3)+1=0$ (B) $(x+3)^2(y+2)+1=0$
(C) $(x-3)(y-3)^2+1=0$ (D) none of these

11. Image of the line $2x+3y=1$ w.r.t the line $3x+2y+1=0$ is:
(A) $46x+9y+37=0$ (B) $46x-9y+37=0$
(C) $46x-9y-37=0$ (D) $46x+9y-37=0$

12. Image of the curve $x^2+2xy+2y^2=1$ w.r.t to the line $x+y-2=0$ is:

(A) $2x^2 + 2xy - y^2 - 12x - 8y - 19 = 0$
(B) $2x^2 - 2xy - y^2 - 12x - 8y - 19 = 0$
(C) $2x^2 + 2xy + y^2 - 12x - 8y + 19 = 0$
(D) $2x^2 - 2xy - y^2 - 12x + 8y - 19 = 0$

Comprehension 5

Let us consider the situation when axes are inclined at an angle 'w'. If the co-ordinates of a point P are (x_1, y_1), then $PN = x_1$, $PM = y_1$, where PM is parallel to y-axis and PN is parallel to x-axis equation of straight line, through P and makes an angle θ with x-axis is $y - y_1 = \frac{\sin\theta}{\sin(w-\theta)}(x - x_1)$,

as $R\theta = y - y_1$, $P\theta = x - x_1$ and $\frac{P\theta}{\sin(w-\theta)} = \frac{R\theta}{\sin\theta}$

This can be written as $y - y_1 = m(x - x_1)$, where $m = \frac{\sin\theta}{\sin(w-\theta)}$

If slope of line is m, then the angle of inclination of line with x-axis is given by $\tan\theta = \frac{m\sin w}{1 + m\cos w}$.

13. The axes being inclined at an angle of 60°, then inclination of the line $y = 2x + 5$ with the x-axis is:

(A) 30°
(B) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$
(C) $\tan^{-1}(2)$
(D) 60°

14. The axes being inclined at an angle of 60°, then angle between the two straight lines $y = 2x + 5$ and $2y + x + 7 = 0$ is:

(A) 90°
(B) $\tan^{-1}\left(\frac{5}{3}\right)$
(C) $\tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$
(D) $\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$

15. The axes being inclined at an angle of 30°, then the equation of straight line which makes an angle of 60° with the positive direction of x-axis and having x-intercept equal to 2 is:

(A) $y - \sqrt{3}x = 2\sqrt{3}$
(B) $\sqrt{3}y = x$
(C) $y + \sqrt{3}x = 2\sqrt{3}$
(D) $y + 2x = 0$

Comprehension 6

A variable line 'L' is drawn through $O(0, 0)$ to meet the lines L_1: $y - x - 10 = 0$ and L_2: $y - x - 20 = 0$ at the points A and B respectively. A point P is taken on L.

16. Locus of P, if $\frac{2}{OP} = \frac{1}{OA} + \frac{1}{OB}$ is:

(A) $3x + 3y = 40$
(B) $3x + 3y + 40 = 0$
(C) $3x - 3y = 40$
(D) $3x - 3y + 40 = 0$

17. Locus of P, if $OP^2 = OA.\ OB$ is:

(A) $(y - x)^2 = 100$
(B) $(y - x)^2 = 50$
(C) $(y - x)^2 = 200$
(D) $(x - y)^2 = 20$

18. Locus of P, if $OP = OA + OB$ is:

(A) $y - x = 10$
(B) $y + x - 20 = 0$
(C) $y - x = 30$
(D) $x - y - 30 = 0$

Comprehension 7

The homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines that passes through the origin. These lines are distinct, if $h^2 > ab$ and coincident, if $h^2 = ab$ and the lines do not exist, if $h^2 < ab$.

19. The difference of the tangent of the angles which the lines $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make with x-axis is:

(A) –1
(B) 0
(C) 1
(D) 2

20. If the pair of straight lines $x^2 - 2hxy - y^2 = 0$ and $x^2 - 2gxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then the value of $(1 - gh)$ is:

(A) 0
(B) 2
(C) –2
(D) –1

21. If two of the three lines given by $ax^3 + bx^2y + cxy^2 + dy^3 = 0 (a \neq 0)$ make complementary angles with x-axis, then:

(A) $a^2 + ac + bd + d^2 = 0$
(B) $a^2 - ac - bd + d^2 = 0$
(C) $a^2 - ac + bd - d^2 = 0$
(D) $a^2 + ac - bd - d^2 = 0$

Comprehension 8

The equations of two equal sides AB and AC of an isosceles triangle ABC are $x + y = 5$ and $7x - y = 3$. If the area of ΔABC is 5 sq. units, then answer the following questions.

22. Which of the following cannot represent BC?

(A) $3y - x = 7$
(B) $y + 3x = 12$
(C) $y + 3x = 20$
(D) $3y + x = -7$

23. If $\left(\frac{7}{2}, \frac{3}{2}\right)$ is one of the possible positions of B, then which of the following cannot be a possible position of C?

(A) $\left(\frac{1}{2},\frac{1}{2}\right)$

(B) $\left(\frac{3}{2},\frac{15}{2}\right)$

(C) $\left(-\frac{3}{2},\frac{13}{2}\right)$

(D) None of these

24. The number of possible triangles ABC must be:

(A) 3

(B) 4

(C) 1

(D) 2

Comprehension 9

Suppose we define the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as $d(P, Q) = \text{maximum}\{|x_2 - x_1|, |y_2 - y_1|\}$, then answer the following questions.

25. For a point P along the line $y = \sqrt{3}\,x$, $d(0, P)$ is:

(A) x

(B) y

(C) x, if $x < 0$

(D) y, if $y > 0$

26. The area of the region bounded by the locus of a point P satisfying $d(P, A) = 4$, where A (1, 2) is:

(A) 64 sq. units

(B) 54 sq. units

(C) 16π sq. units

(D) none of these

27. Suppose the points A and B have co-ordinates (1, 0) and (–1, 0), respectively, then for a variable point P on this plane, the equation $d(P, A) + d(P, B) = 2$ represents:

(A) a line segment joining A and B

(B) an ellipse with foci at A and B

(C) region lying inside a square of area 2 sq. units

(D) region inside a semicircle with AB as diameter

Comprehension 10

For points $P(x_1, y_1)$ and $Q(x_2, y_2)$ of the co-ordinate plane, a new distance $d(P, Q)$ is defined by

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

Let $O(0, 0)$, $A(1, 2)$, $B(2, 3)$ and $C(4, 3)$ are 4 fixed points on xy plane.

28. Let $R(x, y)$, such that R is equidistant from the points O and A with respect to new distance and if $0 \le x < 1$, $0 \le y < 2$, then R lie on a line segment whose equation is:

(A) $x + y = 3$

(B) $x + 2y = 3$

(C) $2x + y = 3$

(D) $2x + 2y = 3$

29. Let $S(x, y)$, such that S is equidistant from points O and B with respect to new distance and if $x \ge 2$ and $0 \le y < 3$, then the locus of S is:

(A) a line segment of finite length

(B) a line of infinite length

(C) a ray of finite length

(D) a ray of infinite length

Comprehension 11

The locus of a moving point is the path traced out by that point under one or more given conditions. Technically, a locus represents the 'set of all points' which lies on it.

A relation $f(x, y) = 0$ between x and y which is satisfied by each point on the locus and such that each point satisfying the equation is on the locus is called the equation of the locus.

30. Locus of centroid of the triangle, whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and (1, 0), where t is a parameter is:

(A) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$

(B) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$

(C) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$

(D) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$

31. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a ΔABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line:

(A) $2x + 3y = 9$

(B) $2x - 3y = 7$

(C) $3x + 2y = 5$

(D) $3x - 2y = 3$

32. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of '$c$' is:

(A) $\frac{1}{2}\left(a_2^2 + b_2^2 - a_1^2 - b_1^2\right)$

(B) $a_1^2 - a_2^2 + b_1^2 - b_2^2$

(C) $\frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2\right)$

(D) $\sqrt{\left(a_1^2 + b_1^2 - a_2^2 - b_2^2\right)}$

Comprehension 12

Let ABC be an acute angled triangle with $BC = a$, $CA = b$ and $AB = c$, where a, b and c are unequal. For any point P lying within this triangle, let D, E, F denote the feet of perpendiculars from P onto the sides BC, CA and AB, respectively.

33. All positions of the point P for which the ΔDEF is isosceles lie on:

(A) the incircle of ΔABC

(B) lines of altitude from A, B and C

(C) arcs of three circles

(D) arcs of three circles and three line segments

34. Let the co-ordinates of the point C, A and B be respectively (0, 0), (7, 0) and (4, 4) and ΔDEF is isosceles with $DE = DF$. Then P lies on the curve:

(A) $\left(x - \frac{69}{4\sqrt{2}+12}\right)^2 + \left(y - \frac{20}{4\sqrt{2}+12}\right)^2 = \left(\frac{20}{4\sqrt{2}+12}\right)^2$

(B) $3\left(x^2 + y^2\right) - 49\left(x + y\right) + 196 = 0$

(C) $x = 4$ or $x + y = 7$ or $4x = 3y$

(D) $x = 4$ or $x^2 + y^2 - 4x - 4y = 0$

35. If the ΔDEF is an equilateral triangle, then P:

(A) coincides with incentre of ΔABC

(B) coincides with orthocentre of ΔABC

(C) lies on the sides of pedal triangle of ΔABC

(D) none of these

Comprehension 13

Consider a ΔOAB formed by the points $O(0, 0)$, $A(2, 0)$ and $B(1, \sqrt{3})$. $P(x, y)$ is an arbitrary interior point of triangle moving is such a way that $d(P, OA) + d(P, AB) + d(P, OB) = \sqrt{3}$, where $d(P, OA)$, $d(P, AB)$, $d(P, OB)$ represent the distance of P from the sides OA, AB and OB, respectively.

36. Area of region representing all possible position of point P is:

(A) $2\sqrt{3}$ (B) $\sqrt{6}$

(C) $\sqrt{3}$ (D) none of these

37. If the point P moves in such a way that $d(P, OA) \le \min\left(d\left(P, OB\right), d\left(P, AB\right)\right)$, then the area of region representing all possible position of point P is:

(A) $\sqrt{3}$ (B) $\sqrt{6}$

(C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{6}}$

38. If the point P moves in such a way that $d(P, OA) \ge \max\left(d\left(P, OB\right)\right), d\left(P, AB\right)\right)$, then area of region representing all possible position of point P is:

(A) $\sqrt{3}$ (B) $\sqrt{6}$

(C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{6}}$

Comprehension 14

One vertex of triangle is a fixed point from where these lines pass.

$$(a-b)x-(b-c)y+(c-a)=0$$

$$(b-c)x-(c-a)y+(a-b)=0$$

$$(c-a)x-(a-b)y+(b-c)=0$$

$$x\cos^2\theta - y\sin^2\theta = 1$$

Another vertex is (3, 4) and the third vertex lies on x-axis at $(\lambda, 0)$.

39. Condition on λ such that it represents triangle is:

(A) $\lambda \ne \frac{7}{5}$ (B) $\lambda \ne \frac{2}{3}$

(C) $\lambda = \frac{7}{5}$ (D) $\lambda = \frac{2}{3}$

40. Locus of centroid of triangle is:

(A) $y = 1$ (B) $x = 1$

(C) $x + y = 1$ (D) none of these

41. Condition on a and k such that $x^2 = 4a(y-k)$ does not intersect the locus of centroid of triangle is:

(A) $a > 0, k < 1$ (B) $a < 0, k < 1$

(C) $a < 0, k > 1$ (D) none of these

Comprehension 15

A (0, 3), B (–2, 0) and C(6, 1) be the vertices of a triangle and $M(\beta, \beta+1)$ be a moving point.

42. M lies on the curve:

(A) $y = x + 1$ (B) $y = x^2$

(C) $x = y + 1$ (D) none of these

43. If M and A lie on same side of BC, then:

(A) $\beta > 2$ (B) $\beta < 2$

(C) $\beta > \frac{-6}{7}$ (D) $\beta < \frac{3}{4}$

44. M lies within ΔABC, if:

(A) $\frac{-6}{7} < \beta < 4$ (B) $-4 < \beta < \frac{-6}{7}$

(C) $\frac{-6}{7} < \beta < \frac{3}{2}$ (D) none of these

Comprehension 16

Consider a fixed point $P(5, 2)$ and also Q and R be two distinct variable points moving on $x - y = 0$ and $y = 0$, respectively.

45. Co-ordinates of Q for which $|PQ + QR + PR|$ become minimum are:

(A) $\left(\frac{29}{10}, \frac{29}{10}\right)$ (B) $\left(\frac{1}{5}, \frac{1}{5}\right)$

(C) (5, 0) (D) none of these

46. The co-ordinates of R for which $|PQ + QR + PR|$ become minimum are:

(A) $\left(\frac{1}{7}, 0\right)$ (B) $\left(\frac{29}{10}, 0\right)$

(C) $\left(\frac{29}{7}, 0\right)$ (D) none of these

47. The minimum value of $|PQ + QR + PR|$ is:

(A) $\sqrt{29}$ (B) $\sqrt{58}$

(C) 1 (D) none of these

Comprehension 17

An equilateral triangle ABC has its centroid at the origin and one side 'AC' is $x + y = 1$.

48. If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of the equilateral triangle, then $x_1x_2 + x_2x_3 + x_3x_1$:

(A) -1 (B) $\frac{-3}{2}$

(C) 1 (D) none of these

49. A point P is taken on the minor arc AC of the circum-circle of ΔABC. Then maximum of $(PA + PB + PC)$ is:

(A) $4\sqrt{2}$ (B) $\sqrt{2}$

(C) $3\sqrt{2}$ (D) $\frac{9}{\sqrt{2}}$

50. Another point P' is taken in the plane of $ABCP$. (P being the point where $PA + PB + PC$ is maximum). If $P'A + P'B + P'C + P'P = \sqrt{6} + \sqrt{8}$. Then locus of P' is:

(A) square (B) circle

(C) a point (D) none of these

Comprehension 18

The vertex A of ΔABC is $(3, -1)$. The equation of median BE and angular bisector CF are $6x + 10y - 59 = 0$ and $x - 4y + 10 = 0$, respectively.

51. Slope of the side BC must be:

(A) $\frac{1}{9}$ (B) $\frac{-2}{9}$

(C) $\frac{1}{7}$ (D) None of these

52. The equation of AB must be:

(A) $x + y = 2$ (B) $x + 4y = 0$

(C) $18x + 13y = 14$ (D) $23x - y = 70$

53. The length of the side AC must be:

(A) $\sqrt{71}$ (B) $\sqrt{83}$

(C) $\sqrt{85}$ (D) None of these

Comprehension 19

$A(1,3)$ and $C\left(\frac{-2}{5}, \frac{-2}{5}\right)$ are the vertices of a ΔABC and the equation of the internal angle bisector of $\angle ABC$ is $x + y = 2$.

54. Equation of side BC is:

(A) $7x + 3y - 4 = 0$ (B) $7x + 3y + 4 = 0$

(C) $7x - 3y + 4 = 0$ (D) $7x - 3y - 4 = 0$

55. Co-ordinates of vertex B are:

(A) $\left(\frac{3}{10}, \frac{17}{10}\right)$ (B) $\left(\frac{17}{10}, \frac{3}{10}\right)$

(C) $\left(\frac{-5}{2}, \frac{9}{2}\right)$ (D) $(1,1)$

56. Equation of side AB is:

(A) $3x + 7y = 24$ (B) $3x + 7y + 24 = 0$

(C) $13x + 7y + 8 = 0$ (D) $13x - 7y + 8 = 0$

Comprehension 20

At times, the methods of co-ordinates become effective in solving problems of properties of triangles. We may choose one vertex of the triangle and one side passing through this vertex at x-axis. Thus, without the loss of generality, we can assume that every ΔABC has a vertex B situated at $B(0, 0)$, vertex C at $(a, 0)$ and A as (h, k).

57. If in ΔABC, $AC = 3$, $BC = 4$ medians AD and BE are perpendicular, then area of ΔABC must be:

(A) $\sqrt{7}$ (B) $\sqrt{11}$

(C) $2\sqrt{2}$ (D) $\sqrt{13}$

58. Suppose the bisector AD of the interior angle A of ΔABC divides side BC into segments $BD = 4$, $DC = 2$. Then we must have:

(A) $b > c$ and $c < 4$ (B) $2 < b < 6$ and $c < 1$

(C) $2 < b < 6$ and $4 < c < 12$ (D) $b < c$

59. If in above question, altitude $AE > \sqrt{10}$ and suppose length of AB and AC are integers, then b will be:

(A) 3 (B) 6

(C) 4 or 5 (D) 7

Comprehension 21

In a ΔABC, $A(2, 3)$ and medians through B and C have equations $x + y - 1 = 0$ and $2y - 1 = 0$.

60. Equation of median through A is:

(A) $x + y = 4$ (B) $5x - 3y = 1$

(C) $5x + 3y = 1$ (D) $5x = 3y$

61. Equation of side BC is:

(A) $5x + 13y + 11 = 0$ (B) $5x - 3y = 1$

(C) $5x = 3y$ (D) $5x + 13y - 11 = 0$

62. Length of altitude from A to side BC is:

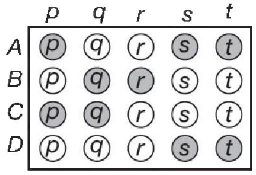
(A) $\frac{60}{\sqrt{194}}$ (B) 5

(C) $\frac{5}{\sqrt{194}}$ (D) $\frac{1}{\sqrt{194}}$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. If the answers for entries in **COLUMN-I** are expressed in the form L: $y = ax + b$, then match them with the entries in **COLUMN-II** for the value of $a + b$.

COLUMN-I	COLUMN-II
(A) If the sides BC, CA and AB of a ΔABC are respectively $x + 2y = 1$, $3x + y + 5 = 0$ and $x - y + 2 = 0$, then the equation of the altitude through B is L_1	(p) 1
(B) The image of the line $x - 2y = 1$ in the line $x + y = 0$ is L_2	(q) $\frac{3}{4}$
(C) A right triangle ABC with sides AB, BC, CA in ratio 5:4:3 moves such that A and B always lie on the positive x and y-axes, respectively. The locus of 'C' is L_3	(r) $\frac{5}{3}$
(D) A (1, 2), B(–1, 5) are two vertices of a ΔABC whose third vertex C lies on the line $2x + y = 2$. The locus of the centroid of the triangle is L_4	(s) –1

2. Regarding the equation $a|x| + b|y| - 6 = 0$, match the following.

COLUMN-I	COLUMN-II
(A) $a = b \neq 0$	(p) Square
(B) $a \neq b, ab \neq 0$	(q) Rhombus
(C) $\lvert a \rvert + \lvert b \rvert = 0$ finite area	(r) 4 straight lines enclosing
(D) $a^2 + b^2 = 13, ab = 6$, $a, b > 0, a > b$	(s) Given equation becomes meaningless

(Continued)

3. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Values of θ for which the point $(\sin^2\theta, \sin\theta)$ lies on or inside the square formed by the lines $xy = 0$ and $4xy - 2x - 2y + 1 = 0$ is/are	(p) 3
(B) If P (x_1, y_1) and Q (x_2, y_2) satisfy the equation $\cot^2(x+y) + \tan^2(x+y) + y^2 + 2y - 1 = 0$, then the minimum distance between P and Q is	(q) $\frac{\pi}{6}$
(C) Number of points (x, y) where $x, y \in N$, which lie inside the region bounded by $\lvert x \rvert + \lvert y \rvert = 5$ is/are	(r) $\frac{\pi}{4}$
(D) Area of the triangle formed by angle bisectors of co-ordinate axes and the line $x = \sqrt{6}$ is	(s) $\frac{\pi}{2}$
	(t) 6

4. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The points on the line $x + y = 2$, which are at a unit distance from the line $4x + 3y = 12$ are	(p) (1, 1)
(B) At what point the origin be shifted such that the equation $x^2+xy-3x-y+2=0$ does not contain any 1st degree term and constant term	(q) (11,–9)
(C) If the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable line is zero, then the line passes through the fixed point, which is	(r) (9, –7)
(D) If O is the origin and P is a variable point on $x^2 = y$, then point which lies on the locus of the midpoint of OP is	(s) (4, 2)
	(t) (2, 4)

5. The family of straight lines $ax+by+c=0$ pasing through a fixed point (s). Given that relation between a, b and c

COLUMN-I	COLUMN-II
(A) a, b, c are in A.P	(p) (–2, –3)
(B) $4a^2 + 9b^2 + 12ab = c^2$	(q) (2, 3)
(C) $a^2 + 4b^2 - 4ab = c^2$	(r) (–1, 2)
(D) $2a + 3b - c = 0$	(s) (1, –2)

6. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The distance between the lines $(x+7y)^2+4\sqrt{2}(x+7y)-42=0$ is	(p) 2
(B) If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is $\|x\| + \|y\| = k$, where k is equal to	(q) 7
(C) If $6x + 6y + m = 0$ is an acute angle bisector of line $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$, then m is equal to	(r) 3
(D) Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 6$ is	(s) 1

(Continued)

7. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The value of k for which $4x^2+8xy+ky^2=9$ is the equation of a pair of straight lines is	(p) 3
(B) If the sum of the slopes of the lines given by $x^2-2Cxy-7y^2=0$ is 4 times their product, then the value of C is	(q) –3
(C) If the gradient of one of the lines $x^2+hxy+2y^2=0$ is twice that of the other, then h is	(r) 2
(D) If the lines $ax^2+2hxy+by^2=0$ are equally inclined to the lines $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$, then the value of λ can be	(s) 4

8. Match the following.
Let ABC be an isosceles triangle with $AB = AC$. If AB lies along $x + y = 10$ and AC lies along $7x - y = 30$ and area of triangle be 20 sq. units.

COLUMN-I	COLUMN-II
(A) Co-ordinate of point B	(p) (10, 0)
(B) Co-ordinate of point C	(q) (4, –2)
(C) Centroid of ΔABC	(r) $\left(\frac{-5}{2}, \frac{5}{2}\right)$
(D) Circumcentre of ΔABC	(s) $\left(3, \frac{13}{3}\right)$
	(t) (0, 10)

9. If $y = m_i x + \dfrac{1}{m_i}\ (i = 1,2,3)$ represent 3 straight lines, whose slopes are the roots of the equation $2m^3 - 3m^2 - 3m + 2 = 0$, then match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Algebraic sum of the intercepts made by the lines on x-axis	(p) $\left(4\sqrt{2} + 9\sqrt{5}\right)/4$
(B) Algebraic sum of the intercepts made by the lines on y-axis	(q) $\dfrac{3}{2}$
(C) Sum of the distances of the lines from the origin	(r) $\dfrac{-21}{4}$
(D) Sum of the lengths of the lines intercepted between the co-ordinate axes	(s) $\left(5\sqrt{2} + 9\sqrt{5}\right)/10$

10. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The area bounded by the curve maximum $\{\lvert x\rvert, \lvert y\rvert\} = \dfrac{1}{2}$ is	(p) 0
(B) If the point (a, a) lies between the lines $\lvert x + y\rvert = 6$, then $[\lvert a\rvert]$ is (where [.] denotes the greatest integer function)	(q) 1
(C) Number of non-zero integral values of b for which the origin and the point (1, 1) lies on the same side of the straight line $a^2x + aby + 1 = 0$, for all $a \in R \sim \{0\}$ is	(r) 2
(D) If (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$, $5x - 6y - 1 = 0$, then $[\alpha]$ is (where [.] denotes the greatest integer function)	(s) -2

11. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The number of integral values of 'a' for which the point $P(a, a^2)$ lie completely inside of the triangle formed by $x = 0$, $y = 0$ and $x + 2y = 3$ is	(p) 1
(B) The ΔABC with $AB = 13$, $BC = 5$ and $AC = 12$ slides on the co-ordinate axis with A and B on the positive x-axis and positive y-axis, respectively. The locus of vertex 'C' is a line $12x - ky = 0$. Then, the value of k is	(q) 4
(C) The reflection of the point $(t-1, 2t+2)$ in a line is $(2t+1, t)$, then the line has slope equals to	(r) 3
(D) In a ΔABC, the bisector of angles B and C lie along the lines $x = y$ and $y = 0$. If A is $(1,2)$, then $\sqrt{10}d(A, BC)$, where $d(A, BC)$ represents the distance from A to BC	(s) 5
	(t) 0

(Continued)

12. Match the following.

COLUMN-I	COLUMN-II
(A) If the pair of lines $6x^2 - \alpha xy - 3y^2 - 24x + 3y + \beta = 0$ intersect on the x-axis, then $\alpha\beta$ is equal to	(p) 40
(B) In ΔABC, $R = 80$ and $r = 30$, then the distance between incentre and circumcentre is equal to	(q) 44
(C) Equations of medians AD and BE in ΔABC are $2x + 3y - 6 = 0$ and $3x - 2y - 10 = 0$. If $AD = 6$ cm and $BE = 11$cm, then the area of ΔABC in cm^2 is equal to	(r) 36
(D) If two of the sides of a triangle inscribed in $xy = 36$ are parallel to $2x + y = 0$ and $6x + y = 0$, then for any point (x, y) on the third side xy is equal to	(s) 48

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. The area enclosed within the curve $|x|+|y|=1$ is __.

2. The points (1, 3) and (5, 1) are two vertices of a rectangle. If the other two vertices lie on the line $y = 2x + c$, then $|c|$ equals ____.

3. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q, respectively. If the point O divides the segment PQ in the ratio $l:m$, then $l + m$ equals _____.

4. If each point on the line $ax + by + c = 0$ is at equal distances from the lines $24x+7y=20$ and $4x+3y=2$, then $|a + b + c|$ equals ____.

5. If the straight lines $7x - 2y + 10 = 0$, $7x + 2y - 10 = 0$ and $y + 2 = 0$ form an isosceles triangle of area is $2k$, then k equals ____.

6. Let P be the point (3, 2). Let Q be the reflection of P about the x-axis. Let R be the reflection of Q about the line $y = -x$ and let S be the reflection of R through the origin. $PQRS$ is a convex quadrilateral. If the area of $PQRS$ is $5k$, then k equals _____.

7. A triangle is formed by the lines $x + y = 0$, $x - y = 0$ and $px + qy = 1$. If $p^2 + q^2 = 1$ $(p^2 - q^2 \neq 0)$ and locus of its circumcentre is $(x^2 + y^2) = k(x^2 - y^2)^2$, then '$k - 1$' equals ____.

8. If the pair of straight lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angle between the other pairs, then $|pq|$ equals ______.

9. The equation $x^2 + y^2 + 2xy + 2gx + 2fy + 4 = 0$ represents a pair of real lines. The number of positive integral values of 'g' less than 8 is ________.

10. Consider 3 lines, L_1: $ax + ky - la - ma^3 = 0$, L_2: $bx + ky - lb - mb^3 = 0$ and L_3: $cx + ky - lc - mc^3 = 0$ in xy plane. If the lines are concurrent, then the value of $a + b + c$ is _______.

11. If the number of integral points exactly in the interior of the triangle with the vertices $(0,0),(0,10)$ and $(10,0)$ is k, then $\frac{k}{36}$ equals _______.

12. Area of the triangle formed by the line $x + y = 3$ and angle bisectors of the pairs of straight lines $x^2 - y^2 + 2y = 1$ is ________.

13. Let (h, k) be a fixed point, where $h > 0, k > 0$. A straight line passing through this point cuts the positive directions of co-ordinates axes at the points P and Q. Then, find $\frac{2\Delta}{hk}$, where Δ is the minimum area of the ΔOPQ.

14. If the line $y=\sqrt{3}x$ cuts the curve $x^4+ax^2y+bxy+cx+dy+6=0$ at A, B, C and D, then the value of $\frac{1}{12}OA.OB.OC.OD$ (where O is the origin) is ______.

15. A point $P(x, y)$ moves in such a way that $[x + y + 1] = [x]$ (where [.] greatest integer function) and $x \in (0, 2)$. Then the area representing all the possible positions of P equals _______.

16. $ABCD$ is a square of side length 1 unit. P and Q are points on AB and BC such that $\angle PDQ = 45°$. Find the perimeter of ΔPBQ.

17. Given a point (2, 1). If the minimum perimeter of a Δ with one vertex at (2, 1), one on the x-axis and one on the line $y = x$ is k, then $[k]$ is equal to (where [ ] denotes the greatest integer function) _________.

18. $ABCD$ and $PQRS$ are two variable rectangles, such that A,B,C and D lie on PQ,QR,RS and SP, respectively and perimeter 'x' of $ABCD$ is constant. If the maximum area of $PQRS$ is 32, then $\frac{x}{4}$ equals _________.

19. The number of points on the line $3x + 4y = 5$, which are at a distance $\sec^2\theta+2\operatorname{cosec}^2\theta,\ \theta\in R$ from the point (1, 3) is ______.

20. The area of the triangular region in the Ist quadrant bounded on the left by the y-axis, bounded above by the line $7x + 4y = 168$ and bounded below by the line $5x + 3y = 121$ is A, then the value of $\frac{3A}{10}$ is __________.

21. In a ΔABC, AB is parallel to y-axis, BC is parallel to x-axis, centroid is at $(2, 1)$. If median through C is $x-y=1$, then the slope of median through A is __________.

22. If the orthocentre of the triangle formed by $2x+3y-1=0$, $x+2y-1=0$, $ax+by-1=0$ is at the origin, then $\frac{b-a}{4}$ equals _________.

23. The area of the rhombus $ABCD$ is 24. The equation of the diagonal BD is $4x + 3y + 2 = 0$. The length of the side of the rhombus is _______.

24. In ΔABC, the equation of altitudes AM and BN are $x+5y-3=0, x+y-k=0$. If the altitude CL is given by $3x-y-1=0$, then find k.

25. The co-ordinate axes are rotated through an angle θ about the origin in anticlockwise sense. If the equation $2x^2+3xy-6x+2y-4=0$ changes to $aX^2+2hXY+bY^2+2gX+2fY+c=0$, then $(a + b)$ is equal to _____.

26. If $A(0, 4)$, $B(0, -4)$ and $|PA - PB| = 6$ and the locus of P is $7y^2-9x^2=9k$. Then k is ____.

27. The locus of the point which is collinear with the points (3, 4) and (–4, 3) is $ax + by + c = 0$, then $a + 3b + c$ equals ______.

28. The intercepts made on the line $x + y = 5\sqrt{2}$ by the lines $y = x \tan \theta$; $\theta = 0, \frac{\pi}{4}, \alpha, \frac{\pi}{2}$ are in A.P., then $\tan\alpha$ equals _____.

29. If the lines $ax+by+p=0, x\cos\alpha+y\sin\alpha-p=0\,(p\neq 0)$ and $x\sin\alpha-y\cos\alpha=0$ are concurrent and the first 2 lines include an angle $\frac{\pi}{4}$, then a^2+b^2 is equal to _____.

30. In an isosceles triangle ABC with $AB = BC$, combined equation of medians through A and C is $2ax^2-bx-ay^2=0$. If $\left(-\sqrt{2},\sqrt{2}\right)$ is an exterior point of side AC, then slope of AC is _____________.

31. If the line $y=\sqrt{3}x$ cuts the curve $x^3+y^3=xy+2$ at the points A, B and C and if O is the origin, then $[OA.\ OB.\ OC]$ (where [.] denotes the greatest integer function) equals ______________.

32. Area of the figure formed by the pairs of lines $xy+2x+5y+10=0$ and $xy-2x+7y-14=0$ in sq. units is ________.

33. The maximum area of the rectangle, whose sides pass through the angular points of a given rectangle of sides 3 and 1 is ________.

34. The length of the largest altitude of the triangle formed by the straight lines $7x-2y+10=0$, $7x+2y-10=0$ and $y+2=0$ is _______.

35. If $A(1, 2)$ and $B(2,-1)$ are two fixed points and 'P' is a variable point on the line $x + y - 2 = 0$, then maximum value of $|PA - PB|^2$ is ______.

36. One side of a square is along $y=2x-17$. The other two vertices lie on $x^2=y$. There are two such squares. If L and l are the sides of larger and smaller squares, then $\frac{L}{l}$ equals _______.

37. If area of the rhombus enclosed by the line $x\pm 3y\pm 3=0$ is k, then find the value k.

38. The vertex A of ΔABC is $(3,-1)$, the equations of median BE and angular bisector CF are $x-y-2=0$ and $x+\left(3+\sqrt{10}\right)y=0$, respectively then co-ordinates of B are (α, β). Find $\alpha + \beta$.

39. A line passing through $(8,2)$ with negative slope makes intercepts axes at P and Q. Then the minimum value of $OP+OQ$ is 2λ, where λ is _______.

ARCHIVE: JEE MAIN

1. **[JEE MAIN 2020: Jan Attempt]**
The length of the perpendicular from the origin, on the normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at the point (2, 2) is:

(A) $\sqrt{2}$ (B) $4\sqrt{2}$

(C) 2 (D) $2\sqrt{2}$

2. **[JEE MAIN 2020: Jan Attempt]**
The locus of the mid-points of the perpendiculars drawn from points on the line, $x = 2y$ to the line $x = y$ is:

(A) $7x - 5y = 0$ (B) $5x - 7y = 0$

(C) $2x - 3y = 0$ (D) $3x - 2y = 0$

3. **[JEE MAIN 2020: Jan Attempt]**
Let two points be $A(1, -1)$ and $B(0, 2)$. If a point $P(x', y')$ be such that the area of $\Delta PAB = 5$ sq. units and it lies on the line, $3x + y - 4\lambda = 0$, then a value of λ is :

(A) 1 (B) 4

(C) –3 (D) 3

4. **[JEE MAIN 2020: Jan Attempt]**
Let C be the centroid of the triangle with vertices (3, –1), (1, 3) and (2, 4). Let P be the point of intersection of the lines $x + 3y - 1 = 0$ and $3x - y + 1 = 0$. Then the line passing through the points C and P also passes through the point :

(A) $(-9, -7)$ (B) $(9, 7)$
(C) $(7, 6)$ (D) $(-9, -6)$

5. **[JEE Main 2019]**
A point on the straight line $3x + 5y = 15$, which is equidistant from the co-ordinate axes will lie only in:
(A) I^{st}, II^{nd} and IV^{th} quadrants
(B) I^{st} and II^{nd} quadrants
(C) IV^{th} quadrant
(D) I^{st} quadrant

6. **[JEE Main 2019]**
Suppose that the points (h, k), $(1, 2)$ and $(-3, 4)$ lie on the line L_1. If a line L_2 passing through the points (h, k) and $(4, 3)$ is perpendicular to L_1, then $\frac{k}{h}$ equals:
(A) $-\frac{1}{7}$ (B) $\frac{1}{3}$
(C) 0 (D) 3

7. **[JEE Main 2019]**
If the two lines $x + (a - 1)y + 1 = 0$ and $2x + a^2y = 1$ $(a \in R - \{0, 1\})$ are perpendicular, then the distance of their point of intersection from the origin is:
(A) $\frac{2}{5}$ (B) $\sqrt{\frac{2}{5}}$
(C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{2}}{5}$

8. **[JEE Main 2019]**
Slope of a line passing through $P(2, 3)$ and intersecting the line $x + y = 7$ at a distance of 4 units from P is:
(A) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ (B) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$
(C) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ (D) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$

9. **[JEE Main 2019]**
Let the equations of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at $(1, 1)$, then the equation of its 3rd side is:
(A) $26x - 122y - 1675 = 0$
(B) $26x + 61y + 1675 = 0$
(C) $122y - 26x - 1675 = 0$
(D) $122y + 26x + 1675 = 0$

10. **[JEE Main 2019]**
The region represented by $|x - y| \leq 2$ and $|x + y| \leq 2$ is bounded by a:
(A) rhombus of area $8\sqrt{2}$ sq. units
(B) square of side length $2\sqrt{2}$ units
(C) square of side 16 sq. units
(D) rhombus of side length 2 units

11. **[JEE Main 2019]**
If the lines are drawn parallel to the line $4x - 3y + 2 = 0$, at a distance of $\frac{3}{5}$ units from the origin, then which one of the following points lies on any of these lines?
(A) $\left(\frac{1}{4}, -\frac{1}{3}\right)$ (B) $\left(-\frac{1}{4}, \frac{2}{3}\right)$
(C) $\left(\frac{1}{4}, \frac{2}{3}\right)$ (D) $\left(-\frac{1}{4}, -\frac{2}{3}\right)$

12. **[JEE Main 2019]**
A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of ΔPQR is a line:
(A) parallel to y-axis (B) with slope $\frac{2}{3}$
(C) parallel to x-axis (D) with slope $\frac{3}{2}$

13. **[JEE Main 2019]**
If the line $3x + 4y - 24 = 0$ intersects the x-axis at the point A and the y-axis at the point B, then the incentre of the ΔOAB, where O is the origin is:
(A) $(3, 4)$ (B) $(2, 2)$
(C) $(4, 4)$ (D) $(4, 3)$

14. **[JEE Main 2019]**
Two sides of a parallelogram are along the lines, $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at $(2, 4)$, then one of its vertex is:
(A) $(2, 1)$ (B) $(2, 6)$
(C) $(3, 5)$ (D) $(3, 6)$

15. **[JEE Main 2019]**
If in a parallelogram $ABDC$, the co-ordinates of A, B and C are respectively $(1, 2)$, $(3, 4)$ and $(2, 5)$, then the equation of the diagonal AD is:
(A) $5x + 3y - 11 = 0$ (B) $5x - 3y + 1 = 0$
(C) $3x - 5y + 7 = 0$ (D) $3x + 5y - 13 = 0$

16. **[JEE Main 2019]**
If the straight line, $2x - 3y + 17 = 0$ is perpendicular to the line passing through the points $(7, 17)$ and $(15, \beta)$, then β equals:
(A) $\frac{35}{3}$ (B) -5
(C) $-\frac{35}{3}$ (D) 5

17. [JEE Main 2019]
If a straight line passing through the point $P(-3, 4)$ is such that its intercepted portion between the co-ordinate axes is bisected at P, then its equation is:

(A) $x - y + 7 = 0$ (B) $4x - 3y + 24 = 0$
(C) $4x + 3y = 0$ (D) $3x - 4y + 25 = 0$

18. [JEE Main 2019]
If a straight line L at a distance of 4 units from the origin makes positive intercepts on the co-ordinate axes and the perpendicular from the origin to this line makes an angle of 60° with the line $x + y = 0$, then an equation of the line L is:

(A) $x + \sqrt{3}y = 8$

(B) $\sqrt{3}x + y = 8$

(C) $(\sqrt{3} + 1)x + (\sqrt{3} - 1)y = 8\sqrt{2}$

(D) $(\sqrt{3} - 1)x + (\sqrt{3} + 1)y = 8\sqrt{2}$

19. [JEE Main 2019]
Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true?

(A) The lines are not concurrent
(B) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$
(C) The lines are all parallel
(D) Each line passes through the origin

20. [JEE Main 2018]
A straight line through a fixed point (2, 3) intersects the co-ordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is:

(A) $2x + 3y = xy$ (B) $3x + 2y = xy$
(C) $3x + 2y = 6xy$ (D) $3x + 2y = 6$

21. [JEE Main 2016]
Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?

(A) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ (B) $(-3, -9)$

(C) $(-3, -8)$ (D) $\left(\frac{1}{3}, -\frac{8}{3}\right)$

22. [JEE Main 2014]
Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the IVth quadrant and it is equidistant from the two axes, then:

(A) $2bc - 3ad = 0$ (B) $2bc + 3ad = 0$
(C) $3bc - 2ad = 0$ (D) $3bc + 2ad = 0$

23. [JEE Main 2014]
Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is:

(A) $4x - 7y - 11 = 0$ (B) $2x + 9y + 7 = 0$
(C) $4x + 7y + 3 = 0$ (D) $2x - 9y - 11 = 0$

24. [JEE Main 2013]
A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is:

(A) $y = x + \sqrt{3}$ (B) $\sqrt{3}y = x - \sqrt{3}$

(C) $y = \sqrt{3}x - \sqrt{3}$ (D) $\sqrt{3}y = x - 1$

25. [JEE Main 2013]
The x-co-ordinate of the incentre of the triangle that has the co-ordinates of midpoints of its sides as (0, 1) (1, 1) and (1, 0) is:

(A) $2 + \sqrt{2}$ (B) $2 - \sqrt{2}$

(C) $1 + \sqrt{2}$ (D) $1 - \sqrt{2}$

26. [AIEEE 2012]
A line is drawn through the point (1, 2) to meet the co-ordinate axes at P and Q such that it forms a ΔOPQ, where O is the origin. If the area of the ΔOPQ is least, then the slope of the line PQ is:

(A) $-\frac{1}{2}$ (B) $-\frac{1}{4}$

(C) -4 (D) -2

27. [AIEEE 2011]
If the lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the Ist quadrant, then the set of all possible values of a in the interval is:

(A) $(-1, 1)$ (B) $(0, \infty)$
(C) $(1, \infty)$ (D) $(-1, \infty)$

28. [AIEEE 2010]
The line L is given by $\frac{x}{5} + \frac{y}{b} = 1$ and it passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is:

(A) $\frac{23}{\sqrt{15}}$ (B) $\sqrt{17}$

(C) $\frac{17}{\sqrt{15}}$ (D) $\frac{23}{\sqrt{17}}$

29. [AIEEE 2009]

The lines $p(p^2 + 1)\,x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for:

(A) exactly two values of p
(B) more than two values of p
(C) no value of p
(D) exactly one value of p

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE 2011]

A straight line L through the point (3, –2) is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, then the equation of L is:

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

2. [JEE 2009]

The locus of the orthocenter of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$ is:

(A) a hyperbola (B) a parabola
(C) an ellipse (D) a straight line

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. [JEE Advanced 2013]

For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$, then:

(A) $a + b - c > 0$ (B) $a - b + c < 0$
(C) $a - b + c > 0$ (D) $a + b - c < 0$

Reasoning Based Questions

This section contains Reasoning type questions, each having four choices (A), (B), (C) and (D) out of which ONLY ONE is correct. Each question contains STATEMENT 1 and STATEMENT 2. You have to mark your answer as

Bubble (A) If both statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

Bubble (B) If both statements are TRUE but STATEMENT 2 is not the correct explanation of STATEMENT 1.

Bubble (C) If STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

Bubble (D) If STATEMENT 1 is FALSE but STATEMENT 2 is TRUE.

1. [IIT JEE 2007, AIEEE 2011]

Lines L_1: $y - x = 0$ and L_2: $2x + y = 0$ intersect the line L_3: $y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

Statement 1: The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$

Because

Statement 2: In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement 1 is true, Statement 2 is true and Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is true, Statement 2 is true and Statement 2 is not a correct explanation for Statement 1.
(C) Statement 1 is true and Statement 2 is false.
(D) Statement 1 is false and Statement 2 is true.

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. [JEE 2008]

Consider the lines given by

$L_1 = x + 3y - 5 = 0$
$L_2 = 3x - ky - 1 = 0$
$L_3 = 5x + 2y - 12 = 0$

Match the statements or expressions in **COLUMN-I** with the statements or expressions in **COLUMN-II** and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in OMR.

COLUMN-I	COLUMN-II
(A) L_1, L_2, L_3 are concurrent, if	(P) $k = -9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(Q) $k = -\frac{6}{5}$
(C) L_1, L_2, L_3 form a triangle, if	(R) $k = \frac{5}{6}$
(D) L_1, L_2, L_3 do not form a triangle, if	(S) $k = 5$

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [JEE Advanced 2014]
For a point P in the plane, let $d_1(P)$ be the distance of the point (P) from the lines $x - y = 0$ and $x + y = 0$, respectively. The area of the region R consisting of all points P lying in the Ist quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$ is:

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	2. A	3. B	4. B	5. D	6. A	7. D	8. B	9. C	10. B
11. D	12. C	13. C	14. C	15. A	16. D	17. D	18. C	19. B	20. C
21. B	22. A	23. C	24. B	25. C	26. B	27. B	28. C	29. B	30. D
31. B	32. C	33. C	34. B	35. C	36. D	37. B	38. A	39. B	40. B
41. D	42. A	43. A	44. B	45. A	46. B	47. A	48. C	49. B	50. D
51. C	52. B	53. C	54. C	55. B	56. A	57. A	58. A	59. A	60. B
61. D	62. B	63. C	64. C	65. A	66. A	67. A	68. D	69. B	70. C
71. A	72. B	73. A	74. C	75. C	76. D	77. C	78. C	79. A	80. C
81. D	82. A	83. C	84. C	85. D	86. C	87. D	88. A	89. B	90. C
91. B	92. B	93. B	94. D	95. C	96. A	97. A	98. C	99. A	100. B
101. B	102. A	103. B	104. A	105. A	106. B	107. C	108. C		

Multiple Correct Choice Type Questions

1. A, B, C	2. C, D	3. A, B, C	4. A, C, D	5. A, B, D
6. B, D	7. B	8. A, B	9. A, C	10. A, B, C, D
11. A, D	12. A, B	13. A, B	14. A, B, C, D	15. A, C
16. A, D	17. A, D	18. A, B, C, D	19. A, C, D	20. A, C
21. B, C	22. A, B	23. A, D	24. A, B, C, D	25. A, C
26. A, D	27. A, D	28. A, B, C	29. B, C	30. A, B
31. A, C	32. B, C	33. A, D	34. B, D	35. A, C
36. A, C	37. A, B	38. A, B, C, D	39. A, B, C, D	40. A, B
41. A, C	42. A, B, C	43. A, B, D	44. A, B	45. A, B
46. A, B	47. B, C	48. A, C	49. A	50. B, C
51. D	52. A, C			

Linked Comprehension Type Questions

1. A	2. C	3. B	4. B	5. A	6. C	7. B	8. A	9. B	10. B
11. A	12. C	13. B	14. D	15. C	16. D	17. C	18. C	19. D	20. B
21. C	22. D	23. C	24. B	25. D	26. A	27. C	28. D	29. D	30. B
31. A	32. A	33. C	34. B	35. D	36. C	37. C	38. C	39. A	40. A
41. B	42. A	43. C	44. C	45. B	46. C	47. B	48. B	49. A	50. C
51. B	52. C	53. C	54. B	55. C	56. A	57. B	58. C	59. C	60. B
61. A	62. A								

Matrix Match/Column Match Type Questions

1. A → (r)	B → (p)	C → (q)	D → (p)
2. A → (p, q, r)	B → (q, r)	C → (s)	D → (q, r)
3. A → (p, q)	B → (s)	C → (t)	D → (t)
4. A → (p, q)	B → (p)	C → (p)	D → (t)
5. A → (s)	B → (p, q)	C → (r, s)	D → (p)
6. A → (p)	B → (s)	C → (q)	D → (r)
7. A → (s)	B → (r)	C → (p, q)	D → (p, q, r, s)
8. A → (p, t)	B → (q)	C → (s)	D → (r)
9. A → (r)	B → (q)	C → (s)	D → (p)
10. A → (q)	B → (p, q, r)	C → (r)	D → (p, s)
11. A → (t)	B → (s)	C → (p)	D → (q)
12. A → (r)	B → (p)	C → (q)	D → (s)

Integer/Numerical Answer Type Questions

1. 2	**2.** 4	**3.** 7	**4.** 7	**5.** 7
6. 3	**7.** 0	**8.** 1	**9.** 7	**10.** 0
11. 1	**12.** 2	**13.** 4	**14.** 8	**15.** 2
16. 2	**17.** 3	**18.** 4	**19.** 2	**20.** 5
21. 4	**22.** 4	**23.** 5	**24.** 1	**25.** 2
26. 7	**27.** 5	**28.** 5	**29.** 2	**30.** 0
31. 2	**32.** 8	**33.** 8	**34.** 7	**35.** 8
36. 4	**37.** 6	**38.** 2	**39.** 9	

ARCHIVE: JEE MAIN

1. D	**2.** B	**3.** D	**4.** B	**5.** B	**6.** B	**7.** B	**8.** C	**9.** A	**10.** B
11. B	**12.** B	**13.** B	**14.** D	**15.** B	**16.** D	**17.** B	**18.** C, D	**19.** B	**20.** B
21. D	**22.** C	**23.** B	**24.** B	**25.** B	**26.** D	**27.** C	**28.** D	**29.** D	

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. B	**2.** D

Multiple Correct Choice Type Problems

1. A, C

Reasoning Based Questions

1. C

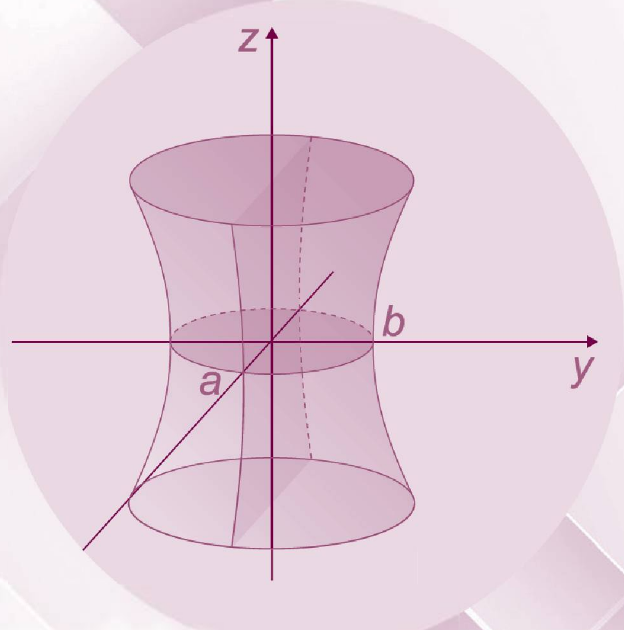
Matrix Match/Column Match Type Questions

1. A → (s) B → (p, q) C → (r) D → (p, q, s)

Integer/Numerical Answer Type Questions

1. 6

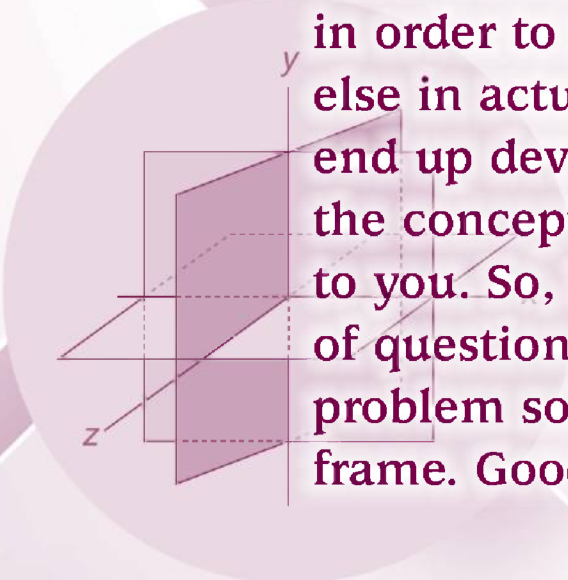
From the Author's Desk

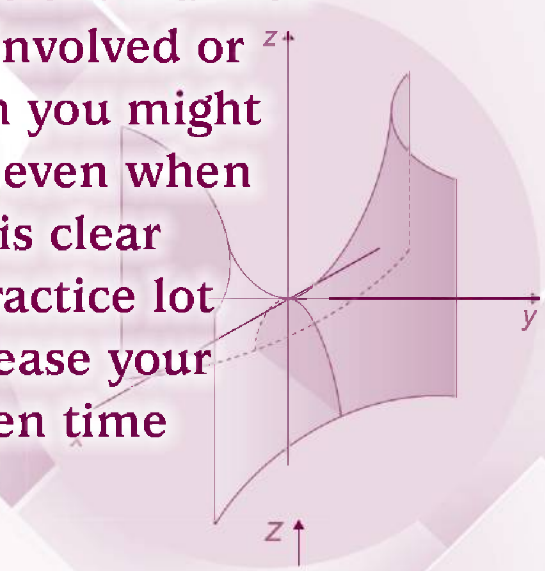


“Dear Learners,

Let’s talk the third and one of the most important chapter of Coordinate Geometry, that is Circle. In Circle, main concepts are related to tangent, normal, chord of contact, relative position of two circles, common tangents, radical axis and family of circles. Circle may be related to the other chapters like Parabola/Ellipse/Hyperbola.

To help you score more in the topic of Circle concept plus calculation part should be handled in an efficient manner in order to reduce the time involved or else in actual exam situation you might end up devoting more time, even when the concept of the question is clear to you. So, I advise you to practice lot of questions in order to increase your problem solving skills in given time frame. Good luck!!”





CHAPTER

3 Circle

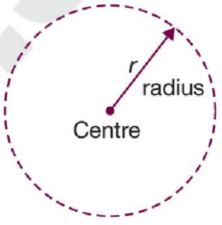
Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Equation of circle and geometrical properties
(b) Position of point and line
(c) Tangent, Normal, chord of contact
(d) Director circle, relative position of two circles, common tangents
(e) Radical axis, common chord, orthogonal circles and family of circles, application of point circle.

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

DEFINITION OF CIRCLE

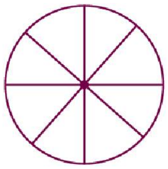


Circle

Locus of a point, which moves in a plane such that its distance from a fixed point is constant is known as circle.

The fixed point is centre and constant distance is radius.

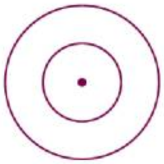
Centre



A point, which bisects each chord passing through it is the centre of that curve (definition of centre valid for all curves).

Geometrical Properties of a Circle

(i) Concentric circles: Circles having same centre.



(ii) Congruent circles: Iff their radii are equal.

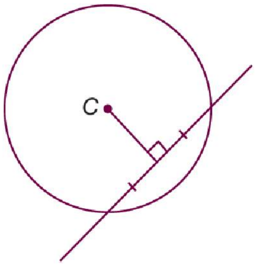
(iii) Congruent arcs: Iff they have same angles at the centre.

(iv) There is unique circle passing through three non-collinear points.

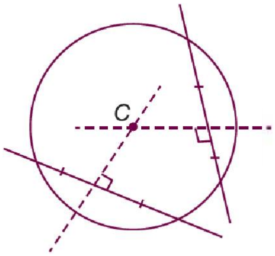
(v) Equal chords of a circle are equidistant from the centre.

(vi) Equal chords of a circle subtend equal angles at the centre.

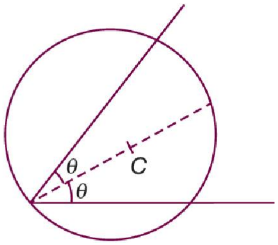
(vii) The perpendicular drawn from the centre of a circle to a chord bisects the chord.



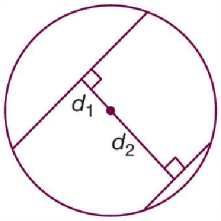
(viii) Perpendicular bisectors of two chords of a circle intersect at its centre.



(ix) If two equal chords are drawn from a point on the circle, then the centre of the circle will lie on angle bisector of these two chords.

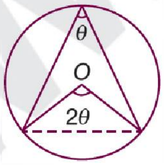


(x) Of any two chords of a circle, the larger will be near to centre.

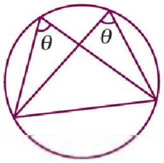


$(d_1 < d_2)$

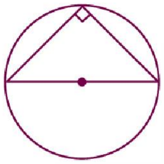
(xi) Angle subtended by a chord at the centre is double the angle subtended by it at any point on circumference.



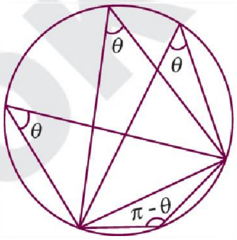
(xii) Angle subtended by a chord on same side of circumference is equal.



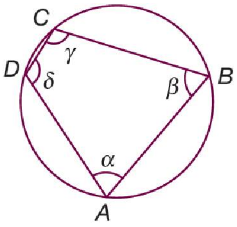
(xiii) Every diameter of a circle subtends a right angle at every point on circumference.



(xiv) Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse ($\theta \to$ acute and $\pi - \theta \to$ obtuse).



CYCLIC QUADRILATERAL

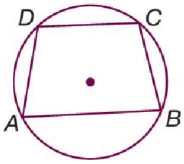


A quadrilateral is called a cyclic quadrilateral, if all of its vertices lie on a circle.

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°, in other words we can say that the opposite angles of a cyclic quadrilateral are supplementary.

$$\alpha + \gamma = \beta + \delta = \pi.$$

If two sides of a cyclic quadrilateral are parallel, then the remaining two sides are equal and the diagonals are also equal. In other words, an **isosceles trapezium** is always cyclic.



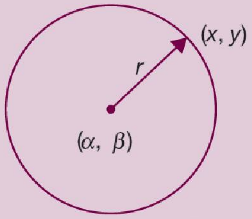
$AB \parallel CD$

$AD = BC$

$ABCD$ is isosceles trapezium.

EQUATION OF CIRCLE

If (α, β) is the centre and r is the radius of the circle, any variable point (x, y) on circle then its equation becomes $(x-\alpha)^2 + (y-\beta)^2 = r^2$



The general quadratic equation in x and y is

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

For this equation to represent a circle, we should compare it with circle.

Circle equation $(x-\alpha)^2 + (y-\beta)^2 = r^2$

$$\Rightarrow \quad x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0$$

Compare it with general two degree equation

$$\frac{1}{a} = \frac{1}{b} = \frac{0}{h} = \frac{-\alpha}{g} = \frac{-\beta}{f} = \frac{\alpha^2 + \beta^2 - r^2}{c}$$

We get,

$$a = b,\ h = 0,\ a\alpha = -g,\ b\beta = -f,\ c = a(\alpha^2 + \beta^2 - r^2)$$

Let $a = b = 1$, we get,

$$\alpha = -g,\ \beta = -f,\ c = (\alpha^2 + \beta^2 - r^2)$$

$$\Rightarrow \quad r^2 = g^2 + f^2 - c \geq 0$$

The general quadratic equation in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle, if

- **(i)** Coefficient of x^2 = Coefficient of y^2 or $a = b \neq 0$
- **(ii)** Coefficient of $xy = 0$ or $h = 0$
- **(iii)** $(g^2 + f^2 - c) \geq 0$ (for a real circle)

GENERAL EQUATION OF CIRCLE

The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f and c are constants.

Centre is $(-g, -f)$ or

$$\left(-\frac{\text{Coeffecient of } x}{2}, -\frac{\text{Coeffecient of } y}{2}\right)$$

radius $r = \sqrt{g^2 + f^2 - c}$

NOTE:

- **(i)** If $g^2 + f^2 - c > 0$, then r is real and positive and the circle is a real circle.
- **(ii)** If $g^2 + f^2 - c = 0$, then radius $r = 0$ and circle becomes a **point circle**.
- **(iii)** If $g^2 + f^2 - c < 0$, then r is imaginary.
  $\therefore$ No real circle possible.
- **(iv)** $x^2 + y^2 + 2gx + 2fy + c = 0$ has three constants and to get the equation of the circle at least three conditions should be known.
- **(v)** If centre is origin (0, 0) and radius is 'r', then equation of circle is $x^2 + y^2 = r^2$ and this is called the standard form.

POINT CIRCLE

The equation $(x-\alpha)^2 + (y-\beta)^2 = 0$ has its locus only at the point (α, β). Such a locus is called a 'point circle' with centre (α, β) and radius equal to zero.

Working Rule

It is the best practice to always make coefficient of x^2 and y^2 as 1 in equation of circle in each problem throughout the chapter.

ILLUSTRATION 1

For what value(s) of λ, the equation $(10 - \lambda^2)x^2 + (\lambda^2 - 8)y^2 + (3 - \lambda)xy - 10x + 4y + 3 = 0$ represent a real circle.

SOLUTION

We must have, coefficient of x^2 = coefficient of y^2

$$\Rightarrow \quad 10 - \lambda^2 = \lambda^2 - 8 \Rightarrow \lambda^2 = 9$$

$$\Rightarrow \quad \lambda = \pm 3 \qquad \ldots(1)$$

Coefficient of $xy = 0 \Rightarrow 3 - \lambda = 0$

$$\Rightarrow \quad \lambda = 3 \qquad \ldots(2)$$

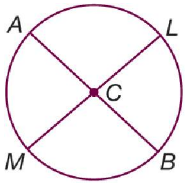
So, the common value from relation (1) and (2) $\lambda = 3$.

ILLUSTRATION 2

If the equations of the two diameters of a circle are $x + y = 6$ and $x + 2y = 4$ and the radius of the circle is 10, then find the equation of the circle.

SOLUTION

Here, radius of the circle = 10



Any two diameters intersects at centre.
Equations of two diameters, say AB and ML of the circle are respectively,

$$x+y=6 \qquad \ldots(1)$$

$$x+2y=4 \qquad \ldots(2)$$

Solving (1) and (2), we get

$$x=8 \text{ and } y=-2$$

Hence, the centre of the circle is $(8, -2)$.
Now, the equation of the required circle is,

$$(x-8)^2+(y+2)^2=10^2 \text{ or } x^2+y^2-16x+4y-32=0$$

ILLUSTRATION 3

Prove that the radii of the circles $x^2+y^2=1$, $x^2+y^2-2x-6y=6$ and $x^2+y^2-4x-12y=9$ are in A.P.

SOLUTION

Given circles are

$$x^2+y^2=1^2 \qquad \ldots(1)$$

$$x^2+y^2-2x-6y-6=0 \qquad \ldots(2)$$

$$x^2+y^2-4x-12y-9=0 \qquad \ldots(3)$$

Let r_1, r_2 and r_3 be the radii of the circles (1), (2) and (3), respectively.

Then, $r_1=1$, $r_2=\sqrt{(-1)^2+(-3)^2+6}=4$ and

$$r_3=\sqrt{(-2)^2+(-6)^2+9}=7$$

Clearly, $r_2-r_1=r_3-r_2 \Rightarrow 2r_2=r_1+r_3=8$
Hence, r_1, r_2 and r_3 are in A.P.

ILLUSTRATION 4

Find the equation of circle, which passes through the points $(1, -2)$ and $(-2, 0)$, whose centre lies on the straight line $x-y=5$.

SOLUTION

Let the circle be $x^2+y^2+2gx+2fy+c=0$
Point $(1, -2)$ lies on it, we get $2g-4f+c+5=0$
Point $(-2, 0)$ lies on it, we get $-4g+c+4=0$
Centre $(-g, -f)$ lies on $x-y=5$, $-g+f=5$,
On solving these three relations we get,

$$g=\frac{19}{2}, f=\frac{29}{2}, c=34$$

$$\therefore \quad x^2+y^2+19x+29y+34=0$$

ILLUSTRATION 5

Find the equation of mirror image of circle $x^2+y^2-4x+2y+4=0$ with respect to line $x-y+4=0$.

SOLUTION

Given circle $C(2, -1)$ $r=1$ mirror image of a circle with respect to a line is a circle whose radius is same and centre is image of given centre relative to line.

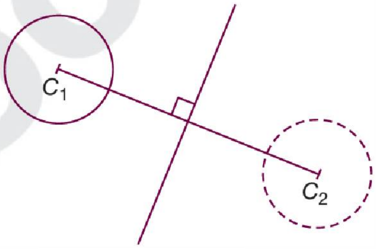


Image of centre $C_1(2, -1)$ relative to line $x-y+4=0$

$$\frac{x-2}{1}=\frac{y+1}{-1}=-\frac{2(2+1+4)}{1+1}=-7$$

$C_2(-5, 6)$; $r_2=r_1=1$, where the radius of both circles are same.
$\therefore$ The required circle is $(x+5)^2+(y-6)^2=1$.

ILLUSTRATION 6

Suppose $f(x, y)=0$ is the equation of the circle such that $f(x, 1)=0$ has equal roots (each equal to 2) and $f(1, x)=0$ also has equal roots (each equal to 0), then find the equation of the circle.

SOLUTION

Let, $f(x, y)=x^2+y^2+2gx+2fy+c$

$$\Rightarrow \quad f(x, 1)=x^2+1+2gx+2f+c=(x-2)^2 \qquad \text{(given)}$$

$$\Rightarrow \quad g=-2, 2f+c+1=4 \qquad \ldots(1)$$

Also, $f(1, x)=1+x^2+2g+2fx+c=x^2$ (given)

$$\Rightarrow \quad f=0, 2g+c+1=0 \qquad \ldots(2)$$

Solving, we get $c=3$

Thus, the equation of circle is $x^2+y^2-4x+3=0$.

ILLUSTRATION 7

Find the equation of the circle, which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as c tends to 1.

SOLUTION

Let $A = (2, 0)$

Given lines are

$$3x + 5y = 1 \qquad \ldots(1)$$

$$(2 + c)x + 5c^2y = 1 \qquad \ldots(2)$$

Multiplying equation (1) by c^2 and subtracting (2) from it, we get

$$(3c^2 - c - 2)\,x = c^2 - 1 \text{ or } x = \frac{c^2-1}{3c^2-c-2}$$

$$\text{Now, } \lim_{c\to1} x = \lim_{c\to1} \frac{(c-1)(c+1)}{(c-1)(3c+2)} = \lim_{c\to1} \frac{c+1}{3c+2} = \frac{2}{5}$$

$\therefore$ x co-ordinate of centre $= \dfrac{2}{5}$

From (1), when $x = \dfrac{2}{5}, y = -\dfrac{1}{25}$

Hence, centre of the circle is $\left(\dfrac{2}{5}, -\dfrac{1}{25}\right)$

Also the circle passes through the point $A(2, 0)$

$\therefore$ Radius of the circle $= \sqrt{\left(2-\dfrac{2}{5}\right)^2\left(0+\dfrac{1}{25}\right)^2}$

Now, the equation of the required circle is

$$\left(x-\frac{2}{5}\right)^2 + \left(y+\frac{1}{25}\right)^2 = \frac{64}{25} + \frac{1}{625}$$

$$\Rightarrow \quad x^2 - \frac{4}{5}x + \frac{4}{25} + y^2 + \frac{2}{25}y + \frac{1}{625} = \frac{64}{25} + \frac{1}{625}$$

$$\Rightarrow \quad 25(x^2 + y^2) - 20x + 2y - 60 = 0$$

INTERCEPTS OF CIRCLE ON CO-ORDINATE AXES

Consider the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$

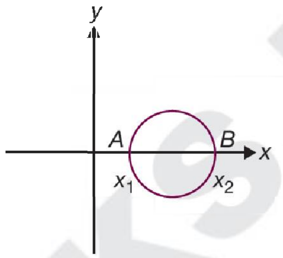
(a) The length of intercept on x-axis (AB) $= 2\sqrt{g^2 - c}$

PROOF:

Circle $x^2 + y^2 + 2gx + 2fy + c = 0$

On x-axis, $y = 0$ and $x^2 + 2gx + c = 0$

$$x_1 + x_2 = -2g;\ x_1x_2 = c$$



$$\text{Now, } AB = |x_1 - x_2| = \sqrt{(x_1+x_2)^2 - 4x_1x_2}$$

$$AB = \sqrt{4g^2 - 4c} = 2\sqrt{g^2 - c}$$

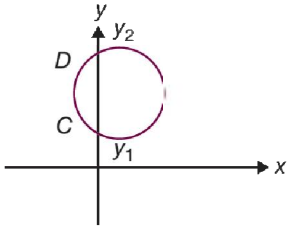
(b) The length of intercept of y-axis (CD) $= 2\sqrt{f^2 - c}$

PROOF:

Circle $x^2 + y^2 + 2gx + 2fy + c = 0$

On y-axis, $x = 0$ and $y^2 + 2fy + c = 0$

$$y_1 + y_2 = -2f;\ y_1y_2 = c$$



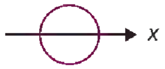
$$\text{Now, } CD = |y_1 - y_2| = \sqrt{(y_1+y_2)^2 - 4y_1y_2}$$

$$CD = \sqrt{4f^2 - 4c} = 2\sqrt{f^2 - c}$$

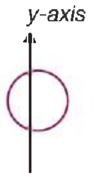
Working Rule

CIRCLE RELATIVE POSITION WITH CO-ORDINATE AXES

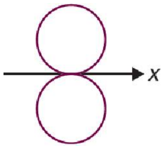
(i) Circle cuts x-axis, if $g^2 - c > 0$



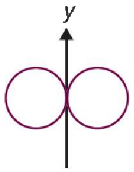
(ii) Circle cuts y-axis, if $f^2 - c > 0$



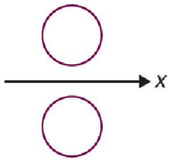
(iii) Circle touches x-axis, if $g^2 - c = 0$



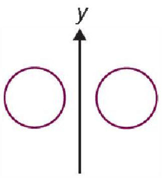
(iv) Circle touches y-axis, if $f^2 - c = 0$



(v) Circle does not cut x-axis, if $g^2 - c < 0$



(vi) Circle does not cut y-axis, if $f^2 - c < 0$



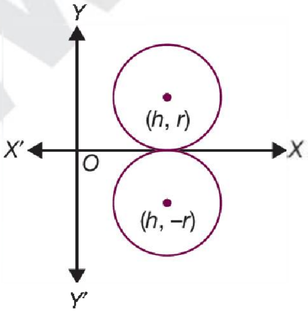
Let us discuss the equation of circle in detail, when it touches co-ordinate axes.

(i) When the circle touches x-axis, then the equation of the circle is

$$(x-h)^2 + (y-k)^2 = k^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky + h^2 = 0$$

Where radius $r = |k|$, $(g^2 - c = 0)$

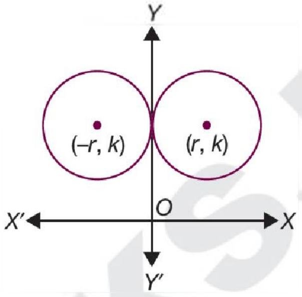


(ii) When the circle touches y-axis, then the equation of circle is

$$(x-h)^2 + (y-k)^2 = h^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky + k^2 = 0$$

Where radius $r = |h|$, $(f^2 - c = 0)$



(iii) When the circle touches both the axes (x-axis and y-axis), then the equation of circle is

$$(x-h)^2 + (y-h)^2 = h^2 \text{ or } x^2 + y^2 - 2hx - 2hy + h^2 = 0$$

Where radius $r = |h| = |k|$, $(g^2 - c = 0; f^2 - c = 0)$

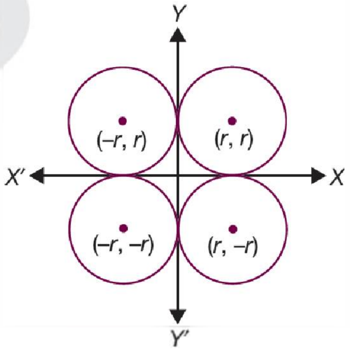


ILLUSTRATION 8

The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect x-axis but cuts y-axis at two distinct points. Find the ranges of the values of k.

SOLUTION

The given circle is $x^2 + y^2 - 6x - 10y + k = 0$

For x-axis, $g^2 - c < 0 \Rightarrow 9 - k < 0 \Rightarrow k > 9$

For y-axis, $f^2 - c > 0 \Rightarrow 25 - k > 0 \Rightarrow k < 25$

Hence, the conditions is $9 < k < 25$

ILLUSTRATION 9

Find the equation of circle touching y-axis at a distance –3 from origin and intercepting a length of 8 units on x-axis.

SOLUTION

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. It touches y-axis at $f^2 = c$

It passes through (0, –3).

$\therefore$ we get $9 - 6f + c = 0$

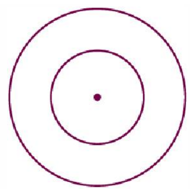
$$(f-3)^2 = 0 \Rightarrow f = 3, c = 9$$

Length of x-intercept $= 2\sqrt{g^2 - c} = 8 \Rightarrow g = \pm 5$

$\therefore$ The required circle is $x^2 + y^2 \pm 10x + 6y + 9 = 0$

EQUATION OF CONCENTRIC CIRCLE

The equation of any circle concentric (having the same centre) with the given circle is



$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$x^2 + y^2 + 2gx + 2fy + \lambda = 0$$

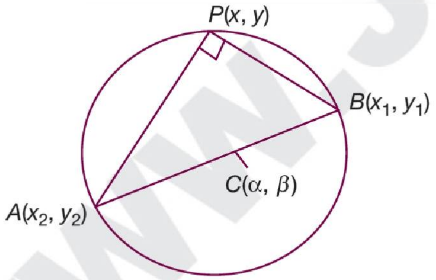
where, λ is any real constant.

Circle equation $(x - \alpha)^2 + (y - \beta)^2 = r^2$, where r is a variable then it represent all concentric circles with centre (α, β).

CIRCLE IN DIAMETER FORM

Every diameter of a circle subtends right angle at circumference.

Let AB is a diameter and point P is on circle, then AP is perpendicular to PB



$$\therefore \quad M_{PA} M_{PB} = -1 \Rightarrow \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

Equation of circle in diameter form,

$$\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$$

- Infinite circles can pass through from two points.
- A circle passing through two points with shortest radius is a circle, whose given points are extremities of diameter.
- A unique circle passes through from three non-collinear points, it is also known as circumcircle of triangle formed by these points and circumcentre is centre of this circle.

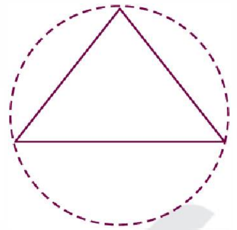
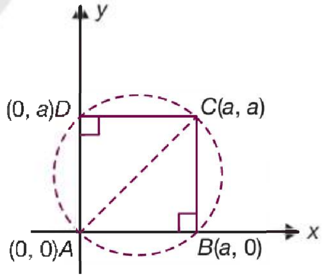


ILLUSTRATION 10

$ABCD$ is a square, whose side is 'a' taking AB and AD as xy-co-ordinate axes, respectively. Prove that the equation of circle circumscribing the square is $x^2 + y^2 = a(x + y)$.

SOLUTION

By taking A at origin and AB on positive x-axis



Clearly, $A(0, 0)$ and $C(a, a)$ is the diameter of circle. Hence, its equation is

$$(x - 0)(x - a) + (y - 0)(y - a) = 0$$

$$\Rightarrow \quad x^2 + y^2 - ax - ay = 0$$

ILLUSTRATION 11

The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and the ordinates are the roots of the equation $y^2 + 2py - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter.

SOLUTION

Given equations are

$$x^2 + 2ax - b^2 = 0 \qquad \ldots(1)$$

$$y^2 + 2py - q^2 = 0 \qquad \ldots(2)$$

Let the roots of equation (1) be α and β and those of equation (2) be γ and δ, then

$$\alpha + \beta = -2a; \ \alpha\beta = -b^2$$

and $\gamma + \delta = -2p; \ \gamma\delta = -q^2$

Let, $A = (\alpha, \gamma)$ and $B = (\beta, \delta)$ be the points

Now, the equation of the circle, whose diameter is AB will be

$$(x-\alpha)(x-\beta)+(y-\gamma)(y-\delta)=0$$

$$\Rightarrow \quad x^2+y^2-(\alpha+\beta)x-(\gamma+\delta)y+a\beta+\gamma\delta=0$$

$$\Rightarrow \quad x^2+y^2+2ax+2py-(b^2+q^2)=0$$

Centre of circle is $(-a, -p)$ and

$$\text{radius} = \sqrt{(-a)^2+(-p)^2+b^2+q^2} = \sqrt{a^2+b^2+p^2+q^2}$$

ILLUSTRATION 12

If $f(x+y)=f(x).f(y)$ for all x and y, $f(1)=2$ and $\alpha_n = f(n), n \in N$, then the find equation of the circle having (α_1, α_2) and (α_3, α_4) as the ends of its one diameter.

SOLUTION

$$f(x+y)=f(x)\cdot f(y) \quad \because f(1)=2$$

Put $x=y=1 \Rightarrow f(2)=2^2 \Rightarrow f(n)=2^n=\alpha_n$

Now, extremities of the diameter (α_1, α_2) and (α_3, α_4) are (2, 4) and (8, 16), respectively.

Hence, the required circle is

$$(x-2)(x-8)+(y-4)(y-16)=0$$

ILLUSTRATION 13

One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are (–3, 4) and (5, 4), then find the area of the rectangle.

SOLUTION

Let $O(h, k)$ be the centre of the circle then $OA^2 = OB^2$

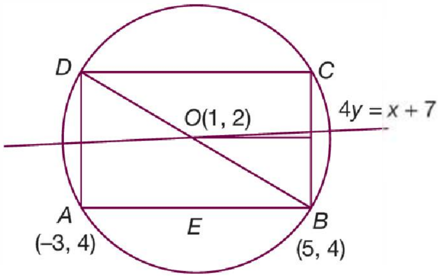
$$\Rightarrow \quad (h+3)^2=(h-5)^2$$

$$\Rightarrow \quad h^2+6h+9=h^2-10h+25$$

$$\Rightarrow \quad 16h=16 \Rightarrow h=1$$

Centre $O(h, k)$ also passes through diameter

$$4y=x+7 \Rightarrow 4k=h+7=8 \Rightarrow k=2$$



Hence, centre O will be (1, 2).

Let, E be the midpoint of AB. Then E is (1, 4)

$\therefore \quad OE = 2; BC = 2OE = 4$

Now $AB = 8$ and $BC = 4$

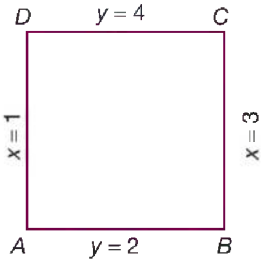
$\Rightarrow \quad$ area of rectangle $ABCD = 4 \times 8 = 32$ sq. units

ILLUSTRATION 14

The sides of a square are $x = 1$, $x = 3$, $y = 2$ and $y = 4$. Find the equation of the circle drawn on the diagonals of the square as its diameter.

SOLUTION

Let $ABCD$ be a square and equation of its sides AD, BC, AB and DC be respectively $x = 1$, $x = 3$, $y = 2$ and $y = 4$. Then $A = (1, 2)$, $B = (3, 2)$, $C = (3, 4)$ and $D = (1, 4)$.



Now the equation of the circle having AC as a diameter is

$$(x-1)(x-3)+(y-2)(y-4)=0$$

$$\Rightarrow \quad x^2+y^2-4x-6y+11=0 \qquad \ldots(1)$$

$\therefore$ Equation (1) is also the equation of the circle having BD as its diameter. In other words, this circle circumscribles given square.

ILLUSTRATION 15

If a variable circle passes through the fixed point $A(p, q)$ and touches x-axis, then find the locus of the other end of the diameter through A.

SOLUTION

Let, extremities of diameter are $A(p, q)$ and $B(a, b)$.

Now, equation of circle is

$$(x-p)(x-a)+(y-q)(y-b)=0$$

$$\Rightarrow \quad x^2+y^2-(p+a)x-(q+b)y+pa+qb=0$$

It touches the x-axis, then $g^2 = c$

We get, $\left(\frac{p+a}{2}\right)^2 = pa+qb$

$$a^2 + p^2 = 2ap + 4qb$$

Locus of point (a, b) replace $a = x$, $b = y$

$$(x - p)^2 = 4qy$$

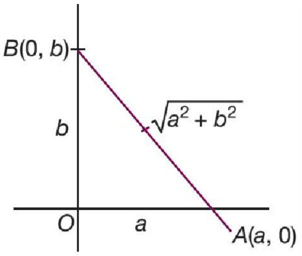
ILLUSTRATION 16

The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle, which has two of its sides along the co-ordinate axes. If the locus of the circumcentre of the triangle is $x + y - xy + k\sqrt{x^2 + y^2} = 0$, find k.

SOLUTION

Given circle is incircle of triangle and lies in Ist quadrant let $A(a, 0)$, $B(0, b)$

Circumcentre of $\Delta OAB = \left(\frac{a}{2}, \frac{b}{2}\right) = (h, k)$



In centre of ΔOAB

$$\left(\frac{(b)(a)+(a)(0)+\sqrt{a^2+b^2}(0)}{a+b+\sqrt{a^2+b^2}}, \frac{(b)(0)+(a)(b)+(0)\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}}\right) = (2, 2)$$

$$\Rightarrow \quad \frac{ab}{a+b+\sqrt{a^2+b^2}} = 2.$$

Now, put $a = 2h$, $b = 2k$

Hence, locus of circumcentre is

$$x + y - xy + \sqrt{x^2 + y^2} = 0 \Rightarrow k = 1$$

ILLUSTRATION 17

$P\left(1, 2\sqrt{2}\right)$ is a point on circle $x^2 + y^2 = 9$, find the points on the given circle, which are at 2 unit distance from the given point P.

SOLUTION

Required point $A(x_1, y_1)$ lies on the circle $x_1^2 + y_1^2 = 9$

Now, at 2 unit distance from point P

Equation of line PA in parametric form

$$\frac{x_1 - 1}{\cos\theta} = \frac{y_1 - 2\sqrt{2}}{\sin\theta} = r;$$

Point $A\left(1 + r\cos\theta,\ 2\sqrt{2} + r\sin\theta\right) = (x_1, y_1)$

It lies on given circle,

$$\Rightarrow \quad x_1^2 + y_1^2 = 9 \Rightarrow r^2 + 1 + 2r\cos\theta + 8 + 4\sqrt{2}r\sin\theta = 9$$

$$\Rightarrow \quad r^2 + 2r\left(\cos\theta + 2\sqrt{2}\sin\theta\right) = 0$$

$$r = -2\left(\cos\theta + 2\sqrt{2}\sin\theta\right)$$

$$r = \pm 2 = -2\left(\cos\theta + 2\sqrt{2}\sin\theta\right)$$

$$r = 2; \cos\theta = -1 \quad \text{Point}\left(-1, 2\sqrt{2}\right)$$

$$r = -2; \cos\theta = \frac{-7}{9} \quad \text{Point}\left(\frac{23}{9}, \frac{10\sqrt{2}}{9}\right)$$

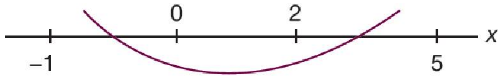
$\therefore$ The points are $\left(-1, 2\sqrt{2}\right), \left(\frac{23}{9}, \frac{10\sqrt{2}}{9}\right)$

ILLUSTRATION 18

If $(a, 0)$ is a point on a diameter, inside of the circle $x^2 + y^2 = 4$, then show that the following are true, equation $x^2 - 4x - a^2 = 0$ has

(a) Exactly one real root in $(-1, 0]$.
(b) Exactly one real root in $(2, 5)$.

SOLUTION



$\because$ $(a, 0)$ is a point on the diameter of the circle,

$$\Rightarrow \quad -2 \le a \le 2$$

So, the maximum value of a^2 is 4.

Let, $f(x) = x^2 - 4x - a^2$

Clearly, $f(-1) = 5 - a^2 > 0$

$$f(2) = -(a^2 + 4) < 0$$

$$f(0) = -a^2 \le 0 \text{ and } f(5) = 5 - a^2 > 0$$

So, graph of $f(x)$ will be as shown. From the graph we can conclude that the given equation has exactly one real root in $(-1, 0]$ and exactly one real root in $(2, 5)$.

Test Your Concepts-I

(Solutions on page H.77)

1. Find the equation of the circle, whose centre is (1, −2) and radius is 4.

2. Find the centre and radius of the circle, whose equation is

(i) $x^2 + y^2 - 4x + 6y + 12 = 0$

(ii) $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

3. Find the equation of the circle and the co-ordinates of the end points of whose diameters are (−1, 2) and (4, −3).

4. Find the circle passing through (1, −2) and touching the x-axis at (3, 0).

5. Find the equation of the circle, which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is (2, − 3).

6. Show that the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is $x^2 + y^2 + 32x + 4y + 235 = 0$.

7. Find the equation of the circle, which passes through the points (1, −2) and (4, −3) and whose centre lies on the line $3x + 4y = 7$.

8. If the equation $px^2 + (2 - q)xy + 3y^2 - 6qx + 30y + 6q = 0$ represent a circle, then find the values of p and q.

9. Find the equation of the circle with centre (α, β) and passing through the origin.

10. Find the radius (in units) of the circle $x^2 + y^2 + 2x\cos\theta + 2y\sin\theta = 8$.

11. Find the equation of the circumcircle of the triangle by the lines $x = 0$, $y = 0$ and $\frac{x}{a} + \frac{y}{b} = 1$.

12. Find the equation of the circle passing the circle passing through the origin with $3x + y = 14$ and $2x + 5y = 18$ as its two diameters.

13. Find the equation of the circle, which passes through (1, 0) and (0, 1) and has its radius as small as possible.

14. The lines $2x - 3y = 5$ and $3x - 4y = 7$ and diameters of a circle have area as 154 sq. units. Find the equation of the circle (use π as $\frac{22}{7}$).

15. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along the diameters of a circle of circumference 10π, then find the equation of the circle.

16. If an equilateral triangle is inscribed in the circle $x^2 + y^2 = a^2$, then find the length of its side.

17. Find the image of the centre of the circle $x^2 + y^2 = a^2$ with respect to the mirror $x + y = 1$.

18. Find the centre of the circle inscribed in the square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$.

19. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the co-ordinate axes, find its vertices.

20. If a line segment $AM = a$, moves in the plane XOY remaining parallel to OX, so that the left end point A slides along the circle $x^2 + y^2 = a^2$, then find the locus of M.

21. Find the point that is diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$.

22. Find the circle passing through the point (−1, 0) and touching the Y-axis at (0, 2).

Answer Keys

1. $x^2 + y^2 - 2x + 4y - 11 = 0$

2. **(i)** (2, −3), 1 **(ii)** $\left(\frac{4}{3}, \frac{5}{3}\right), \frac{4\sqrt{2}}{3}$

3. $x^2 + y^2 - 3x + y - 10 = 0$

4. $(x - 3)^2 + (y + 2)^2 = 4$

5. $(x - 2)^2 + (y + 3)^2 = (\sqrt{109})^2$

7. $15x^2 + 15y^2 - 94x + 18y + 55 = 0$

8. $p = 3$ and $q = 2$

9. $x^2 + y^2 - 2\alpha x - 2\beta y = 0$

10. 3 units

11. $x^2 + y^2 - ax - by = 0$

12. $x^2 + y^2 - 8x - 4y = 0$

13. $x^2 + y^2 - x - y = 0$

14. $x^2 + y^2 - 2x + 2y = 47$

15. $x^2 + y^2 - 2x + 2y - 23 = 0$

16. $b = \sqrt{3}a$

17. (1, 1)

18. (4, 7)

19. (−6, −9), (−6, 5), (8, −9) and (8, 5)

20. $x^2 + y^2 = 2ax$

21. (−3, −4)

22. $x^2 + y^2 + 5x - 4y + 4 = 0$

PARAMETRIC EQUATION OF CIRCLE

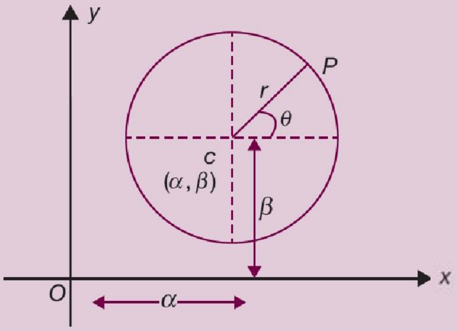
Equation of circle in

Cartesian form: $(x-\alpha)^2+(y-\beta)^2=r^2$

Parametric form: $\begin{cases} x=\alpha+r\cos\theta \\ y=\beta+r\sin\theta \end{cases}$

Any point $P(x, y)=(\alpha+r\cos\theta, \beta+r\sin\theta)$ lies on circle.

Where θ is a variable and r is fixed.



NOTE: If θ is constant and r is variable, then it becomes the equation of straight line.

ILLUSTRATION 19

Convert $\left.\begin{matrix} x=-2+4\cos\theta \\ y=1+4\sin\theta \end{matrix}\right\}$ into Cartesian form.

SOLUTION

$x+2=4\cos\theta;\ y-1=4\sin\theta$

Now $(x+2)^2+(y-1)^2=(4\cos\theta)^2+(4\sin\theta)^2=16$

$\Rightarrow$ $(x+2)^2+(y-1)^2=16$, it is equation of circle in cartesian form.

ILLUSTRATION 20

Write the parametric equation of circle, $x^2+y^2-4x+6y-5=0$.

SOLUTION

For given circle

Centre $(2, -3)$, $r=\sqrt{4+9+5}=3\sqrt{2}$

Parametric equation is $x=2+r\cos\theta=2+3\sqrt{2}\cos\theta$

$y=-3+r\sin\theta=-3+3\sqrt{2}\sin\theta$

ILLUSTRATION 21

A point is moving along the circle $x^2+y^2-2x-2y-2=0$. If it's initial position is $(2, 1+\sqrt{3})$, then find its final position after it has moved by $\frac{\pi}{2}$ in anticlockwise sense.

SOLUTION

For given circle

Centre $(1, 1)$ $r=\sqrt{1+1+2}=2$;

Initial position $(2, 1+\sqrt{3})$

$=(1+2\cos 60°, 1+2\sin 60°)$

Final position $(1+2\cos(60°+90°), 1+2\sin(60°+90°))$

$=(1-2\sin 60°, 1+2\cos 60°)=(1-\sqrt{3}, 2)$

POSITION OF POINT

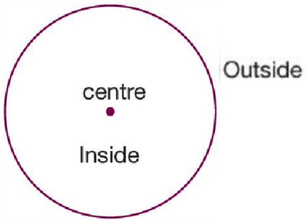
$S: x^2+y^2+2gx+2fy+c=0$ (Equation of circle)

We are looking for position of point $P(x_1, y_1)$ relative to circle S

$S_1:\ x_1^2+y_1^2+2gx_1+2fy_1+c$

Expression S_1 is also known as power of point (x_1, y_1).

We know that the centre $(-g, -f)$ always lies inside the circle.



So, $S_1=g^2+f^2-2g^2-2f^2+c=-g^2-f^2+c=-r^2<0$

$\therefore$ For all inside points, $S_1<0$. (Power of point <0)

Condition	Location of point
$S_1<0$	Inside
$S_1=0$	On the circle
$S_1>0$	Outside

ILLUSTRATION 22

How are the points (0, 1), (3, 1) and (1, 3) situated with respect to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$?

SOLUTION

Given equation is $x^2 + y^2 - 2x - 4y + 3 = 0$

Let $A = (0, 1)$, $B = (3, 1)$ and $C = (1, 3)$

For point $A(0, 1)$,

$$S_1 = x_1^2 + y_1^2 - 2x_1 - 4y_1 + 3 = 0^2 + 1^2 - 2.0 - 4.1 + 3 = 0$$

Hence, point A lies on the circle.

For point $B(3, 1)$,

$$S_1 = x_1^2 + y_1^2 - 2x_1 - 4y_1 + 3 = 9 + 1 - 6 - 4 + 3 = 3 > 0$$

Hence, point B lies outside the circle.

For point $C(1, 3)$,

$$S_1 = x_1^2 + y_1^2 - 2x_1 - 4y_1 + 3 = 1 + 9 - 2 - 12 + 3 = -1 < 0$$

Hence, point C lies inside the circle.

ILLUSTRATION 23

If $P(2, 8)$ is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$, which neither touches nor intersects the axes, then find set for p.

SOLUTION

For internal point $p(2, 8)$

$$S_1 = 4 + 64 - 4 + 32 - p < 0$$

$$\Rightarrow \quad p > 96 \qquad \ldots(1)$$

and x intercept $= 2\sqrt{g^2 - c} = 2\sqrt{1 + p}$

$$\therefore \quad 1 + p < 0 \Rightarrow p < -1 \qquad \ldots(2)$$

$\Rightarrow \quad y$ intercept $= 2\sqrt{f^2 - c} = 2\sqrt{4 + p}$

$$\Rightarrow \quad p < -4 \qquad \ldots(3)$$

$\therefore$ Common value of $p \in \phi$

ILLUSTRATION 24

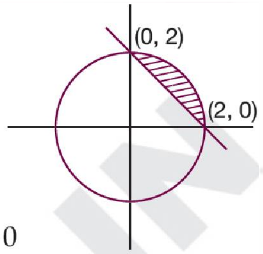
For what value of a, the point $(a, a + 1)$ lies inside the region bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ in the Ist quadrant.

SOLUTION

The point $(a, a + 1)$ lies inside circle, then $S_1 < 0$

$$\Rightarrow \quad a^2 + (a + 1)^2 - 4 < 0$$

$$\Rightarrow \quad 2a^2 + 2a - 3 < 0$$



$$\Rightarrow \quad a^2 + a - \frac{3}{2} < 0$$

$$\Rightarrow \quad \left(a + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{7}}{2}\right)^2 < 0$$

$$\Rightarrow \quad \left(\frac{a + 1 - \sqrt{7}}{2}\right)\left(\frac{a + 1 + \sqrt{7}}{2}\right) < 0$$

$$a \in \left(\frac{-1 - \sqrt{7}}{2}, \frac{\sqrt{7} - 1}{2}\right) \qquad \ldots(1)$$

For line $x + y - 2 = 0$, $O(0, 0)$ and $P(a, a + 1)$ lie opposite side of line

$$(0 + 0 - 2)\ (a + a + 1 - 2) < 0 \Rightarrow a > \frac{1}{2} \qquad \ldots(2)$$

Final common values (1) and (2), we get

$$a \in \left(\frac{1}{2}, \frac{\sqrt{7} - 1}{2}\right)$$

POSITION OF LINE RELATIVE TO CIRCLE

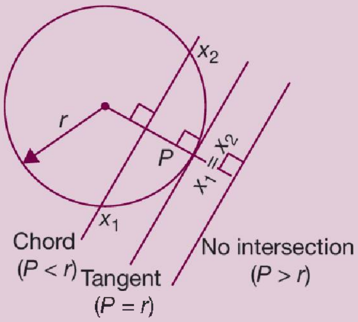
Circle $S : x^2 + y^2 + 2gx + 2fy + c = 0$

Straight line $y = mx + c$

Ist Method

Length of perpendicular from centre to line $= p$

$p > r \Rightarrow$	Line is outside the circle
$p = r \Rightarrow$	Line touches circle (tangent)
$p < r \Rightarrow$	Line is the chord of circle (secant)
$p = 0 \Rightarrow$	Line is diameter of circle



IInd Method (always applicable for all IInd degree curve)

Solve straight line and circle

$$x^2 + (mx + c)^2 + 2gx + 2f(mx + c) + c = 0$$

$$\Rightarrow \quad Ax^2 + Bx + C = 0$$

It is quadratic in x.

$D > 0 \Rightarrow$ Line is the chord of circle (secant)
$D = 0 \Rightarrow$ Line touches circle (tangent)
$D < 0 \Rightarrow$ Line is outside the circle (non-intersecting line)
Point of tangency = Foot of perpendicular from centre to tangent

Remark(s)

We can use the following direct formula for calculating the length of perpendicular and foot of perpendicular.

Length of perpendicular from a point $A(x_1, y_1)$ on the line $ax + by + c = 0$ is $p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Foot of perpendicular drawn from the point $A(x_1, y_1)$ to line $ax + by + c = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

ILLUSTRATION 25

Find the relative position of straight line $y = x + 5$ to circle $x^2 + y^2 - 4x - 6y + 9 = 0$.

SOLUTION

Centre $(2, 3)$, $r = \sqrt{4 + 9 - 9} = 2$

Straight line, $x - y + 5 = 0$

$$p = \left|\frac{2 - 3 + 5}{\sqrt{2}}\right| = 2\sqrt{2}$$

Clearly $p > r$.

Hence, there is no intersection with circle and the given straight line lies completely outside of circle.

ILLUSTRATION 26

Show that the line $3x - 4y - c = 0$ will meet the circle having centre at $(2, 4)$ and the radius 5 in real and distinct points if $-35 < c < 15$.

SOLUTION

Line should be chord, then $p < r$

If the perpendicular is drawn from the centre to line $< r$, then

$$\frac{|6 - 16 - c|}{\sqrt{9 + 16}} < 5$$

$$\Rightarrow \quad |c + 10| < 25$$

$$\Rightarrow \quad -25 < c + 10 < 25$$

$$\Rightarrow \quad -35 < c < 15$$

ILLUSTRATION 27

If straight line $3x + 4y = \lambda$ is a tangent to circle $x^2 + y^2 - 10x = 0$, then find λ.

SOLUTION

For given circle

Centre $C(5, 0)$ $r = 5$; for line $3x + 4y = \lambda$ to be tangent

$$P = r$$

$$\frac{|15 + 0 - \lambda|}{\sqrt{9 + 16}} = 5 \Rightarrow |15 - \lambda| = 25$$

$$\Rightarrow \quad 15 - \lambda = \pm 25$$

Taking positive sign, $15 - \lambda = 25 \Rightarrow \lambda = -10$

Taking negative sign, $15 - \lambda = -25 \Rightarrow \lambda = 40$

ILLUSTRATION 28

If the straight line $y = 2x + c$ is the chord of circle, $x^2 + y^2 = 5$, then find c.

SOLUTION

For given circle

$$C(0, 0);\ r = \sqrt{5}$$

For chord $p < r$,

$$\Rightarrow \quad \left|\frac{0 - 0 + c}{\sqrt{4 + 1}}\right| < \sqrt{5} \Rightarrow |c| < 5 \Rightarrow -5 < c < 5$$

ILLUSTRATION 29

Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y = 0$ which are:

(i) Perpendicular
(ii) Parallel to the line $3x - 4y - 7 = 0$

SOLUTION

(i) Required perpendicular tangent $4x + 3y + k = 0$

Centre $C(1, 2)$ $r = \sqrt{5}$

$$\left|\frac{4+6+k}{\sqrt{16+9}}\right| = \sqrt{5}$$

$\Rightarrow 10 + k = \pm 5\sqrt{5} \Rightarrow k = -10 \pm 5\sqrt{5}$

$\therefore$ The equation of tangent is

$$4x + 3y - 10 \pm 5\sqrt{5} = 0$$

(ii) Required parallel tangents are $3x - 4y + \lambda = 0$

Centre $C(1, 2)$ $r = \sqrt{5}$

$$P = r \Rightarrow \left|\frac{3-8+\lambda}{\sqrt{9+16}}\right| = \sqrt{5}$$

$\Rightarrow |\lambda - 5| = 5\sqrt{5} \Rightarrow \lambda = 5 \pm 5\sqrt{5}$

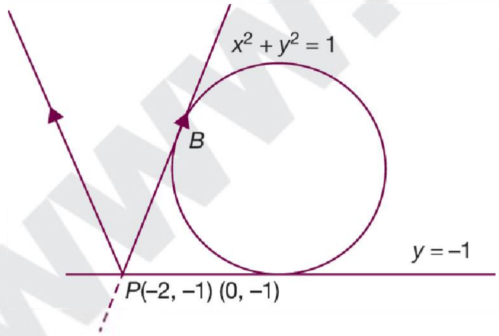
$\therefore$ The equation of tangent is

$$3x - 4y + 5 \pm 5\sqrt{5} = 0$$

ILLUSTRATION 30

A ray of light incident at the point $(-2, -1)$ gets reflected from the tangent at $(0, -1)$ to the circle $x^2 + y^2 = 1$. If the reflected ray touches the circle, then find the equation of the line along which the incident ray moved.

SOLUTION



Any line through $(-2, -1)$ is $y + 1 = m(x + 2)$

It touches the circle, centre $(0, 0)$; $r = 1$

For tangency $P = r$

$$\Rightarrow \quad \left|\frac{2m-1}{\sqrt{1+m^2}}\right| = 1$$

$$|2m - 1| = \sqrt{1+m^2}$$

$\Rightarrow \quad 4m^2 + 1 - 4m = 1 + m^2$

$\Rightarrow \quad 3m^2 - 4m = 0$

$\Rightarrow \quad m = 0, \frac{4}{3}$

$\therefore$ Equation of PB is $y + 1 = \frac{4}{3}(x + 2)$

$\Rightarrow \quad 4x - 3y + 5 = 0$

For $m = 0$; we get $y = -1$ (it can not be PB)

Let incident line slope is m_0

Tangent $y = -1$ makes equal angles with incident and reflected ray.

$$\left|\frac{m_0 - 0}{1+0}\right| = \left|\frac{0 - \frac{4}{3}}{1+0}\right| \Rightarrow |m_0| = \frac{4}{3}$$

we get $m_0 = \frac{4}{3}, \frac{-4}{3}$ but $m_0 \neq \frac{4}{3}$

Incident ray $y + 1 = \frac{-4}{3}(x + 2)$

$\Rightarrow \quad 3y + 3 + 4x + 8 = 0$

$\Rightarrow \quad 4x + 3y + 11 = 0$

ILLUSTRATION 31

A circle touches both x-axis and the line $4x - 3y + 4 = 0$. Its centre is in III$^{\text{rd}}$ quadrant and lies on the lines $x - y - 1 = 0$. Find the equation of the circles.

SOLUTION

Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Its centre is $C(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$

$\because$ circle touches the x–axis

$\therefore \quad g^2 - c = 0$ or $g^2 = c$

Again, circle touches the line $4x - 3y + 4 = 0$

$$p = r \Rightarrow \frac{|-4g + 3f + 4|}{5} = \sqrt{g^2 + f^2 - c} = \sqrt{f^2} = |f|$$

$(\because g^2 = c)$

$$\Rightarrow \quad -4g+3f+4=\pm 5f$$

$$\therefore \quad \begin{cases} 4g+2f=4 & \text{or} \quad 2g+f=2 & \ldots(1) \\ -4g+8f=-4 & \text{or} \quad g-2f=1 & \ldots(2) \end{cases}$$

Again, centre $C(-g,-f)$ lies on the line $x-y-1=0$

$$\therefore \quad -g+f=1 \qquad \ldots(3)$$

Solving (1) and (3), we get $g=\frac{1}{3}, f=\frac{4}{3}, c=g^2=\frac{1}{9}$

Thus, centre $C=\left(\frac{-1}{3},\frac{-4}{3}\right)$, which lies in the III$^{\text{rd}}$ quadrant. (Acceptable case)

Solving (2) and (3), we get

$$f=-2,\ g=-3$$

∴ Centre $O=(3,2)$ which lies in the I$^{\text{st}}$ quadrant. Hence, it is rejected.

Thus, for the required circle, $g=\frac{1}{3}, f=\frac{4}{3}, c=\frac{1}{9}$

∴ The equation of the required circle will be

$$x^2+y^2+\frac{2}{3}x+\frac{8}{3}y+\frac{1}{9}=0$$

$$\Rightarrow \quad 9(x^2+y^2)+6x+24y+1=0$$

ILLUSTRATION 32

If $4\ell^2-5m^2+6\ell+1=0$, then show that the line $\ell x+my+1=0$ touches a fixed circle. Find the radius and centre of the circle.

SOLUTION

Let the circle be $(x-\alpha)^2+(y-\beta)^2=r^2$

The line $\ell x+my+1=0$ touches the circle, if

$$r=\left|\frac{l\alpha+m\beta+1}{\sqrt{l^2+m^2}}\right| \qquad (\because\ r=p)$$

$$\Rightarrow \quad r^2(\ell^2+m^2)=(\ell\alpha+m\beta+1)^2$$

$$\Rightarrow \quad (r^2-\alpha^2)\ell^2+(r^2-\beta^2)m^2-2\alpha\beta\ell m-2\ell\alpha-2m\beta-1=0 \qquad \ldots(1)$$

However, given relation is

$$4\ell^2-5m^2+6\ell+1=0 \qquad \ldots(2)$$

It is possible to find α, β, r if (1) and (2) are identical. The condition is

$$\frac{r^2-\alpha^2}{4}=\frac{r^2-\beta^2}{-5}=\frac{-2\alpha\beta}{0}=\frac{-2\alpha}{6}=\frac{-2\beta}{0}=\frac{-1}{1} \qquad \ldots(3)$$

$$\therefore \quad \beta=0,\ \alpha=3$$

and $\frac{r^2-\alpha^2}{4}=-1$ which implies, $r^2-3^2=-4$, i.e., $r=\sqrt{5}$

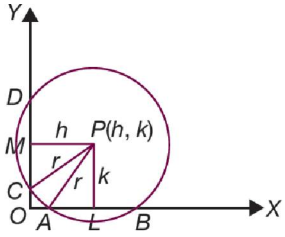
We get $\alpha=3, \beta=0, r=\sqrt{5}$

∴ The line touches the fixed circle, whose centre $=(\alpha,\beta)=(3,0)$ and radius $(\sqrt{5})$.

ILLUSTRATION 33

Two rods of length a and b slide along the axes, which are rectangular, in such a manner that their ends are concyclic. Prove that the locus of the centres of the circle passing through these ends is the curve $4(x^2-y^2)=a^2-b^2$.

SOLUTION



$$AB=a,\ CD=b$$

$$AL=\frac{a}{2},\ CM=\frac{b}{2}$$

Let the centres of the circles are (h,k).

$$PL=k,\ PM=h$$

In ΔPAL, $PA^2=PL^2+AL^2$

$$r^2=k^2+\frac{a^2}{4} \qquad \ldots(1)$$

In ΔPCM, $PC^2=PM^2+CM^2$

$$\Rightarrow \quad r^2=h^2+\frac{b^2}{4} \qquad \ldots(2)$$

From (1) and (2), we get,

$$h^2+\frac{b^2}{4}=k^2+\frac{a^2}{4}$$

∴ The locus of (h,k) is $4(x^2-y^2)=a^2-b^2$

Test Your Concepts-II

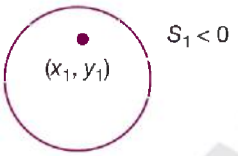
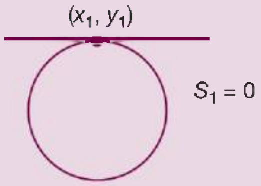
(Solutions on page H.80)

1. Find the parametric equations of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$.
2. Find the equations of the circle in Cartesian form. Also, find the centre and the radius of the circle $x = a + c\cos\theta$, $y = b + c\sin\theta$.
3. Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$.
4. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle.
5. Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at $(4, -1)$.
6. Find the tangent lines to the circle $x^2 + y^2 - 6x + 4y = 12$, which are parallel to the line $4x + 3y + 5 = 0$.
7. Find the value of c, for which $x + y = 2$ touches the circle $x^2 + y^2 - 2cx = 0$ and (1, 0) lies inside the circle.

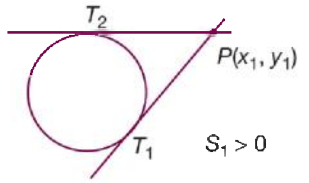
Answer Keys

1. $x = 2 + 2\cos\theta$, $y = 1 + 2\sin\theta$
2. centre (a, b), radius c.
3. **(i)** inside **(ii)** outside
4. $\frac{3}{4}$
5. $11x - 2y - 46 = 0$
6. $4x + 3y - 31 = 0$ and $4x + 3y + 19 = 0$
7. $c = 2(\sqrt{2} - 1)$

TANGENT/PAIR OF TANGENTS ON CIRCLE

Location of Point	Number of Tangent	Equation of Tangent
(i) (Point lies inside the circle) 	0	Not possible
(ii) (Point lies on the circle) 	1	**Ist Method (Preferred)** Equation of tangent at point (x_1, y_1) $T = 0$ $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ **IInd Method** Equation of tangent $y - y_1 = m(x - x_1)$ Length of perpendicular from centre to line = Radius $P = R$(radius). On solving, we get m. **IIIrd Method** Equation of tangent $y - y_1 = m(x - x_1)$ Solve tangent with equation of circle and make quadratic equation in x or y then make $D = 0$. On solving, we get m.

(Continued)

Location of Point	Number of Tangent	Equation of Tangent
(iii) (Point lies outside the circle)  	2 (Pair of tangents)	**Ist Method (Preferred)** Tangent $y - y_1 = m(x - x_1)$ Length of perpendicular from centre to tangent = Radius $P = R$(radius), we get two values of m_1, m_2 **IInd Method** Equation of tangent $y - y_1 = m(x - x_1)$ $D = 0$ with equation of circle. On solving, we get two values m_1 and m_2 **IIIrd Method** Equation of pair of tangents $SS_1 = T^2$ Where $S : x^2 + y^2 + 2gx + 2fy + c$ $S_1 : x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ $T : xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

TANGENT FOR STANDARD CIRCLE

Slope Form

The equation of tangent of slope 'm' to the circle $x^2 + y^2 = a^2$ is given by

$$y = mx \pm a\sqrt{1+m^2}$$

Parametric Form

The equation of tangent at (a cos θ, a sin θ) or θ to the circle $x^2 + y^2 = a^2$ is given by

$$x \cos\theta + y \sin\theta = a$$

ILLUSTRATION 34

Consider the circle $x^2 + y^2 - 30x + 6y + 109 = 0$. Write the equation of tangent point $P(4, -1)$ on the circle.

SOLUTION

Position of point $P(4, -1)$

$$S_1 = 16 + 1 - 120 - 6 + 109 = 0$$

$\therefore$ P lies on the circle.

Equation of tangent, $T = 0$

$$xx_1 + yy_1 - 15(x + x_1) + 3(y + y_1) + 109 = 0$$

$$P(4, -1) = (x_1, y_1)$$

$$4x - y - 15(x + 4) + 3(y - 1) + 109 = 0$$

$$\Rightarrow \quad -11x + 2y + 46 = 0$$

ILLUSTRATION 35

Write the equation of tangent form point $P(0, 1)$ on the circle $x^2 + y^2 - 2x + 4y = 0$.

SOLUTION

$$S_1 = 0 + 1 - 0 + 4 = 5 > 0$$

Point $P(0, 1)$ lies outside the circle.

Let the tangent $y - 1 = m(x - 0)$

$$\Rightarrow \quad y = mx + 1$$

$$\Rightarrow \quad mx - y + 1 = 0$$

Centre $(1, -2)$ $r = \sqrt{5}$

Tangency condition $P = r \Rightarrow \left|\dfrac{m+2+1}{\sqrt{1+m^2}}\right| = \sqrt{5}$

$$\Rightarrow \quad m^2 + 9 + 6m = 5 + 5m^2$$

$$\Rightarrow \quad 4m^2 - 6m - 4 = 0$$

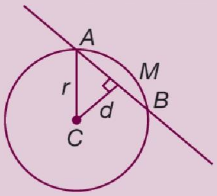
$$\Rightarrow \quad (m - 2)(2m + 1) = 0$$

We get $m = 2$, $m = -\dfrac{1}{2}$

$\therefore$ The required tangents are $2x - y + 1 = 0$ and $x + 2y - 2 = 0$.

LENGTH OF CHORD (SECANT)

Given chord cuts circle at points A and B and perpendicular from centre to chord is M, hence it is midpoint of AB.



$\therefore \quad AM^2 + CM^2 = AC^2$

$AM^2 + d^2 = r^2$

$\Rightarrow \quad AM = \sqrt{r^2 - d^2}$

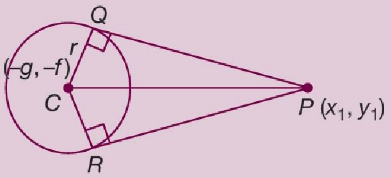
$AB = 2AM = 2\sqrt{r^2 - d^2}$

Length of the intercept made by the circle on the line

$= \text{Length of chord} = AB = 2\sqrt{r^2 - d^2}$

LENGTH OF TANGENT

The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is given by



$PQ = PR$

$PQ^2 + QC^2 = CP^2$ (from Pythagoras)

$PQ^2 + r^2 = (x_1 + g)^2 + (y_1 + f)^2$

$PQ^2 + g^2 + f^2 - c = x_1^2 + g^2 + 2gx_1 + y_1^2 + 2y_1 f + f^2$

$PQ = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Length of tangent $PQ = \sqrt{S_1}$

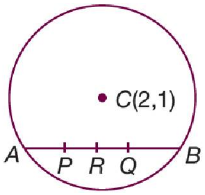
NOTE: Always make coefficient of x^2 and y^2 as 1, before using length of tangent formula.

ILLUSTRATION 36

If a chord of the circle $x^2 + y^2 - 4x - 2y - c = 0$ is trisected at the points $\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\left(\frac{8}{3}, \frac{8}{3}\right)$, then find c.

SOLUTION

Midpoint of $PQ = R\left(\frac{3}{2}, \frac{3}{2}\right)$



$PQ = \sqrt{\frac{49}{9} + \frac{49}{9}} = \frac{7}{3} \cdot \sqrt{2}$

$\therefore \quad AB = 7\sqrt{2}$

$CR^2 = \left(2 - \frac{3}{2}\right)^2 + \left(1 - \frac{3}{2}\right)^2 = \frac{1}{2}$ and $AR = \frac{7}{\sqrt{2}}$

$(\text{Radius})^2 = CA^2 = CR^2 + AR^2$

$r^2 = \frac{1}{2} + \frac{49}{2} = 25$

$\therefore \quad g^2 + f^2 - c = r^2$

$4 + 1 + c = 25$

Hence, $c = +20$

ILLUSTRATION 37

Find the length of the tangent drawn from the point (5, 1) to the circle $2x^2 + 2y^2 + 12x - 8y - 6 = 0$.

SOLUTION

Given circle is $x^2 + y^2 + 6x - 4y - 3 = 0$

$$\text{Length of tangent} = \sqrt{S_1} = \sqrt{25 + 1 + 30 - 4 - 3}$$

$$= \sqrt{49} = 7$$

ILLUSTRATION 38

If the length of the tangent from a point (f, g) to the circle $x^2 + y^2 = 4$ be four times the length of the tangent from it to the circle $x^2 + y^2 = 4x$, then show that $15f^2 + 15g^2 - 64f + 4 = 0$.

SOLUTION

Given relation, $\ell_1 = 4\ell_2$

$\Rightarrow \quad \sqrt{f^2 + g^2 - 4} = 4\sqrt{f^2 + g^2 - 4f}$

$\Rightarrow \quad f^2 + g^2 - 4 = 16f^2 + 16g^2 - 64f$

$\Rightarrow \quad 15f^2 + 15g^2 - 64f + 4 = 0$. Hence proved

ILLUSTRATION 39

Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circles $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then find the equation of L_1.

SOLUTION

Centre of circle, $c\left(\frac{1}{2}, \frac{-3}{2}\right)$; $L_1 : y = mx$

Intercepts are equal, which means length of perpendicular from centre to these chords are equal.

WWW.EBOOKS.IN

$$\frac{\left|\frac{1}{2}-\frac{3}{2}-1\right|}{\sqrt{1+1}}=\frac{\left|\frac{m}{2}+\frac{3}{2}\right|}{\sqrt{1+m^2}}$$

$\Rightarrow \quad 2\sqrt{2}\ \sqrt{1+m^2} = |m+3|$

$8m^2+8=m^2+9+6m$

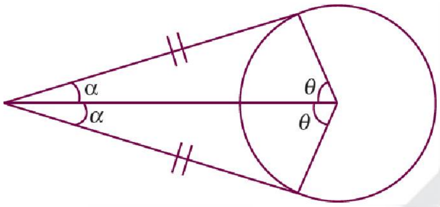
$\Rightarrow \quad 7m^2-6m-1=0$

$\Rightarrow \quad (7m+1)(m-1)=0$, we get $m=1, \dfrac{-1}{7}$

$\therefore$ The required two chords are $y=x$ and $7y+x=0$.

GEOMETRICAL PROPERTIES OF CIRCLE RELATED TO TANGENTS AND CHORDS

(a) A tangent to a circle is perpendicular to the radius through the point of contact.



(b) If two tangents are drawn to a circle from an external point, then: they

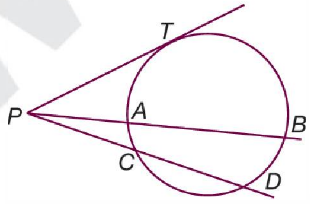
(i) are equal.

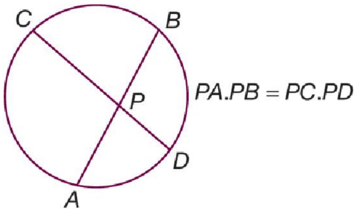
(ii) subtend equal angles at the centre.

(iii) are equally inclined to the segment, joining the centre to that point.

(c) Two chords AB and CD of a circle intersect inside or outside the circle at point P and a tangent PT is drawn from P to circle, then:

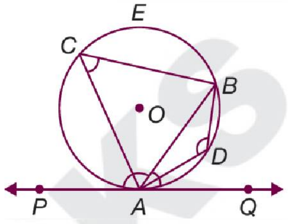
$$PT^2=PA\times PB=PC\times PD.$$





(d) If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the alternate segments.

$$\angle BAQ=\angle ACB \text{ and } \angle BAP=\angle ADB$$

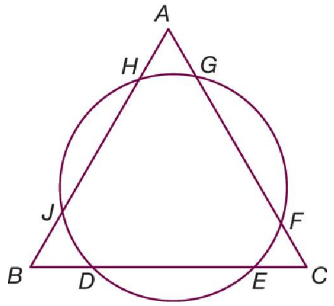


Converse

If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

ILLUSTRATION 40

In the adjoining figure, the circle meets the sides of an equilateral triangle at six points. If $AG=2$, $GF=13$, $FC=1$ and $HJ=7$, then find DE.



SOLUTION

Let, $BD=x$, $EC=y$ and $AH=z$

Now, $AG\times AF=AH\times AJ$

$\Rightarrow \quad 2\times 15=z\times(z+7)\Rightarrow z^2+7z-30=0$

$\Rightarrow \quad z=3$

Now, $AC=AB \Rightarrow 16=3+7+BJ \Rightarrow BJ=6$

Now, $BJ\times BH=BD\times BE$

So, $\quad 6\times 13=x\times(16-y) \qquad \ldots(1)$

Also, $CE\times CD=CF\times CG$

$\Rightarrow \quad y(16-x)=1\times 14 \qquad \ldots(2)$

Solving (1) and (2), we get,

$$x = 10 - \sqrt{22},\ y = 6 - \sqrt{22};\ DE = 16 - x - y = 2\sqrt{22}$$

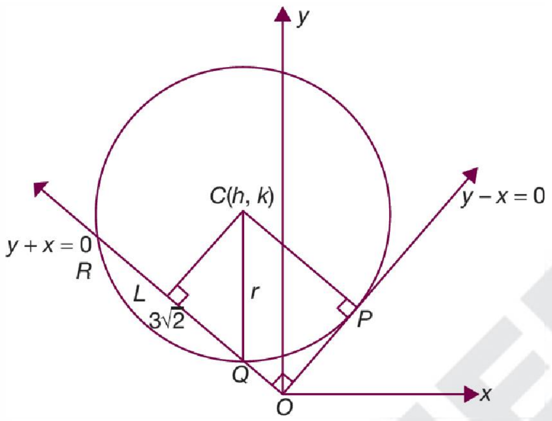
ILLUSTRATION 41

The line $y = x$ touches a circle at P so that $OP = 4\sqrt{2}$, where O is the origin. The point $(-10, 2)$ lies inside the circle and the length of the chord $x + y = 0$ is $6\sqrt{2}$. Find the equation of the circle.

SOLUTION

$OPCL$ is a rectangle in which

$$OP = 4\sqrt{2}$$



Also, $QR = 6\sqrt{2} \Rightarrow QL = 3\sqrt{2}$

In ΔCQL, $CQ^2 = CL^2 + QL^2$

$$= \left(4\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2$$

$$= 32 + 18 = 50$$

Also, the circle touches the line $x - y = 0$.

Hence, $\pm\dfrac{h-k}{\sqrt{2}} = r = 5\sqrt{2}$

$$\Rightarrow \quad h - k = \pm 10 \qquad \ldots(1)$$

Also, $CL = 4\sqrt{2}$

$$\Rightarrow \quad \frac{h+k}{\sqrt{2}} = \pm 4\sqrt{2}$$

$$\Rightarrow \quad h + k = \pm 8 \qquad \ldots(2)$$

From (1) and (2), the possible centres of the circle are $(9, -1)(1, -9), (-1, 9)(-9, 1)$.

Hence, the equations of the circles are:

$$(x - 9)^2 + (y + 1)^2 = 50 \qquad \ldots(3)$$

$$(x - 1)^2 + (y + 9)^2 = 50 \qquad \ldots(4)$$

$$(x + 1)^2 + (y - 9)^2 = 50 \qquad \ldots(5)$$

$$(x + 9)^2 + (y - 1)^2 = 50 \qquad \ldots(6)$$

$\because$ $(-10, 2)$ lies inside the circle, for $(-10, 2)$, LHS < 50

This is true for (6).

Hence, the required equation of the circle is

$$(x + 9)^2 + (y - 1)^2 = 50,$$

i.e., $x^2 + y^2 + 18x - 2y + 32 = 0$.

ILLUSTRATION 42

If $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$ and a line through $P(a, n)$ cuts the circle $x^2 + y^2 = 4$ in A and B, if $PA \cdot PB = 4\lambda$, then find λ.

SOLUTION

Given $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$

$$\Rightarrow \quad 1 + \frac{n}{1!}ax + \frac{n(n-1)}{2!}a^2x^2 + \ldots = 1 + 8x + 24x^2 + \ldots$$

Equating the coefficients of x and x^2, we get

$$na = 8 \Rightarrow a^2n^2 = 64 \qquad \ldots(1)$$

and $$\frac{n(n-1)}{2}a^2 = 24 \qquad \ldots(2)$$

Dividing (2) by (1), we get

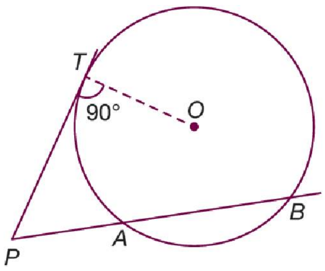
$$\frac{n-1}{2n} = \frac{3}{8}$$

$$\Rightarrow \quad 4n - 4 = 3n \Rightarrow n = 4$$

From (1), $a = 2$

Given circle is

$$x^2 + y^2 - 4 = 0 \qquad \ldots(4)$$



Let, $P \equiv (a, n)$, then $P \equiv (2, 4)$

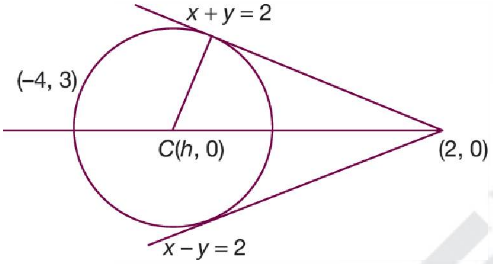
Now, $PA \cdot PB = PT^2 = S_1 = 4 + 16 - 4 = 16 = 4\lambda$, we get $\lambda = 4$.

ILLUSTRATION 43

Find the equation of the circles passing through (–4, 3) and touching the lines $x + y = 2$ and $x - y = 2$.

SOLUTION

The centre of the circle C is equidistant from the two tangents $x + y = 2$ and $x - y = 2$.
$\therefore$ It lies on the bisector of the angle between them, which is $y = 0$.
[The other bisector $x = 2$ is to be rejected
$\because$ (–4, 3) is a point in the II$^{\text{nd}}$ quadrant, which lies on the circle]



$\therefore$ Let $C = (h, 0)$

The circle passes through (–4, 3).

$$\therefore \quad (h+4)^2 + 3^2 = \left[\left|\frac{h-2}{\sqrt{2}}\right|\right]^2$$

$$\Rightarrow \quad h^2 + 20h + 46 = 0$$

$$\therefore \quad h = -10 \pm 3\sqrt{6}$$

$\therefore$ There are 2 circles.

Circle I

$$C_1 = (-10 + 3\sqrt{6}\,, 0)$$

$$r_1 = \left|\frac{-10 + 3\sqrt{6} - 2}{\sqrt{2}}\right|$$

$\therefore$ Its equation is $(x + 10 - 3\sqrt{6}\,)^2 + y^2 = (6\sqrt{2} - 3\sqrt{3}\,)^2$

Circle II

$$C_2 = (-10 - 3\sqrt{6}\,, 0) \text{ and } r_2 = 6\sqrt{2} + 3\sqrt{3}$$

$\therefore$ The equation of the circle is

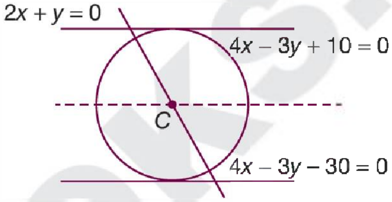
$$(x + 10 + 3\sqrt{6}\,)^2 + y^2 = (6\sqrt{2} + 3\sqrt{3}\,)^2$$

ILLUSTRATION 44

Find the equation of the circle with centre on the line $2x + y = 0$ and touching the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$.

SOLUTION

$\because$ the two given lines are tangents to the given circle and they are parallel, we have



$$\text{Radius} = \frac{1}{2}\left|\frac{c_1 - c_2}{\sqrt{A^2 + B^2}}\right| = \frac{1}{2}\left|\frac{10 + 30}{\sqrt{16 + 9}}\right| = 4 \text{ units}$$

We know that, if $ax + by + c = 0$ and $ax + by + c' = 0$ are two parallel tangent to a circle, then the equation of the line parallel to given lines and passing through the centre is given by

$$ax + by + \frac{c + c'}{2} = 0$$

Hence, the centre of the circle lies on $4x - 3y - 10 = 0$.
Also the centre lies on $2x + y = 0$.

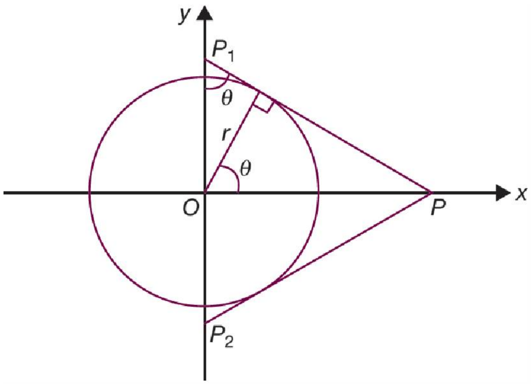
Hence, the co-ordinates of the centre are (1, –2).

$\therefore$ The equation of the circle is $(x - 1)^2 + (y + 2)^2 = 16$.

ILLUSTRATION 45

Tangents are drawn to the circle $x^2 + y^2 = 50$ from a point P lying on the x-axis. These tangents meet the y-axis at points P_1 and P_2. Find the possible co-ordinates of P so that area of ΔPP_1P_2 is minimum.

SOLUTION



$$OP = 5\sqrt{2}\ \sec\theta = r\sec\theta$$

$$OP_1 = 5\sqrt{2}\ \operatorname{cosec}\theta = r\operatorname{cosec}\theta$$

$$\Delta PP_1P_2 = 2(\Delta OPP_1) = 2\left(\frac{1}{2}OP\cdot OP_1\right)$$

$$= \frac{(5\sqrt{2})^2}{\cos\theta\sin\theta} = \frac{100}{\sin 2\theta}$$

$$(\Delta PP_1P_2)_{\min} = 100$$

So, $\theta = \frac{\pi}{4} \Rightarrow OP = 10;$

$$\pi = \frac{3\pi}{4};\ OP = -10$$

Hence, $P \equiv (10, 0), (-10, 0)$

ILLUSTRATION 46

The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $(2 + \sqrt{3}, 3)$ by 2 units, assuming the x-axis as horizontal, find the equation of the circle in the new position.

SOLUTION

Given circle is

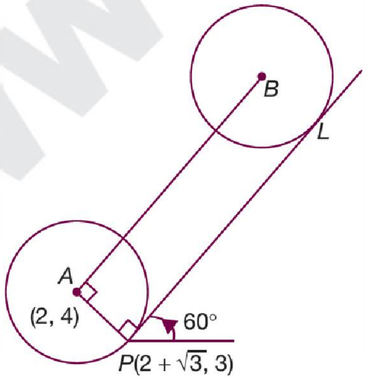
$$x^2 + y^2 - 4x - 8y + 16 = 0 \qquad \ldots(1)$$

Let $P \equiv (2 + \sqrt{3}, 3)$,

Equation of tangent to the circle (1) at $P(2 + \sqrt{3}, 3)$ will be

$$(2 + \sqrt{3})x + 3y - 2(x + 2 + \sqrt{3}) - 4(y + 3) + 16 = 0$$

$$\Rightarrow \quad \sqrt{3}\,x - y - 2\sqrt{3} = 0 \qquad \ldots(2)$$



If line (2) makes an angle θ with the positive direction of x-axis, then $\tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°.$

Let, A and B be the centres of the circles in old and new positions respectively, then

$$A = (2, 4) \text{ and } B = (2 + 2\cos 60°, 4 + 2\sin 60°)$$

$$(\because AB = 2)$$

Thus, $B = (3, 4 + \sqrt{3})$

Radius of the circle $= \sqrt{2^2 + 4^2 - 16} = 2$

$\therefore$ Equation of the required circle is

$$(x - 3)^2 + (y - 4 - \sqrt{3})^2 = 4$$

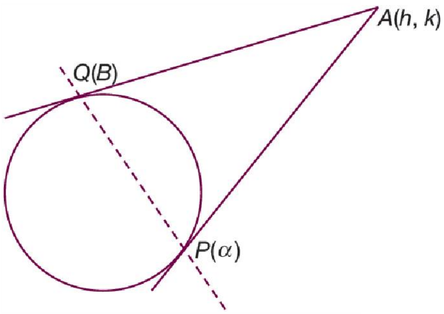
ILLUSTRATION 47

Find the locus of the point of intersection of the pair of tangent drawn to a circle $x^2 + y^2 = a^2$ at $P(\alpha)$ and $Q(\beta)$, where $|\alpha - \beta| = 120°$

SOLUTION

$$P(\alpha) = (a\cos\alpha, a\sin\alpha)$$

$$Q(\beta) = (a\cos\beta, a\sin\beta)$$



Equation of PQ

$$y - a\sin\alpha = \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}(x - a\cos\alpha)$$

$$\Rightarrow \quad x(\sin\alpha - \sin\beta) + y(\cos\beta - \cos\alpha) = a\sin(\alpha - \beta)$$

$$\Rightarrow \quad 2x\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$+ 2y\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$= 2a\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = a\cos\left(\frac{\alpha-\beta}{2}\right)$$

Given $|\alpha - \beta| = 120°$

$$x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\cos(\pm 60^\circ)=\frac{a}{2} \qquad \ldots(1)$$

From $A(h, k)$, PQ is chord of contact

$$T=0 \Rightarrow hx+ky=a^2 \qquad \ldots(2)$$

(1) and (2) represents same line, compare them

$$\frac{h}{\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{a^2}{a/2}=2a$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)=\frac{h}{2a};\ \sin\left(\frac{\alpha+\beta}{2}\right)=\frac{k}{2a}$$

Square and add,

$$1=\frac{h^2}{4a^2}+\frac{k^2}{4a^2} \Rightarrow h^2+k^2=4a^2$$

Locus $h=x$; $k=y$, we get $x^2+y^2=4a^2$

ILLUSTRATION 48

Obtain the locus of the point of intersection of the tangents to the circle $x^2+y^2=a^2$, which include an angle α.

SOLUTION

Let (x_1, y_1) be the point of intersection of a pair of tangents to the circle $x^2+y^2=a^2$. The combined equation of this pair of tangents is

$$SS_1=T^2$$

i.e., $(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)=(xx_1+yy_1-a^2)^2$

$$\Rightarrow \quad (y_1^2-a^2)x^2-2x_1y_1xy+(x_1^2-a^2)y^2+2a^2x_1x$$
$$+2a^2y_1y-a^2(x_1^2+y_1^2)=0$$

If this pair of straight lines includes an angle α, we have

$$\tan\alpha=\frac{2\sqrt{x_1^2y_1^2-(y_1^2-a^2)(x_1^2-a^2)}}{y_1^2-a^2+x_1^2-a^2}$$

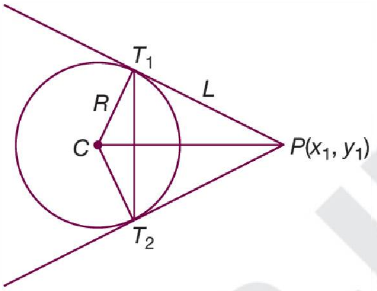
Furthermore, $(x_1^2+y_1^2-2a^2)^2\tan^2\alpha=4a^2(x_1^2+y_1^2-a^2)$

$\therefore$ The locus of (x_1, y_1)

$$(x^2+y^2-2a^2)^2\tan^2\alpha=4a^2(x^2+y^2-a^2).$$

CHORD OF CONTACT

A line joining the two points of contacts of two tangents drawn from a point outside the circle is called chord of contact from that point.



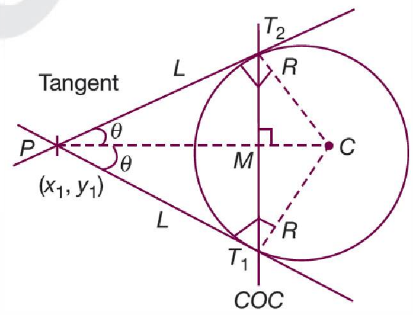
If two tangents PT_1 and PT_2 are drawn from the point $P(x_1, y_1)$ to the circle

$$S \equiv x^2+y^2+2gx+2fy+c=0,$$

Equation of the chord of contact T_1T_2 is $T=0$

$$\Rightarrow \quad xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

Given circle S: $x^2+y^2+2gx+2fy+c=0$



For point $P(x_1, y_1)$,

$$S_1:\ x_1^2+y_1^2+2gx_1+2fy_1+c>0$$

Length of tangent $L=\sqrt{S_1}$

$$PT_1=PT_2=L$$

From ΔPT_2C, we get $\tan\theta=\dfrac{R}{L}$

Angle between of tangent $=2\tan^{-1}\dfrac{R}{L}$

$$=\begin{cases}\tan^{-1}\dfrac{2LR}{L^2R^2} & \text{if } \dfrac{R}{L}<1\\[2ex] \pi+\tan^{-1}\dfrac{2LR}{L^2-R^2} & \text{if } \dfrac{R}{L}>1\end{cases}$$

Length of chord of contact $=2L\sin\theta$

$$=T_1T_2=\frac{2LR}{\sqrt{L^2+R^2}}$$

Area of $\Delta(PT_1T_2)$

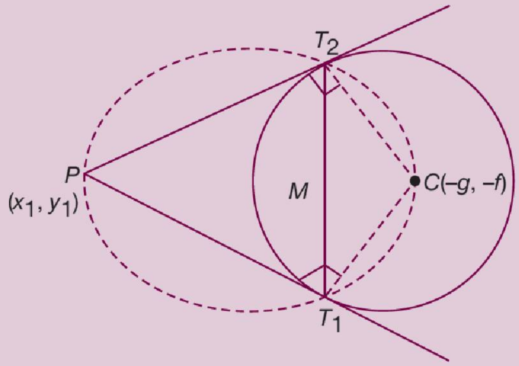
$$= \frac{1}{2}(PM)T_1T_2 = \frac{1}{2}(L\cos\theta)(2L\sin\theta)$$

$$= L^2\sin\theta\cos\theta = \frac{RL^3}{R^2+L^2}$$

Area formed by triangle formed by pair of tangents and chord of contact

$$\Delta(PT_1T_2) = \frac{RL^3}{R^2+L^2}$$

Area of quadrilateral formed by pair of tangents and pair of radii $PT_1\ CT_2 = LR$



Circle circumscribing the triangle formed by pair of tangents and chord of contact passes through the centre of given circle.

Circle circumscribing the $\Delta PT_1T_2 =$ Circle taking PC at diameter.

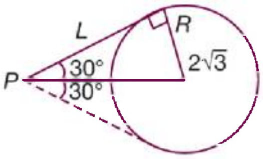
$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$$

ILLUSTRATION 49

Angle between the tangents drawn from a point P to the circle $x^2 + y^2 - 4x - 8y + 8 = 0$ is 60°, then find the length of the tangent drawn from P to the circle.

SOLUTION

Let draw a diagram



$$R = \sqrt{4+16-8} = 2\sqrt{3}$$

$$\tan\theta = \frac{R}{L} = \frac{1}{\sqrt{3}}$$

$\Rightarrow$ $L = \sqrt{3}R$, $L = 6$. (lengh of tangent)

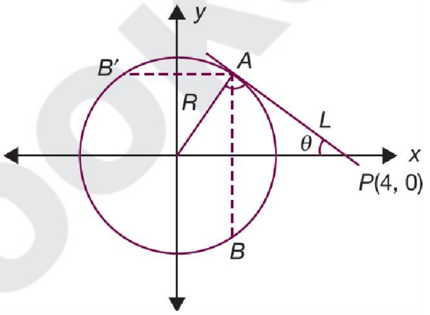
ILLUSTRATION 50

Tangent is drawn from the point $P(4, 0)$ to the circle $x^2 + y^2 = 8$ touches it at the point A in the Ist quadrant. Find the co-ordinates of another point B on the circle such that $AB = 4$.

SOLUTION

$$\tan\theta = \frac{R}{L} = \frac{2\sqrt{2}}{\sqrt{S_1}} = \frac{2\sqrt{2}}{\sqrt{16+0-8}} = 1$$

$\Rightarrow$ $\theta = 45°$



$\therefore$ Point $A(2\sqrt{2}\cos 45°,\ 2\sqrt{2}\sin 45°) = A(2, 2)$

B lies on circle

Let $B(2\sqrt{2}\cos\theta,\ 2\sqrt{2}\sin\theta)$ $A(2, 2)$

$$AB = 4 \Rightarrow 16 = AB^2$$

$$16 = (2 - 2\sqrt{2}\cos\theta)^2 + (2 - 2\sqrt{2}\sin\theta)^2$$

$$16 = 8 + 8 - 8\sqrt{2}(\cos\theta + \sin\theta)$$

$\Rightarrow$ $\cos\theta + \sin\theta = 0$

$\Rightarrow$ $\tan\theta = -1$

$$\theta = \frac{-\pi}{4}, \frac{3\pi}{4}$$

As we required $AB = 4$, then $B(2, -2)$ or $(-2, 2)$.

ILLUSTRATION 51

Tangents are drawn from $P(4, 5)$ to circle $2x^2 + 2y^2 - 8x - 4y - 22 = 0$ find the length of tangent, chord of contact, angle between tangents, area formed by pair of tangents and chord of contact.

SOLUTION

Given circle is $x^2 + y^2 - 4x - 2y - 11 = 0$; radius $R = 4$

Length of tangent $L = \sqrt{S_1} = 2$ from point $P(4, 5)$

WWW.EBOOKS.IN

$$\theta = \tan^{-1}\left(\frac{R}{L}\right) = \tan^{-1} 2$$

Angle between tangents $= 2\theta = 2\tan^{-1} 2$

$$= 180^\circ - 53^\circ = 127^\circ$$

Length of chord of contact $\ell_{coc} = \dfrac{2LR}{\sqrt{L^2 + R^2}} = \dfrac{8}{\sqrt{5}}$

Area of area formed by pair of tangents and chord of contact $= \dfrac{RL^3}{R^2 + L^2} = \dfrac{4.8}{16+4} = \dfrac{8}{5}$

ILLUSTRATION 52

Tangents PA and PB are drawn to circle $(x + 3)^2 + (y - 2)^2 = 1$ from point P lying on $y^2 = 4x$, then find the locus of circumcentre of ΔPAB.

SOLUTION

Point on curve $y^2 = 4x$, $P(t^2, 2t)$; $C(-3, 2)$ = centre of given circle.

Circumcentre of $\Delta PAB = \left(\dfrac{t^2 - 3}{2}, t + 1\right)$ = midpoint of PC.

Eliminate 't', we get locus $2x + 3 = (y - 1)^2$

ILLUSTRATION 53

The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through a fixed point, find it.

SOLUTION

The chord of contact of tangent from (α, β) is

$$\alpha x + \beta y = 1 \qquad \ldots(1)$$

Also, (α, β) lies on $2x + y = 4$,

So, $2\alpha + \beta = 4$

$$\Rightarrow \quad \frac{\alpha}{2} + \frac{\beta}{4} = 1 \qquad (\because \alpha x + \beta y = 1)$$

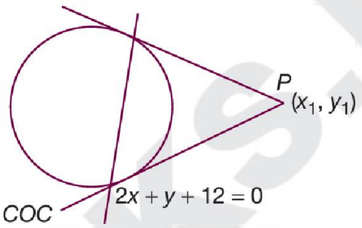
$$x = \frac{1}{2},\ y = \frac{1}{4}$$

Hence, chord of contacts pass through $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$ (fixed point)

ILLUSTRATION 54

Find the co-ordinates of the point of intersection of tangents at the points, where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$.

SOLUTION



Let the point of intersection of tangents is $P(x_1, y_1)$

Equation of chord of contact $T = 0$

$$xx_1 + yy_1 - 2(x + x_1) + \frac{3}{2}(y + y_1) - 1 = 0$$

$$\Rightarrow \quad (x_1 - 2)x + y(y_1 + \frac{3}{2}) - 2x_1 + \frac{3}{2}y_1 - 1 = 0$$

Given line $2x + y + 12 = 0$ (both lines represent the same straight line), then comparing their coefficient

$$\frac{x_1 - 2}{2} = \frac{y_1 + \frac{3}{2}}{1} = \frac{-2x_1 + \frac{3}{2}y_1 - 1}{12}$$

$$\Rightarrow \quad x_1 = 2y_1 + 5 \text{ and } 12x_1 - 24 = -4x_1 + 3y_1 - 2$$

Or $\quad 16x_1 - 3y_1 - 22 = 0$

On solving we get, $x_1 = 1, y_1 = -2$

$\therefore \quad P(1, -2)$

ILLUSTRATION 55

Find the equation of chord AB of circle $x^2 + y^2 = 2$ passing through $P(2, 2)$ such that $\dfrac{PB}{PA} = 3$.

SOLUTION

Any line passing through $P(2, 2)$ will be of the form,

$$\frac{x - 2}{\cos\theta} = \frac{y - 2}{\sin\theta} = r;$$

Any point on it $(2 + r\cos\theta, 2 + r\sin\theta)$

When this line cuts the circle $x^2 + y^2 = 2$

$$\Rightarrow \quad (r\cos\theta + 2)^2 + (r\sin\theta + 2)^2 = 2$$

$$\Rightarrow \quad r^2 + 4(\sin\theta + \cos\theta)\,r + 6 = 0;\ \text{let } PA = r_1;\ PB = r_2$$

Given $\dfrac{PB}{PA} = \dfrac{r_2}{r_1} = 3 \Rightarrow r_2 = 3r_1$

$r^2 + 4(\sin\theta + \cos\theta)r + 6 = 0$ has roots r_1 and r_2

$r_2 + r_1 = 4r_1 = -4(\sin\theta + \cos\theta)$

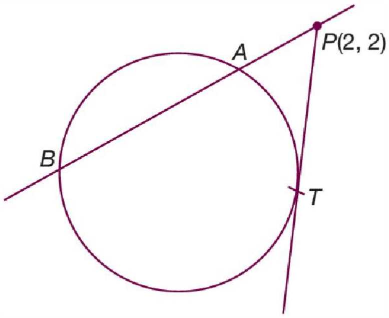
$r_1 r_2 = 3{r_1}^2 = 6 \Rightarrow {r_1}^2 = 2$

$(-\sin\theta - \cos\theta)^2 = 2$

$\Rightarrow \quad \sin 2\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

So, the required chord will be $y - 2 = 1\,(x - 2) \Rightarrow y = x$.

Alternate



$$PA \cdot PB = PT^2 = 2^2 + 2^2 - 2 = 6 \qquad \ldots(1)$$

$$\frac{PB}{PA} = 3 \qquad \ldots(2)$$

From (1) and (2), we have

$PA = \sqrt{2},\ PB = 3\sqrt{2}$

$\Rightarrow \quad AB = 2\sqrt{2}$

Now, the diameter of the circle is $2\sqrt{2}$ (as radius is $\sqrt{2}$).

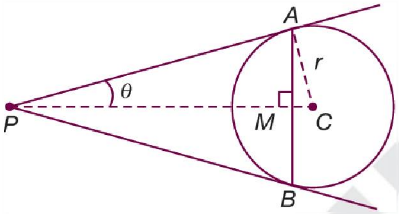
Hence chord AB is diameter of circle the line passes through the centre. Now equation of chord $y = x$.

ILLUSTRATION 56

From a point P outside a circle with centre at C, tangents PA and PB are drawn such that $\dfrac{1}{(CA)^2} + \dfrac{1}{(PA)^2} = \dfrac{1}{16}$, then find the length of chord AB.

SOLUTION

$\tan\theta = \dfrac{r}{PA}$



Given $\dfrac{1}{r^2} + \dfrac{1}{PA^2} = \dfrac{1}{16}$

$\Rightarrow \quad \dfrac{\cot^2\theta + 1}{(PA)^2} = \dfrac{1}{16}$

$\Rightarrow \quad PA\sin\theta = 4 = AM$

Length of chord $= AB = 2AM = 8$

ILLUSTRATION 57

The chord of contact of tangents from a point 'P' to a circle passes through Q. If ℓ_1 and ℓ_2 are the lengths of the tangents from P and Q to the circle, then find PQ.

SOLUTION

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ and circle is $x^2 + y^2 = a^2$

From $p(x_1, y_1)$ to the given circle, chord of contact is $xx_1 + yy_1 = a^2$

$\because$ It passes through $Q(x_2, y_2)$

$$\therefore \quad x_1x_2 + y_1y_2 = a^2 \qquad \ldots(1)$$

$$PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2(x_1x_2 + y_1y_2)$$

$$= (x_1^2 + y_1^2 - a^2) + (x_2^2 + y_2^2 - a^2)$$

Now, length of tangents from point $P(x_1, y_1)$ and $Q(x_2, y_2)$ on circle.

$$l_1 = \sqrt{x_1^2 + y_1^2 - a^2},\ l_2 = \sqrt{x_2^2 + y_2^2 - a^2}$$

Hence, $PQ = \sqrt{\ell_1^2 + \ell_2^2}$

ILLUSTRATION 58

The chord of contact of the circle $x^2 + y^2 = b^2$ is generated by a point on the circle $x^2 + y^2 = a^2$ and the chord of contact touches the circle $x^2 + y^2 = c^2$. Prove that a, b and c are in G.P.

SOLUTION

From any point $P(a\cos\theta, a\sin\theta)$ on circle $x^2+y^2=a^2$, chord of contact to circle $x^2+y^2=b^2$ is $T=0$

$\Rightarrow \quad x(a\cos\theta)+y(a\sin\theta)=b^2$

It is tangent to circle $x^2+y^2=c^2$ Centre $(0, 0)$ $R=c$ then $P=R$

$$\Rightarrow \quad \frac{|0+0-b^2|}{\sqrt{a^2\cos^2\theta+a^2\sin^2\theta}}=c$$

$\Rightarrow \quad b^2=ac$

$\therefore$ a, b, c are in G.P.

Alternate

The line $y=mx+c\sqrt{1+m^2}$ ...(1)

touches the circle $x^2+y^2=c^2$ for all m.

Let $P(a\cos\theta, a\sin\theta)$ be a point on $x^2+y^2=a^2$. The equation of the chord of contact of the point P with respect to the circle $x^2+y^2=b^2$ is

$$ax\cos\theta+ay\sin\theta=b^2 \quad \text{...(2)}$$

$\because$ (1) and (2) represent the same line.

$$a\sin\theta=-\frac{a\cos\theta}{m}=\frac{b^2}{c\sqrt{1+m^2}}$$

$\Rightarrow \quad m=-\cot\theta$ and $b^2=ac\sqrt{1+m^2}\,.\sin\theta$

Hence, $b^2=ac\sqrt{1+\cot^2\theta}\,.\sin\theta$

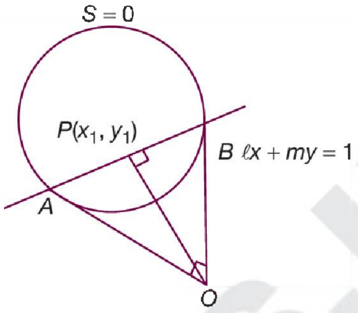
$\Rightarrow \quad b^2=ac \Rightarrow a, b$ and c are in G.P.

ILLUSTRATION 59

Let $S \equiv x^2+y^2+2gx+2fy+c=0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S, which subtends a right angle at the origin.

SOLUTION

Let, $\ell x+my=1$ be any chord of $S=0$, which subtends a right angle at O.



Let, A and B are the points of intersection of the chord and circle.

The join equation of the line OA and OB can be obtained by homogenising the equation $S=0$ with the help of $\ell x+my=1$.

$\therefore$ Their joining equation is

$$x^2+y^2+(2gx+2fy)(\ell x+my)+c(\ell x+my)^2=0 \quad \text{...(1)}$$

$\because$ The angle between OA and OB is 90°.

$\therefore$ The sum of coefficient of x^2+y^2 in (1) $=0$

Now $2+2g\ell+2fm+c(\ell^2+m^2)=0$ ...(2)

Let $P(x_1, y_1)$ be the foot of perpendicular from O on AB.

Then $\ell x_1+my_1=1$ and $mx_1=\ell y_1$.

$$\therefore \quad \ell=\frac{x_1}{x_1^2+y_1^2} \text{ and } m=\frac{y_1}{x_1^2+y_1^2}$$

Substituting in (2) we get,

$$2+2\frac{gx_1+fy_1}{x_1^2+y_1^2}+c\frac{x_1^2+y_1^2}{(x_1^2+y_1^2)^2}=0$$

$\Rightarrow \quad 2(x_1^2+y_1^2)+2(gx_1+fy_1)+c=0$

Hence, the locus of $P(x_1, y_1)$ is

$$2(x^2+y^2)+2(gx+fy)+c=0.$$

Test Your Concepts-III

(Solutions on page H.80)

1. If the line passing through $P=(8, 3)$ meets the circle $S=x^2+y^2-8x-10y+26=0$ at A and B, then find $PA \cdot PB$.
2. If the lengths of the tangents from the point (1, 2) to the circles $x^2+y^2+x+y-4=0$ and $3x^2+3y^2-x-y-\lambda=0$ are in the ratio 4:3, then find λ.

3. Find the area of the triangle formed by the line joining the origin to the points of intersection(s) of the line $x\sqrt{5}+2y=3\sqrt{5}$ and the circle $x^2+y^2=10$.
4. Find the locus of a point which moves such that the tangents from it to the two circles $x^2+y^2-5x-3=0$ and $3x^2+3y^2+2x+4y-6=0$ are equal.
5. Find the equations of the tangents from the point $A(3, 2)$ to the circle $x^2+y^2=4$ and hence find the angle between the pair of tangents.
6. If the tangents drawn from the point $P(1, 8)$ to the circle $x^2+y^2-6x-4y-11=0$ touch the circle at the points A and B, then find the equation of the circumcircle of the ΔPAB.
7. If the circle $x^2+y^2=a^2$ cuts off a chord of length $2b$ from the line $y=mx+c$, then find the condition.
8. Find the length of the tangent drawn from any point on the circle $x^2+y^2+2gx+2fy+\alpha=0$ to the circle $x^2+y^2+2gx+2fy+\beta=0$.
9. Find the area of circle, in which a chord of length $2\sqrt{3}$ units subtends an angle of 120° at its centre.
10. If the tangent at the point P on the circle $x^2+y^2+6x+6y=2$ meets the straight line $5x-2y+6=0$ at point Q on the y-axis, then find the length of PQ.

Answer Keys

1. 5

2. $\lambda=\frac{27}{4}$

3. 5

4. $17x+4y+3=0$

5. $\tan^{-1}\left(\frac{12}{5}\right)$

6. $x^2+y^2-4x-10y+19=0$

7. $c^2=(a^2-b^2)(m^2+1)$

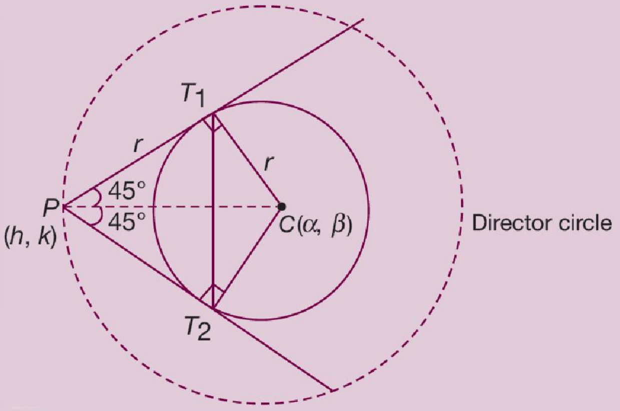
8. $\sqrt{\beta-\alpha}$

9. 4π sq units

10. 5 units

DIRECTOR CIRCLE

Locus of a point from which mutually perpendicular tangents are drawn to given curve is called as director circle of that curve.



From $P(h, k)$ two perpendicular tangents PT_1 and PT_2 are drawn.

$$\angle T_1PC=\angle T_2PC=45^\circ$$

$$PT_1=T_1C=r$$

Given circle $(x-\alpha)^2+(y-\beta)^2=r^2$; centre (α, β), radius $=r$

$$\because\ PC=\sqrt{2}r \Rightarrow (h-\alpha)^2+(k-\beta)^2=\left(\sqrt{2}r\right)^2$$

Now locus of $P(h, k)$ is

$$\boxed{(x-\alpha)^2+(y-\beta)^2=2r^2}\ \text{Director circle}$$

* Circle and its director circle are concentric

* $r_{\text{director circle}}=\sqrt{2}\,r_{\text{circle}}$

ILLUSTRATION 60

Find the equation of director circle of the circle $(x-2)^2+(y+1)^2=2$.

SOLUTION

Both circles are concentric, $C(2, -1)$;

$$\text{radius } r'=\sqrt{2}\,r=\sqrt{2}\cdot\sqrt{2}=2$$

Director circle $(x-2)^2+(y+1)^2=4$

$$\Rightarrow\quad x^2+y^2-4x+2y+1=0$$

ILLUSTRATION 61

Find the equation of director circle of the circle, whose diameters are $2x + 3y + 12 = 0$ and $x - 4y - 5 = 0$ and area is 154 square units. $\left(\text{use } \pi \text{ as } \dfrac{22}{7}\right)$

SOLUTION

Diameters intersect at centre (–3, –2) (Point of intersection of given diameters)

$$\text{Area} = \pi r^2 = 154$$

$$\Rightarrow \quad \frac{22}{7} r^2 = 154 \Rightarrow r = 7$$

Director circle whose centre C(–3, –2) and radius $r' = 7\sqrt{2}$

$$(x+3)^2 + (y+2)^2 = 98$$

ILLUSTRATION 62

If the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular, then find the relation.

SOLUTION

Origin lies on director circle $O(0, 0)$ and centre $C(r, h)$.

$$R' = \sqrt{2}\, R = \sqrt{2}\,\sqrt{r^2 + h^2 - h^2}$$

$$OC = R' = \sqrt{2}\, r$$

$$r^2 + h^2 = 2r^2$$

$$h^2 = r^2 \Rightarrow r = \pm h$$

Alternate

The combined equation of the tangents drawn from (0, 0) to $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ is $SS_1 = T^2$

$$(x^2 + y^2 - 2rx - 2hy + h^2)h^2 = (-rx - hy + h^2)^2$$

This equation represents a pair of perpendicular straight lines.

If the coefficient of x^2 + coefficient of $y^2 = 0$,

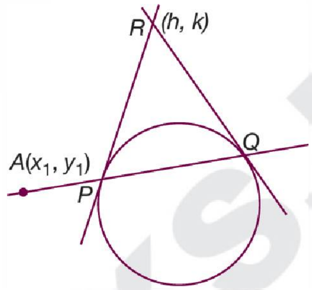
i.e., $2h^2 - r^2 - h^2 = 0$

$\Rightarrow \quad r^2 = h^2$

$\Rightarrow \quad r = \pm h$

POLE AND POLAR

Let any straight line through the given point $A(x_1, y_1)$ intersect the given circle $S = 0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R, then the locus of point R is called **polar** of the point A and point A is called the **pole**, with respect to the given circle.



Circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Pole (x_1, y_1)

The equation of the polar is

$$T = 0 \Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

(i) If the point is outside the circle, then the equation of polar and chord of contact is same. So, the chord of contact is polar.

(ii) If the point is inside the circle, then the chord of contact does not exist but polar exists.

(iii) If the point lies on the circle then the polar and tangent on that point are same.

(iv) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P and Q are conjugate points of each other w.r.t. the given circle.

(v) If pole of a line w.r.t. a circle lies on IInd line, then the pole of IInd line lies on Ist line and hence both lines are the conjugate lines of each other w.r.t. the given circle.

Pole of a Given Line with respect to a Circle

To find the pole of a line, we assume that, the coordinates of the pole. Then from these co-ordinates, we find the polar. This polar and the given line represent the same line. Then by comparing the coefficients of similar terms, we can get the co-ordinates of the pole. The pole of $\ell x + my + n = 0$ w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\dfrac{-\ell a^2}{n}, \dfrac{-ma^2}{n}\right)$

ILLUSTRATION 63

Find the equation of the polar of the point (2, –1) with respect to the circle $x^2 + y^2 - 3x + 2y + 8 = 0$.

SOLUTION

Equation of polar $T = 0$

$$\Rightarrow \quad xx_1 + yy_1 - \frac{3}{2}(x + x_1) + (y + y_1) + 8 = 0$$

$$x_1 = 2, y_1 = -1$$

$$2x - y - \frac{3}{2}(x + 2) + y - 1 + 8 = 0$$

or $\quad 4x - 2y - 3x - 6 + 2y - 2 + 16 = 0$

or $\quad x + 8 = 0$

ILLUSTRATION 64

Find the pole of the line $3x + 5y + 17 = 0$, with respect to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$.

SOLUTION

Let the pole is (x_1, y_1).
Polar $T = 0$

$$xx_1 + yy_1 + 2(x + x_1) + 3(y + y_1) + 9 = 0$$

$$x(x_1 + 2) + y(y_1 + 3) + 2x_1 + 3y_1 + 9 = 0$$

Given line is $3x + 5y + 17 = 0$

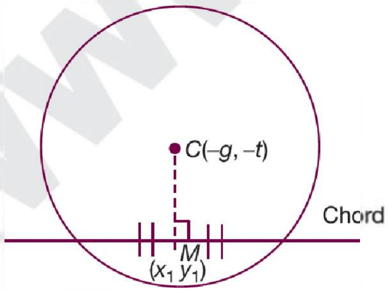
Both represent the same straight line

$$\frac{x_1 + 2}{3} = \frac{y_1 + 3}{5} = \frac{2x_1 + 3y_1 + 9}{17}$$

$$5x_1 = 3y_1 - 1 \text{ and } 11x_1 - 9y_1 + 7 = 0$$

We get $x_1 = 1$ and $y_1 = 2$ and pole $(1, 2)$.

CHORD WITH GIVEN MIDDLE POINT



Here, we are going to discuss about the chord, whose midpoint is known.

As we know that every chord is bisected by perpendicular from centre to given chord, the equation of the chord of the circle S: $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its midpoint $M(x_1, y_1)$ is

$$(y - y_1) = \frac{-1}{m_{CM}}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

On simplification, this can be put in the form,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$

which is designated by $T = S_1$.

Working Rule

Ist Approach

Equation of chord, whose midpoint $M(x_1, y_1)$ is $T = S_1$

OR

IInd Approach

Equation of chord, whose midpoint $M(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{m_{CM}}(x - x_1)$$

$$\Rightarrow \quad y - y_1 = -\frac{(x_1 + g)}{(y_1 + f)}(x - x_1)$$

NOTE:

If the point (x_1, y_1) lies inside the circle, then the chord passing through $(x_1\ y_1)$ and

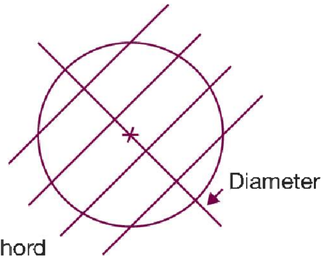
(i) At maximum distance from centre.

OR

(ii) Of the shortest length.
It is the same chord, whose midpoint must be the given point (x_1, y_1).
Hence, the equation of chord $T = S_1$.

DIAMETER OF CIRCLE

The locus of middle points of a system of parallel chords of a circle is called the diameter of a circle.



The diameter of the circle $x^2 + y^2 = r^2$ corresponding to the system of parallel chords $y = mx + c$ is $x + my = 0$.

PROOF:

Let the midpoint is $M(h, k)$.

Chord equation $T = S_1$

$$xh + ky - r^2 = h^2 + k^2 - r^2$$

$$y = \left(-\frac{h}{k}\right)x + \left(\frac{h^2 + k^2}{k}\right)$$

$$\therefore \quad m = -\frac{h}{k}$$

Locus $m = -\frac{x}{y}$ $\Rightarrow x + my = 0$, which is the diameter of the circle.

NOTE:

(i) Every diameter passes through the centre of the circle.

(ii) A diameter is perpendicular to the system of parallel chords.

(iii) Length of diameter of a circle is double the radius of circle.

ILLUSTRATION 65

Find the co-ordinates of the point from which the tangents drawn to the circle $x^2 + y^2 - 6x - 4y + 3 = 0$ such that the midpoint of its chord of contact is (1, 1).

SOLUTION

Let the required point be $P(x_1, y_1)$. The equation of the chord of contact P with respect to the given circle is $T = 0$.

$$xx_1 + yy_1 - 3(x + x_1) - 2(y + y_1) + 3 = 0$$

$$\Rightarrow \quad (x_1 - 3)x + (y_1 - 2)\,y = 3x_1 + 2y_1 - 3 \qquad \ldots(1)$$

The equation of the chord with midpoint (1, 1) is $T = S_1$

$$x + y - 3(x + 1) - 2(y + 1) + 3 = 1 + 1 - 6 - 4 + 3$$

$$\Rightarrow \quad 2x + y = 3 \qquad \ldots(2)$$

Equation (1) and (2) represent the same straight line.

Equating the ratios of the coefficients of x, y and the constant terms,

$\frac{x_1 - 3}{2} = \frac{y_1 - 2}{1} = \frac{3x_1 + 2y_1 - 3}{3}$ and solving for x and y we get, $x_1 = -1$, $y_1 = 0$. Hence required point (–1, 0).

ILLUSTRATION 66

Find the locus of middle point of chord of circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which pass through a fixed point (a, b).

SOLUTION

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

Let the middle point of a chord be $P(x_1, y_1)$.

Then, the equation of chord is $T = S_1$

$$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f\,(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

It passes through (a, b), which satisfies the condition,

$ax_1 + by_1 + g(a + x_1) + f(b + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$

Locus of (x_1, y_1) as replace x_1 by x and y_1 by y

$$ax + by + g(a + x) + f(b + y) = x^2 + y^2 + 2gx + 2fy$$

$$\Rightarrow \quad x^2 + y^2 + x(g - a) + y(f - b) - ag - bf = 0$$

ILLUSTRATION 67

Consider the circle, $x^2 + y^2 + 6x + 3y - 11 = 0$ find the chord passing through (1, –1) of the shortest length.

SOLUTION

$$S_1 = 1 + 1 + 6 - 3 - 11 = -6$$

$S_1 < 0$; point lies inside. Then, the chord of shortest length, when (1, –1) is the midpoint of chord. Equation of chord $T = S_1$

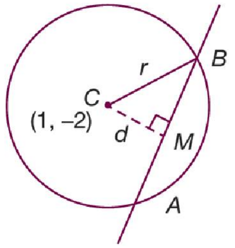
$$x(1) + y(-1) + 3(x + 1) + \frac{3}{2}(y - 1) - 11$$
$$= 1 + 1 + 6 - 3 - 11$$

$$\Rightarrow \quad 8x + y - 7 = 0$$

ILLUSTRATION 68

Find the length and the midpoint of the chord $4x - 3y + 5 = 0$ w.r.t the circle $x^2 + y^2 - 2x + 4y - 20 = 0$.

SOLUTION



Midpoint of chord = Foot of perpendicular from centre to given chord

$$\frac{x-1}{4}=\frac{y+2}{-3}=-\frac{(4+6+5)}{16+9}=-\frac{3}{5};$$

We get mid point $M\left(-\frac{7}{5},-\frac{1}{5}\right)$

Now length of perpendicular from centre C to chord

$$d=CM=\left|\frac{4+6+5}{\sqrt{16+9}}\right|=3$$

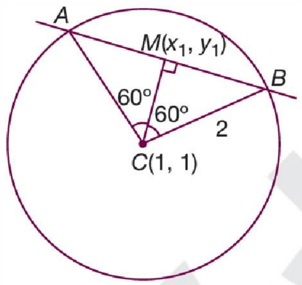
Length of chord $=2MB=2\sqrt{r^2-d^2}$

$$=2\sqrt{25-9}=8$$

ILLUSTRATION 69

Find the locus of the midpoint of the chord of the circle $x^2+y^2-2x-2y-2=0$, which makes an angle of 120° at the centre.

SOLUTION



For given circle, Centre (1, 1) and radius = 2 = CB

$\angle BCM=60°$ and $CB\cos 60°=CM$

$\Rightarrow\quad 1=CM$

$\Rightarrow\quad (x_1-1)^2+(y_1-1)^2=1$

$$x_1^2+y_1^2-2x_1-2y_1+1=0$$

Now replace x_1 by x and y_1 by y

Locus of $M(x_1,y_1)$ $x^2+y^2-2x-2y+1=0$

ILLUSTRATION 70

From point (p, q), two distinct chords are drawn to circle $x^2+y^2=px+qy$ $(pq\neq 0)$, which are bisected by x-axis. Find the condition on p, q. **[IIT–2004]**

SOLUTION

Let the midpoint $M(\alpha,0)$ be on – x-axis.

Required equation of chord $T=S_1$

$$\Rightarrow\quad x(\alpha)+y(0)-\frac{p}{2}(x+\alpha)-\frac{q}{2}(y+0)=\alpha^2+0-p\alpha-0$$

It passes through point (p, q)

$$p\alpha-\frac{p}{2}(p+\alpha)-\frac{q^2}{2}=\alpha^2-p\alpha$$

$\Rightarrow\quad 2p\alpha-p^2-p\alpha-q^2=2\alpha^2-2p\alpha$

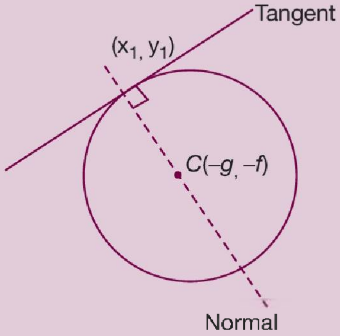
$\Rightarrow\quad 2\alpha^2-3p\alpha+p^2+q^2=0$ (it is quadratic in α)

It is given that, there are two distinct chords, which means two midpoints, so α has two distinct values, hence discriminant $=B^2-4AC>0$

$\therefore\quad D>0\Rightarrow 9p^2-4(2)(p^2+q^2)>0\Rightarrow p^2>8q^2$ is required condition.

NORMAL OF CIRCLE

- The normal at any point on a curve is a line, which is perpendicular to the tangent to the curve at point of contact.
- Every normal of circle always passes through centre.



- Equation of normal to the circle $x^2+y^2+2gx+2fy+c=0$ at (x_1,y_1) is $y-y_1=\left(\frac{y_1+f}{x_1+g}\right)(x-x_1)$
- The greatest and the least distance of a point A from a circle with centre C and radius r lies along their normal. Hence, they are $AC+r$ (greatest distance AQ) and $AC-r$ (least distance AP), respectively.

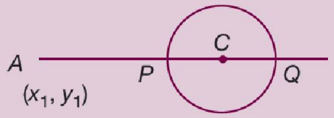


ILLUSTRATION 71

Find equation of normal to circle $x^2 + y^2 - 4x - 6y + 9 = 0$, whose slope is $\sqrt{3}$. Also find the co-ordinates of point, where it cuts the circle.

SOLUTION

Centre C(2, 3) and radius $r = 2$

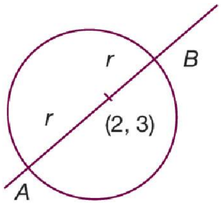
As we know, every normal passes through centre of circle so normal has equation.

$$y - 3 = \sqrt{3}\,(x - 2)$$

$$\Rightarrow \quad y = \sqrt{3}x + 3 - 2\sqrt{3}$$

By solving normal with circle, we get the points $A(3, 3 + \sqrt{3}\,)$ and $B(1, 3 - \sqrt{3}\,)$

Alternate



Equation of Normal

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = \pm r \text{ (Parametric form)}$$

where $r = 2$; $\theta = 60°$

Point $(2 \pm 2\cos 60°, 3 \pm 2\sin 60°)$ or $(2 \pm 1, 3 \pm \sqrt{3}\,)$

Point $A(3, 3 + \sqrt{3}\,)$ and $B(1, 3 - \sqrt{3}\,)$

ILLUSTRATION 72

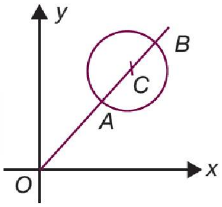
In the circle $x^2 + y^2 - 12x - 4y + 30 = 0$, find the maximum/minimum distance of origin.

SOLUTION

Origin (0, 0), $S_1 = 30 > 0$, which means that the origin lies outside the circle.

Centre C(6, 2), $r = \sqrt{36 + 4 - 30} = \sqrt{10}$

$OC = 2\sqrt{10}$



Minimum/maximum distance from origin lie along normal passing through origin.

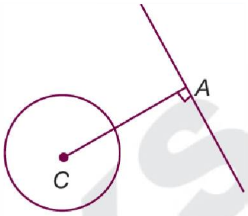
Maximum distance $OB = OC + r = 2\sqrt{10} + \sqrt{10} = 3\sqrt{10}$

Minimum distance $OA = OC - r = 2\sqrt{10} - \sqrt{10} = \sqrt{10}$

ILLUSTRATION 73

Find point on line $x + y + 13 = 0$, which is nearest to circle $x^2 + y^2 + 4x + 6y - 5 = 0$.

SOLUTION



$C(-2, -3)$; $r = \sqrt{4 + 9 + 5} = 3\sqrt{2}$

Length of perpendicular from centre to given line

$$P = \left|\frac{-2-3+13}{\sqrt{2}}\right| = 4\sqrt{2}$$

Clearly $p > r$, which implies that the line is away from given circle. (no intersection)

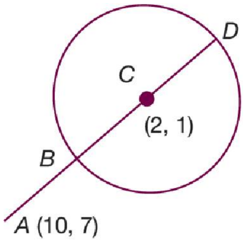
For nearest point, A = foot of perpendicular from the centre C(–2, –3) to the line.

$$\frac{x+2}{1} = \frac{y+3}{1} = -\frac{(8)}{2} = -4 \text{, we get } A(-6, -7)$$

ILLUSTRATION 74

Find the longest and the shortest distances from (10, 7) to the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.

SOLUTION



Draw the diameter through $A(10, 7)$, meeting the circle at B and D. AD is the longest distance and AB is the shortest distance of the point A to the circle.

The equation of the diameter is

$$4(y - 1) = 3(x - 2)$$

By solving the two equations, we get the co-ordinates of the points D and B as (–2, –2) and (6, 4).

$$\therefore \quad AD = \sqrt{(10+2)^2 + (7+2)^2} = \sqrt{144 + 81} = 15$$

$$AB = \sqrt{(10-6)^2 + (7-4)^2} = \sqrt{16 + 9} = 5$$

Alternate

Centre (2, 1), radius $(r) = 5$, $A(10, 7)$, $C(2, 1)$; $AC = 10$

Longest and shortest distance from $A(10, 7)$ lie along normal.

Longest distance $= AD = AC + r = 10 + 5 = 15$

Shortest distance $= AB = AC - r = 10 - 5 = 5$

Test Your Concepts-IV

(Solutions on page H.82)

1. Find the equation of the normal at the point $(4, -1)$ of the circle $x^2 + y^2 - 40x + 10y + 153 = 0$.
2. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point $(5, 6)$.
3. Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose midpoint is $(1, -1)$.
4. Find the locus of midpoints of the chords of the circle $x^2 + y^2 - 2x - 6y - 10 = 0$, which pass through the origin.
5. If the straight line $ax + by = 2$; $a, b \neq 0$ touches the circle $x^2 + y^2 - 2x = 3$ and it is normal to the circle $x^2 + y^2 - 4y = 6$, then find the respective values of a and b.
6. Find the locus of midpoints of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point $(c, 0)$.
7. If $x + 2y = 5$ is the tangent to the circle $x^2 + y^2 = 5$, then find the equation of normal at their point of contact.
8. If the line $2x - y + 1 = 0$ is tangent to the circle at the point $(2, 5)$ and the centre of the circles lies on $x - 2y = 4$, then find the radius of the circle.

Answer Keys

1. $x + 4y = 0$

2. $14x - 5y - 40 = 0$

3. $4x + 3y - 1 = 0$

4. $x^2 + y^2 - x - 3y = 0$

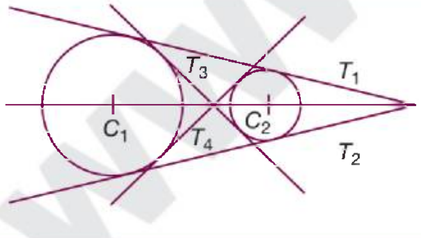
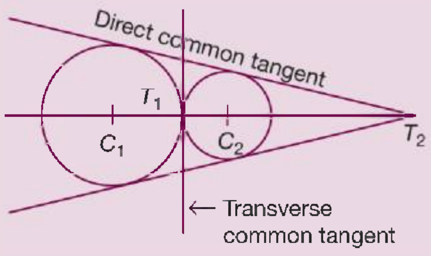
5. $a = -\frac{4}{3}, b = 1$

6. $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$

7. $y = 2x$

8. $3\sqrt{5}$ units

RELATIVE POSITION OF TWO CIRCLES

Position of Two Circle	Number of Common Tangents	Condition
	4 T_1 and T_2 → Two Direct common tangents (External common tangents) (Longitudinal common tangents) T_3 and T_4 → Two Indirect common tangents (Internal common tangents) (Transverse common tangents)	$C_1C_2 > r_1 + r_2$ Circles are away from each other. (no intersection)
 Direct common tangent T_1 C_1 C_2 T_2 ← Transverse common tangent	3 (Two direct common tangents and 1 indirect common tangent)	$C_1C_2 = r_1 + r_2$ Externally touches each other.

(Continued)

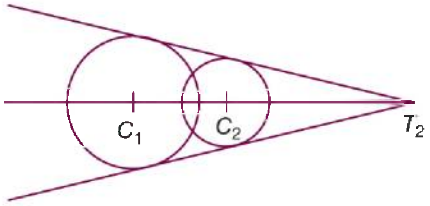
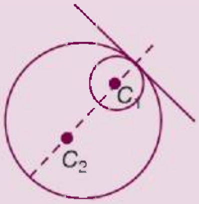
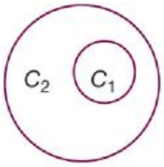
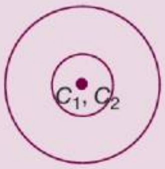
Position of Two Circle	Number of Common Tangents	Condition
	2	$\lvert r_1 - r_2 \rvert < C_1C_2 < r_1 + r_2$ Circles intersect.
	1	$C_1C_2 = \lvert r_1 - r_2 \rvert$ Circles touch internally.
	0	$0 < C_1C_2 < \lvert r_1 - r_2 \rvert$ One circle lies completely inside other circle.
	0	$C_1C_2 = 0$ Concentric circles.

ILLUSTRATION 75

Comment on the relative position of given circles.

(i) $x^2 + y^2 - 2x - 4y = 0$
$x^2 + y^2 - 8y - 4 = 0$

(ii) $x^2 + y^2 - 2x - 6y + 9 = 0$
$x^2 + y^2 - 10x + 16 = 0$

SOLUTION

(i) The given circles are $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$.

Centres of given circles are $C_1(1,2)$ and $C_2(0, 4)$.

$$r_1 = \sqrt{5}\,;\ r_2 = 2\sqrt{5}$$

$$C_1C_2 = \lvert r_1 - r_2 \rvert$$

$\therefore$ The circles internally touch each other.

(ii) The given circles are $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 - 10x + 16 = 0$.

Centres of the given circles are $C_1(1,3)$ and $C_2(5, 0)$.

$$r_1 = 1;\ r_2 = 3$$

$$C_1C_2 > r_1 + r_2$$

$\therefore$ Both the circles are away from each other.

ILLUSTRATION 76

If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then find r.

SOLUTION

Let d be the distance between the centres $C_1(1, 3)$ and $C_2(4, -1)$ of two circles of radii r_1 and r_2. Such that, $d = 5$, $r_1 = r$ and $r_2 = 3$.

These circle intersect at two distinct point if,

$$\lvert r_1 - r_2 \rvert < d < r_1 + r_2$$

Thus, $\lvert r - 3 \rvert < 5 < r + 3$

$\Rightarrow \quad -2 < r < 8$ and $r > 2 \Rightarrow 2 < r < 8$

ILLUSTRATION 77

Find the number of common tangents that can be drawn to the circles

$$x^2 + y^2 - 4x - 6y - 3 = 0 \text{ and } x^2 + y^2 + 2x + 2y + 1 = 0$$

SOLUTION

The two circles are

$$x^2 + y^2 - 4x - 6y - 3 = 0 \quad \text{and} \quad x^2 + y^2 + 2x + 2y + 1 = 0$$

Centre: $C_1 = (2, 3)$ $\qquad C_2 = (-1, -1)$

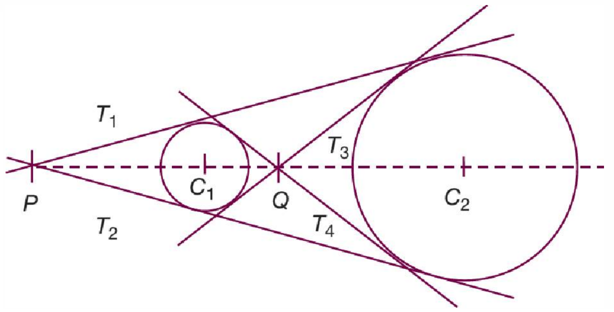
Radii: $r_1 = 4$ $\qquad r_2 = 1$

We have $C_1C_2 = 5 = r_1 + r_2$ and they touch externally.

$\therefore$ There are 3 common tangents to the given circles.

Common Tangents

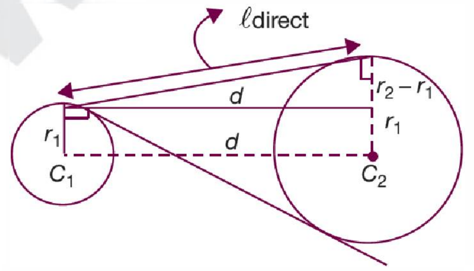
The points Q and P (points of intersection of indirect and direct common tangents) divides C_1C_2 internally and externally in the ratio $r_1:r_2$.



- P divides C_1C_2 in ratio $r_1:r_2$ externally.
- Q divides C_1C_2 in ratio $r_1:r_2$ internally.
- P and Q are harmonic conjugate of each other with respect to line segment C_1C_2. Hence, PC_1, PQ and PC_2 are in H.P. (if $r_1 < r_2$)

Equation of Direct Tangent T_1, T_2	Equation of Indirect Tangents T_3, T_4
Let $P(x_1, y_1)$ $y - y_1 = m(x - x_1)$ Now use condition for tangency $P = r$ with respect to any circle, get two values of m.	Let $Q(x_2, y_2)$ $y - y_2 = m(x - x_2)$ Now use condition for tangency $P = r$ with respect to any circle, get two values of m.

LENGTH OF DIRECT TANGENT

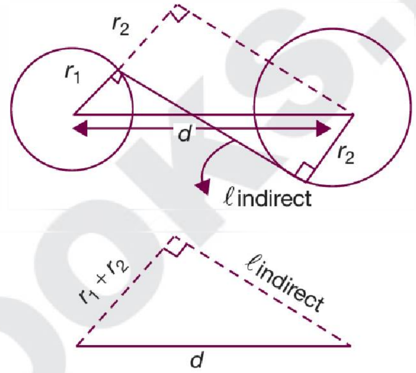


Length of direct common tangent

$$\ell_{\text{direct}} = \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$$

$$= \sqrt{d^2 - (r_1 - r_2)^2}\text{, where } C_1C_2 = d$$

LENGTH OF INDIRECT TANGENT



$$\ell_{\text{indirect}} = \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2} = \sqrt{d^2 - (r_1 + r_2)^2}$$

Clearly, we can say that $\ell_{\text{direct}} > \ell_{\text{indirect}}$

ILLUSTRATION 78

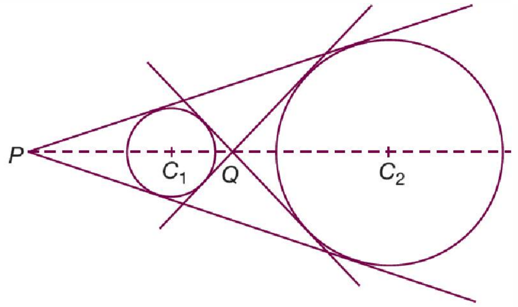
Consider the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$. Find the equation of common tangent and their length.

SOLUTION

For given circles, let get centres and radius.

First circle $C_1(1, 3)$ $r_1 = 1$, for second circle $C_2(-3, 1)$, $r_2 = 3$

$$d = C_1C_2 = \sqrt{16 + 4} = 2\sqrt{5} > r_1 + r_2$$



P divides C_1C_2 in 1:3 externally

$$\therefore \quad P\left(\frac{-3-3}{1-3}, \frac{1-9}{1-3}\right) = (3, 4)$$

Q divides C_1C_2 in 1:3 internally

$$\therefore \quad Q\left(\frac{-3+3}{1+3},\frac{1+9}{1+3}\right)=\left(0,\frac{5}{2}\right)$$

From $P(3, 4)$, tangent $y - 4 = m(x - 3)$

$\Rightarrow \quad mx - y + 4 - 3m = 0.$

$C_1(1, 3)$ and $r_1 = 1.$

Condition for tangency $P = r \Rightarrow \left|\frac{m-3+4-3m}{\sqrt{m^2+1}}\right| = 1$

$\Rightarrow \quad (1 - 2m)^2 = m^2 + 1$

$\Rightarrow \quad 3m^2 - 4m = 0 \Rightarrow m = 0; m = \frac{4}{3}$

Direct tangents $y = 4$ and $3y = 4x$

From $Q(0, \frac{5}{2})$ tangent $y - \frac{5}{2} = m(x - 0)$

$\Rightarrow \quad 2mx - 2y + 5 = 0$

$C_1(1, 3), r_1 = 1$

Condition for tangency

$$P = r \Rightarrow \left|\frac{2m-6+5}{\sqrt{4m^2+4}}\right| = 1 \text{ or } (4m^2 + 1 - 4m) = 4m^2 + 4$$

We get, $m = -\frac{3}{4}$ or $m \to \infty$

Indirect tangent $x = 0$ and $4y + 3x - 10 = 0$

We have $d = c_1c_2 = 2\sqrt{5}, r_1 = 1, r_2 = 3$

$$\ell_{\text{direct tangent}} = \sqrt{d^2-(r_1-r_2)^2}$$
$$= \sqrt{20-4} = 4$$
$$\ell_{\text{indirect tangent}} = \sqrt{d^2-(r_1+r_2)^2}$$
$$= \sqrt{20-16} = 2$$

ILLUSTRATION 79

Prove that the circles $x^2 + y^2 - 4x - 6y + 9 = 0$ and $x^2 + y^2 + 2x + 2y - 7 = 0$ touch. Find the co-ordinate of the point of contact and also the equation of the common tangent at point of contact.

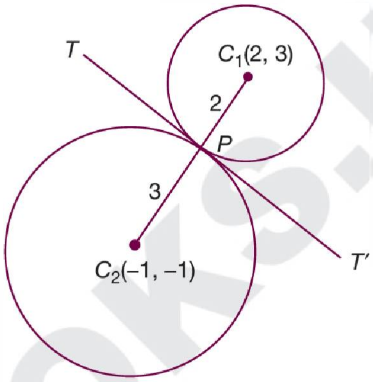
SOLUTION

The centres of the given circles are $C_1(2, 3)$ and $C_2(-1, -1)$.

The radii of the circles are $r_1 = \sqrt{4+9-9} = 2$ and $r_2 = \sqrt{1+1+7} = 3$.

Furthermore, $C_1C_2 = \sqrt{(2+1)^2+(3+1)^2} = 5.$

$\because r_1 + r_2 = 2 + 3 = 5 = C_1C_2$, the circles touch externally at a point $P(\alpha, \beta)$.



Furthermore, $C_1P : C_2P = 2:3$ and so

$$\alpha = \frac{2(-1)+3(2)}{2+3} = \frac{4}{5}, \beta = \frac{2(-1)+3(3)}{2+3} = \frac{7}{5}$$

i.e., P is the point $\left(\frac{4}{5},\frac{7}{5}\right)$

Let TPT' be the common tangent at P.

Slope of C_1C_2 is $\frac{4}{3}$

Hence, the slope of TPT', which is perpendicular to C_1C_2 is $\frac{-3}{4}$

$\therefore$ The equation of TPT' is $\left(y-\frac{7}{5}\right) = -\left(\frac{3}{4}\right)\left(x-\frac{4}{5}\right)$

i.e., $\quad 4(5y - 7) + 3(5x - 4) = 0$

$\Rightarrow \quad 3x + 4y - 8 = 0$

ILLUSTRATION 80

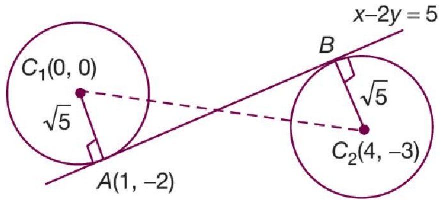
Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find its point of contact.

SOLUTION

First circle $x^2 + y^2 = 5$, centre $C_1(0, 0)$ and radius $r_1 = \sqrt{5}$

Second circle $x^2 + y^2 - 8x + 6y + 20 = 0$, $C_2(4, -3)$, $r_2 = \sqrt{5}$

$\therefore r_1 = r_2$ and $C_1C_2 > r_1 + r_2$



The equation of the tangent at $A(1, -2)$ to first circle is $x - 2y = 5$.

It is also tangent to second circle.

Let the point of contact be B, it is foot of perpendicular from $C_2(4, -3)$ on tangent $x - 2y - 5 = 0$

$$\frac{x-4}{1} = \frac{y+3}{-2} = -\frac{(4+6-5)}{1+4} = -1$$

$\therefore \quad B(3, -1)$

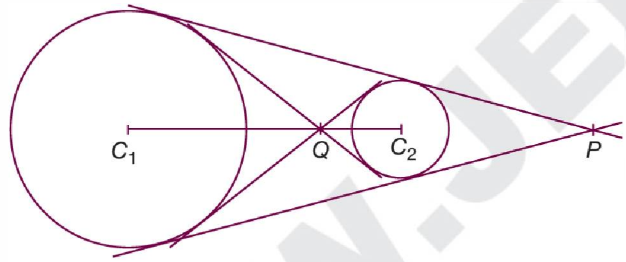
Required point of contact is $(3, -1)$.

ILLUSTRATION 81

Find the equations to the common tangents to the circles $x^2 + y^2 = 25$ and $(x - 12)^2 + y^2 = 9$.

SOLUTION

For $x^2 + y^2 = 25$, $C_1(0, 0)$ and $r_1 = 5$



$$(x - 12)^2 + y^2 = 9,\ C_2(12, 0);\ r_2 = 3$$

$$C_1C_2 = 12$$

$$C_1C_2 > r_1 + r_2$$

$\therefore$ The two circles lie completely outside of each other.

$\therefore$ There are four common tangents to the circles — two direct and two transverse tangents.

Let $C_1\,C_2$ be divided internally and externally in the ratio 5:3.

Internally at the point $Q\left(\frac{15}{2}, 0\right)$ and externally at the point $P(30, 0)$.

The straight line through $\left(\frac{15}{2}, 0\right)$ with a slope m is $y = m\left(x - \frac{15}{2}\right)$, or $2mx - 2y - 15m = 0$

This straight line will be a tangent to the circle $x^2 + y^2 = 25$, using condition for tangency $p = r_1$

$\left|\frac{15m}{\sqrt{4m^2+4}}\right| = 5$, on solving $m = \pm\frac{2}{\sqrt{5}}$

$\therefore$ The transverse common tangents are given by $y = \pm\frac{2}{\sqrt{5}}\left(x - \frac{15}{2}\right)$

Now, the straight line through $(30, 0)$ with a slope m is given by

$$y = m'(x - 30)$$

i.e., $\quad m'x - y - 30\,m' = 0$

This straight line will be a tangent to the circle $x^2 + y^2 = 25$, using condition for tangency $p = r_1$

$\left|\frac{30m'}{\sqrt{m'^2+1}}\right| = 5$; on solving $m' = \pm\frac{1}{\sqrt{35}}$

$\therefore$ The equation of the two direct common tangents are

$$y = \pm\frac{1}{\sqrt{35}}(x - 30)$$

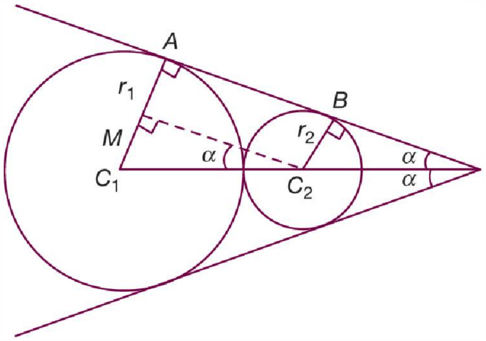
ILLUSTRATION 82

Two circles with radii r_1 and r_2, $r_1 > r_2 \geq 2$, touch each other externally. If 'θ' be the angle between the direct common tangents, then find θ.

SOLUTION

Let understand it, by drawing it.

Angle between direct common tangents $= \theta = 2\alpha$



Both circle externally touch $C_1C_2 = r_1 + r_2$ draw a perpendicular C_2M from C_2 to C_1A

$$\sin\alpha = \frac{C_1M}{C_1C_2} = \frac{r_1 - r_2}{r_1 + r_2}$$

Now $\theta = 2\sin^{-1}\left(\dfrac{r_1 - r_2}{r_1 + r_2}\right)$

ILLUSTRATION 83

Prove that $x^2 + y^2 = a^2$ and $(x - 2a)^2 + y^2 = a^2$ are two equal circles touching each other. Find the equation of circle of the same radius touching both the circles.

SOLUTION

Given circles are

$$x^2 + y^2 = a^2 \qquad \ldots(1)$$

$$(x - 2a)^2 + y^2 = a^2 \qquad \ldots(2)$$

Let A and B be the centres and r_1 and r_2 the radii of the two circles (1) and (2), respectively. Then

$$A(0, 0),\ B(2a, 0),\ r_1 = a,\ r_2 = a$$

Now, $AB = \sqrt{(0 - 2a)^2 + 0^2} = 2a = r_1 + r_2$

Hence, the two circles touch each other externally.
Let the equation of the circle having same radius 'a' and touching the circles (1) and (2) externally

$$(x - \alpha)^2 + (y - \beta)^2 = a^2 \qquad \ldots(3)$$

Its centre C is (α, β) and radius $r_3 = a$.

$\because$ Circle (3) touches the circle (1),

$$AC = r_1 + r_3 = 2a.$$

$$\Rightarrow \quad AC^2 = 4a^2 \Rightarrow \alpha^2 + \beta^2 = 4a^2 \qquad \ldots(4)$$

Again since circle (3) touches the circle (2),

$$BC = r_2 + r_3 \Rightarrow BC^2 = (r_2 + r_3)^2$$

$$\Rightarrow \quad (2a - \alpha)^2 + \beta^2 = (a + a)^2$$

$$\Rightarrow \quad \alpha^2 + \beta^2 - 4a\,\alpha = 0$$

$$\Rightarrow \quad 4a^2 - 4a\alpha = 0 \qquad \text{(From (4))}$$

$\Rightarrow \quad \alpha = a$ and from (4), we have $\beta = \pm\sqrt{3}a$

Hence, the required circles are

$$(x - a)^2 + \left(y \mp a\sqrt{3}\right)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ax \mp 2\sqrt{3}ay + 3a^2 = 0$$

ILLUSTRATION 84

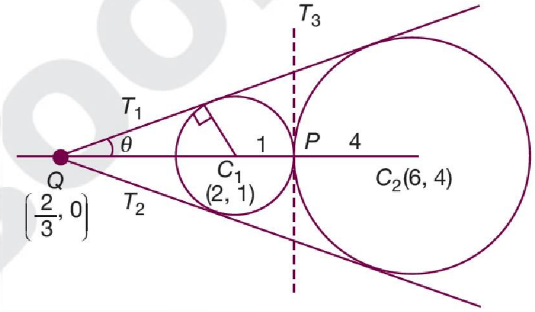
Find the co-ordinates of the point, at which the circles $x^2 + y^2 - 4x - 2y + 4 = 0$ and $x^2 + y^2 - 12x - 8y + 36 = 0$ touch each other. Also find the equations of common tangents touching the circles in distinct points.

SOLUTION

First Circle	**Second Circle**
Centre $C_1(2, 1)$	$C_2(6, 4)$
Radius $r_1 = \sqrt{4 + 1 - 4} = 1$	$r_2 = \sqrt{36 + 16 - 36} = 4$

$$C_1C_2 = 5 = 1 + 4 = r_1 + r_2$$

$\therefore$ The two circles touch externally.



P, the point of contact, divides C_1C_2 internally in the ratio 1 : 4. Hence, P is $\left(\dfrac{14}{5}, \dfrac{8}{5}\right)$ and Q which divides C_1C_2 externally in the same ratio is $\left(\dfrac{2}{3}, 0\right)$.

The tangent T_1 and T_2 from Q to the circle I will also be tangent to the circle II and hence it will be a common tangent touching the two circles in two distinct points.
If θ be the angle that this tangent (actually there are two) makes with C_1C_2, then

$$\sin\theta = \frac{1}{C_1Q} = \frac{1}{\sqrt{1 + \frac{16}{9}}} = \frac{3}{5}$$

$$\therefore \quad \tan\theta = \frac{3}{4}$$

Now if 'm' be the slope of one of the common tangents from Q, then

$$\frac{3}{4} = \tan\theta = \left|\frac{m - \frac{3}{4}}{1 + m\frac{3}{4}}\right| \quad \left(\text{slope of } C_1C_2 = \frac{3}{4}\right)$$

On solving, we get

$$m = \frac{24}{7}\,;\, m = 0$$

$\therefore$ The two common tangents are

$$y = 0 \text{ and } 24x - 7y - 16 = 0.$$

Alternate

T_1 and T_2 are common tangents touching both circles at distinct point and T_3 touches both circles at point P. We need to find equation of T_1 and T_2.

$$Q\left(\frac{2}{3}, 0\right)$$

Equation of tangent

$$y - 0 = m\left(x - \frac{2}{3}\right)$$

$$\Rightarrow \quad 3mx - 3y - 2m = 0$$

Condition of tangency $P = r_1$; $C_1 = (2, 1)$

$$\left|\frac{6m - 3 - 2m}{\sqrt{9m^2 + 9}}\right| = 1$$

$$\Rightarrow \quad (4m - 3)^2 = 9m^2 + 9$$

$$\Rightarrow \quad 7m^2 - 24m = 0$$

$$m = \frac{24}{7}, 0$$

Equation of common tangents $y = 0$ and $24x - 7y - 16 = 0$

ILLUSTRATION 85

Find the locus of the point of intersection of the tangents at the ends of a chord of the circle $x^2 + y^2 = a^2$, which touches the circle $x^2 + y^2 - 2ax = 0$.

SOLUTION

Let (x_1, y_1) be one such point. Chord of contact of tangents from (x_1, y_1) to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$. This line is a tangent to second circle $x^2 + y^2 - 2ax = 0$, whose centre is $(a, 0)$ and radius $= a$.

Condition for tangency $P = r$

$$\therefore \quad \left|\frac{ax_1 - a^2}{\sqrt{x_1^2 + y_1^2}}\right| = a$$

$$\therefore \quad (x_1 - a)^2 = x_1^2 + y_1^2 \text{ or } y_1^2 = a^2 - 2ax_1$$

$$\therefore \quad \text{Locus of } (x_1, y_1) \text{ is } y^2 = a(a - 2x).$$

ILLUSTRATION 86

Let 'C' be a curve, which is the locus of the point of the intersection of the lines $x = 2 + m$ and $my = 4 - m$. A circle S: $(x - 2)^2 + (y + 1)^2 = 25$ intersects the curve C at four points namely, P, Q, R and S. If 'O' is the centre of the curve 'C', then find $\dfrac{\left(OP^2 + OQ^2 + OR^2 + OS^2\right)}{25}$

SOLUTION

Rewriting the equations of given lines

$$x - 2 = m \text{ and } y + 1 = \frac{4}{m}$$

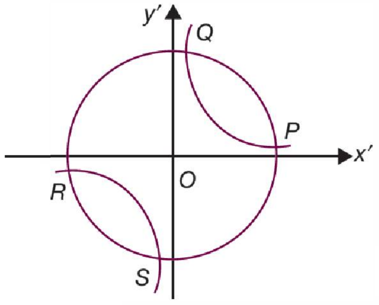
$$\Rightarrow \quad (x - 2)(y + 1) = 4 \Rightarrow x'y' = 4$$

Where, $x' = x - 2$, $y' = y + 1$.

Centre of C as $(2, -1) = 0$

Now, let shift origin to centre of C as $(2, -1)$

New equations are $x'y' = 4$ and $x'^2 + y'^2 = 25$



Any point on curve $x'y' = 4$ $P\left(2t_1, \dfrac{2}{t_1}\right)$

centre $O(0, 0)$

P also lies on circle $4t_1^2 + \dfrac{4}{t_1^2} = 25$

$\Rightarrow OP = 5 = \text{constant} = r$

$$\therefore \quad OP = OQ = OR = OS = r = 5$$

$$\Rightarrow \quad OP^2 + OQ^2 + OR^2 + OS^2 = 4r^2 = 100$$

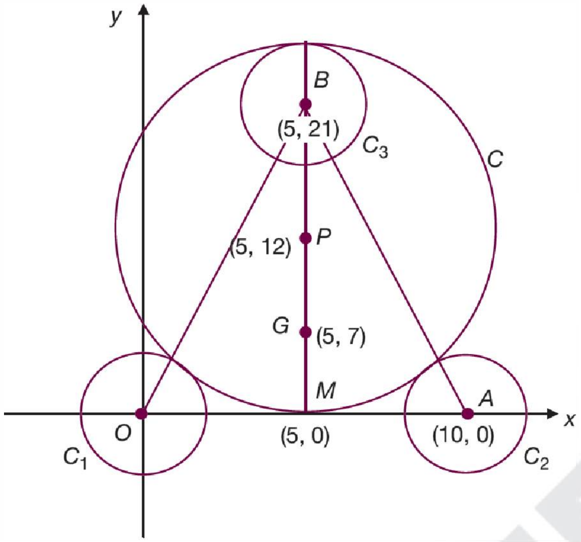
$$\text{Now, } \frac{OP^2 + OQ^2 + OR^2 + OS^2}{4} = 25$$

ILLUSTRATION 87

Let C_1: $x^2 + y^2 = 1$; C_2: $(x - 10)^2 + y^2 = 1$ and C_3: $x^2 + y^2 - 10x - 42y + 457 = 0$ be three circles. A circle C has been drawn to touch the circles C_1 and C_2 externally and C_3 internally. Now, if the circles C_1, C_2 and C_3 start rolling on the circumference of circle C in anticlockwise direction with constant speed, then find the locus of centroid of the triangle formed by joining the centres of rolling circles C_1, C_2 and C_3.

SOLUTION

Centre of C lies on perpendicular bisector of OA. Hence, $x = 5$.



Now, $OP = 1 + r = \sqrt{OM^2 + PM^2}$

$$\Rightarrow \quad 1 + r = \sqrt{25 + r^2}$$

$$\Rightarrow \quad 1 + r^2 + 2r = 25 + r^2$$

$$\Rightarrow \quad r = 12$$

Centre of circle C is $P(5, 12)$

$\therefore$ The equation of circle C is $(x - 5)^2 + (y - 12)^2 = 12^2$.

This circle also touches x–axis at (5, 0).

Now circles C_1, C_2 and C start rolling with constant speed in anticlockwise then relative position of vertices of ΔOAB remain unchanged.

$PG = 5 \qquad P(5, 12)$

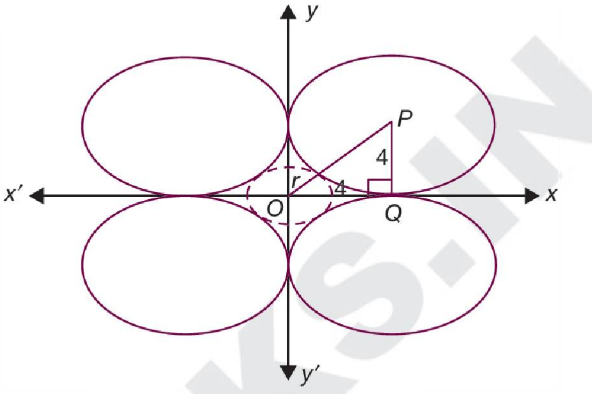
Centroid $G(x, y)$

$$(x - 5)^2 + (y - 12)^2 = 25$$

ILLUSTRATION 88

If the equations of four circles are $(x \pm 4)^2 + (y \pm 4)^2 = 4^2$, then find the radius of the smallest circle touching all the four circles.

SOLUTION



Using Pythagoras

$$OP^2 = OQ^2 + PQ^2$$

$$(r + 4)^2 = 4^2 + 4^2 = 32$$

$$\Rightarrow \quad r + 4 = 4\sqrt{2}$$

$$r = 4\sqrt{2} - 4 = 4\left(\sqrt{2} - 1\right)$$

ILLUSTRATION 89

Find the equation of the circle of minimum radius, which contains the three circles

$$x^2 + y^2 - 4y - 5 = 0,$$

$$x^2 + y^2 + 12x + 4y + 31 = 0 \text{ and}$$

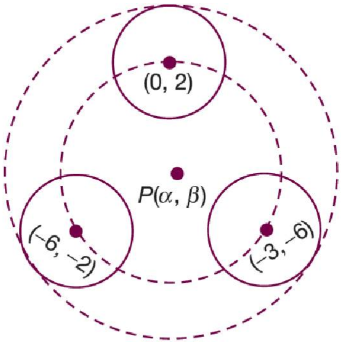
$$x^2 + y^2 + 6x + 12y + 36 = 0$$

SOLUTION

For the circle $x^2 + y^2 - 4y - 5 = 0$, centre $C_1(0, 2)$ and radius $r_1 = \sqrt{(-2)^2 + 5} = 3$

For the circle $x^2 + y^2 + 12x + 4y + 31 = 0$, centre $C_2(-6, -2)$ and radius $r_2 = \sqrt{6^2 + 2^2 - 31} = 3$

For the circle $x^2 + y^2 + 6x + 12y + 36 = 0$, centre $C_3(-3, -6)$ and radius $r_3 = \sqrt{3^2 + 6^2 - 36} = 3$.



Let, $P(\alpha, \beta)$ be the centre of the circle passing through the centres of the three given circles. Then,

$$a^2 + (\beta - 2)^2 = (\alpha + 3)^2 + (\beta + 6)^2$$

$$= (\alpha + 6)^2 + (\beta + 2)^2, \{= r^2\}$$

$$\Rightarrow \quad 4 - 4\beta = 6\alpha + 9 + 12\beta + 36 = 12\alpha + 36 + 4\beta + 4$$

$$\therefore \quad 6\alpha + 16\beta + 41 = 0 \text{ and } 6\alpha - 8\beta - 5 = 0$$

On solving we get, $\therefore\ \alpha = \dfrac{-31}{18}, \beta = \dfrac{-23}{12}$

Required circle is concentric with circumcircle of triangle formed by centres of given circles.

∴ The centre of the circle passing through centres of given circle is $\left(\dfrac{-31}{18}, \dfrac{-23}{12}\right)$ and

$$\text{radius } r = \sqrt{\alpha^2 + (\beta - 2)^2}$$

$$= \sqrt{\left(-\frac{31}{18}\right)^2 + \left(-\frac{23}{12} - 2\right)^2} = \frac{5}{36}\sqrt{949}$$

∴ The radius of the required circle $= 3 + \dfrac{5}{36}\sqrt{949}$

and centre $= \left(\dfrac{-31}{18}, \dfrac{-23}{12}\right)$

∴ The equation of the required circle is

$$\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$$

Test Your Concepts-V

(Solutions on page H.83)

1. Find the number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$.

2. Find the co-ordinates of the point, at which the circle $x^2 + y^2 - 4x - 2y - 4 = 0$ and $x^2 + y^2 - 12x - 8y - 36 = 0$ touch each other.

3. Find the number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$.

4. Two circles $x^2 + y^2 + 2hx = r_1^2 - h^2$ and $x^2 + y^2 + 2kx = r_2^2 - k^2$ touch externally, find relation.

5. Find the number of common tangents to the circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 - 12x - 16y + 91 = 0$.

6. The line $y = mx$ intersects the circle $x^2 + y^2 - 2x - 2y = 0$ and $x^2 + y^2 + 6x - 8y = 0$ at points A and B (points being other than origin). Find the range of 'm' such that the origin divides AB internally.

7. Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$.

Answer Keys

1. 3

2. No such point exists

3. 1

4. $h^2 + k^2 = (r_1 + r_2)^2$

5. 4 tangents

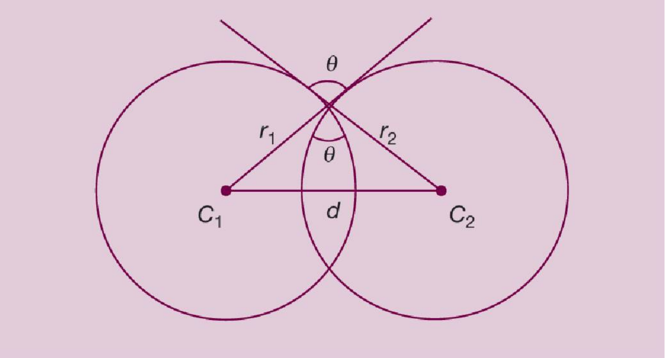
6. $-1 < m < \dfrac{3}{4}$

ANGLE BETWEEN TWO INTERSECTING CIRCLES

The angle between the tangents or normals of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If any two circles are

$$S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the angle between them,



Now

$$\cos\theta=\frac{r_1^2+r_2^2-d^2}{2r_1r_2}=\frac{2g_1g_2+2f_1f_2-c_1-c_2}{2\sqrt{g_1^2+f_1^2-c_1}\sqrt{g_2^2+f_2^2-c_2}}$$

Where, r_1 and r_2 are the radii of the circles, where $C_1C_2=d=\sqrt{(-g_1+g_2)^2+(-f_1+f_2)^2}$.

$$r_1=\sqrt{g_1^2+f_1^2-c_1}$$

$$r_2=\sqrt{g_2^2+f_2^2-c_2}$$

ORTHOGONAL CIRCLES

Two circles $S_1=0$ and $S_2=0$ are said to intersect orthogonally, if the tangents at their point of intersection include a right angle.
For orthogonal, $\theta=90^\circ$

$$\cos\theta=\frac{r_1^2+r_2^2-d^2}{2r_1r_2}=\frac{2g_1g_2+2f_1f_2-c_1-c_2}{2\sqrt{g_1^2+f_1^2-c_1}\sqrt{g_2^2+f_2^2-c_2}}=0$$

$$\Rightarrow \quad r_1^2+r_2^2=d^2 \text{ or } 2(g_1g_2+f_1f_2)=c_1+c_2$$

The condition for two circles to be orthogonal is $\boldsymbol{2g_1g_2+2f_1f_2=c_1+c_2}$

ILLUSTRATION 90

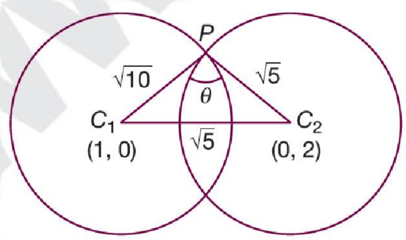
Find the angle at which the circles $(x-1)^2+y^2=10$ and $x^2+(y-2)^2=5$ intersect.

SOLUTION

The equations of the circles are

$$(x-1)^2+y^2=10 \qquad \ldots(1)$$

$$x^2+(y-2)^2=5 \qquad \ldots(2)$$



The centres of these circles are $C_1(1, 0)$ and $C_2(0, 2)$, respectively and let r_1 and r_2 be their radii.

Then, $r_1=\sqrt{10}$ and $r_2=\sqrt{5}$

$$C_1C_2=\sqrt{(1-0)^2+(0-2)^2}=\sqrt{5}$$

Suppose, if the circles (1) and (2) intersect at P, then

$$\cos\angle C_1PC_2=\frac{C_1P^2+C_2P^2-C_1C_2^2}{2C_1\cdot P\cdot C_1P}=\frac{r_1^2+r_2^2-d^2}{2r_1r_2}$$

$$=\frac{10+5-5}{2\times\sqrt{50}}=\frac{10}{2\times5\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$\therefore \quad \angle C_1PC_2=45^\circ=\frac{\pi}{4}$$

ILLUSTRATION 91

For what value of k the circles $x^2+y^2+5x+3y+7=0$ and $x^2+y^2-8x+8y+k=0$ cut orthogonally.

SOLUTION

For two circles to be orthogonal, $2(g_1g_2+f_1f_2)=c_1+c_2$

$$\Rightarrow \quad 2\left(\frac{5}{2}(-4)+\frac{3}{2}(4)\right)=7+k\Rightarrow k=-15$$

ILLUSTRATION 92

Obtain the equation of the circle, which is orthogonal to both the circles $x^2+y^2+3x-5y+6=0$ and $4x^2+4y^2-28x+29=0$ and whose centre lies on the line $3x+4y+1=0$.

SOLUTION

Given circles are S_1: $x^2+y^2+3x-5y+6=0$ and S_2: $4x^2+4y^2-28x+29=0$.

$$\Rightarrow \quad x^2+y^2-7x+\frac{29}{4}=0$$

Let the required circle be S: $x^2+y^2+2gx+2fy+c=0$

$\because$ The circle S cuts the circles S_1 and S_2 orthogonally.

$$\therefore \quad 2g\left(\frac{3}{2}\right)+2f\left(-\frac{5}{2}\right)=c+6\Rightarrow 3g-5f=c+6 \quad \ldots(1)$$

$$2g\left(-\frac{7}{2}\right)+2f\,.0=c+\frac{29}{4}\Rightarrow -7g=c+\frac{29}{4} \quad \ldots(2)$$

On (1) – (2), we get,

$$10g-5f=-\frac{5}{4}\Rightarrow 8g-4f=-1 \qquad \ldots(3)$$

Given line is $3x+4y+1=0$

$\because$ Centre $(-g,-f)$ of circle S lies on S given line.

$$\therefore \quad -3g-4f=-1 \qquad \ldots(4)$$

On solving these relation (3) and (4), we get

$$g=0, f=\frac{1}{4} \text{ and } c=-\frac{29}{4}$$

$\therefore$ The required circle is

$$x^2+y^2+\frac{1}{2}y-\frac{29}{4}=0$$

$$\Rightarrow \quad 4(x^2+y^2)+2y-29=0$$

ILLUSTRATION 93

The centre of the circle $S=0$ lies on the line $2x-2y+9=0$ and $S=0$ cuts orthogonally the circle $x^2+y^2=4$. Show that $S=0$ passes through two fixed points and find their co-ordinates.

SOLUTION

Let the circle $S=0$ be represented by

$$x^2+y^2+2gx+2fy+c=0.$$

Its centre is $(-g,-f)$, and lies on given line $2x-2y+9=0$.

$$\Rightarrow \quad 2(-g)-2(-f)+9=0$$

$$\Rightarrow \quad 2g-2f=9 \qquad \ldots(1)$$

The circle $S=0$ cuts the circle $x^2+y^2-4=0$ orthogonally. Hence,

$$2g(0)+2(f)(0)=c-4 \Rightarrow c=4 \qquad \ldots(2)$$

Now using relation (1) and (2), we get required circle

$$\Rightarrow \quad S=x^2+y^2+(9+2f)x+2fy+4=0$$

$$\Rightarrow \quad x^2+y^2+9x+4+2f(x+y)=0$$

This passes through the points of intersection of the line.

$x+y=0$ and the circle $x^2+y^2+9x+4=0$

$$\Rightarrow \quad x^2+(-x)^2+9x+4=0$$

$$\Rightarrow \quad 2x^2+9x+4=0$$

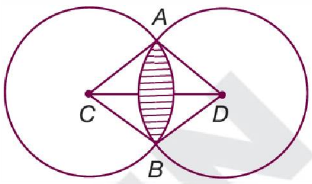
$$\Rightarrow \quad x=-\frac{1}{2},-4;\ y=\frac{1}{2},4 \text{ respectively}$$

$\therefore$ The fixed points are $\left(-\frac{1}{2},\frac{1}{2}\right)$ and $(-4,4)$.

ILLUSTRATION 94

If two circles each of unit radius intersect orthogonally and the common area of the circles is $\frac{\pi}{\lambda}-1$, then find the value of λ.

SOLUTION



Let consider two circles of radius 1, they are intersecting at A, B, C, D.

$$AB=BC=CD=DA=1$$

They are diagonal then $\theta=90°$.

$ABCD$ is square.

$\therefore$ The required area = Area of sector ABC + Area of sector ABD – area of square $ABCD$

$$=2\left(\frac{\pi}{4}\right)-1=\frac{\pi}{2}-1$$

Hence $\lambda=2$.

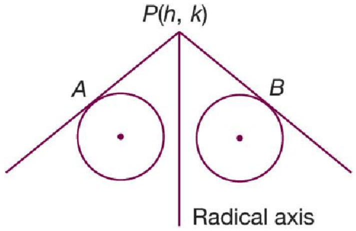
RADICAL AXIS

The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and it is called the radical axis.

Consider the two circles,

$$S_1: x^2+y^2+2g_1x+2f_1y+c_1=0$$

$$S_2: x^2+y^2+2g_2x+2f_2y+c_2=0$$



Let $P(h,k)$ is a point and PA and PB are the lengths of two tangents on the circles from point P, then from definition,

$$PA=PB \Rightarrow \sqrt{S_1}=\sqrt{S_2}$$

$$\Rightarrow \quad \sqrt{h^2+k^2+2g_1h+2f_1k+c_1}$$

$$=\sqrt{h^2+k^2+2g_2h+2f_2k+c_2}$$

$$\Rightarrow \quad 2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2=0$$

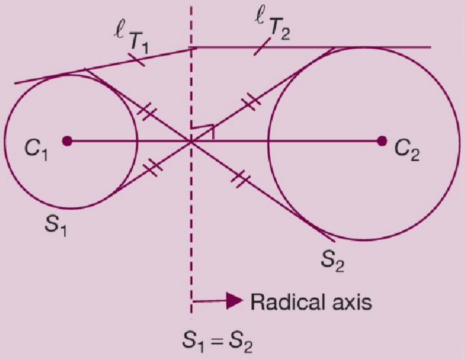
$\therefore$ Locus of (h,k) is $2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0$

The equation of radical axis is $S_1-S_2=0$.

NOTE:

To get the equation of the radical axis, first make the coefficient of x^2 and y^2 as 1 in both equations of given circles.

(a) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through midpoint of it.



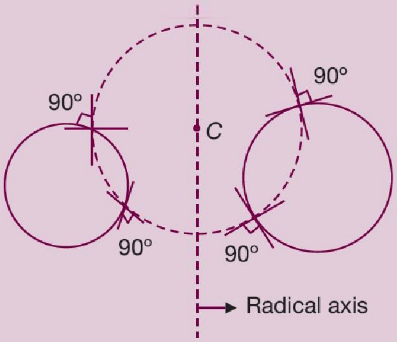
Equation of radical axis $\boxed{S_1 - S_2 = 0}$

(b) Line segment joining the centres C_1 and C_2 is bisected by radical axis only when they have same radii.

(c) Radical axis always bisects all possible common tangents.

(d) If the circles are concentric, then the radical axis does not always exist but if the circles are not concentric, then the radical axis always exists.

(e) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis of the two fixed circles.



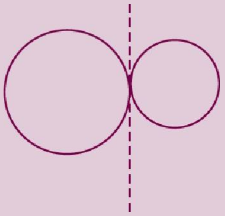
(f) If we take centre on radical axis and a circle is drawn, then it will cut the given circle orthogonally.

SPECIAL CASES

(i) Both circles touch externally

Radical axis (common tangent at point of contact)

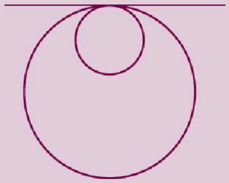
$S_1 - S_2 = 0$



(ii) Both circles touch internally

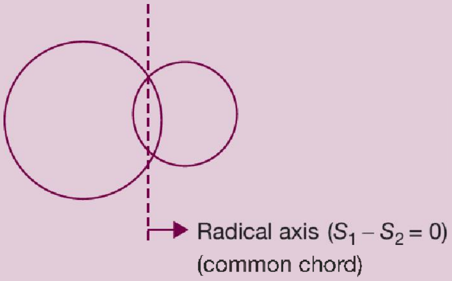
Radical axis (common tangent at point of contact)

$S_1 - S_2 = 0$



(iii) For intersecting circles (common chord)

The chord joining the points of intersection of two given circles is known as **common chord**.



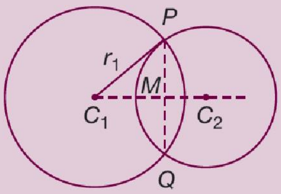
Let the circles are,

$$S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots(1)$$

$$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \ldots(2)$$

The equation of the common chord of two circles is $S_1 - S_2 = 0$

$$\Rightarrow 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$$



The length of the common chord

$$PQ = 2PM = 2\sqrt{C_1P^2 - C_1M^2}$$

Where, C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.

ILLUSTRATION 95

Find locus of centre of variable circle, which is orthogonal to the circles $x^2 + y^2 + 4x - 6y - 9 = 0$ and $x^2 + y^2 - 4x + 6y - 4 = 0$.

SOLUTION

Let variable circle $S: x^2 + y^2 + 2gx + 2fy + c = 0$

$$S_1: x^2 + y^2 + 4x - 6y - 9 = 0$$

$$S_2: x^2 + y^2 - 4x + 6y - 4 = 0$$

S and S_1 orthogonal $2(2g - 3f) = c - 9$

S and S_2 orthogonal $2(-2g + 3f) = c - 4$

$$8g - 12f = -5$$

Now replace

$$g = -x \qquad f = -y \qquad -8x + 12y = -5$$

$$8x + 12y - 5 = 0$$

It is radical axis of S_1 and S_2

ILLUSTRATION 96

Show that the circles $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$ touches each other. Find the co-ordinates of the point of contact and the equation of the common tangent at the point of contact.

SOLUTION

The given circles are

$$S_1: x^2 + y^2 - 10x + 4y - 20 = 0 \qquad \text{...(1)}$$

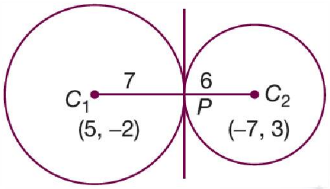
$$S_2: x^2 + y^2 + 14x - 6y + 22 = 0 \qquad \text{...(2)}$$

For the circle (1), the co-ordinates of the centre C_1 are $(5, -2)$ and radius, $r_1 = \sqrt{(25+4+20)} = 7$

For the circle (2), the co-ordinates of the centre C_2 are $(-7, 3)$ and radius $r_2 = \sqrt{(49+9-22)} = 6$

Now, $C_1C_2 = \sqrt{\left[(5+7)^2 + (-2-3)^2\right]} = 13 = r_1 + r_2$

Hence, the two circles (1) and (2) touch externally.



If P is the point of contact of the two circle (1) and (2), then P will divide C_1C_2 internally in the ratio $r_1:r_2$ i.e., 7 : 6.

∴ Co-ordinates of P are

$$\left(\frac{(7)(-7)+(6)(5)}{7+6}, \frac{(7)(3)+(6)(-2)}{7+6}\right)$$

$$\Rightarrow \quad = \left(-\frac{19}{13}, \frac{9}{13}\right)$$

∵ The two circles touch each other, $S_1 - S_2 = 0$ is the common tangent at the point of contact.

∴ The equation of the common tangent is

$$S_1 - S_2 = -24x + 10y - 42 = 0$$

$$\Rightarrow \quad 12x - 5y + 21 = 0$$

ILLUSTRATION 97

Two circles each of radius 5 units touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find the equation of the two circles.

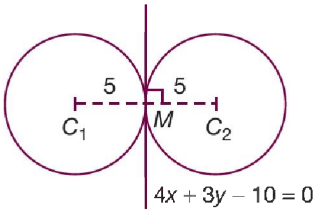
SOLUTION

The radius of the required circle is 5 units.

The two circles touch each other externally only at the midpoint of the line joining their centres.

Also, we know that the common tangent is $4x + 3y = 10$ and the line joining the centres are perpendicular to common tangent $4x + 3y = 10$.

Hence, slope of C_1C_2 is $\tan\theta = \frac{3}{4}$, where C_1C_2 is the line joining the centres. Midpoint of C_1C_2 is $M(1, 2)$.



Hence, the two centres C_1 and C_2 are obtained by using the parametric form.

$$\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=\pm r \Rightarrow \frac{x-1}{\left(\frac{4}{5}\right)}=\frac{y-2}{\left(\frac{3}{5}\right)}=\pm 5$$

$$x=1\pm 4;\ y=2\pm 3$$

Hence, the centres of the required circles are

$C_1(5, 5)$ and $C_2(-3, -1)$.

We get the equations of the circles as

$$(x-5)^2+(y-5)^2=25 \text{ and } (x+3)^2+(y+1)^2=25.$$

ILLUSTRATION 98

Find the angle subtended by the common chord of $x^2+y^2-4x-4y=0$ and $x^2+y^2=16$ at the origin.

SOLUTION

The equation of the common chord of the circle $x^2+y^2-4x-4y=0$ and $x^2+y^2=16$ is radical axis $S_1-S_2=0 \Rightarrow x+y=4$, which meets the circle $x^2+y^2=16$ at the points $A(4, 0)$ and $B(0, 4)$. Obviously, OA is perpendicular to OB. Hence, the common chord AB makes a right angle at the centre of the circle $x^2+y^2=16$.

ILLUSTRATION 99

Find the equation of the circle described on the common chord of the circles $x^2+y^2-4x-5=0$ and $x^2+y^2+8y+7=0$ as diameter.

SOLUTION

Equation of the common chord is $S_1-S_2=0$

$$\Rightarrow \quad x+2y+3=0.$$

Equation of the circle through the two circles is $S_1+\lambda S_2=0$.

$$\Rightarrow \quad x^2+y^2-\frac{4}{1+\lambda}x+\frac{8\lambda y}{1+\lambda}+\frac{7\lambda-5}{1+\lambda}=0$$

Its centre $\left(\frac{2}{1+\lambda}, -\frac{4\lambda}{1+\lambda}\right)$ lies on $x+2y+3=0$.

$$\Rightarrow \quad \frac{2}{1+\lambda}-\frac{8\lambda}{1+\lambda}+3=0 \Rightarrow 2-8\lambda+3+3\lambda=0$$

$$\Rightarrow \quad \lambda=1$$

Hence, the required circle is $x^2+y^2-2x+4y+1=0$.

ILLUSTRATION 100

Prove that the circle $x^2+y^2-6x-4y+9=0$ bisects the circumference of the circle $x^2+y^2-8x-6y+23=0$.

SOLUTION

The given circles are

$$S_1: x^2+y^2-6x-4y+9=0 \quad \ldots(1)$$

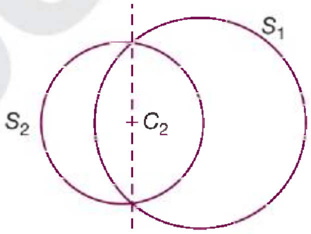
$$S_2: x^2+y^2-8x-6y+23=0 \quad \ldots(2)$$

Equation of the common chord of circles (1) and (2), which is also the radical axis of the circles S_1 and S_2 is

$$S_1-S_2=0$$

$$\Rightarrow \quad 2x+2y-14=0$$

$$\Rightarrow \quad x+y-7=0 \quad \ldots(3)$$



Centre of the circle S_2 is (4, 3). Clearly, line (3) passes through the point (4, 3) and hence the line (3) is the equation of the diameter of the circle (2). Hence, circle (1) bisects the circumference of circle (2).

ILLUSTRATION 101

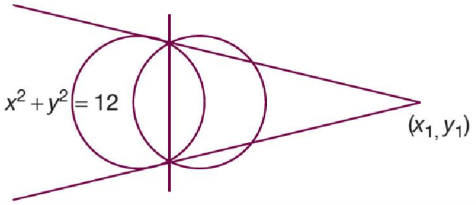
If tangents be drawn to the circle $x^2+y^2=12$ at its points of intersection, with the circle $x^2+y^2-5x+3y-2=0$, find the co-ordinates of the point of intersection of the tangents.

SOLUTION

Let the tangents to the circle $x^2+y^2=12$ meet at the point (x_1, y_1).

Equation of the chord of contact is

$$xx_1+yy_1-12=0 \quad \ldots(1)$$



Equation of the common chord of two circles $x^2 + y^2 - 12 = 0$ and $x^2 + y^2 - 5x + 3y - 2 = 0$ is radical axis

$$S_1 - S_2 = -5x + 3y + 10 = 0 \qquad \ldots(2)$$

∵ (1) and (2) are identical as chord of contact and radical axis are same straight line.

$$\frac{x_1}{-5} = \frac{y_1}{3} = \frac{-12}{10}$$

$$\Rightarrow \quad x_1 = 6 \text{ and } y_1 = \frac{-18}{5}$$

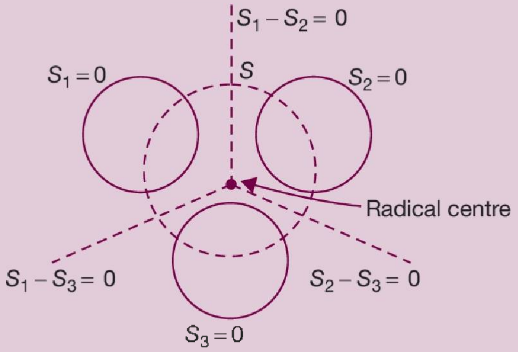
Hence, the required point is $\left(6, \frac{-18}{5}\right)$.

RADICAL CENTRE

(i) Point of concurrency of radical axes.

(ii) A point from which equal length of tangents can be drawn to given circles.

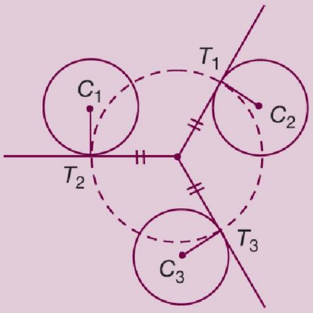
To get the radical centre of three circles—$S_1 = 0$, $S_2 = 0$ and $S_3 = 0$, we have to solve any two radical axes, i.e., $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$.



(a) From radical centre,

$$\ell_{T_1} = \ell_{T_2} = \ell_{T_3} \Rightarrow \sqrt{S_1} = \sqrt{S_2} = \sqrt{S_3}$$

(b) By taking radical centre as centre and length of tangent as radius, a circle S is drawn. It cuts the given three circles S_1, S_2, S_3 orthogonally.



(c) If three circles are drawn on three sides of a triangle taking them as diameter, then its orthocentre will be its radical centre.

EBOOKS.IN

ILLUSTRATION 102

From the circles, $x^2 + y^2 + 3x + 2y + 1 = 0$, $2x^2 + 2y^2 - 2x + 12y + 10 = 0$ and $3x^2 + 3y^2 + 15x - 24y + 45 = 0$, find the radical centre and circle, which is orthogonal to these circles.

SOLUTION

Given circles are

$$S_1 : x^2 + y^2 + 3x + 2y + 1 = 0$$

$$S_2 : x^2 + y^2 - x + 6y + 5 = 0$$

$$S_3 : x^2 + y^2 + 5x - 8y + 15 = 0$$

Equation of Radical axis 1 = $S_1 - S_2 = 0 \Rightarrow x - y - 1 = 0$

Equation of Radical axis 2 = $S_2 - S_3 = 0 \Rightarrow 3x - 7y + 5 = 0$

Equation of Radical axis 3 = $S_3 - S_1 = 0 \Rightarrow x - 5y + 7 = 0$

All these axes are concurrent at (3, 2) = radical centre.

$$\ell_{T_1} = \ell_{T_2} = \ell_{T_3} = \sqrt{27}$$

Required circle orthogonal to given circle centre as radical centre and radius as length of tangent $(x - 3)^2 + (y - 2)^2 = 27$.

ILLUSTRATION 103

Show that each of the circles, $S_1 = x^2 + y^2 + 4y - 1 = 0$, $S_2 = x^2 + y^2 + 6x + y + 8 = 0$ and $S_3 = x^2 + y^2 - 4x - 4y - 37 = 0$ touches the other two and determine the points of contact and the point of concurrence of the tangents at these points.

SOLUTION

S_1 has centre C_1 (0, –2) and radius $r_1 = \sqrt{5}$

S_2 has centre $C_2 \left(-3, -\frac{1}{2}\right)$ and radius $r_2 = \frac{1}{2}\sqrt{5}$

S_3 has centre C_3 (2, 2) and radius $r_3 = 3\sqrt{5}$

$$C_1C_2 = \sqrt{9 + \frac{9}{4}} = \frac{3}{2}\sqrt{5} = r_1 + r_2$$

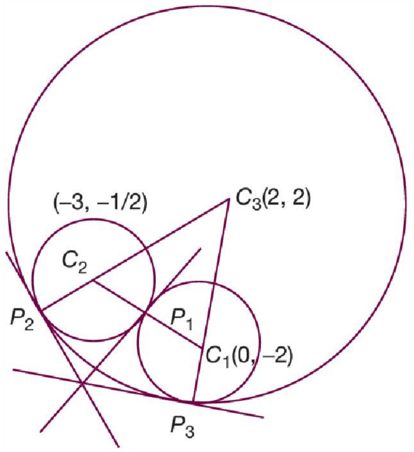
⇒ S_1 and S_2 touch externally

$$C_2C_3 = \sqrt{25 + \frac{25}{4}} = \frac{5}{2}\sqrt{5} = r_3 - r_2$$

⇒ S_2 and S_3 touch internally

$C_3C_1 = \sqrt{4+16} = 2\sqrt{5} = r_3 - r_1$

$\Rightarrow$ S_3 and S_1 touch internally



The point of contact P_1 divides C_1C_2 in the ratio $r_1{:}r_2 = 2{:}1$.

$$\therefore \quad P_1\left(\frac{-6+0}{2+1}, \frac{-1-2}{2+1}\right) = (-2, -1).$$

The point of contact P_2 divides C_2C_3 in the ratio $r_2{:}r_3 = 1{:}6$ externally.

$$\therefore \quad P_2\left(\frac{2-(-18)}{1-6}, \frac{2-(-3)}{1-6}\right) = (-4, -1).$$

The point of contact P_3 divides C_3C_1 in the ratio $r_3 : r_1 = 3{:}1$ externally.

$$\therefore \quad P_3\left(\frac{0-2}{3-1}, \frac{-6-2}{3-1}\right) = (-1, -4).$$

The common tangent at $P_1 = 2x - y + 3 = 0$

(i.e., $S_1 - S_2 = 0$) ...(1)

The common tangent at $P_2 = 2x + y + 9 = 0$

(i.e., $S_2 - S_3 = 0$) ...(2)

The common tangent at $P_3 = x + 2y + 9 = 0$

(i.e., $S_3 - S_1 = 0$) ...(3)

(1) and (2) intersect at (–3, –3), which satisfies (3)

Hence, (–3, –3) is the point of concurrence of these tangents = radical centre.

Test Your Concepts-VI

(Solutions on page H.84)

1. If the circles of same radius r and centres at (3, 4) and (7, 8) cut orthogonally, then find r.
2. Find the locus of the centres of the circles, which cut the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 - y^2 - 5x + 4y - 2 = 0$ orthogonally.
3. Find the radical centre of circles $(x - p)^2 + y^2 = p^2$, $x^2 + (y - q)^2 = q^2$ and $x^2 + y^2 = p^2 + q^2$.
4. The radical axis of two circles and the line joining their centres are perpendicular. State whether true or false.
5. If two equal circles of radius r, intersect such that each passes through the centre of the other, then find the length of their common chord.
6. If a circle S_1 bisects the circumference of another circle S_2, then their radical axis passes through the centre of ___.
7. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2ax + 2by + d = 0$, then find the relation.
8. Find the radical centre of the three circles $x^2 + y^2 = a^2$, $(x - c)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = a^2$.
9. Find the point from which the tangents to the circles $x^2 + y^2 - 8x + 40 = 0$, $5x^2 + 5y^2 - 25x + 80 = 0$ and $x^2 + y^2 - 8x + 16y + 160 = 0$ are of equal length.

Answer Keys

1. 4 units

2. $9x - 10 + 11 = 0$

3. $\left(\frac{p^2+q^2}{2q}, \frac{p^2+q^2}{2q}\right)$

4. True

5. $\sqrt{3}r$

6. S_2

7. $2a(g - a) + 2b(f - b) = (c - d)$

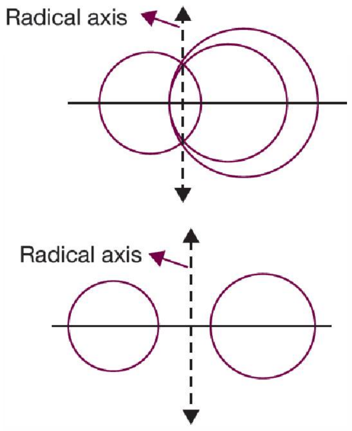
8. $\left(\frac{c}{2}, \frac{b}{2}\right)$

9. $\left(8, \frac{-15}{2}\right)$

COAXIAL SYSTEM OF CIRCLES

A system of more than two circles is said to be coaxial, when they have a common radical axis, i.e., when radical axis of each pair of circles of the system is the same.

The radical axis of any pair of the circles is perpendicular to the line joining their centres, which follows that the centres of the circles of a coaxial system must be collinear.



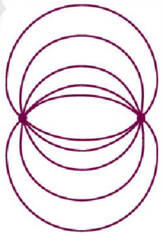
NOTE:

$S - S' = 0$ gives common radical axis of the system and in $S + \lambda u = 0$, family of circles, $u = 0$ is common radical axis of the system represents coaxial system.

Also, $x^2 + y^2 + 2\lambda x + c = 0$ and $x^2 + y^2 + 2\mu y + c = 0$ represent coaxial system (where λ and μ are parameters) having the y-axis and the x-axis as their common radical axis of the system, respectively.

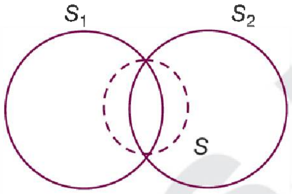
FAMILY OF CIRCLES

There can be countless circles possible passing through two points.



Two distinct circles can intersect at two point and number of circles passing through their point of intersection can be infinite. All these circles are known as family of circles.

(a) Equation of family of circle which passes through point of intersection of circles S_1 and S_2 is given by $S: S_1 + \lambda S_2 = 0 \;\; \lambda \in R - \{-1\}$



(b) Equation of family of circles, which pass through point of intersection of circles S and Line L is given by $S' : S + \lambda L = 0$

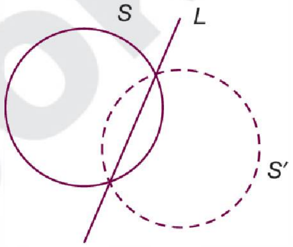


ILLUSTRATION 104

Find the equation of the circles, which pass through the ends of the common chords of two circles $2x^2 + 2y^2 + 8x + 4y - 7 = 0$ and $x^2 + y^2 - 8x - 4y - 5 = 0$ and touch the line $x = 7$.

SOLUTION

The equation to the common chord of the two given circle is $S_1 - S_2 = 0$

$$\Rightarrow \quad 2x^2 + 2y^2 + 8x + 4y - 7 - 2(x^2 + y^2 - 8x - 4y - 5) = 0$$

$$\text{or} \quad 24x + 12y + 3 = 0 \quad \text{or} \quad 8x + 4y + 1 = 0$$

Any circle through the points of intersection of this line and the circle

$$x^2 + y^2 - 8x - 4y - 5 = 0 \text{ is of the form,}$$

$$x^2 + y^2 - 8x - 4y - 5 + \lambda(8x + 4y + 1) = 0$$

This meets $x = 7$, at points whose ordinates are given by

$$49 + y^2 - 56 - 4y - 5 + \lambda(56 + 4y + 1) = 0$$

$$\text{i.e.,} \quad y^2 + 4(\lambda - 1)y + 57\lambda - 12 = 0$$

This quadratic has equal roots, since $x = 7$ is to be a tangent.

$$16(\lambda - 1)^2 - 4(57\lambda - 12) = 0$$

i.e., $4\lambda^2 - 65\lambda + 16 = 0 \Rightarrow (4\lambda - 1)(\lambda - 16) = 0$

$$\Rightarrow \quad \lambda = \frac{1}{4}\ ;\ \lambda = 16$$

For $\lambda = \frac{1}{4}$,

circle is $x^2 + y^2 - 8x - 4y - 5 + \frac{1}{4}(8x + 4y + 1) = 0$

$$\Rightarrow \quad 4\left(x^2 + y^2\right) - 24x - 12y - 19 = 0$$

For $\lambda = 16$,

circle is $x^2 + y^2 - 8x - 4y - 5 + 16(8x + 4y + 1) = 0$

$$\Rightarrow \quad x^2 + y^2 + 120x + 60y + 11 = 0$$

Alternate

Any circle through the points of intersection of given line and the circle

$x^2 + y^2 - 8x - 4y - 5 = 0$ is of the form $x^2 + y^2 - 8x - 4y - 5 + \lambda(8x + 4y + 1) = 0$

$$\Rightarrow \quad x^2 + y^2 + 8(\lambda - 1)x + 4(\lambda - 1)y + \lambda - 5 = 0$$

Centre $(-4(\lambda - 1), -2(\lambda - 1))$

$P = r$ for line $x - 7 = 0$

$$\left|\frac{4 - 4\lambda - 7}{1}\right| = \sqrt{16(\lambda - 1)^2 + 4(\lambda - 1)^2 - \lambda + 5}$$

Squaring both sides,

$$\Rightarrow \quad 16\lambda^2 + 9 + 24\lambda = 16\lambda^2 + 16 - 32\lambda + 4\lambda^2 + 4 - 8\lambda - \lambda + 5$$

$$\Rightarrow \quad 4\lambda^2 - 65\lambda + 16 = 0$$

$$\Rightarrow \quad (4\lambda - 1)(\lambda - 16) = 0$$

$$\Rightarrow \quad \lambda = \frac{1}{4}, \lambda = 16$$

For $\lambda = \frac{1}{4}$; circle is $4\left(x^2 + y^2\right) - 24x - 12y - 19 = 0$

For $\lambda = 16$, circle is $x^2 + y^2 + 120x + 60y + 11 = 0$

ILLUSTRATION 105

Circles are drawn to cut two circles $x^2 + y^2 + 6x + 5 = 0$ and $x^2 + y^2 - 6y + 5 = 0$ orthogonally. All such circles will pass through the fixed points, find them.

SOLUTION

The radical axis of the given circles is $x + y = 0$. Let $P(\lambda, -\lambda)$ be any point on the above radical axis.

The length of the tangent drawn from P to any of the given circles is

$$\ell = \sqrt{\lambda^2 + \lambda^2 + 6\lambda + 5}$$

A circle having centre at P and radius equal to l will be orthogonal to both the given circles.

Equation of such a circle, is $(x - \lambda)^2 + (y + \lambda)^2 = \ell^2 = \lambda^2 + \lambda^2 + 6\lambda + 5$

i.e., $x^2 + y^2 + 2\lambda^2 - 2\lambda x + 2\lambda y = 2\lambda^2 + 6\lambda + 5$

i.e., $(x^2 + y^2 - 5) - 2\lambda(x - y + 3) = 0$

which represents a family of circles passing through the intersection points of

$$x^2 + y^2 - 5 = 0 \qquad \ldots(1)$$

and $x - y + 3 = 0 \qquad \ldots(2)$

Eliminating y we get,

$x = -1, -2$ and the corresponding $y = 2, 1$.

Hence, the required points are $(-1, 2)$ and $(-2, 1)$.

ILLUSTRATION 106

Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

SOLUTION

The equation of the circle through the intersection of the given circles is

$$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0 \qquad \ldots(1)$$

Where $(-10x - 10y = 0)$ is the equation of radical axis of the given circles

Equation (1) can be rearranged as,

$$x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally

$$\Rightarrow \quad 2gg_1 + 2ff_1 = c + c_1$$

Hence, $2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4 \Rightarrow \lambda = -2$

Hence the required circle is

$$x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$$

i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

ILLUSTRATION 107

If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points, then find the value of a.

SOLUTION

Any second degree curve passing through the intersection of the given curves is

$$ax^2 + 4xy + 2y^2 + x + y + 5$$
$$+ \lambda (ax^2 + 6xy + 5y^2 + 2x + 3y + 8) = 0.$$

If it is a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$

$$a(1 + \lambda) = 2 + 5\lambda \text{ and } 4 + 6\lambda = 0$$

$$\Rightarrow \quad a = \frac{2+5\lambda}{1+\lambda} \text{ and } \lambda = -\frac{2}{3}$$

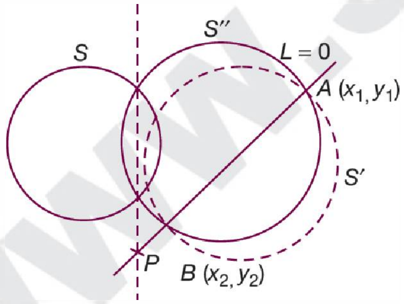
$$\Rightarrow \quad a = \frac{2-\frac{10}{3}}{1-\frac{2}{3}} = -4$$

ILLUSTRATION 108

A fixed circle is cut by a family of circles all of which pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Prove that the chord of intersection of the fixed circle with any circle of the family passes through a fixed point.

SOLUTION

Let $S = 0$ be the equation of fixed circle.
Line $AB = L$; Circle AB as diameter $= S'$
Family of circle passing through A and B is given by $S'' : S' + \lambda L = 0$



Common chord of fixed circle and variable circle S'' is $S - S'' = 0$

$$S - (S' + \lambda L) = 0$$

$$S - S' - \lambda L = 0$$

$$L' - \lambda L = 0$$

L' is radical axis of fixed circle S and circle S' whose diameter is AB.
Fixed point P is point of intersection of given line $L = 0$ and radical axis $L' = 0$.
Hence proved.

ILLUSTRATION 109

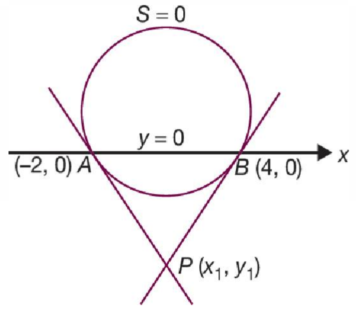
Show that the equation $x^2 + y^2 - 2x - 2ay - 8 = 0$ represents, for different values of 'a' a system of circles passing through two fixed point A and B on the x-axis and find the equation of that circle of the system the tangents to which at A and B meet on the line $x + 2y + 5 = 0$.

SOLUTION

The given equation can be written in the form $(x^2 + y^2 - 2x - 8) + a(-2y) = 0$ is the equation of a system of circles passing through the points of intersection of the fixed circle $x^2 + y^2 - 2x - 8 = 0$ and the fixed line $y = 0$ (i.e., x-axis).
These points of intersection are therefore fixed, they are given by

$$x^2 - 2x - 8 = 0 \text{ or } (x + 2)(x - 4) = 0$$

and hence they are $A(-2, 0)$ and $B(4, 0)$



Let the tangent at A and B to variable circle $S = x^2 + y^2 - x - 2ay - 8 = 0$ intersect at $P(x_1, y_1)$. Then AB is the chord of contact given by the tangents drawn from $P(x_1, y_1)$ to the circle $S = 0$.
The equation of AB should be

$$xx_1 + yy_1 - (x + x_1) - a(y + y_1) - 8 = 0$$

However, the equation AB is $y = 0$

Both are identical $x_1 - 1 = 0$ and $x_1 + ay_1 + 8 = 0$

$$\therefore \quad x_1 = 1 \text{ and } y_1 = -\frac{9}{a}$$

$$\therefore \quad P(x_1, y_1) = P\left(1, -\frac{9}{a}\right)$$

However, it is given that $P(x_1, y_1)$ lies on the line $x + 2y + 5 = 0$.

$$\therefore \quad 1 - \frac{18}{a} + 5 = 0$$

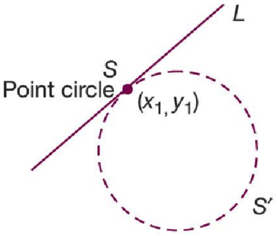
$\therefore \quad a = 3$

Hence, the required circle is

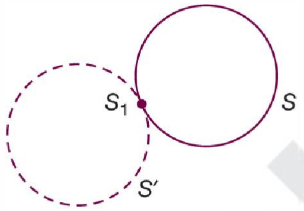
$$x^2 + y^2 - 2x - 6y - 8 = 0.$$

APPLICATION OF POINT CIRCLE AND MISCELLANEOUS CASES

1. Equation of circle, which touches line L at point (x_1, y_1) is given by $S' : S + \lambda L = 0$, where point circle $S : (x - x_1)^2 + (y - y_1)^2 = 0$.



2. Equation of circle, which touches given circle S at point (x_1, y_1) is given by S': $S + \lambda S_1 = 0$, where point circle S_1: $(x - x_1)^2 + (y - y_1)^2 = 0$.

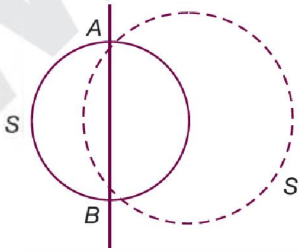


3. Equation of circle passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $S' : S + \lambda L = 0$,

   where circle whose diameter is AB.

   $$S: (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

   and straight line AB (L): $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$



4. Equation of circle circumscribing a triangle, whose equation of sides $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are given, is given by

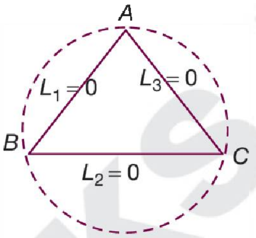
   $$L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0;$$

   Now satisfying following conditions.

   (i) Coefficient of x^2 = coefficient of y^2

   (ii) Coefficient of $xy = 0$

   Solve the above two relations and get μ and λ



   Alternate

   Solve the equations of sides and get vertices A, B and C, now these points lie on general equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and get variables g, f and c.

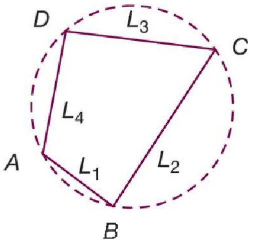
5. Equation of circle circumscribing quadrilateral, whose side's equations are given by $L_1L_3 + \lambda L_2L_4 = 0$.

   Now satisfy following conditions

   (i) Coefficient of x^2 = coefficient of y^2

   (ii) Coefficient of $xy = 0$.

   Now solve above relations and get λ



   Alternate

   Solve equations of given straight lines to get the points A, B, C and D. Now, these points lie on general equation of circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ get the variables g, f and c.

ILLUSTRATION 110

Find the equation of circle, which touches the straight line $x + 2y = 1$ at point $(3, -1)$ and (a) passes through origin (b) radius is 2.

SOLUTION

Required circle is $S + \lambda L = 0$, where S is a point circle.

$$(x - 3)^2 + (y + 1)^2 + \lambda(x + 2y - 1) = 0$$

(a) It passes through origin (0, 0)

We get $\lambda = 10$,

Required circle is $x^2 + y^2 + 4x + 22y = 0$.

(b) $r = 2$; required circle $S + \lambda L = 0$, where S is a point circle.

$$(x-3)^2 + (y+1)^2 + \lambda(x+2y-1) = 0$$

$$x^2 + y^2 + (\lambda - 6)x + y(2\lambda + 2) + 10 - \lambda = 0$$

$$r^2 = 2^2 = \left(\frac{\lambda - 6}{2}\right)^2 + (\lambda + 1)^2 - 10 + \lambda$$

$$16 = \lambda^2 - 12\lambda + 36 + 4\lambda^2 + 8\lambda + 4 - 40 + 4\lambda$$

$$5\lambda^2 = 16$$

$$\lambda = \pm\frac{4}{\sqrt{5}}$$

$\therefore$ The required circle is

$$(x-3)^2 + (y-1)^2 \pm \frac{4}{\sqrt{5}}(x + 2y - 1) = 0.$$

ILLUSTRATION 111

Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords, in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the co-ordinates of this point.

SOLUTION

The equation of the family of circles passing through $A(3, 7)$ and $B(6, 5)$ is $S + \lambda L = 0$

Where S is circle whose diameter is AB and L is line AB

$$(x-3)(x-6) + (y-7)(y-5) + \lambda(2x + 3y - 27) = 0$$

i.e., $x^2 + y^2 + (2\lambda - 9)x + (3\lambda - 12)y + (53 - 27\lambda) = 0$ ...(1)

The equation of the given circle is

$x^2 + y^2 - 4x - 6y - 3 = 0$ ...(2)

$\therefore$ The equation of common chord = radical axis

$$(2\lambda - 9 + 4)x + (3\lambda - 12 + 6)y + (53 - 27\lambda + 3) = 0$$

$\Rightarrow$ $\lambda(2x + 3y - 27) - (5x + 6y - 56) = 0$,

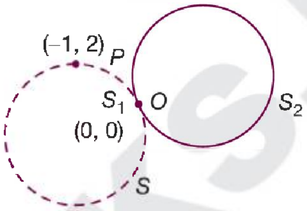
which passes through the point of intersection of $2x + 3y - 27 = 0$ and $5x + 6y - 56 = 0$ for all values of λ.

$\therefore$ The fixed point is $\left(2, \frac{23}{3}\right)$.

ILLUSTRATION 112

Find the equations of the circle, which passes through the point (–1, 2) and touches the circle $x^2 + y^2 - 8x + 6y = 0$ at origin.

SOLUTION



Required circle $S = S_1 + \lambda S_2 = 0$.

Where, S_1 is point circle $(x-0)^2 + (y-0)^2 = 0$

S_2 is given circle, $S_2 = x^2 + y^2 - 8x + 6y = 0$

$$S = (x-0)^2 + (y-0)^2 + \lambda(x^2 + y^2 - 8x + 6y) = 0$$

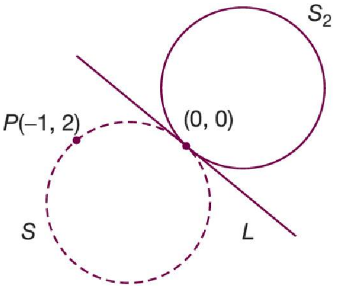
$P(-1, 2)$ lies on it

$$1 + 4 + \lambda(1 + 4 + 8 + 12) = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{5}$$

$\therefore$ The required circle is $4x^2 + 4y^2 + 8x - 6y = 0$.

Alternate



Given circle S_2: $x^2 + y^2 - 8x + 6y = 0$.

It has tangent at (0, 0)

$$L: 0 + 0 - 4(x+0) + 3(y+0) = 0$$

$$L: 4x - 3y = 0$$

Required circle $S_2 + \lambda L = 0$

$$x^2 + y^2 - 8x + 6y + \lambda(4x - 3y) = 0$$

It passes through point (–1, 2)

$$1 + 4 + 8 + 12 + \lambda(-4 - 6) = 0$$

$$\lambda = \frac{5}{2}$$

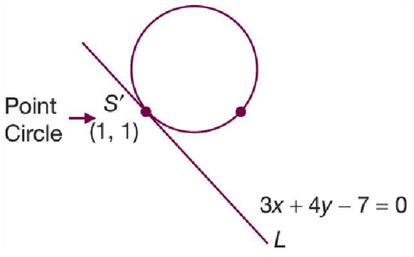
Required circle $2x^2 + 2y^2 + 4x - 3y = 0$

ILLUSTRATION 113

Find equation of a circle $S(x, y) = 0$, $S(2, 3) = 16$, which touches the line $3x + 4y - 7 = 0$ at $(1, 1)$.

SOLUTION

The required circle will be as shown in figure.



Any circle which touches $3x + 4y - 7 = 0$ at $(1, 1)$ will be of the form,

$$S(x, y): S' + \lambda L = 0$$

$$\Rightarrow \quad (x-1)^2 + (y-1)^2 + \lambda(3x + 4y - 7) = 0$$

$$\because \quad S(2, 3) = 16 \Rightarrow 1 + 4 + \lambda(6 + 12 - 7) = 16 \Rightarrow \lambda = 1$$

Required circle is

$$(x-1)^2 + (y-1)^2 + 3x + 4y - 7 = 0$$

$$\Rightarrow \quad x^2 + y^2 + x + 2y - 5 = 0$$

ILLUSTRATION 114

If a circle $S(x, y) = 0$ touches at the point $(2, 3)$ of the line $x + y = 5$ and $S(1, 2) = 0$, then find the radius of such circle.

SOLUTION

Desired equation of the circle is $S' + \lambda L = 0$, where S' is point circle.

$$(x-2)^2 + (y-3)^2 + \lambda(x + y - 5) = 0$$

$$1 + 1 + \lambda(1 + 2 - 5) = 0 \Rightarrow \lambda = 1$$

We get $(x-2)^2 + (y-3)^2 + (x + y - 5) = 0$

$$x^2 + y^2 - 3x - 5y + 8 = 0$$

Hence, radius $= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 8} = \frac{1}{\sqrt{2}}$

ILLUSTRATION 115

Find the equation of the circle circumscribing the triangle whose sides are $3x - y - 9 = 0$, $5x - 3y - 23 = 0$ and $x + y - 3 = 0$.

SOLUTION

Required circle is

$$L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$$

$$(3x - y - 9)(5x - 3y - 23)$$
$$+ \lambda(5x - 3y - 23)(x + y - 3)$$
$$+ \mu(x + y - 3)(3x - y - 9) = 0$$

$$\Rightarrow \quad (5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2$$
$$+ xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu)$$
$$+ y(50 - 14\lambda - 6\mu) + 27\mu - 207 + 69\lambda = 0$$

Coefficient of x^2 = Coefficient of y^2

$$5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu$$

$$\Rightarrow \quad 8\lambda + 4\mu + 12 = 0 \Rightarrow 2\lambda + \mu + 3 = 0$$

Coefficient of $xy = 0$

$$\Rightarrow \quad 2\lambda + 2\mu - 14 = 0 \Rightarrow \lambda + \mu = 7$$

We get $\lambda = -10$ and $\mu = 17$

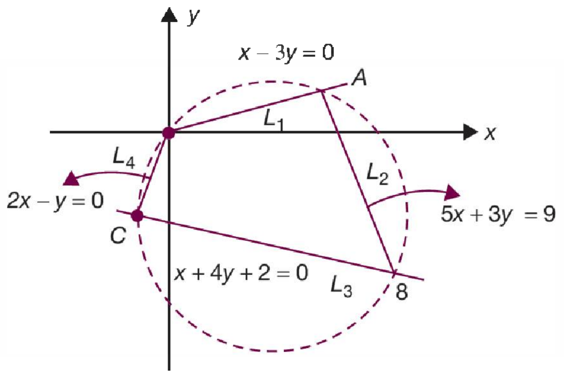
$\therefore$ The required circle is $16x^2 + 16y^2 - 40x + 88y - 24 = 0$

$$\Rightarrow \quad 2x^2 + 2y^2 - 5x + 11y - 3 = 0$$

ILLUSTRATION 116

Find the equation of the circle circumscribing the quadrilateral, whose sides are $5x + 3y = 9$, $x = 3y$, $2x = y$ and $x + 4y + 2 = 0$.

SOLUTION



Required circumcircle $L_1L_3 + \lambda L_2L_4 = 0$

$$(x - 3y)(x + 4y + 2) + \lambda(2x - y)(5x + 3y - 9) = 0$$

$$\Rightarrow \quad x^2 - 12y^2 + xy - 6y + 2x$$
$$+ \lambda(10x^2 + 6xy - 18x - 5xy - 3y^2 + 9y) = 0$$

$$\Rightarrow \quad x^2(1+10\lambda)-(12+3\lambda)y^2+xy(1+\lambda)$$
$$+(2-18\lambda)x+(9\lambda-6)y=0$$

Coefficient of x^2 = Coefficient of y^2

$$\Rightarrow \quad 1+10\lambda=-12-3\lambda \Rightarrow \lambda=-1$$

Coefficient of $xy=0$

$$\Rightarrow \quad 1+\lambda=0 \Rightarrow \lambda=-1$$

$\therefore$ The required circle is $9x^2+9y^2-20x+15y=0$.

ILLUSTRATION 117

If the curves $\frac{x^2}{4}+y^2=1$ and $\frac{x^2}{a^2}+y^2=1$ for suitable value of a cut on 4 concyclic points, then find the radius of the smallest circle passing through these 4 points.

SOLUTION

Let equation of the circle which passes through point of intersection of given curves, be

$$\left(\frac{x^2}{4}+y^2-1\right)+\lambda\left(\frac{x^2}{a^2}+y^2-1\right)=0$$

$$\Rightarrow \quad x^2\left(\frac{1}{4}+\frac{\lambda}{a^2}\right)+y^2(1+\lambda)=1+\lambda$$

$$\Rightarrow \quad x^2\left(\frac{a^2+4\lambda}{4a^2}\right)+y^2(1+\lambda)=1+\lambda$$

$$\Rightarrow \quad x^2\left(\frac{a^2+4\lambda}{4a^2(1+\lambda)}\right)+y^2=1\text{, then } \frac{a^2+4\lambda}{4a^2(1+\lambda)}=1$$

(Coefficient of x^2 = Coefficient of y^2 for circle)

$$\Rightarrow \quad 4\lambda=3a^2+4a^2\lambda \Rightarrow \lambda=\frac{3a^2}{4(1-a^2)}$$

Clearly, the circle is $x^2+y^2=1$

$\therefore$ The radius is 1 unit.

Test Your Concepts-VII

(Solutions on page H.85)

1. Find the circumcircle of the quadrilateral formed by the lines $x=a$, $x=2a$, $y=a$ and $y=-a$.
2. Find the equations of the circles passing through the points of intersection of the circles $x^2+y^2-2x-4y-4=0$ and $x^2+y^2-10x-12y+40=0$ and whose radius is 4.
3. Find the equation of circle, which touches $2x-y+3=0$ and pass through the points of intersection of the line $x+2y-1=0$ and the circle $x^2+y^2-2x+1=0$.
4. Find the equation of circle passing through the points $A(1, 1)$ and $B(2, 2)$ and whose radius is 1.
5. Find the equation of the circle passing through the point $(2, 1)$ and touching the line $x+2y-1=0$ at the point $(3, -1)$.

Answer Keys

1. $x^2+y^2-3ax+a^2=0$
2. $2x^2+2y^2-18x-22y+69=0$ and $x^2+y^2-2y-15=0$
3. $x^2+y^2+4y-1=0$ and $x^2+y^2-4x-4y+3=0$
4. $x^2+y^2-2x-4y+4=0$ and $x^2+y^2-4x-2y+4=0$
5. $3x^2+3y^2-23x-4y+35=0$

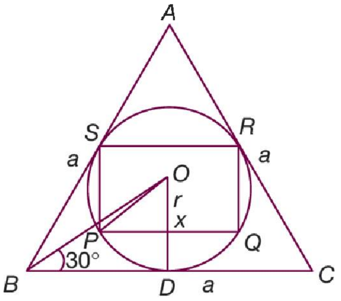
SOLVED PROBLEMS

PROBLEM 1

A circle is inscribed in an equilateral triangle of side a. Find the area of any square inscribed in the circle.

SOLUTION

The given triangle is an equilateral triangle with side a.



Let the radius of the circle be $OD = r$

In ΔBOD, $\angle OBD = 30^\circ$ $(\because \Delta ABC$ is equilateral)

Here, $BD = \dfrac{a}{2}$

$$\therefore \quad \tan 30^\circ = \frac{r}{\frac{a}{2}} \Rightarrow r = \frac{a}{2\sqrt{3}}$$

$PQRS$ is a square inscribed in the circle.

Diagonal of square = Diameter of circle

$$\sqrt{2}x = \frac{a}{\sqrt{3}} \Rightarrow x = \frac{a}{\sqrt{6}}$$

Area of square = Side × Side

$$= \frac{a}{\sqrt{6}} \times \frac{a}{\sqrt{6}} = \frac{a^2}{6} \text{ sq. units}$$

PROBLEM 2

The equation of circumcircle of a ΔABC is $x^2 + y^2 + 3x + y - 6 = 0$. If $A(1, -2)$, $B(-3, 2)$ and the vertex C varies, then find the locus of orthocentre of ΔABC.

SOLUTION

Equation of circumcircle is

$$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{17}{2} = r^2$$

$$C = \left(-\frac{3}{2} + r\cos\theta, -\frac{1}{2} + r\sin\theta\right)$$

Circumcentre of ΔABC is $\left(-\dfrac{3}{2}, -\dfrac{1}{2}\right)$

Centroid $G\left(-\dfrac{7}{6} + \dfrac{r}{3}\cos\theta, -\dfrac{1}{6} + \dfrac{r}{3}\sin\theta\right)$

In a triangle centroid divides circumcentre $\left(-\dfrac{3}{2}, -\dfrac{1}{2}\right)$ and orthocentre (h, k) in 1:2 ratio, centroid G $\left(\dfrac{h-3}{3}, \dfrac{k-1}{3}\right) = G\left(\dfrac{-7}{6} + \dfrac{r}{3}\cos\theta, \dfrac{-1}{6} + \dfrac{r}{3}\sin\theta\right)$

Locus of orthocentre $\left(h + \dfrac{1}{2}\right)^2 + \left(k - \dfrac{1}{2}\right)^2 = r^2 = \dfrac{17}{2}$

$$h = x, k = y \Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{17}{2}$$

PROBLEM 3

Find the range of parameter 'a', for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circles.

SOLUTION

The given circles are S_1: $(x - 1)^2 + (y - 1)^2 = 1$ and S_2: $(x - 8)^2 + (y - 1)^2 = 4$

The line $y - 2x - a = 0$ will lie between these circles, if the centre of the circles lie on opposite sides of the line, i.e., $(1 - 2 - a)(1 - 16 - a) < 0 \Rightarrow a \in (-15, -1)$ ...(1)

Line wouldn't touch or intersect the circles if, $p > r$ for both circles $\dfrac{|1-2-a|}{\sqrt{5}} > 1$ and $\dfrac{|1-16-a|}{\sqrt{5}} > 2$

$\Rightarrow \quad |1 + a| > \sqrt{5}$ and $|15 + a| > 2\sqrt{5}$

$\Rightarrow \quad a > \sqrt{5} - 1$

or $\quad a < -\sqrt{5} - 1$ and $a > 2\sqrt{5} - 15$ or $a < -2\sqrt{5} - 15$

$a \in (-\infty, -\sqrt{5} - 1) \cup (\sqrt{5} - 1, \infty)$ ...(2)

$a \in (-\infty, -2\sqrt{5} - 15) \cup (2\sqrt{5} - 15, \infty)$ ...(3)

Hence, the common values of 'a' (1) and (2) and (3) are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

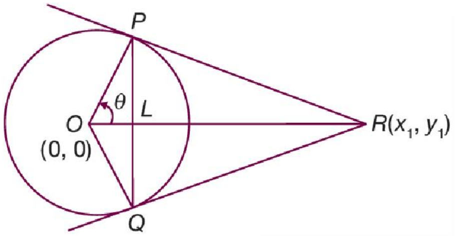
PROBLEM 4

Show that the area of the triangle formed by the pair of tangent from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ and the chord joining their points of contact with the circle is

$$a(x_1^2 + y_1^2 - a^2)^{3/2}(x_1^2 + y_1^2)^{-1} \text{ sq. units.}$$

SOLUTION

Let, $R\ (x_1, y_1)$ be the point from which the tangents RP and RQ are drawn to the circle $x^2 + y^2 = a^2$. Let, OR meet PQ at L, where $O\ (0, 0)$ is the centre of the circle.



Then, $\Delta RPQ = 2\Delta RLP = 2(\Delta OPR - \Delta OLP)$

However, $\Delta OPR = \dfrac{1}{2} OP \cdot PR = \dfrac{1}{2}\, a \cdot \sqrt{x_1^2 + y_1^2 - a^2}$

and $\Delta OLP = \dfrac{1}{2}(OP \cos\theta)\,(OP \sin\theta) = \dfrac{1}{2} a^2 \cos\theta \sin\theta,$

where $\angle PQR = \theta$

However, from ΔOPR, $\sin\theta = \dfrac{PR}{OR} = \dfrac{\sqrt{x_1^2 + y_1^2 - a^2}}{\sqrt{x_1^2 + y_1^2}}$

and $\cos\theta = \dfrac{OP}{OR} = \dfrac{a}{\sqrt{x_1^2 + y_1^2}}$

Hence, $\Delta LOP = \dfrac{1}{2} a^2 \dfrac{\sqrt{x_1^2 + y_1^2 - a^2}}{\sqrt{x_1^2 + y_1^2}} \dfrac{a}{\sqrt{x_1^2 + y_1^2}}$

$$= \frac{1}{2} a^3 \frac{\sqrt{x_1^2 + y_1^2 - a^2}}{x_1^2 + y_1^2}$$

and $\Delta RPQ = 2(\Delta OPR - \Delta OLP)$

$$= 2\left[\frac{1}{2} a\sqrt{x_1^2 + y_1^2 - a^2} - \frac{1}{2} a^3 \frac{\sqrt{x_1^2 + y_1^2 - a^2}}{x_1^2 + y_1^2}\right]$$

$$= a\sqrt{x_1^2 + y_1^2 - a^2}\left[1 - \frac{a^2}{x_1^2 + y_1^2}\right]$$

$$= a\sqrt{x_1^2 + y_1^2 - a^2}\ \frac{(x_1^2 + y_1^2 - a^2)}{x_1^2 + y_1^2}$$

$$= \frac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2} \text{ sq. units.}$$

PROBLEM 5

P is a point on the circle $x^2 + y^2 = a^2$. The chord of contact of P w.r.t. the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$ and a, b and c are the roots of $x^3 - 14x^2 + 56x + k = 0$. Find '$a$'.

SOLUTION

From any point $P(a \cos\theta, a \sin\theta)$ on circle $x^2 + y^2 = a^2$, chord of contact to circle $x^2 + y^2 = b^2$ is $T = 0$

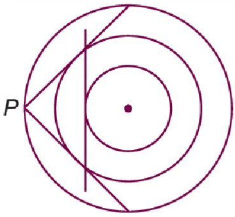
$\Rightarrow \quad x(a \cos\theta) + y(a \sin\theta) = b^2$

It is tangent to circle $x^2 + y^2 = c^2$ Centre $(0, 0)$ $R = 0$ then $P = R$

$$\Rightarrow \quad \frac{|0 + 0 - b^2|}{\sqrt{a^2\cos^2\theta + a^2\sin^2\theta}} = c$$

$\Rightarrow \quad b^2 = ac$

$\therefore$ a, b, c are in G.P.



Clearly $a > b > c$

Sum of roots $a + b + c = 14$

$ab + bc + ca = 56$

$b(a + c + b) = 56$

$b(14) = 56$

$b = 4;\ a + c = 10;\ b^2 = ac = 16$

We get, $a, b, c \rightarrow 2, 4, 8$ or $8, 4, 2$

However, a is greatest.

$\therefore \quad a = 8$

PROBLEM 6

Let a circle be given by $2x(x - a) + y(2y - b) = 0$ $(a \neq 0,\ b \neq 0)$. Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from $\left(a, \dfrac{b}{2}\right)$.

SOLUTION

The given circle is $2x(x-a)+y(2y-b)=0$

$$\Rightarrow \quad x^2+y^2-ax-\frac{by}{2}=0$$

Let AB be the chord, which is bisected by x-axis at a point M. Let its co-ordinates be $M(h,\,0)$ and $S=x^2+y^2-ax-\frac{by}{2}=0.$

$\therefore$ Equation of chord AB is $T=S_1$

$$hx+0-\frac{a}{2}(x+h)-\frac{b}{4}(y+0)=h^2+0-ah-0$$

$\because$ It passes through $\left(a,\frac{b}{2}\right)$, we have

$$ah-\frac{a}{2}(a+h)-\frac{b^2}{8}=h^2-ah$$

$$\Rightarrow \quad h^2-\frac{3}{2}ah+\frac{a^2}{2}+\frac{b^2}{8}=0$$

Now, there are two chords bisected by the x-axis. So, there must be two distinct real roots of h.

$$\therefore \quad B^2-4AC>0$$

$$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2-(4)(1)\left(\frac{a^2}{2}+\frac{b^2}{8}\right)>0 \Rightarrow \frac{a^2}{4}-\frac{b^2}{2}>0$$

$$\Rightarrow \quad a^2>2b^2$$

PROBLEM 7

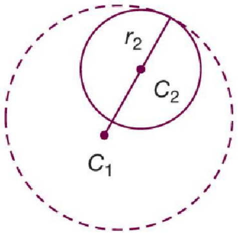
Find the equation of a circle having the lines $x^2+2xy+3x+6y=0$ as its normal and having size just sufficient to contain the circle $x(x-4)+y(y-3)=0$.

SOLUTION

Pair of normals are $(x+2y)\,(x+3)=0$

$\therefore$ Normals are $x+2y=0$ and $x+3=0$

Point of intersection of normals is the centre of the required circle, i.e., $C_1\left(-3,\frac{3}{2}\right)$ and centre of given circle is $C_2\left(2,\frac{3}{2}\right)$ and radius $r_2=\sqrt{4+\frac{9}{4}}=\frac{5}{2}$



Let, r_1 be the radius of required circle.

$$\Rightarrow \quad r_1=C_1C_2+r_2=\sqrt{(-3-2)^2+\left(\frac{3}{2}-\frac{3}{2}\right)^2}+\frac{5}{2}=\frac{15}{2}$$

Hence, equation of the required circle is

$$x^2+y^2+6x-3y-45=0.$$

PROBLEM 8

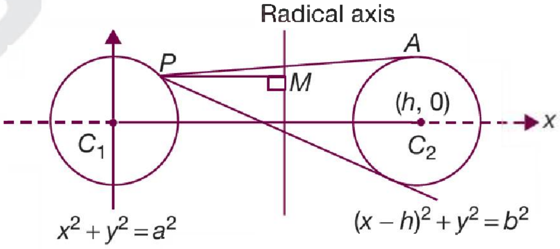
Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of the perpendicular distance of the point from the radical axis of the two circles and the distance between their centres.

SOLUTION

Without the loss of generality, let consider two circles whose line joining their centres lie along x-axis

First circle $\quad x^2+y^2=a^2$

Second circle $\quad (x-h)^2+y^2=b^2$



We have to prove that

$$PA^2=2\,PM.\,C_1\,C_2$$

$$P=(a\cos\theta,\,a\sin\theta)$$

$$PA=\sqrt{a^2-b^2+h^2-2ah\cos\theta}$$

Radical axis $S_2-S_1=0$

$$\Rightarrow \quad -2hx+h^2+a^2-b^2=0$$

$$\Rightarrow \quad x=\frac{h^2+a^2-b^2}{2h}$$

$$PM=\frac{h^2+a^2-b^2}{2h}-a\cos\theta$$

$$=\frac{h^2+a^2-b^2-2ah\cos\theta}{2h}$$

$$PM\cdot C_1\,C_2=\frac{h^2-a^2+b^2-2ah\cos\theta}{2}\cdot h$$

(where, $C_1\,C_2=h$)

$$PM\cdot C_1\,C_2=\frac{PA^2}{2} \Rightarrow PA^2=2\,PM\cdot C_1\,C_2$$

PROBLEM 9

The equation of three circles are given as $x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 15 = 0$ and $x^2 + y^2 + 10y + 24 = 0$. Determine the co-ordinates of the point P such that the tangents drawn from it to the circles are equal in length. Also find the equation of the circle through P, which touches the line $x + y = 5$ at the point $(6, -1)$.

SOLUTION

We know that the point from which the lengths of tangents are equal in length is radical centre of the given three circles. Now, the radical axis of the first two circles is,

$$(x^2 + y^2 - 1) - (x^2 + y^2 - 8x + 15) = 0$$

$$\Rightarrow \quad x - 2 = 0 \qquad \ldots(1)$$

and radical axis of the second and third circles is

$$(x^2 + y^2 - 8x + 15) - (x^2 + y^2 + 10y + 24) = 0$$

$$\Rightarrow \quad 8x + 10y + 9 = 0 \qquad \ldots(2)$$

Solving (1) and (2), the co-ordinates of radical centre, $P\left(2, -\dfrac{5}{2}\right)$

Now centre of the circle touching the line $x + y = 5$ at $(6, -1)$ lie on the line

$S + \lambda L = 0$, where S is a point circle.

$$(x - 6)^2 + (y + 1)^2 + \lambda(x + y - 5) = 0$$

$P\left(2, -\dfrac{5}{2}\right)$ lies on it

$$16 + \frac{9}{4} + \lambda\left(-\frac{1}{2} - 5\right) = 0$$

$$\frac{11}{2}\lambda = \frac{73}{4} \Rightarrow \lambda = \frac{73}{22}$$

Required circle, i.e., $x^2 + y^2 - \dfrac{191}{22}x + \dfrac{117}{22}y + \dfrac{449}{22} = 0$

$$\Rightarrow \quad 22x^2 + 22y^2 - 191x + 117y + 449 = 0$$

PROBLEM 10

Find the distance from the centre of the circle $x^2 + y^2 = 2x$ to the straight line passing through the point of intersection of two circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x + 7y - 25 = 0$.

SOLUTION

The point of intersection of two circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x + 7y - 25 = 0$ will be the common chord of these circles.

The common chord of circle is given by $S_1 - S_2 = 0$

$$\Rightarrow \quad 8x - 15y + 26 = 0$$

The centre of the given circle $x^2 + y^2 - 2x = 0$ is $(1, 0)$

The distance of $8x - 5y + 26 = 0$ from point $(1, 0)$ is

$$\left|\frac{8 + 26}{\sqrt{64 + 225}}\right| = \frac{34}{17} = 2 \text{ units}$$

PROBLEM 11

A and B are two fixed points and P moves such that $PA = nPB$, where $n \neq 1$. Show that the locus of P is a circle and for different values of n all the circles have a common radical axis.

SOLUTION

Without the loss of generality let $A = (a, 0)$, $B = (-a, 0)$ and $P(h, k)$

So, $PA = nPB$

$$\Rightarrow \quad (h - a)^2 + k^2 = n^2[(h + a)^2 + k^2]$$

$$\Rightarrow \quad (1 - n^2)h^2 + (1 - n^2)k^2 - 2ah(1 + n^2) + (1 - n^2)a^2 = 0$$

$$\Rightarrow \quad h^2 + k^2 - 2ah\left(\frac{1 + n^2}{1 - n^2}\right) + a^2 = 0$$

Hence, locus of P is

$x^2 + y^2 - 2ax\left(\dfrac{1 + n^2}{1 - n^2}\right) + a^2 = 0$, which is a circle for different values of n.

Let n_1 and n_2 are two different values of n so that their radical axis is $x = 0$, i.e., y-axis. Hence, for different values of n, the circles have a common radical axis.

Hence proved.

PROBLEM 12

The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + ax + by + c = 0$ in P and Q. The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ in R and S. If P, Q, R and S are concyclic, prove that, $\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$.

SOLUTION

The equation of the circle through the 1st line and the 1st circle, i.e., through P and Q is

$$x^2+y^2+ax+by+c+\lambda_1(Ax+By+C)=0 \quad \ldots(1)$$

The equation of the circle through R and S is

$$x^2+y^2+a'x+b'y+c'+\lambda_2(A'x+B'y+C')=0 \quad \ldots(2)$$

∵ P, Q, R, S are concyclic, (1) and (2) are identical.

$$\Rightarrow \quad 1=\frac{a+\lambda_1 A}{a'+\lambda_2 A'}=\frac{b+\lambda_1 B}{b'+\lambda_2 B'}=\frac{c+\lambda_1 C}{c'+\lambda_2 C'}$$

$$\Rightarrow \quad \lambda_1 A-\lambda_2 A'+a-a'=0$$

$$\lambda_1 B-\lambda_2 B'+b-b'=0$$

$$\lambda_1 C-\lambda_2 C'+c-c'=0$$

These relations can be considered as three straight lines in variable λ_1 and λ_2, we are looking.

For common solution it mean they should be concurrent then determinant formed by their coefficients must be zero.

Eliminating λ_1 and λ_2, we get $\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0$

PROBLEM 13

Two circles, each of radius 5 units touch each other at (1, 2). If the equation of their common tangent is $4x+3y=10$, find the equations of the two circles.

SOLUTION

The equation of the family of circles touching the line $4x+3y-10=0$ at the point (1, 2) is $(x-1)^2+(y-2)^2+\lambda(4x+3y-10)=0$

$$\Rightarrow \quad x^2+y^2+2x(2\lambda-1)+2y\left(\frac{3\lambda}{2}-2\right)+(5-10\lambda)=0$$

Since its radius is 5 units,

$$(2\lambda-1)^2+\left(\frac{3\lambda}{2}-2\right)^2-(5-10\lambda)=25 \Rightarrow \lambda=\pm 2$$

Hence, the required circles are

$$(x-1)^2+(y-2)^2\pm 2(4x+3y-10)=0$$

$$\Rightarrow \quad (x+3)^2+(y+1)^2=25 \text{ and } (x-5)^2+(y-5)^2=25$$

PROBLEM 14

If the origin is one of the limiting point of a system coaxial circles of which $x^2+y^2+2gx+2fy+c=0$ is a member, then find the co-ordinates of the other limiting point.

SOLUTION

One limiting point is the origin, namely the point circle $(x-0)^2+(y-0)^2=0$.

i.e., $x^2+y^2=0$

This is one member of the coaxial system and another member is

$$x^2+y^2+2gx+2fy+c=0$$

Hence, the equation of the radical axis of the system is

$$(x^2+y^2+2gx+2fy+c)-(x^2+y^2)=0$$

i.e., $2gx+2fy+c=0$

The equation of the coaxial system is thus

$$x^2+y^2+\lambda\,(2gx+2fy+c)=0$$

Its centre is $(-g\lambda, -f\lambda)$ and its radius $\sqrt{g^2\lambda^2+f^2\lambda^2-c\lambda}$

For a limiting point, the radius is 0.

Hence, $g^2\lambda^2+f^2\lambda^2-c\lambda=0$, giving

$\lambda\,[\lambda(g^2+f^2)-c]=0$, so that $\lambda=0$,

$$\lambda=\frac{c}{g^2+f^2}$$

One limiting point is (0, 0) and the other is $\left(-\frac{gc}{g^2+f^2}, -\frac{fc}{g^2+f^2}\right)$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is:

(A) a^2 (B) $\frac{a^2}{4}$

(C) $\frac{a^2}{3}$ (D) $\frac{a^2}{6}$

2. AB is a line segment of length 48 cm and C is its mid-point. AB, AC and CB are the semicircles described, then the radius of the circle inscribed in the space enclosed by three semicircles is:

(A) $3\sqrt{2}$ (B) 6

(C) 8 (D) 10

3. Two circles passing through the points $A(1, 2)$ and $E(2, 1)$ touch the line $4x + 8y - 7 = 0$ at B and D. The possible co-ordinates of point C such that $ABCD$ is a parallelogram are:

(A) $\left(\frac{15}{2}, -\frac{9}{2}\right)$ (B) $(0, 3)$

(C) $\left(\frac{1}{2}, \frac{5}{2}\right)$ (D) $\left(-\frac{5}{2}, \frac{3}{4}\right)$

4. In the xy plane, the length of the shortest path from $(0, 0)$ to $(12, 16)$, that does not go inside the circle $(x-6)^2 + (y-8)^2 = 25$ is:

(A) $10\sqrt{3}$ (B) $10\sqrt{5}$

(C) $10\sqrt{3} + \frac{5\pi}{3}$ (D) $10 + 5\pi$

5. $S = 0$ and $S' = 0$ are two circles of radii 4 and 6 and they touch externally. $L = 0$ is a direct common tangent. If p and q are radii of two circles, each of which touches externally both $S = 0$ and $S' = 0$ and also touches the line $L = 0$, then harmonic mean of p and q is:

(A) 1 (B) $\frac{2}{5}$

(C) $\frac{5}{6}$ (D) $\frac{12}{5}$

6. If $x^2 + y^2 = 16$, $x^2 + y^2 = 36$ are two circles and P and Q move respectively on these circles such that $PQ = 4$, then the locus of midpoint of PQ is a circle of radius:

(A) $\sqrt{20}$ (B) $\sqrt{22}$

(C) $\sqrt{30}$ (D) $\sqrt{32}$

7. An isosceles triangle with base 24 and legs 15 each is inscribed in a circle with centre at $(-1, 1)$, then locus of the centroid of that Δ is:

(A) $4\left(x^2 + y^2\right) + 8x - 8y - 73 = 0$

(B) $2\left(x^2 + y^2\right) + 4x - 4y - 31 = 0$

(C) $2\left(x^2 + y^2\right) + 4x - 4y - 21 = 0$

(D) $4\left(x^2 + y^2\right) + 8x - 8y - 161 = 0$

8. For all values of $m \in R$, the line $y - mx + m - 1 = 0$ cuts the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ at an angle:

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

9. A square of side 6 is given. C_1 is the circle circumscribing the square. A regular octagon is formed by cutting off each corner of that square. C_2 is the circle circumscribing the octagon. The locus of point of intersection of perpendicular tangents of C_1 and C_2 is a circle of radius:

(A) $3\sqrt{6 - 2\sqrt{2}}$ (B) $2\sqrt{8 - 2\sqrt{2}}$

(C) $4\sqrt{\sqrt{2} - 1}$ (D) $6\sqrt{3 - 2\sqrt{2}}$

10. The equation of the circle passing through $(3, 0)$ and $(0, 4)$ and having the minimum radius is:

(A) $x^2 + y^2 = 4$ (B) $x^2 + y^2 - 2x + 4y = 0$

(C) $x^2 + y^2 - 2x - 2y + 2 = 0$ (D) $x^2 + y^2 - 3x - 4y = 0$

11. The points $(2k, 3k)$, $(1, 0)$ $(0, 1)$ and $(0, 0)$ lie on a circle for:

(A) all values of k (B) $0 \le k \le 1$

(C) $k < 0$ (D) $k = \frac{5}{13}$

12. The normal of the circle $(x-2)^2+(y-1)^2=1$, which bisects the chord cut off by the line $x-2y+3=0$ is:

(A) $2x+y+3=0$ (B) $2x+y-4=0$

(C) $2x+y-5=0$ (D) $2x+y-7=0$

13. The length of the common chord of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+bx+ay+c=0$ is:

(A) $\sqrt{\dfrac{(a+b)^2-8c}{2}}$ (B) $\sqrt{\dfrac{(a-b)^2-8c}{2}}$

(C) $\sqrt{\dfrac{(a-b)^2+8c}{2}}$ (D) $\sqrt{\dfrac{(a+b)^2+8c}{2}}$

14. The equation of the circle with centre (4, 1) and having $3x+4y-1=0$ as tangent is:

(A) $x^2+y^2-8x=0$

(B) $x^2+y^2-8x-2y+8=0$

(C) $x^2+y^2-8x+2y+8=0$

(D) $x^2+y^2-8x+4=0$

15. The radius of the least circle passing through the point (8, 4) and cutting the circle $x^2+y^2=40$ orthogonally is:

(A) $\sqrt{5}$ (B) $\sqrt{7}$

(C) $2\sqrt{5}$ (D) $4\sqrt{5}$

16. The line $2x-y+1=0$ is tangent to the circle at the point (2, 5) and the centre of the circles lies on $x-2y=4$. The radius of the circle is:

(A) $3\sqrt{5}$ (B) $5\sqrt{3}$

(C) $2\sqrt{5}$ ((D) $5\sqrt{2}$

17. The locus of the midpoints of the chords of the circle $x^2+y^2-ax-by=0$, which subtends a right angle at $\left(\dfrac{a}{2},\dfrac{b}{2}\right)$ is:

(A) $ax+by=0$

(B) $ax+by=a^2+b^2$

(C) $x^2+y^2-ax-by-\dfrac{a^2+b^2}{8}=0$

(D) $x^2+y^2-ax-by+\dfrac{a^2+b^2}{8}=0$

18. A rhombus is inscribed in the region common to the two circles $x^2+y^2-4x-12=0$ and $x^2+y^2+4x-12=0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is:

(A) $8\sqrt{3}$ sq. units (B) $4\sqrt{3}$ sq. units

(C) $16\sqrt{3}$ sq. units (D) none of these

19. In a right ΔABC, right angled at A, on the leg AC as diameter, a semicircle is described. If the chord joining A with the point of intersection D of the hypotenuse and the semicircle, then the length of AC is:

(A) $\dfrac{AB\cdot AD}{\sqrt{AB^2+AD^2}}$ (B) $\dfrac{AB\cdot AD}{AB+AD}$

(C) $\sqrt{AB\cdot AD}$ (D) $\dfrac{AB\cdot AD}{\sqrt{AB^2-AD^2}}$

20. The locus of the midpoints of the chords of the circle $x^2+y^2+4x-6y-12=0$, which subtend an angle of $\dfrac{\pi}{3}$ radians at its circumference is:

(A) $(x-2)^2+(y+3)^2=6.25$

(B) $(x+2)^2+(y-3)^2=6.25$

(C) $(x+2)^2+(y-3)^2=18.75$

(D) $(x+2)^2+(y+3)^2=18.75$

21. The value of 'c' for which the set, $\{(x,y)\,|\,x^2+y^2+2x\le 1\}\cap\{(x,y)\,|\,x-y+c\ge 0\}$ contains only one point in common is:

(A) $(-\infty,-1]\cup[3,\infty)$ (B) $[-1,3]$

(C) $\{-3\}$ (D) $\{-1\}$

22. P is a point (a, b) in the Ist quadrant. If the two circles which pass through P and touch both the coordinate axes cut at right angles, then:

(A) $a^2-6ab+b^2=0$ (B) $a^2+2ab-b^2=0$

(C) $a^2-4ab+b^2=0$ (D) $a^2-8ab+b^2=0$

23. If (α,β) is a point on the circle, whose centre is on the x-axis and which touches the line $x+y=0$ at $(2,-2)$, then the greatest value of α is:

(A) $4-\sqrt{2}$ (B) 6

(C) $4+2\sqrt{2}$ (D) $4+\sqrt{2}$

24. Three concentric circles of which the biggest is $x^2+y^2=1$, have their radii in A.P. If the line $y=x+1$ cuts all the circles in real and distinct points. The interval in which the common difference of the A.P. will lie is:

(A) $\left(0,\dfrac{1}{4}\right)$ (B) $\left(0,\dfrac{1}{2\sqrt{2}}\right)$

(C) $\left(0,\dfrac{2-\sqrt{2}}{4}\right)$ (D) none of these

25. The equations of four circles are $(x\pm a)^2+(y\pm a)^2=a^2$ the radius of a circle touching all the four circles is:

(A) $\left(\sqrt{2}+2\right)a$ (B) $2\sqrt{2}a$

(C) $\left(\sqrt{2}+1\right)a$ (D) $\left(2+\sqrt{2}\right)a$

26. If a line segment $AM = a$, moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, then the locus of M is:

(A) $x^2 + y^2 = 4a^2$

(B) $x^2 + y^2 = 2ax$

(C) $x^2 + y^2 = 2ay$

(D) $x^2 + y^2 - 2ax - 2ay = 0$

27. B and C are fixed points having co-ordinates (3, 0) and (– 3, 0), respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation:

(A) $x^2 + y^2 = 1$
(B) $x^2 + y^2 = 2$
(C) $9(x^2 + y^2) = 1$
(D) $9(x^2 + y^2) = 4$

28. Tangents are drawn to a unit circle with centre at the origin from each point on the line $2x + y = 4$. Then the equation to the locus of the midpoint of the chord of contact is:

(A) $2\,(x^2 + y^2) = x + y$
(B) $2\,(x^2 + y^2) = x + 2\,y$
(C) $4\,(x^2 + y^2) = 2x + y$
(D) none of these

29. The common chord of two intersecting circles C_1 and C_2 can be seen from their centres at the angles of 90° and 60°, respectively. If the distance between their centres is equal to $\sqrt{3}\ + 1$, then the radii of C_1 and C_2 are:

(A) $\sqrt{3}$ and 3
(B) $\sqrt{2}$ and $2\sqrt{2}$
(C) $\sqrt{2}$ and 2
(D) $2\ \sqrt{2}$ and 4

30. If the line $x \cos \theta + y \sin \theta = 2$ is the equation of a transverse common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6\ \sqrt{3}\ x - 6y + 20 = 0$, then the value of θ is:

(A) $\frac{5\pi}{6}$
(B) $\frac{2\pi}{3}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{6}$

31. The locus of the midpoints of the chords of the circle $x^2 + y^2 - 2x - 4y - 11 = 0$, which subtends 60° at the centre is:

(A) $x^2 + y^2 - 4x - 2y - 7 = 0$
(B) $x^2 + y^2 + 4x + 2y - 7 = 0$
(C) $x^2 + y^2 - 2x - 4y - 7 = 0$
(D) $x^2 + y^2 + 2x + 4y + 7 = 0$

32. The locus of the foot of the perpendicular, from the origin to chord of the circle $x^2 + y^2 - 4x - 6y - 3 = 0$, which subtend a right angle at the origin is:

(A) $x^2 + y^2 - 2x - 3y - 3 = 0$
(B) $2x^2 + 2y^2 - 4x - 6y - 9 = 0$
(C) $3x^2 + 3y^2 - 12x - 18y - 6 = 0$
(D) $2x^2 + 2y^2 - 4x - 6y - 3 = 0$

33. Chord AB of the circle $x^2 + y^2 = 100$ passes through the point (7, 1) and subtends an angle of 60° at the circumference of the circle. If m_1 and m_2 are the slopes of two such chords, then the value of $m_1 m_2$ is:

(A) –1
(B) 1
(C) $\frac{7}{12}$
(D) –3

34. An equation of a line through the point (1, 2), whose distance from the point (3, 1) has the greatest value is:

(A) $y = 2x$
(B) $y = x + 1$
(C) $x + 2y - 5 = 0$
(D) $y = 3x - 1$

35. Tangents are drawn to $x^2 + y^2 - 1 = 0$ from any arbitrary point P on the circle C_1: $x^2 + y^2 - 4 = 0$. These tangents meet the circle 'C_1' again in A and B. Locus of point of intersection of tangents drawn to C_1 at A and B is:

(A) $x^2 + y^2 = 10$
(B) $x^2 + y^2 = 16$
(C) $x^2 + y^2 = 5$
(D) $x^2 + y^2 = 25$

36. Let PQ and RS be tangents at the extremities of a diameter PR of a circle of radius r such that PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals:

(A) $\sqrt{PQ.RS}$
(B) $\frac{PQ + RS}{2}$
(C) $\frac{2PQ.RS}{PQ + RS}$
(D) $\sqrt{\frac{(PQ)^2 + (RS)^2}{2}}$

37. The centres of two circles are $5\sqrt{2}$ units apart. The circles are orthogonal and their radii are integers $\neq 1$. The area of the region, in square units, common to the two circles, is:

(A) $\frac{25}{4}(\pi - 2)$
(B) $\frac{25}{2}(\pi - 2)$
(C) $\frac{25}{4}(\pi - 3)$
(D) $\frac{25}{2}(\pi - 3)$

38. Let α, β and γ are the parametric angles of three points P, Q and R, respectively lying on the circle $x^2 + y^2 = 1$. If the lengths of chords AP, AQ and AR are in G.P., where A is (1, 0) then (let $0 \leq \alpha, \beta$ and $\gamma \leq 2\pi$):

(A) $\sin\frac{\alpha+\gamma}{4}\cos\frac{\alpha-\gamma}{4} \geq \sin\frac{\beta}{2}$

(B) $\sin\frac{\alpha+\gamma}{4}\cos\frac{\alpha-\gamma}{4} \leq \sin\frac{\beta}{2}$

(C) $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} \geq \sin\frac{\beta}{2}$

(D) $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} \leq \sin\frac{\beta}{2}$

39. Two perpendicular tangents are drawn from a point P to a circle C_1. Another circle C_2 touches C_1 and also these perpendicular tangents from P. If r_1 and r_2 be the radii of circles C_1 and C_2, respectively then $\frac{r_1}{r_2}$ is:

(A) $\sqrt{2}+1$
(B) $3+2\sqrt{2}$
(C) $\sqrt{2}-1$
(D) $3-2\sqrt{2}$

40. Five circles C_1, C_2, C_3, C_4, C_5 with radii r_1, r_2, r_3, r_4, r_5, respectively ($r_1 < r_2 < r_3 < r_4 < r_5$) be such that C_i and C_{i+1} touch each other externally for all $i = 1, 2, 3, 4$. If all the five circles touch each of the two straight lines L_1 and L_2 and $r_1 = 2$ and $r_5 = 32$, then r_3 is:

(A) 8
(B) 17
(C) $\frac{64}{17}$
(D) dependent on r_2 and t_4

41. Let C_1, C_2 and C_3 be three parallel chords of a circle on the same side of its centre. The distance between C_1 and C_2 is same as the distance between C_2 and C_3. If lengths of C_1, C_2 and C_3 be respectively 20, 16 and 8. Then the radius of the circle is:

(A) 12
(B) $5\frac{\sqrt{22}}{2}$
(C) $5\frac{\sqrt{65}}{3}$
(D) data insufficient

42. A circle C_1 is drawn having any point P on x-axis as its centre and passing through the centre of circle C: $x^2 + y^2 = 1$. A common tangent to C_1 and C touches circles at Q and R respectively, then $Q(x, y)$ also satisfies:

(A) $x^2 = 1$
(B) $x^2 + y^2 = 1$
(C) $y^2 - 1 = 0$
(D) none of these

43. The equation of chord of the circle $x^2 + y^2 - 6x - 4y - 12 = 0$, which passes through the origin such that the origin divides it in the ratio 3:2 is:

(A) $y + x = 0, 7y + 17x = 0$
(B) $y + 3x = 0, 7y + 3x = 0$
(C) $4x + y = 0, 9y + 8x = 0$
(D) none of these

44. Point P is moving along the circumference of the circle $(x-4)^2 + (y-8)^2 = 16$. Then it breaks away from it and moving along a tangent to the circle, cuts the x-axis at the point (8, 0). The co-ordinates of the point on the circle at which the moving point broke away can be:

(A) (8, 8)
(B) (0, 8)
(C) (4, 4)
(D) none of these

45. Let X be a point on a straight line segment AB such that $AB.BX = (AX)^2$. Let C be point on circle with centre A and radius AB such that $BC = AX$, then angle $\angle BAC$ is:

(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{5}$
(D) none of these

46. A variable circle cuts the lines $px + qy + r = 0$ and $qx - py + s = 0$ making intercepts 10 and 6 respectively, then the locus of the centre of the circle is:

(A) a circle
(B) a parabola
(C) an ellipse
(D) a hyperbola

47. Consider, $A = \{(x, y) : x^2 + y^2 \leq 9\}$,

$B = \{(x, y) : |x - 1 - x^2| \leq |x^2 - 3x + 4|\}$. Number of integrals (x, y) in the set $A \cap B$ is/are:

(A) 20
(B) 18
(C) 16
(D) 22

48. From point (3, 4) two tangents are drawn to the circle $x^2 + y^2 = 4$, then the radius of circle which is enclosed between pair of tangent drawn from P and the given circle is:

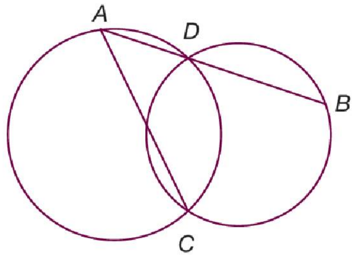
(A) $\frac{2}{7}$
(B) $\frac{3}{7}$
(C) $\frac{4}{7}$
(D) $\frac{6}{7}$

49. P is a point on the circumference of a circle of radius r units. Chords QR are drawn parallel to the tangent at P. The maximum area of the ΔPQR is:

(A) $\frac{\sqrt{3}}{4}r^2$
(B) $\frac{3\sqrt{3}}{4}r^2$
(C) $\frac{3}{4}r^2$
(D) $\frac{3\sqrt{3}}{2}r^2$

50. Let $S_1 : (x-1)^2 + (y-2)^2 = 25$ and $S_2 : (x-1)^2 + (y-7)^2 = 9$ be two circles such that they intersect at C and D (see the figure). AB is any chord drawn through D, A

and C are joined. If the length of the chord AC of circle $S_1 = 0$ is π unit, then the length of AB is:



(A) 2π (B) 3π

(C) π (D) $\frac{3\pi}{4}$

51. Tangents TP and TQ are drawn from a point T to the circle $x^2 + y^2 = a^2$. If the point T lies on the line $px + qy = r$, then the locus of the centre of the circumcircle of ΔTPQ is:

(A) $2px + qy = r$ (B) $2px + 2qy = r$

(C) $px + qy + r = 0$ (D) $px - qy + r = 0$

52. The range of values of 'a' for which the point $(a, 4)$ is outside the circle $x^2 + y^2 + 10x = 0$ and $x^2 + y^2 - 12x + 20 = 0$ is:

(A) $(-\infty,-8)\cup(-2,6)\cup(6,\infty)$

(B) $(-8, -2)$

(C) $(-\infty,-8)\cup(-2,\infty)$

(D) none of these

53. The locus of the centre of a circle touching the circle $x^2 + y^2 - 4y - 2x = \left(2\sqrt{3} - 1\right)$ internally and tangents on which from (1, 2) is making a 60° angle with each other, will be:

(A) $(x+2)^2 + (y-1)^2 = 7$

(B) $(x-2)^2 + (y+3)^2 = 5$

(C) $(x-1)^2 + (y-2)^2 = 3$

(D) none of these

54. Two circles $S_1 = 0, S_2 = 0$ of equal radius 'r' intersect such that the centre of one circle passes through the centre of the other circle. Another circle ($S = 0$) touches $S_1 = 0$ internally and $S_2 = 0$ externally and also touches the line joining the centres of the two circles $S_1 = 0$ and $S_2 = 0$. Then the radius of $S = 0$ is:

(A) $\frac{\sqrt{3}r}{2}$ (B) $\frac{\sqrt{3}r}{4}$

(C) $\frac{\sqrt{3}r}{8}$ (D) none of these

55. Two perpendicular tangents are drawn to the circle $x^2 + y^2 = 1$, then the maximum value of ordinate of the point from where tangents are drawn is:

(A) $2\sqrt{2}$ (B) 4

(C) 3 (D) none of these

56. Equation of the straight line meeting the circle with centre at origin and radius equal to 5 in two points at equal distances of 3 units from the point $A(3, 4)$ is:

(A) $6x + 8y = 41$ (B) $6x - 8y + 41 = 0$

(C) $8x + 6y + 41 = 0$ ((D) $8x - 6y + 41 = 0$

57. PA and PB are the tangents to a circle S touching at points A and B, C is a point on S in between A and B as shown in the figure. L.C.M. is a tangent to S intersecting PA and PB in points L and M, respectively. Then the perimeter of the ΔPLM depends on:

(A) A, B, C and P (B) P but not on C

(C) P and C only (D) the radius of S only

58. The difference between the radii of largest and the smallest circles which have their centres on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$ and passes through the point (a, b) lying outside the given circle is:

(A) $\left((a+1)^2 + (b+1)^2\right)^{1/2}$

(B) $\left((a+1)^2 + (b+1)^2\right)^{1/3}$

(C) 6

(D) 3

59. Given that, $x^2 + y^2 = 14x + 6y + 6$, then the greatest possible value of $3x + 4y$ is:

(A) 70 (B) 73

(C) 71 (D) 72

60. The equation of the tangent to the circle $x^2 + y^2 - 80x - 60y + 2100 = 0$ at the point nearest to (0, 0) is:

(A) $4x + 3y = 150$ (B) $4x - 3y = 150$

(C) $3x + 4y = 15$ (D) $3x - 4y = 15$

61. In ΔABC, the equation of the side BC is $2x - y = 3$ and its circumcentre and orthocentre are $(2, 4)$ and $(1, 2)$ respectively, then radius of the circumcircle is:

(A) $\sqrt{\frac{51}{5}}$ (B) $\sqrt{\frac{61}{5}}$

(C) $\sqrt{\frac{31}{5}}$ (D) $\sqrt{\frac{41}{5}}$

62. C_1 is a circle of radius 1 unit touching the x-axis and y-axis. C_2 is another circle of radius > 1 and touching the axes as well as circle C_1. Then the radius of C_2 is:

(A) $3-2\sqrt{2}$ (B) $3+2\sqrt{2}$

(C) $3+2\sqrt{3}$ (D) $2+2\sqrt{3}$

63. The equation of the circle having the pair of lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having the size just sufficient to contain the circle $x(x-4) + y(y-3) = 0$ is:

(A) $x^2 + y^2 - 6x + 3y = 0$

(B) $x^2 + y^2 + 6x - 3y - 45 = 0$

(C) $x^2 + y^2 + 6x - 3y - 5 = 0$

(D) $x^2 + y^2 - 6x - 3y - 45 = 0$

64. $P(1, 1)$ is a fixed point on the circle $x^2 + y^2 - 6x - 4y + 8 = 0$ and A, B are moving on the circumference of the same circle such that $PA = PB = d(>0)$. The equation of the secant line AB, then d is maximum is:

(A) $2x + y - 3 = 0$ (B) $x + y - 8 = 0$

(C) $2x + y - 13 = 0$ (D) $x - 2y + 1 = 0$

65. The equation of the circle touching the pair of lines $7x^2 - 18xy + 7y^2 = 0$ and the circle $x^2 + y^2 - 8x - 8y = 0$ and contained in the given circle is:

(A) $x^2 + y^2 + 12x + 12y + 64 = 0$

(B) $x^2 + y^2 - 12x - 12y + 64 = 0$

(C) $x^2 + y^2 + 12x - 12y + 80 = 0$

(D) $x^2 + y^2 - 12x - 12y + 80 = 0$

66. If r_1 and r_2 are the radii of smallest and largest circles, which passes through (5, 6) and touches the circle $(x-2)^2 + y^2 = 4$, then $r_1 r_2$ is:

(A) $\frac{4}{41}$ (B) $\frac{41}{4}$

(C) $\frac{5}{41}$ (D) $\frac{41}{6}$

67. If $C_1 : x^2 + y^2 = \left(3+2\sqrt{2}\right)^2$ be a circle and PA and PB are pair of tangents on C_1, where P is any point on the director circle of C_1, then the radius of smallest circle which touches C_1 externally and also the two tangents PA and PB is:

(A) $2\sqrt{2}-3$ (B) $2\sqrt{2}-1$

(C) $2\sqrt{2}+1$ (D) 1

68. From a point $R(5, 8)$ two tangents RP and RQ are drawn to a given circle $S = 0$, whose radius is 5. If the circumcentre of the ΔPQR is (2, 3), then the equation of circle $S = 0$ is:

(A) $x^2 + y^2 + 2x + 4y - 20 = 0$

(B) $x^2 + y^2 + x + 2y - 10 = 0$

(C) $x^2 + y^2 - x - 2y - 20 = 0$

(D) $x^2 + y^2 - 4x - 6y - 12 = 0$

69. On the line segment joining (1, 0) and (3, 0) an equilateral triangle is drawn having its vertex in the IVth quadrant, then radical centre of the circles described on its sides as diameter is:

(A) $\left(3, -\frac{1}{\sqrt{3}}\right)$ (B) $\left(3, -\sqrt{3}\right)$

(C) $\left(2, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(2, -\sqrt{3}\right)$

70. If the tangents are drawn from any point on the line $x + y = 3$ to the circle $x^2 + y^2 = 9$, then the chord of contact passes through the point:

(A) (3, 5) (B) (3, 3)

(C) (5, 3) (D) none of these

71. The ΔPQR is inscribed in the circle $x^2 + y^2 = 25$ such that P lies on the major arc QR. If Q and R have coordinates (3, 4) and (– 4, 3) respectively, then $\angle QPR$ is:

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

72. Equation of chord of the circle $x^2 + y^2 - 3x - 4y = 0$, which passes through the origin such that the origin divides it in the ratio 4:1 is:

(A) $x = 0$ (B) $24x + 7y = 0$

(C) $7x + 24y = 0$ (D) $7x - 24y = 0$

73. If the radius of the circumcircle of the ΔTPQ, where PQ is chord of contact corresponding to point T with respect to circle $x^2 + y^2 - 2x + 4y - 11 = 0$ is 6 units, then minimum distance of T from the director circle of the given circle is:

(A) 6 (B) 12

(C) $6\sqrt{2}$ (D) $12-4\sqrt{2}$

74. The exhaustive range of values of 'a' such that the angle between the pair of tangents drawn from (a, a) to the circle $x^2+y^2-2x-2y-6=0$ lies in the range $\left(\frac{\pi}{3}, \pi\right)$ is:

(A) $(1, \infty)$

(B) $(-5,-3)\cup(3,5)$

(C) $\left(-\infty,-2\sqrt{2}\right)\cup\left(2\sqrt{2},\infty\right)$

(D) $(-3,-1)\cup(3,5)$

75. In ΔABC, equation of side BC is $x-y=0$. Circumcentre and orthocentre of the triangle are (2, 3) and (5, 8), respectively. Equation of circumcircle of the triangle is:

(A) $x^2+y^2-4x+6y-27=0$

(B) $x^2+y^2-4x-6y-27=0$

(C) $x^2+y^2+4x+6y-27=0$

(D) $x^2+y^2+4x-6y-27=0$

76. A circle touches the lines $y=\dfrac{x}{\sqrt{3}}, y=x\sqrt{3}$ and has unit radius. If the centre of this circle lies in the Ist quadrant, then one possible equation of this circle is:

(A) $x^2+y^2-2x\left(\sqrt{3}+1\right)-2y\left(\sqrt{3}+1\right)+8+4\sqrt{3}=0$

(B) $x^2+y^2-2x\left(\sqrt{3}+1\right)-2y\left(\sqrt{3}+1\right)+5+4\sqrt{3}=0$

(C) $x^2+y^2-2x\left(\sqrt{3}+1\right)-2y\left(\sqrt{3}+1\right)+7+4\sqrt{3}=0$

(D) $x^2+y^2-2x\left(\sqrt{3}+1\right)-2y\left(\sqrt{3}+1\right)+6+4\sqrt{3}=0$

77. $\lim\limits_{x\to0}\left(\dfrac{1+cx}{1-cx}\right)^{\frac{1}{x}}=4.\ x_1=\lim\limits_{x\to0}\left(\dfrac{1+2cx}{1-2cx}\right)^{\frac{1}{x}}, r=\lim\limits_{x\to0}\left(\dfrac{2+cx}{2-cx}\right)^{\frac{1}{x}}.$

The equations of the circles which touch x-axis at $(x_1, 0)$ and radius r are:

(A) $x^2+y^2-16x\pm4y+64=0$

(B) $x^2+y^2-16x\pm2y+64=0$

(C) $x^2+y^2-32x\pm4y+256=0$

(D) $x^2+y^2-32x\pm2y+256=0$

78. Two circles of unequal radii intersect in two distinct points P and Q. A line through P cuts first circle at A and second circle at B. Let M be the midpoint of AB. If the line MQ meets the first circle at X and the second circle at Z, then M divides XZ internally in the ratio:

(A) 1:1 (B) 1:4

(C) 3:4 (D) 1:2

79. Let C_n be the circle of radius $\left(\frac{2}{3}\right)^n$ centred at the origin for $n=0, 1, 2,\ldots\ldots$ and A_n be the area of the region inside the circle C_{2n} and outside the circle C_{2n+1} for $n=0, 1, 2,\ldots\ldots$then find $\sum\limits_{n=0}^{\infty}A_n$.

(A) $\dfrac{3\pi}{5}$ (B) $\dfrac{7\pi}{15}$

(C) $\dfrac{9\pi}{13}$ (D) $\dfrac{10\pi}{13}$

80. All chords through an external point to the circle $x^2+y^2=16$ are drawn having length l, where l is positive integer then sum of the squares of the distances from centre of circle to these chords is:

(A) 154 (B) 124

(C) 172 (D) 128

81. Two circles passing through the points $A(1, 2)$ and $E(2, 1)$ touch the line $4x+8y-7=0$ at B and D. The possible co-ordinate of point C such that $ABCD$ is a parallelogram is:

(A) $\left(\frac{15}{2}, -\frac{9}{2}\right)$ (B) (0, 3)

(C) $\left(\frac{1}{2}, \frac{5}{2}\right)$ (D) $\left(-\frac{5}{2}, \frac{3}{4}\right)$

82. Tangents are drawn to the circle $x^2+y^2=12$ at the points, where it is met by the circle $x^2+y^2-5x+3y-2=0$. The point of intersection of these tangents is:

(A) $\left(4, -\frac{16}{5}\right)$ (B) $\left(6, -\frac{18}{5}\right)$

(C) $\left(2, -\frac{17}{3}\right)$ (D) $\left(6, -\frac{8}{5}\right)$

83. Equation of the straight line meeting the circle $x^2+y^2=100$ in two points, each point at a distance of 4 units from the point (8, 6) on the circle is:

(A) $4x+3y-50=0$ (B) $4x+3y-100=0$

(C) $4x+3y-46=0$ (D) none of these

84. A light ray gets reflected from the line $x=-2$. If the reflected ray touches the circle $x^2+y^2=4$ and point of incident is $(-2, -4)$, then the equation of incident ray is:

(A) $3x+4y+22=0$ (B) $4x+3y+20=0$

(C) $x+2y+10=0$ (D) $x+y+6=0$

85. Two circles of radii 9 and 4 respectively touch each other externally at P. If T_1T_2 be the direct common tangent to the two circles then length of median from P to the side T_1T_2 of the ΔPT_1T_2 is:

(A) 5 (B) 6

(C) 2 (D) $\sqrt{13}$

86. A point P lies inside the circles $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 8x + 7 = 0$. The point P starts moving under the conditions that its path encloses greatest possible area and it is at a fixed distance from any arbitrarily chosen fixed point in its region. The locus of P is:

(A) $4x^2 + 4y^2 - 12x + 1 = 0$ (B) $4x^2 + 4y^2 + 12x - 1 = 0$

(C) $x^2 + y^2 - 3x - 2 = 0$ (D) $x^2 + y^2 - 3x + 2 = 0$

87. There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0$, $n \in I$, having exactly two common tangents. Then the number of possible values of n is:

(A) 2 (B) 8

(C) 9 (D) 7

88. From a point 'P' on the line $2x + y + 4 = 0$, which is nearest to the circle $x^2 + y^2 - 12y + 35 = 0$, tangents are drawn to given circle. The area of quadrilateral $PACB$ (where 'C' is the centre of circle and PA and PB are the tangents) is:

(A) 8 (B) $\sqrt{110}$

(C) $\sqrt{19}$ (D) none of these

89. If $A(0, 2)$ and $B(-1, 1)$ are two points on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ and C is a variable point on the same circle, then the locus of the orthocentre of ΔABC is:

(A) $x^2 + y^2 + 8x - 10y + 25 = 0$

(B) $x^2 + y^2 + 8x - 10y + 16 = 0$

(C) $x^2 + y^2 - 10x + 8y + 25 = 0$

(D) $x^2 + y^2 - 10x + 8y + 16 = 0$

90. The equation of the circle symmetric to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ about the line $x - y = 3$ is:

(A) $x^2 + y^2 - 10x + 4y + 28 = 0$

(B) $x^2 + y^2 + 6x + 8 = 0$

(C) $x^2 + y^2 - 14x - 2y + 49 = 0$

(D) $x^2 + y^2 + 8x + 2y + 16 = 0$

91. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre at (2, 1), then the radius of the circle is:

(A) 3 (B) $\sqrt{2}$

(C) $\sqrt{3}$ (D) 2

92. PA and PB are the two tangents drawn from $P(5,5)$ to the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ with A and B as the points of contact. Then the equation of the line which divides the line segments PA and PB in the ratio 1:2 is:

(A) $3x + 2y - 28 = 0$ (B) $3x + 2y - 24 = 0$

(C) $3x + 2y - 26 = 0$ (D) $3x + 2y - 27 = 0$

93. Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord AB is:

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$

(C) $2\sqrt{6}$ (D) $6\sqrt{2}$

94. If the line $|y| = x - \alpha$, $\alpha > 0$ does not meet the circle $x^2 + y^2 - 10x + 21 = 0$, then α belongs to:

(A) $\left(-\infty, 5 - 2\sqrt{2}\right) \cup \left(5 + 2\sqrt{2}, \infty\right)$

(B) $\left(5 - 2\sqrt{2}, 5 + 2\sqrt{2}\right)$

(C) $\left(-\infty, 5 - 2\sqrt{2}\right) \cup (7, \infty)$

(D) $\left(5 - 2\sqrt{2}, 7\right)$

95. Two concentric circles are such that the smaller divides the larger into two regions of equal area. If the radius of the smaller circle is 2, then the length of the tangent from any point 'P' on the larger circle to the smaller circle is:

(A) 1 (B) $\sqrt{2}$

(C) 2 (D) None of these

96. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is:

(A) $\dfrac{16}{\sqrt{5}}$ (B) 8

(C) $4\sqrt{6}$ (D) $\dfrac{8\sqrt{5}}{5}$

97. Tangent to the curve $y = x^2 + 6$ at a point $P(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then, the co-ordinates of Q are:

(A) (– 6, –11) (B) (–9, – 13)

(C) (– 10, – 15) (D) (– 6, – 7)

98. If the length of a common internal tangent to two circles is 7 and that of a common external tangent is 11, then the product of the radii of the two circles is:

(A) 36 (B) 9

(C) 18 (D) 4

99. If P is the midpoint of chord AB of the circle $x^2 + y^2 = 16$ and AB subtends a right angle at the centre of the circle, then the locus of P is:

(A) $x^2 + y^2 = 8$ (B) $x^2 + y^2 = 32$

(C) $x^2 + y^2 = 4$ (D) $x^2 + y^2 = 64$

100. Equation of the circle of radius '$\sqrt{2}$' containing the point (3, 1) and touching the line $|x-1| = |y-1|$ is:

(A) $x^2 + y^2 - 4x - 2y + 3 = 0$

(B) $x^2 + y^2 - 6x + 4y + 11 = 0$

(C) $x^2 + y^2 - 6x - 2y + 8 = 0$

(D) $x^2 + y^2 + 4x + 4y + 6 = 0$

101. The circles $x^2 + y^2 - 6x + 6y + 17 = 0$ and $x^2 + y^2 - 6x - 2y + 1 = 0$, a common exterior tangent is drawn thus forming a curvilinear triangle. The radius of the circle inscribed in this triangle is:

(A) $\frac{3}{2}\left(2+\sqrt{3}\right)$ (B) $\frac{1}{2}\left(2-\sqrt{3}\right)$

(C) $\frac{1}{2}\left(2+\sqrt{3}\right)$ (D) $\frac{3}{2}\left(2-\sqrt{3}\right)$

102. Equation of circle touching the line $|x-2| + |y-3| = 4$ will be:

(A) $(x-2)^2 + (y-3)^2 = 12$

(B) $(x-2)^2 + (y-3)^2 = 4$

(C) $(x-2)^2 + (y-3)^2 = 10$

(D) $(x-2)^2 + (y-3)^2 = 8$

103. C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0), respectively. C_3 is a circle of unit radius, which passes through the centres of the circles C_1 and C_2 and has its centre above x-axis. Equation of the common tangent to C_1 and C_3, which does not pass through C_2, is:

(A) $x - \sqrt{3}y + 2 = 0$ (B) $\sqrt{3}x - y + 2 = 0$

(C) $\sqrt{3}x - y - 2 = 0$ (D) $x + \sqrt{3}y + 2 = 0$

104. A circle of radius '5' touches the co-ordinate axes in the Ist quadrant. If the circle makes one complete roll on x-axis along the positive direction, then its equation in new position is:

(A) $x^2 + y^2 - 10(2\pi + 1)x - 10y + 100\pi^2 + 100\pi + 25 = 0$

(B) $x^2 + y^2 + 10(2\pi + 1)x - 10y + 100\pi^2 + 100\pi + 25 = 0$

(C) $x^2 + y^2 - 10(2\pi + 1)x + 10y + 100\pi^2 + 100\pi + 25 = 0$

(D) $x^2 + y^2 + 10(2\pi + 1)x + 10y + 100\pi^2 + 100\pi + 25 = 0$

105. From origin, chords are drawn to the circle $x^2 + y^2 - 2y = 0$. The locus of the midpoints of these chords is:

(A) $x^2 + y^2 - y = 0$ (B) $x^2 + y^2 - x = 0$

(C) $x^2 + y^2 - 2x = 0$ (D) $x^2 + y^2 - x - y = 0$

106. Circles are drawn through the point (2, 0) to cut intercept of length '5' units on the x-axis. If their centres lie in the Ist quadrant, then their equation is:

(A) $x^2 + y^2 - 9x + 2ky + 14 = 0, k > 0$

(B) $3x^2 + 3y^2 + 27x - 2ky + 42 = 0, k > 0$

(C) $x^2 + y^2 - 9x - 2ky + 14 = 0, k > 0$

(D) $x^2 + y^2 - 2ky - 9y + 14 = 0, k > 0$

107. A line meets the co-ordinate axes in A and B. A circle is circumscribed about the ΔAOB. If m and n are the distances of the tangent to the circle at the origin from the points A and B respectively, then the diameter of the circle is:

(A) $m(m + n)$ (B) $m + n$

(C) $n(m + n)$ (D) $2(m + n)$

108. The equation of the circle passing through the point (2, – 1) and having two diameters along the pair of lines $2x^2 + 6y^2 - x + y - 7xy - 1 = 0$ is:

(A) $x^2 + y^2 + 10x + 6y - 19 = 0$

(B) $x^2 + y^2 + 10x - 6y + 19 = 0$

(C) $x^2 + y^2 - 10x - 6y - 19 = 0$

(D) $x^2 + y^2 - 10x + 6y - 19 = 0$

109. If from any point on the circle $x^2 + y^2 = a^2$, tangents are drawn to the circle $x^2 + y^2 = b^2$ $(a > b)$, then the angle between tangents is:

(A) $\sin^{-1}\left(\frac{b}{a}\right)$ (B) $2\sin^{-1}\left(\frac{a}{b}\right)$

(C) $2\sin^{-1}\left(\frac{b}{a}\right)$ (D) $\sin^{-1}\left(\frac{a}{b}\right)$

110. An equilateral triangle has two vertices (–2, 0) and (2, 0) and its 3rd vertex lies below the x-axis. The equation of the circumcircle of the triangle is:

(A) $\sqrt{3}(x^2 + y^2) - 4y + 4\sqrt{3} = 0$

(B) $\sqrt{3}(x^2 + y^2) - 4y - 4\sqrt{3} = 0$

(C) $\sqrt{3}(x^2 + y^2) + 4y + 4\sqrt{3} = 0$

(D) $\sqrt{3}(x^2 + y^2) + 4y - 4\sqrt{3} = 0$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. A circle passes through the points (– 1, 1), (0, 6) and (5, 5). The point(s) on this circle, where the tangent(s) at which is/are parallel to the straight line joining the origin to its centre is/are:

(A) $(1, -5)$ (B) $(5, 1)$

(C) $(-5, -1)$ (D) $(-1, 5)$

2. If a circle $S = 0$ passes through the common points of family of circles $x^2 + y^2 + \lambda x - 4y + 3 = 0$ $(\lambda \in R)$ and have a minimum area, then:

(A) area of $S = 0$ is π sq. units

(B) radius of director circle of $S = 0$ is $\sqrt{2}$

(C) radius of director circle of $S = 0$ for x-axis is 1 unit

(D) $S = 0$ never cuts $|2x| = 1$

3. The system of circles $x^2 + y^2 + 2\lambda x - 2y + 1 = 0$ $(\lambda \in R)$:

(A) possesses only one limiting point

(B) each of the member touches y-axis

(C) there are exactly two members touching x-axis

(D) the radius of the greatest circle of the system touching $y = x$ is $\sqrt{2} + 1$

4. A ray of light passing through (–3, 1) reflects on x-axis at A and the reflected ray is tangential to the circle $x^2 + y^2 = 1$ at P.

(A) The reflected ray is $3x - 4y + 5 = 0$

(B) $A = \left(\frac{5}{3}, 0\right)$

(C) Area of ΔAPO is $\frac{2}{3}$ sq. units ('O' is origin)

(D) $P = \left(\frac{-3}{5}, \frac{4}{5}\right)$

5. The real numbers a and b are distinct. From the circles,

$$S_1 : (x - a)^2 + (y - b)^2 = a^2 + b^2$$

$$S_2 : (x - b)^2 + (y - a)^2 = a^2 + b^2$$

which of the following assertion(s) is/are true?

(A) The line $y = x$ is an axis of symmetry for the circles

(B) The circles intersect at the origin and a point, P (say), which lies on the line $y = x$

(C) The line $y = x$ is the radical axis of the pair of circles

(D) The circles are orthogonal for all $a \neq b$

6. All the members of the family of the circles $x^2 + y^2 - 4\sin\theta\, x + 6\cos\theta\, y + 5\cos^2\theta + 1 = 0$, $\theta \in R$:

(A) are concyclic

(B) are congruent

(C) are forming a coaxial system

(D) have their centres on a fixed ellipse

7. If two vertices of an equilateral triangle are (–1, 0) and (1, 0), then the equation of its circumcircle is:

(A) $\sqrt{3}x^2 + \sqrt{3}y^2 + 2y - \sqrt{3} = 0$

(B) $2x^2 + 2y^2 + \sqrt{3}y - 2 = 0$

(C) $\sqrt{3}x^2 + \sqrt{3}y^2 - 2y - \sqrt{3} = 0$

(D) $2x^2 + 2y^2 - \sqrt{3}y - 2 = 0$

8. Number of points of intersections of director circle of $x^2 + y^2 = 4$ and the following curve is/are:

(A) $|x| + |y| = 4$ are 4

(B) $|x| + |y| = 2$ are 8

(C) $|x| + |y| = 6$ is 0

(D) $|x| + |y| = 2\sqrt{2}$ are 4

9. If C_1 and C_2 are two circles of radii a and b $(a < b)$ touching both the co-ordinate axes and have their centres in the Ist quadrant, then identify the correct statements among the following.

(A) If C_1 and C_2 touch each other, then $\frac{b}{a} = 3 + 2\sqrt{2}$

(B) If C_1 and C_2 are orthogonal, then $\frac{b}{a} = 2 + \sqrt{3}$

(C) If C_1, C_2 intersect in such a way that their common chord has maximum length, then $\frac{b}{a} = 3$

(D) If C_2 passes through the centre of C_1, then $\frac{b}{a} = 2 + \sqrt{2}$

10. The circles $x^2 + y^2 + 2x + 4y - 20 = 0$ and $x^2 + y^2 + 6x - 8y + 10 = 0$ are such that:

(A) the number of common tangents on them is 2

(B) orthogonal

(C) the length of their common tangents is $5\left(\frac{12}{5}\right)^{\frac{1}{4}}$

(D) the length of their common chord is $5\left(\sqrt{\frac{3}{2}}\right)$

11. Consider the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. Let '$O$' be the centre of the circle and tangent at $A(7, 3)$ and $B(5, 1)$ meeting at C. If $S = 0$ represents the family of circles passing through A and B, then:

(A) area of quadrilateral $OACB = 4$

(B) the radical axis for the family of circles $S = 0$ is $x + y = 10$

(C) the smallest possible circle of the family $S = 0$ is $x^2 + y^2 - 12x - 4y + 38 = 0$

(D) the co-ordinates of C are $(7, 1)$

12. In a ΔABC, let $BC = 13$, $CA = 14$ and $AB = 15$. If I is the incentre of the ΔABC and if the incircle meets the sides AB, BC and CA at the points P, Q and R, respectively, then which of the following options are correct?

(A) $AP = 8$

(B) Area of the ΔAPR is $\frac{128}{5}$ sq. units

(C) Area of the quadrilateral $APIR$ is 32 sq. units

(D) $AI = 4\sqrt{5}$

13. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are the tangents to the same circle, then:

(A) radius of the circle $= \frac{3}{4}$

(B) radius of the circle $= \frac{3}{2}$

(C) centre of the circle lies on $12x - 16y + 1 = 0$

(D) centre of the circle lies on $12x - 16y + 31 = 0$

14. C_1: $x^2 + y^2 = 25$ and C_2: $x^2 + y^2 - 2x - 4y - 7 = 0$ be two circles intersecting at the points A and B.

(A) Equation of common chord must be $x + 2y - 9 = 0$

(B) Equation of common chord must be $x + 2y + 7 = 0$

(C) Tangents at A and B to the circle C_1 intersect at $\left(\frac{25}{9}, \frac{50}{9}\right)$

(D) Tangents at A and B to the circle C_1 intersect at $(1, 2)$

15. If α, β and γ are the parameters of the points A, B and C on circle $x^2 + y^2 = a^2$ and if the ΔABC be equilateral, then:

(A) $\sum \cos\alpha = 0$
(B) $\sum \sin\alpha = 0$
(C) $\sum \tan\alpha = 0$
(D) $\sum \cot\alpha = 0$

16. Let S_1 and S_2 be the circles passing through (2, 3) and touching the co-ordinate axis and S be the circle having common points of S_1 and S_2 as the centre and radius equal to geometric mean of radius of S_1 and S_2, then:

(A) $S \equiv 0$ cuts y-axis is but not the x-axis

(B) $S \equiv 0$ cuts $y = x$

(C) radius of director circle of $S \equiv 0$ is $\sqrt{26}$

(D) A.M. of radius of S_1 and S_2 is 5

17. If $I_1 \equiv (1, -2)$ and $I_2 \equiv (-3, 2)$ are the excentres of ΔABC and excentre I_3 lies on the circle $x^2 + y^2 + 3x + y - 6 = 0$, then:

(A) locus of incentre of ΔABC is given by $x^2 + y^2 + x - y - 5 = 0$

(B) locus of incentre of ΔABC is given by $x^2 - y^2 + x + y + 5 = 0$

(C) centre of locus lies on the line $y = -x$

(D) origin lies inside the locus

18. Let $S_2 = 0$ is the mirror image of $S_1 : x^2 + y^2 - 4x - 6y + 12 = 0$ w.r.t the line $L_1 : 10^4x + (10^4 + 10)y + (10^4 + 20) = 0$. Let $L_2 : 2^{11}x + (2^{11} + 2^{12})y + (2^{11} + 2^{13}) = 0$ be a line, then the equations of line passing through the point of intersection of the line $L_2 = 0$ with radical axes of $S_1 = 0$, $S_2 = 0$ and making equal intercepts with the co-ordinate axes is/are:

(A) $x - y - 3 = 0$
(B) $x + y + 1 = 0$
(C) $2x - 2y + 1 = 0$
(D) $2x + 2y + 3 = 0$

19. Common tangents are drawn to two circles $x^2 + y^2 = 1$ and $(x - 4)^2 + (y - 4)^2 = 4$. If the intersection points of transverse common tangents with the direct common tangents are A, B, C and D, then:

(A) $ABCD$ is a cyclic quadrilateral

(B) equation of circumcircle of ΔABC is $x^2 + y^2 - 4x - 4y = 0$

(C) equation of circumcircle of ΔABC is $x^2 + y^2 - 5x - 5y = 0$

(D) $ABCD$ is a square

20. Let x and y be variables satisfying the equation $x^2 + y^2 + 8x - 10y - 40 = 0$.

Let, $a = \max\left\{(x+2)^2 + (y-3)^2\right\}$ and $b = \min\left\{(x+2)^2 + (y-3)^2\right\}$, then:

(A) $a + b = 8$
(B) $a + b = \sqrt{2}$
(C) $a - b = 4\sqrt{2}$
(D) $ab = 73$

21. The radius of the circle, which touches the lines $x+y=2, x-y=2$ and the circle $x^2+y^2=1$ is:

(A) $\sqrt{2}-1$ (B) $\sqrt{2}+1$

(C) $3\left(\sqrt{2}-1\right)$ (D) $3\left(\sqrt{2}+1\right)$

22. If the conics, whose equations are

S_1:

$\left(\sin^2\theta\right)x^2+\left(2h\tan\theta\right)xy+\left(\cos^2\theta\right)y^2+32x+16y+19=0$

S_2:

$\left(\cos^2\theta\right)x^2-\left(2h'\cot\theta\right)xy+\left(\sin^2\theta\right)y^2+16x+32y+19=0$,

intersect in four concylic points, where $\theta\in\left[0,\frac{\pi}{2}\right]$, then:

(A) $h \cdot h^1=1$ (B) $h-h^1=0$

(C) $\theta=\frac{\pi}{4}$ (D) $\theta=\frac{3\pi}{4}$

23. The centre of the circle of radius $\sqrt{10}$ and having $3(x-1)^2+10(x-1)(y-2)+3(y-2)^2=0$, as a pair of tangents is:

(A) $(6,-3),(-4,\ 7)$ (B) $(6,3),(4,-7)$

(C) $\left(\frac{7}{2},\frac{9}{2}\right),\left(\frac{-3}{2},\frac{-1}{2}\right)$ (D) $\left(\frac{7}{2},\frac{-9}{2}\right),\left(\frac{3}{2},\frac{1}{2}\right)$

24. Equations of the circles concentric with the circle $x^2+y^2-2x-4y=0$ and touching the circle $x^2+y^2+2x=1$, will be:

(A) $x^2+y^2-2x-4y=0$

(B) $x^2+y^2-2x-4y+3=0$

(C) $x^2+y^2-2x-4y-13=0$

(D) $x^2+y^2-2x-4y-1=0$

25. If $16\ell^2+9m^2=24\ell m+6\ell+8m+1$ and S be the equation of circle having $\ell x+my+1=0$ is tangent, then:

(A) equation of director circle of S is $x^2+y^2-6x-8y-25=0$

(B) radius of circle is 5

(C) perpendicular distance from centre of S to $x-y+1=0$ is $\sqrt{2}$

(D) equation of circle S is $x^2+y^2+6x+8y=0$

26. A point $P(x, y)$ is called a lattice point, if $x,y\in I$ (set of integers), then the total number of lattice points in the interior of the circle $x^2+y^2=a^2$, $a\neq 0$ cannot be:

(A) 1996 (B) 1998

(C) 1999 (D) 2001

27. Consider the circle $x^2+y^2-8x-18y+93=0$ with centre 'C' and point $P(2, 5)$ outside it. From the point P, pairs of tangents PQ and PR are drawn to the circle with S as the midpoint of QR. The line joining P to C intersects the given circle at A and B. Which of the following hold(s) good?

(A) CP is the A.M. of AP and BP

(B) PR is the G.M. of PS and PC

(C) PS is the H.M. of PA and PB

(D) The angle between the two tangents from P is $\tan^{-1}\left(\frac{3}{4}\right)$

28. Q is any point on the circle $x^2+y^2=9$. If QN is perpendicular from Q on the x-axis, then the locus of the point of trisection of QN is:

(A) $4x^2+9y^2=36$ (B) $9x^2+4y^2=36$

(C) $9x^2+y^2=9$ (D) $x^2+9y^2=9$

29. $\left(1,2\sqrt{2}\right)$ is a point on the circle $x^2+y^2=9$. Locate the points on the given circle, which are at 2 unit distance from $\left(1,2\sqrt{2}\right)$.

(A) $\left(-1,2\sqrt{2}\right)$ (B) $\left(2\sqrt{2},1\right)$

(C) $\left(\frac{23}{9},\frac{10\sqrt{2}}{9}\right)$ (D) $\left(\frac{23}{9},\frac{10}{9}\right)$

30. If the line $3x-4y-\lambda=0$ touches the circle $x^2+y^2-4x-8y-5=0$ at (a,b), then find $\lambda+a+b$.

(A) 20 (B) 22

(C) –30 (D) –28

31. Tangents are drawn to the circle $x^2+y^2=32$ from a point A lying on the x-axis. If the tangents cut the y-axis at points B and C, then find the possible co-ordinates of A such that the area of ΔABC is minimum, are:

(A) $\left(8\sqrt{2},0\right)$ (B) $(8,0)$

(C) $(-8,0)$ (D) $\left(-8\sqrt{2},0\right)$

32. Let $C_1=x^2+y^2=r_1^2$ and $C_2=x^2+y^2=r_2^2\left(r_1<r_2\right)$ be two circles. Let A be the fixed point $\left(r_1,0\right)$ on C_1 and B be a variable point on C_2. Let the line BA meet the circle C_2 again at E, then:

(A) the maximum value of BE is $2r_2$

(B) the minimum value of BE is $2\sqrt{r_2^2-r_1^2}$

(C) if O is origin, least value of $OA^2+OB^2+BE^2$ is $5r_2^2-3r_1^2$

(D) if O is origin, greatest value of $OA^2+OB^2+BE^2$ is $r_1^2+5r_2^2$

33. If Q and S are two points on the circle $x^2+y^2=4$, such that the tangents QP and SR are parallel and if PS and QR intersect at T, then $\left(\frac{QT}{PQ}\right)^2+\left(\frac{ST}{RS}\right)^2+PQ.RS \neq$

(A) 5 (B) 10

(C) 16 (D) 17

34. In a variable ΔABC, where the base BC is fixed and $\angle BAC=\alpha$ (a constant):

(A) the locus of centroid of ΔABC lies on a circle

(B) the locus of incentre of ΔABC lies on a circle

(C) the locus of orthocentre of ΔABC lies on a circle

(D) the locus of excentre opposite to 'A' lies on a circle

35. If A and B are two points on the circle $x^2+y^2-4x+6y-3=0$, which are the farthest and the nearest points, respectively from the point (7, 2), then:

(A) $A\equiv\left(2-2\sqrt{2},-3-2\sqrt{2}\right)$

(B) $A\equiv\left(2+2\sqrt{2},-3+2\sqrt{2}\right)$

(C) $B\equiv\left(2+2\sqrt{2},-3+2\sqrt{2}\right)$

(D) $B\equiv\left(2-2\sqrt{2},-3-2\sqrt{2}\right)$

36. If two circles $x^2+y^2-6x-12y+1=0$ and $x^2+y^2-4x-2y-11=0$ cut a third circle orthogonally, then the radical axis of the two circles passes through:

(A) $(1,1)$

(B) $(0,6)$

(C) centre of the third circle

(D) midpoint of the line joining the centres of the given circles

37. The equation of a circle C_1 is $x^2+y^2=4$. If the locus of the intersection of orthogonal tangents to the circle is the curve C_2 and the locus of the intersection of perpendicular tangents to the curve C_2 is the curve C_3, then:

(A) C_3 is a circle

(B) the area enclosed by the curve C_3 is 8π

(C) C_2 and C_3 are circles with the same centre

(D) none of the above

38. Consider the circles $C_1\equiv x^2+y^2-2x-4y-4=0$ and $C_2\equiv x^2+y^2+2x+4y+4=0$ and the line $L\equiv x+2y+2=0$, then:

(A) L is the radical axis of C_1 and C_2

(B) L is the common tangent of C_1 and C_2

(C) L is the common chord of C_1 and C_2

(D) L is perpendicular to the joining centres of C_1 and C_2

39. The equation of a circle is $S_1\equiv x^2+y^2=1$. If the orthogonal tangents to S_1 meet at another circle S_2 and the orthogonal tangents to S_2 meet at the third circle S_3, then:

(A) radius of S_2 and S_3 are in the ratio $1:\sqrt{2}$

(B) radius of S_2 and S_3 are in the ratio $1:2$

(C) the circles S_1, S_2 and S_3 are concentric

(D) none of the above

40. If a chord of the circle $x^2+y^2-4x-2y-c=0$ is trisected at the points $\left(\frac{1}{3},\frac{1}{3}\right)$ and $\left(\frac{8}{3},\frac{8}{3}\right)$, then:

(A) length of the chord is $7\sqrt{2}$

(B) $c=20$

(C) radius of the circle is 25

(D) midpoint of the chord is (1, 1)

41. If a circle touches the line $x+y-2=0$ at (1, 1) and cuts the circle $x^2+y^2+4x+5y-6=0$ at P and Q, then:

(A) PQ can never be parallel to the given line $x+y-2=0$

(B) PQ can never be perpendicular to the given line $x+y-2=0$

(C) PQ always passes through (6, –4)

(D) PQ always passes through (–6, 4)

42. If $al^2-bm^2+2dl+1=0$, where a, b and d are fixed real numbers such that $a+b=d^2$, then the line $lx+my+1=0$ touches a fixed circle. Then the fixed circle is that:

(A) which cuts the x-axis orthogonally

(B) with radius equal to b

(C) on which the length of the tangent from the origin is $\sqrt{d^2-b}$

(D) with centre $(d, 0)$

43. Point M moved along the circle $(x-4)^2+(y-8)^2=20$. Then it broke away from it and moving along a tangent to the circle cuts the x-axis at the point (–2, 0). The co-ordinate of the point on the circle at which the moving point broke away can be:

(A) $\left(\frac{-3}{5},\frac{46}{5}\right)$ (B) $\left(\frac{-2}{5},\frac{44}{5}\right)$

(C) (6, 4) (D) (3, 5)

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

To the circle $x^2+y^2=4$, two tangents are drawn from $P(-4, 0)$, which touch the circle at Q and R and a rhombus $PQP'R$ is completed.

1. Circumcentre of ΔPQR is at:

(A) (–2, 0) (B) (2, 0)

(C) $\left(\frac{\sqrt{3}}{2},0\right)$ (D) (1, 0)

2. Ratio of area of $\Delta PQP'$ and $\Delta P'QR$ is:

(A) 2:1 (B) 1:2

(C) $\sqrt{3}:2$ (D) 1:1

3. Area of the rhombus $PQP'R$ is:

(A) $6\sqrt{3}$ (B) $2\sqrt{3}$

(C) $3\sqrt{3}$ (D) $\frac{\sqrt{3}}{2}$

Comprehension 2

The circle $x^2+y^2-4x-4y+4=0$ inscribed in a ΔOAB, which has two of its sides along the co-ordinate axes.

4. Equation of AB is:

(A) $x+y=2(2+\sqrt{2})$ (B) $x-y=2(2-\sqrt{2})$

(C) $x+y=2\sqrt{2}$ (D) $x+y=4$

5. Orthocentre of ΔOAB is:

(A) $(0, 2+\sqrt{2})$ (B) $(2, 2+\sqrt{2})$

(C) (0, 0) (D) $(2+\sqrt{2}, 2+\sqrt{2})$

6. If the locus of the circumcentre of the ΔOAB (for line AB variable) is $x+y-xy+k\sqrt{x^2+y^2}=0$, then find k.

(A) –1 (B) 0

(C) 1 (D) 2

Comprehension 3

Two circles touch each other externally. Equation of the pair of direct common tangents is $x^2+4xy+y^2=0$ and the equation of the smaller circle is $x^2+y^2-4x+4y+6=0$.

7. Length of the direct common tangents of the two circles is:

(A) $4\sqrt{2}$ (B) $6\sqrt{3}$

(C) $2\sqrt{6}$ (D) $3\sqrt{2}$

8. Area of the triangle formed by the common tangents of the two circles is:

(A) $2\sqrt{6}$ sq. units (B) $6\sqrt{3}$ sq. units

(C) $4\sqrt{3}$ sq. units (D) $4\sqrt{2}$ sq. units

9. Radius of the circle, which touches the two circles externally and the direct common tangent is:

(A) $\frac{3\sqrt{2}}{4+2\sqrt{3}}$ (B) $\frac{6}{\sqrt{2}+\sqrt{3}}$

(C) $\frac{3\sqrt{2}}{2\sqrt{2}+\sqrt{3}}$ (D) $\frac{6\sqrt{2}}{\sqrt{2}+2\sqrt{3}}$

Comprehension 4

Equations of two circles are $x^2+y^2+2ax+c=0$ and $x^2+y^2+2by+c=0$.

10. Circles will touch one another, if a^2, $2c$ and b^2 are in:

(A) A.P. (B) G.P.

(C) H.P. (D) none of these

11. If their common chord is parallel to $3x-2y=0$, then the value of $\frac{a}{b}$ is:

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$

(C) 1 (D) none of these

12. If a is variable and c is constant, then radical axis of family of circles $x^2 + y^2 + 2ax + c = 0$ is:

(A) x-axis (B) y-axis

(C) $x - y = 0$ (D) $x + y = 0$

Comprehension 5

P is a point on the circle $C_1 : q^2\left(x^2 + y^2\right) = a^2p^2$

Q is a point on the circle $C_2 : x^2 + y^2 = a^2$

13. If the co-ordinates of P are (h, k), then the locus of the point which divides the join of PQ in the ratio $p:q$ is a circle C_3, whose centre is at the point:

(A) $\left(\frac{hp}{p+q}, \frac{kq}{p+q}\right)$ (B) $\left(\frac{h}{p+q}, \frac{k}{p+q}\right)$

(C) $\left(\frac{hq}{p+q}, \frac{kq}{p+q}\right)$ (D) $\left(\frac{hp}{p+q}, \frac{kp}{p+q}\right)$

14. Locus of the centre of the circle C_3 as P moves on the circle C_1 is a circle C_4.

(A) Non-concentric with C_1

(B) Non-concentric with C_2

(C) Having radius equal to the radius of C_3

(D) Having area equal to the area of C_1

15. If the point (p, q) lies on the line $y = 2x$, then $\frac{\text{Radius of } C_1}{\text{Radius of } C_4}$ is:

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$

(C) 3 (D) 2

Comprehension 6

Consider the two circles C_1: $x^2 + y^2 = r_1^2$ and C_2: $x^2 + y^2 = r_2^2$ $(r_2 < r_1)$. Let A $(r_1, 0)$ be a fixed point on C_1 and 'B' be a variable point on C_2. The line BA meets the circle C_2 again at C.

16. Maximum value of BC^2 is:

(A) $4r_1^2$ (B) $4r_2^2$

(C) $4r_2^2 - 4r_1^2$ (D) $r_1^2 - r_2^2$

17. Locus of midpoint of AB is:

(A) $\left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}$ (B) $\left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{2}$

(C) $\left(x - \frac{r_2}{2}\right)^2 + y^2 = \frac{r_1^2}{2}$ (D) $\left(x - \frac{r_2}{2}\right)^2 + y^2 = \frac{r_1^2}{4}$

18. Locus of midpoint of AB, when BC^2 is maximum is:

(A) $x^2 + y^2 = (r_1 + r_2)^2$ (B) $x^2 + y^2 = (r_1 - r_2)^2$

(C) $x^2 + y^2 = \frac{(r_1 - r_2)^2}{4}$ (D) none of these

Comprehension 7

Tangents PA and PB are drawn to the circle $(x - 4)^2 + (y - 5)^2 = 4$ from the point P on the curve $y = \sin x$, where A and B lie on the circle. Consider the function $y = f(x)$ represented by the locus of the center of the circumcircle of ΔPAB.

19. Range of $y = f(x)$ is:

(A) $[-2, 1]$ (B) $[-1, 4]$

(C) $[0, 2]$ (D) $[2, 3]$

20. Fundamental period of $y = f(x)$ is:

(A) 2π (B) 3π

(C) π (D) not defined

21. Which of the following is true?

(A) $f(x) = 4$ has real roots

(B) $f(x) = 1$ has real roots

(C) Range of $y = f^{-1}(x)$ is $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$

(D) None of these

Comprehension 8

Given equation of two intersecting circles are $S_1 = 0$ and $S_2 = 0$.

Equation of family of circles passing through the intersection points of $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$, (where $\lambda \neq -1$).

Equation of common chord is $S_1 - S_2 = 0$.

Equation of chord of contact for circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with respect to external point (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

22. The equation of the circle described on the common chord of the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 + 2y = 0$ as diameter is:

(A) $x^2 + y^2 + x - y = 0$ (B) $x^2 + y^2 - x - y = 0$

(C) $x^2 + y^2 - x + y = 0$ (D) $x^2 + y^2 + x + y = 0$

23. Let P be any moving point on the circle $x^2 + y^2 - 2x - 1 = 0$. AB be the chord of contact of this point with respect to the circle $x^2 + y^2 - 2x = 0$. The locus of the circumcentre of the ΔPAB (C being centre of the circles) is:

(A) $2x^2 + 2y^2 - 4x + 1 = 0$

(B) $x^2 + y^2 - 4x + 2 = 0$

(C) $x^2 + y^2 - 4x + 1 = 0$

(D) $2x^2 + 2y^2 - 4x + 3 = 0$

24. The common chord of the circle $x^2 + y^2 + 6x + 8y - 7 = 0$ and a circle passing through the origin and touching the line $y = x$, always passes through the point:

(A) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (B) (1, 1)

(C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (D) none of these

Comprehension 9

The line $x + 2y + a = 0$ intersects the circle $x^2 + y^2 - 4 = 0$ at two distance points A and B. Another line $12x - 6y - 41 = 0$ intersect the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ at two distinct points C and D.

25. The value of a so that the line $x + 2y + a = 0$ intersect the circle $x^2 + y^2 - 4 = 0$ at two distance points A and B is:

(A) $-2\sqrt{5} < a < 2\sqrt{5}$ (B) $0 < a < 2\sqrt{5}$

(C) $-\sqrt{5} < a < \sqrt{5}$ (D) $0 < a < 2\sqrt{5}$

26. The value of 'a' for which the four points A, B, C and D are concyclic is:

(A) 1 (B) 3

(C) 4 (D) 2

27. The equation of circle passing through the points A, B, C and D is:

(A) $5x^2 + 5y^2 - 8x - 16y - 36 = 0$

(B) $5x^2 + 5y^2 + 8x - 16y - 36 = 0$

(C) $5x^2 + 5y^2 + 8x + 16y - 36 = 0$

(D) $5x^2 + 5y^2 - 8x - 16y + 36 = 0$

Comprehension 10

The two circles $C_1: x^2 + y^2 = 4$ and $C_2: x^2 + y^2 - 4x = 0$ intersect at points $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$. Let two particles P and Q move on the circumference of the circle with the constant and equal speed (with particle P on C_1 and particle Q on C_2). Both the particles start from point A.

28. When particle P is at a point $C(-1, -\sqrt{3})$ and the position of particle Q be D, then the area of ΔACD is:

(A) $4\sqrt{3}$ (B) 12

(C) $2\sqrt{3}$ (D) 48

29. The locus of a point which is equidistant from every position of particle P and Q at any time t when P and Q move in opposite direction (on in clockwise and other in anticlockwise sense) will be a:

(A) fixed point only (B) straight line

(C) parabola (D) hyperbola

30. The locus of a point which is equidistant from every position of particles P and Q at any time is a (given that both particle moves in same sense) is a:

(A) point (B) straight line

(C) parabola (D) hyperbola

Comprehension 11

An equation of the family of circles passing through a given pair of points (x_1, y_1) and (x_2, y_2) can be taken as $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$, k being a real parameter. If a member of this family satisfies some other conditions, then it enables us to determine k.

31. The number of values of $\lambda \in R$ for which there exists exactly one circle passing through the points $(2, -3)$ and $(\lambda, 2\lambda - 1)$ and touching the line $16x - 2y + 27 = 0$ is:

(A) 0 (B) 1

(C) 2 (D) infinitely many

32. There exist exactly two circles that pass through $(3, -5)$ and $(5, -3)$ and touch the line $2x + 2y + 13 = 0$. Let the ratio of radii of the two circles be $\frac{m}{n}$ with $m(> 0)$ and $n(> 0)$ having no common factors except 1. Then $(m + n)$ equals:

(A) 2 (B) 3

(C) 5 (D) 7

33. Consider the set S of all possible circles that pass through $(4, -3)$ and $(-3, 4)$ and touch the line $x + y - 5\sqrt{2} = 0$. Then which of the following statements is correct?

(A) S consists of exactly one circle, having radius 5

(B) S consists of exactly one circle, having radius 10

(C) S consists of exactly two circles, the ratio of radii being 1:2

(D) S consists of exactly two circles, the ratio of radii being 2:3

Comprehension 12

$P(a, 5a)$ and $Q(4a, a)$ are two points. Two circles are drawn through these points touching the y-axis.

34. Centre of these circles is at:

(A) (a, a), $(2a, 3a)$

(B) $\left(\frac{205a}{18}, \frac{29a}{3}\right)$, $\left(\frac{5a}{2}, 3a\right)$

(C) $\left(3a, \frac{29a}{3}\right), \left(\frac{205a}{9}, \frac{29a}{18}\right)$

(D) none of these

35. Angle of intersection of these circles is:

(A) $\tan^{-1}\left(\frac{4}{3}\right)$ (B) $\tan^{-1}\frac{40}{9}$

(C) $\tan^{-1}\left(\frac{84}{187}\right)$ (D) $\frac{\pi}{4}$

36. If C_1, C_2 are the centres of these circles then area of ΔOC_1C_2, where O is the origin, is:

(A) a^2 (B) $5a^2$

(C) $10a^2$ (D) $20a^2$

Comprehension 13

Two perpendicular tangents are drawn from a point P to a circle C_1. A circle C_2 is drawn touching the circle C_1 and also the perpendicular tangents from P. r_1 and r_2 are the radii of circles C_1 and C_2 respectively.

37. Let $r_1 = 3 + 2\sqrt{2}$, then r_2 is:

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

38. If A and B are points of contact of tangents to the circle C_1, then area enclosed between the tangents PA, PB circle C_1 and C_2 is:

(A) lesser than area of circle C_2

(B) greater than area of circle C_2

(C) equal to area of circle C_2

(D) will be greater than area of circle C_2, if $r_1 > 4$ and less than area of circle C_2, if $2 < r_1 < 4$

39. If $r_1 = 3 + 2\sqrt{2}$, then the length of common tangents to the circle C_1 and C_2 intercepted between the perpendicular tangents from P is:

(A) $\frac{1}{2}\left(\sqrt{2}+1\right)$ (B) $\left(\sqrt{2}-1\right)$

(C) $2\left(\sqrt{2}+1\right)$ (D) $4\left(2\sqrt{2}-1\right)$

Comprehension 14

P is a variable point on the line $2x + y = 4$ and the tangents from P to the circle $x^2 + y^2 = 1$ touch it at A and B. The chord AB always passes through the fixed point Q.

40. The locus of circumcentre S of ΔPAB is:

(A) $4(x^2 + y^2) = 2x + y$ (B) $2x + y = 2$

(C) $x^2 + y^2 = 21$ (D) none of these

41. The co-ordinates of Q are:

(A) $\left(-\frac{1}{2}, 3\right)$ (B) $\left(\frac{1}{2}, \frac{1}{4}\right)$

(C) $\left(\frac{1}{2}, -2\right)$ (D) $\left(\frac{3}{4}, \frac{1}{2}\right)$

42. The midpoint of AB always lies on:

(A) $x - 2y = 0$

(B) $x^2 + y^2 = 6x - y$

(C) $4(x^2 + y^2) = 2x + y$

(D) $2x + y = 2$

Comprehension 15

Reflection of a rational point with respect to a line having its equation with rational coefficients gives a rational point. Reflection of a rational point under opposite scenario gives an irrational point.

43. Number of rational points on a circle with centre $\left(\sqrt{2}, \sqrt{3}\right)$ and with any radius is:

(A) 1 (B) 2

(C) variable (D) ∞

44. Number of rational points on a circle with centre $\left(\sqrt{2}, -\sqrt{2}\right)$ and passing through (1, 2) are:

(A) 1 (B) 2

(C) variable (D) ∞

45. Number of rational points on a circle with centre $\left(\sqrt{2}, -\sqrt{2}\right)$ and passing through (1, –1) are:

(A) 1 (B) 2

(C) variable (D) ∞

Comprehension 16

A point $P(x, y)$ in a plane is called lattice point, if $x, y \in Z$ and a rational point if $x, y \in Q$.

Every lattice point is a rational point.

46. The number of lattice points inside the circle $x^2 + y^2 = 16$ is:

(A) 16 (B) 45

(C) 28 (D) 36

47. A rational point on $x^2+y^2=1$ is of the form:

(A) $\left(\dfrac{m-n}{m+n}, \dfrac{2\sqrt{mn}}{m+n}\right)$, $m, n \in Z, m+n=0$

(B) $\left(\dfrac{m^2-n^2}{m^2+n^2}, \dfrac{2mn}{m^2+n^2}\right)$, $m, n \in Q, m^2+n^2 \neq 0$

(C) $\left(\dfrac{m}{m+n}, \dfrac{n}{m+n}\right)$, $m, n \in Q, m+n \neq 0$

(D) $\left(\dfrac{2\sqrt{mn}}{m+n}, \dfrac{m-n}{m+n}\right)$, $m, n \in Z, m+n \neq 0$

48. For a circle whose centre is not a rational point, maximum number of rational points on it is:

(A) 1 (B) 2

(C) 3 (D) 4

Comprehension 17

In a circle with centre 'O', *PA* and *PB* are two chords drawn from point *P* which lies on circle *PC* is the chord that bisects the $\angle APB$. The tangent to the circle at C is drawn meeting *PA* and *PB* extended at *Q* and *R*, respectively. If *QC* = 3, *QA* = 2 and *RC* = 4, then answer the following questions.

49. The length of *PQ*:

(A) $\dfrac{9}{2}$ (B) $\dfrac{2}{3}$

(C) $\dfrac{8}{3}$ (D) $\dfrac{3}{2}$

50. The value of *PQ*:*PR* is:

(A) 4:3 (B) 3:4

(C) 2:3 (D) 3:2

51. The length of *RB* is:

(A) $\dfrac{1}{2}$ (B) $\dfrac{2}{3}$

(C) $\dfrac{3}{4}$ (D) $\dfrac{8}{3}$

Comprehension 18

P is any point on the circle $x^2+y^2=9$. From *P*, point pair of tangents drawn to circle $x^2+y^2=4$ *cuts* $x^2+y^2=9$ at *A* and *B*, then locus of point of intersection of tangents drawn to *A* and *B* is called *C*. Six point (x_i, y_i), where *i* = 1, 2,...6 are taken on circle $x^2+y^2=4$ such that $\sum_{i=1}^{6} x_i = 8$ and $\sum_{i=1}^{6} y_i = 4$. Line *L* joining the orthocentre of triangle form by any three points and centroid of triangle form by other three points passes through (*h*, *k*).

52. Find the equation of *C*:

(A) $x^2+y^2=27^2$ (B) $x^2+y^2=\dfrac{27}{4}$

(C) $x^2+2y^2=27^2$ (D) none of these

53. Find the equation of *L*:

(A) $x+y=4$ (B) $x+y=3$

(C) $x+y=7$ (D) $x+y=5$

54. Curve *L* cut at point *D* and *E*, then intersection point of tangent drawn at *D* and *E* is:

(A) (7, 7) (B) (8, 8)

(C) (6, 6) (D) (9, 9)

Comprehension 19

Circle *C* whose radius is 1 unit, touches the *x*-axis at point *A*. The centre *Q* of *C* lies in Ist quadrant. The tangent from origin *O* to the circle touches it at *T* and a point *P* lies on it such that ΔOAP is a right angle triangle at *A* and its perimeter is 8 units.

55. The length *QP* is:

(A) $\dfrac{1}{2}$ (B) $\dfrac{4}{3}$

(C) $\dfrac{5}{3}$ (D) $\dfrac{1}{3}$

56. Equation of circle C is:

(A) $\left(x-(2+\sqrt{3})\right)^2+(y-1)^2=1$

(B) $\left(x-(\sqrt{3}+\sqrt{2})\right)^2+(y-1)^2=1$

(C) $\left(x-\sqrt{3}\right)^2+(y-1)^2=1$

(D) $\left(x+\sqrt{3}\right)^2+(y-1)^2=1$

57. Equation of tangent *OT* is:

(A) $x-\sqrt{3}y=0$ (B) $x-\sqrt{2}y=0$

(C) $y-\sqrt{3}x=0$ (D) $x-y=0$

Comprehension 20

Consider the relation $4\ell^2-5m^2+6\ell+1=0$, where $\ell, m \in R$, then the line $\ell x+my+1=0$ touches a fixed circle.

58. Centre and radius are:

(A) (2, 0), 3 (B) (–3, 0), $\sqrt{3}$

(C) (3, 0), $\sqrt{5}$ (D) none of these

59. Tangents PA and PB are drawn to the above fixed circle from the points P on the line $x + y - 1 = 0$. Then chord of contact AB passes though the fixed point is:

(A) $\left(\frac{1}{2}, -\frac{5}{2}\right)$ (B) $\left(\frac{1}{3}, \frac{4}{3}\right)$

(C) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (D) none of these

Comprehension 21

Consider two circles S_1 and S_2, whose equations are respectively $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x = 0$. Let the direct common tangents of circles touches S_1 at A and B and S_2 at C and D. Further, these direct common tangents meet at P and intersect a transverse common tangent at Q and R.

60. Centre of incircle of ΔPCD is:

(A) $(-2, 0)$ (B) $(-1, 0)$

(C) $(0, 0)$ (D) $(1, 0)$

61. Orthocentre of ΔPQR is:

(A) $(-2, 0)$ (B) $(-1, 0)$

(C) $(0, 0)$ (D) $(1, 0)$

62. Centre of circle through A, B, C and D and centre of circle touching the circles S_1 and S_2 internally is:

(A) $(-1, 0)$, $(-2, 0)$ (B) $(-1, 0)$, $(0, 0)$

(C) $(1, 0)$, $(-1, 0)$ (D) none of these

Comprehension 22

Let $ABCD$ is a rectangle with $AB = a$ and $BC = b$. A circle is drawn passing through A and B and touching side CD. Another circle is drawn passing through B and C and touching the side AD. Let r_1 and r_2 be the radii of these two circles, respectively.

63. r_1 equals:

(A) $\frac{4b^2 - a^2}{8b}$ (B) $\frac{4b^2 + a^2}{8b}$

(C) $\frac{4a^2 + b^2}{8a}$ (D) $\frac{a^2 - 4b^2}{8a}$

64. $\frac{r_1}{r_2}$ equals:

(A) $\frac{a}{b}\left(\frac{4b^2 + a^2}{4a^2 + b^2}\right)$ (B) $\frac{b}{a}\left(\frac{4a^2 + b^2}{4b^2 + a^2}\right)$

(C) $\frac{a}{b}\left(\frac{4b^2 - a^2}{4a^2 - b^2}\right)$ (D) $\frac{b}{a}\left(\frac{a^2 + 4b^2}{4a^2 + b^2}\right)$

65. Minimum value of $(r_1 + r_2)$ equals:

(A) $\frac{5}{8}(a - b)$ (B) $\frac{5}{8}(a + b)$

(C) $\frac{3}{8}(a - b)$ (D) $\frac{3}{8}(a + b)$

Comprehension 23

A circle C_1 of radius r units rolls outside the circle C_2: $x^2 + y^2 + 2rx = 0$ touching it externally. The line of centres has an inclination 60°.

66. The point of contact of C_1 and C_2 is:

(A) $\left(-r, r\sqrt{3}\right)$ (B) $\left(-r, -r\sqrt{3}\right)$

(C) $\left(\frac{-r}{2}, \frac{-r\sqrt{3}}{2}\right)$ (D) $\left(\frac{-r}{2}, \frac{r\sqrt{3}}{2}\right)$

67. The equation of direct common tangents is:

(A) $\sqrt{3}x - y + r\left(2 \pm \sqrt{3}\right) = 0$

(B) $\sqrt{3}x - y + r\left(\sqrt{3} \pm 2\right) = 0$

(C) $\sqrt{3}x - y + 2r\left(2 \pm \sqrt{3}\right) = 0$

(D) $\sqrt{3}x - y + 2r\left(\sqrt{3} \pm 2\right) = 0$

68. The transverse common tangent is:

(A) $x + \sqrt{3}y + r = 0$ (B) $x + \sqrt{3}y + 2r = 0$

(C) $x + \sqrt{3}y - r = 0$ (D) $x + \sqrt{3}y + 2r = 0$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. If $x^2+y^2-4x-4y+7=0$, then find the maximum values of the following.

COLUMN-I	COLUMN-II
(A) $x+y$	(p) $4+\sqrt{2}$
(B) $x-y$	(q) $\frac{9+4\sqrt{2}}{2}$
(C) xy	(r) $\frac{4+\sqrt{7}}{3}$
(D) $\frac{x}{y}$	(s) $\sqrt{2}$

2. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The number of common tangents to the circle $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ is	(p) 1
(B) If the circumference of the circle $x^2+y^2+8x+8y-b=0$ is bisected by the circle $x^2+y^2-2x+4y+a=0$, then $\frac{\lvert a+b\rvert}{14}$ equals to	(q) 2
(C) $f(x, y)=0$ is a circle such that $f(0, \lambda)=0$ and $f(\lambda, 0)=0$ have equal roots and $f(1, 1)=-2$, then the radius of the circle is	(r) 3
(D) If a circle passes through the points of intersection of the lines $x+y+1=0$ and $x+\lambda y-3=0$ with the co-ordinate axes, then λ equals to	(s) 4

3. A variable straight line through $A(-1, 1)$ is drawn to cut the circle $x^2+y^2=1$ at the points B and C. A point P is chosen on the line ABC, which satisfies the condition in **COLUMN-I.** Let d be the minimum distance of the origin from the locus of P, which is given in **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) AB, AP and AC are in A.P.	(p) 0
(B) AB, AP and AC are in G.P.	(q) $\frac{1}{\sqrt{2}}$
(C) AB, AP and AC are in H.P.	(r) $\sqrt{2}$
(D) AB, $\frac{AP}{2}$ and AC are in A.P.	(s) $\sqrt{2}-1$

4. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If each of 4 circles touches the three lines $x+2y-1=0, 2x-3y+5=0$ and $3x+y-1=0$ of radii a, b, c, d and $a>b>c>d$, then $\frac{d}{a}+\frac{d}{b}+\frac{d}{c}$ equals	(p) 5
(B) If $A(-2,0)$ and $B(2,0)$ are two points, then the number of integral values of a if AB subtends obtuse angle at the point $(a, a+1)$ is	(q) 2
(C) If the lengths of the tangents from two points A and B to a circle are 3 and 4 and A and B are conjugate points then AB equals	(r) 4
(D) A rectangle has dimensions 4 and 2. If two circles are constructed on these sides as diameters and d is the length of common chord, then $\sqrt{5}d$ equals	(s) 1

5. If the circle $x^2+y^2+2gx+2fy+c=0$ cuts each of the circles $x^2+y^2=4$, $x^2+y^2-6x-8y+10=0$ and $x^2+y^2+2x-4y-2=0$ at the extremities of diameter, then match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) $g+f-c$	(p) -1
(B) gf	(q) 9
(C) g^2+f^2-c	(r) 6
(D) g^2+f^2+c	(s) 17

6. Let C_1 and C_2 be two circles whose equations are $x^2+y^2-2x=0$ and $x^2+y^2+2x=0$. If $P(\lambda, \lambda)$ is a variable point, then match **Column-I** with **Column-II**.

COLUMN-I	COLUMN-II
(A) P lies inside C_1 but outside C_2	(p) $\lambda \in (-\infty, -1) \cup (0, \infty)$
(B) P lies inside C_2 but outside C_1	(q) $\lambda \in (-\infty, -1) \cup (1, \infty)$
(C) P lies outside C_1 and outside C_2	(r) $\lambda \in (-1, 0)$
(D) P does not lie inside C_2	(s) $\lambda \in (0, 1)$

7. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I ($c > 0$)	COLUMN-II
(A) The circle $x^2+y^2+2x+c=0$ and $x^2+y^2+2y+c=0$ touch each other	(p) If $c=1$
(B) The circles $x^2+y^2+2x+3y+c=0$ and $x^2+y^2-x+2y+c=0$ intersect orthogonally	(q) If $c<2$
(C) The circle $x^2+y^2=9$ contains the circle $x^2+y^2-2x+1-c^2=0$	(r) If $c=\dfrac{1}{2}$
(D) The circle $x^2+y^2=9$ is contained in the circle $x^2+y^2-6x-8y+25-c^2=0$	(s) If $c>8$

8. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The number of common tangents that can be drawn to the two circles C_1: $x^2+y^2-4x-6y-3=0$ C_2: $x^2+y^2+2x+2y+1=0$ is	(p) 2
(B) The common chord of $x^2+y^2-4x-4y=0$ and $x^2+y^2=16$ subtends at the origin an angle equal to $\dfrac{\pi}{k}$, then k equals	(q) 0
(C) Shortest distance from the point $(2, -7)$ to the circle $x^2+y^2-14x-10y-151=0$ is	(r) 3
(D) If real number x and y satisfy $(x+5)^2+(y-12)^2=(14)^2$, then the minimum value of $\sqrt{x^2+y^2}$ is	(s) 1

9. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The line $x+2y=5$ touches the circle $x^2+y^2-4x-8y+15=0$ at	(p) $\left(\dfrac{1}{2}, \sqrt{2}\right)$
(B) A and B are two points on the circle $x^2+y^2-4x-8y-1=0$, $O(0, 0)$ and OA, OB are tangents to the circle, then the circumcentre of the ΔOAB is	(q) $(1, 2)$
(C) Transverse common tangents of the circles $x^2+y^2-4x-2x+4y+4=0$, $x^2+y^2-4x-2y+1=0$ meet at	(r) $\left(\dfrac{1}{2}, -\sqrt{2}\right)$
(D) Circle passes through $(0, 0)$, $(1, 0)$ touching the circle $x^2+y^2=9$, then the centre of the circle is	(s) $(0, -1)$

10. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The radical axis of two circles	(p) Subtends a right angle at a point of intersection
(B) The common tangent to two intersecting circles of equal radii	(q) Is perpendicular to the line joining the centres
(C) The common chord of two intersecting circles	(r) Is parallel to the line joining the centres
(D) The line joining the centres of two circles intersecting orthogonally	(s) Is bisected by the line joining the centres

11. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If a circle passes through $A(1,0)B(0,-1)$ and $C\left(\frac{1}{\sqrt{3}},\sqrt{\frac{2}{3}}\right)$ such that the tangent at B makes an angle θ with line AB, then $\tan\theta$ equals	(p) –4
(B) From a point $(h, 0)$, common tangents are drawn to the circles $x^2+y^2=1$ and the $(x-2)^2+y^2=4.$ The value of h can be	(q) –2
(C) If the common chord of the circle $x^2+y^2=8$ and $(x-a)^2+y^2=8$ subtends right angle at the origin, then a can be	(r) 1
(D) If the tangents drawn from $(4, k)$ to the circle $x^2+y^2=10$ are at right angles, then k can be	(s) 2
	(t) 4

12. Let C_1 and C_2 be two circles whose equations are $x^2 + y^2 = 16$ and $x^2 + y^2 - 2x\lambda + \lambda^2 - 4 = 0$, then match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) For $\lambda = +4$,	(p) Three common tangents of C_1 and C_2
(B) For $\lambda = -2$,	(q) Four common tangents of C_1 and C_2
(C) For $\lambda = +6$,	(r) Two common tangents of C_1 and C_2
(D) For $\lambda = -8$,	(s) One common tangent of C_1 and C_2

13. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The ΔPQR is inscribed in the circle $x^2 + y^2 = 169$. If Q and R have co-ordinates (5, 12) and (–12, 5), respectively, then find $\angle QPR$	(p) $\frac{\pi}{4}$
(B) The angle between the line joining the origin to the point of intersection of the line $4x + 3y = 24$ with circle $(x-3)^2 + (y-4)^2 = 25$ is	(q) $\frac{\pi}{2}$
(C) Two parallel tangents drawn to a given circle are cut by a third tangent. Double the angle subtended by the portion of third tangent between the given tangents at the centre is	(r) π
(D) A chord is drawn joining the point of contact of tangents drawn from a point P. If the chord subtends an angle $\frac{\pi}{2}$ at the centre, then the angle included between the tangents at P is	(s) $\frac{\pi}{6}$

14. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Number of circles touching given three non-concurrent lines is/are	(p) 1
(B) Number of circles touching $y = x$ at (2, 2) and also touching $x + 2y - 4 = 0$ is/are	(q) 2
(C) Number of circles touching lines $x \pm y = 2$ and passing through the point (4, 3) is/are	(r) 4
(D) Number of circles intersecting given three circles orthogonally is/are	(s) Infinite

15. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If $ax + by - 5 = 0$ is the equation of the chord of the circle $(x-3)^2 + (y-4)^2 = 4$, which passes through (2, 3) and at the greatest distance from the centre of the circle, then $\|a + b\|$ is equal to	(p) 0
(B) Let O be the origin and P be a variable point on the circle $x^2 + y^2 + 2x + 2y = 0$. If the locus of midpoint of OP is $x^2 + y^2 + 2gx + 2fy + c = 0$, then $(g + f)$ is equal to	(q) 3
(C) The x-co-ordinates of the centre of the smallest circle which cuts the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$ orthogonally is	(r) 2
(D) If $px + qy + r = 0$ is the radical axis of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$, then $2p + q$ equals	(s) 1

(*Continued*)

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If the area of a part of the circle $x^2 + y^2 - 4x - 6y + 12 = 0$ above the line $4x + 7y - 29 = 0$ is Δ, then find $[\Delta]$ (where [.] is G.I.F.)

2. Let $S_1 = x^2 + y^2 - 4x - 8y + 4 = 0$ and S_2 be its image in the line $y = x$. If the radius of the circle touching $y = x$ at (1, 1) and orthogonal to S_2 is $\frac{3}{\sqrt{\lambda}}$, then find $\lambda^2 + 2$.

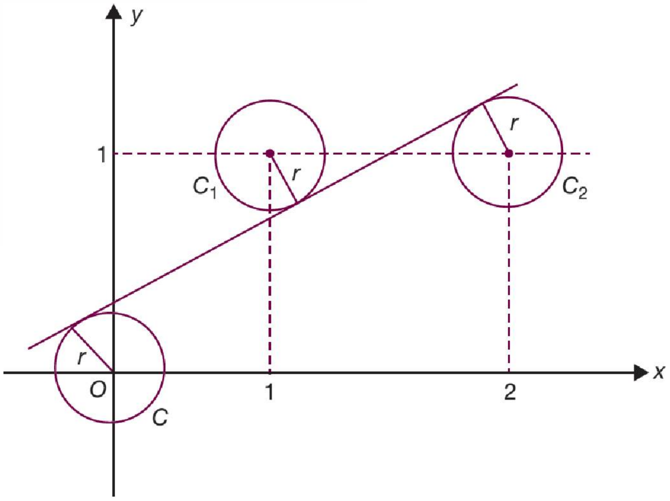
3. The centre of each of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is a ring, whose area is $\lambda\pi$. Find $\frac{\lambda}{10}$.

4. If the radius of the circle touching the pair of lines $7x^2 - 18xy + 7y^2 = 0$ and the circle $x^2 + y^2 - 8x - 8y = 0$ and contained in the given circle is equal to k, then find k^2.

5. Circles are drawn through (1, 1) touching the circle $x^2 + y^2 - 6x + 2y + 1 = 0$. If r_1 and r_2 are the radii of the smallest and the largest circles, then find the value of $(r_2 + r_1)^2 - (r_2 - r_1)^2$.

6. Line segments AC and BD are the diameters of the circle of radius 1. If $\angle BDC = 60°$, then find the length of the line segment AB.

7. As shown in figure, three circles having the same radius r, have their respective centres at (0, 0), (1, 1) and (2, 1). If they have a common tangent line, as shown in figure, then find the value of $10\sqrt{5}\,r$.



8. If the radii of the circles, which pass through the point (2, 3) and cut off equal chords of length 6 units along the lines $y - x - 1 = 0$ and $y + x - 5 = 0$ is 'r', then find $[r]$.

9. Find the number of integral values of α for which the point $(\alpha - 1, \alpha + 1)$ lies in the larger segment of the circle $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$.

10. Find the radius of the smallest circle that can be drawn to pass through the point (0, 4) and which touches the x-axis.

11. Let $M(-1, 2)$ and $N(1, 4)$ be two points in a plane rectangular co-ordinate system XOY. P is a moving point on the x-axis. When $\angle MPN$ takes its maximum value, then find the x-co-ordinate of point P.

12. If r be the radius of incircle of triangle formed by joining the centres $(x-a)^2 + (y-b)^2 = 9$ and $(x-a)^2 + (y-b-7)^2 = 16$ and the circle touching above two circles and having radius as 5 units, find r^2.

13. Three points $P(a, b)$, $Q(c, d)$ and $R(e, f)$ satisfy the inequality $x^2 + y^2 - 6x - 8y < 0$ such that 'P' lies at a least distance from $A(-2, 4)$ and Q and R lies at a maximum distance from A, where a, b, c, d, e and $f \in I$. The internal bisector of 'P' of ΔPQR intersects the tangents at origin and the point $\left(\frac{c+e}{2} + 1, d\right)$ to the circle $x^2 + y^2 - 6x - 8y = 0$. If the area of triangle formed by these three lines is Δ, then find $\frac{3\Delta}{50}$.

14. Tangents are drawn to the circle $x^2 + y^2 = 1$ at the points, where it is met by the circles $x^2 + y^2 - (\lambda + 6)x + (8 - 2\lambda)y - 3 = 0$ where λ is being parameter. If the locus of point of intersection of these tangents is $px + qy + 10 = 0$, then find $p + q$.

15. From $P(-1, 1)$, tangents are drawn to $x^2 + y^2 + 2y = 0$ and A and B are the points of contact. $PATB$ parallelogram is completed. If d is the distance between the circumcentre of ΔPAB and ΔTAB, then find the integral part of d.

16. If the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ and the line $12x - 6y - 41 = 0$ intersect at P, Q and the circle $x^2 + y^2 = 4$ and the line $x + 2y + t = 0$ intersect at R and S and if P, Q, R and S are concyclic, then find the value of t.

17. If $\left(m_i, \frac{1}{m_i}\right), m_i > 0,\ i = 1, 2, 3, 4$ are four distinct points on a circle, then find the value of $\prod_{i=1}^{4} m_i$

18. Two circles which touch both the axes and lying in the Ist quadrant intersect at (x_1, y_1), where x_1 and y_1 belongs to the set $\{1, 2, 3, 4, 5\}$. If the length of the common chord is $2\sqrt{2}$, then find the number of such pairs of circles.

19. If the line $3x - 4y - 8 = 0$ divides the circumference of the circle with centre $(2, -3)$ in the ratio 1:2, then find the radius of the circle.

20. If r_1 and r_2 are the radii of smallest and largest circles which pass through (5, 6) and touches the circle $(x-2)^2 + y^2 = 4$ and $r_1 r_2 = \frac{41}{\lambda}$, then find λ.

21. The radius of circumcircle of the ΔTPQ, where PQ is a chord of contact corresponding to point T with respect to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ is 6 units. Then the minimum distance of T from the director circle of the given circle is $12 - 4\sqrt{k}$, then find k.

22. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum.

23. The minimum distance between the circle $x^2 + y^2 = 9$ and the curve $2x^2 + 10y^2 + 6xy = 1$ is ______.

24. From a point 'P' outside the circle $x^2 + y^2 = r_2$, the tangents PA and PB are drawn. If $\frac{1}{r^2} + \frac{1}{(PA)^2} = \frac{1}{16}$, then find the length of chord of contact of the pair of tangents.

25. Let the circle $C_1: x^2+(y-4)^2=12$ intersects the circle $C_2: (x-3)^2+y^2=13$ at A and B. A quadrilateral $ACBD$ is formed by tangents at A and B to both circles. Find the diameter of circumcircle of the quadrilateral $ACBD$.

26. If the circumference of the circle $x^2 + y^2 - 2x + 8y - a = 0$ is bisected by the circle $x^2 + y^2 + 4x + 22y + b = 0$, then find $\dfrac{a+b}{10}$.

27. Let S_1 and S_2 denote the circles $x^2+y^2+10x-24y-87=0$ and $x^2+y^2-10x-24y+153=0$, respectively. Let m be the smallest positive value of 'a' for which the line $y = ax$ contains the centre of a circle, which touches S_2 externally and S_1 internally. Given that, $m^2=\dfrac{p}{q}$, where p and q are relatively prime integers, if $(p + q)$ is equal to 13^k, then find the value of k.

28. Find the intercept made by the circle $x^2 + y^2 - 14x - 10y + 23 = 0$ on the line $7x + y - 4 = 0$.

29. The equation of circle is $S_1 \equiv x^2+y^2=1$. The orthogonal tangents to S_1 meet at another circle S_2 and the orthogonal tangents to S_2 meet at the third circle S_3. If the ratio of radii of S_2 and S_3 are $1:\mu$, then find $1+\mu^2$.

30. A circle of radius 1 unit touches positive x-axis and positive y-axis at A and B respectively. A variable line passing through the origin intersects the circle at two points namely, D and E. If the area of the ΔDEB is maximum when the slope of the line is m, then find m^{-2}.

31. Circle $(x-1)^2+(y-1)^2=1$ touches x and y axes at A and B respectively. A line $y = mx$ intersects this circle at P and Q. If area of ΔPQB is maximum, then find the value of $\left[m+\dfrac{1}{2}\right]$ (where [.] denotes the greatest integer function).

32. The orthocentre and circumcentre of a ΔABC are (1, 2) and (2, 4), respectively. If the base BC has the equation $2x-y-3=0$ and the radius of the circumcircle of ΔABC is k, then find $\sqrt{\dfrac{5}{61}}k$.

33. If (α, β) is the centre of the circle, which passes through the intersection of $x^2 + y^2 - 2x - 6y + 6 = 0$ and $x^2 + y^2 + 2x - 6y + 6 = 0$ and intersects $x^2 + y^2 + 4x + 6y + 4 = 0$ orthogonally, then find $|\alpha+\beta|$.

34. If the sum of the squares of the length of the chords intercepted by the $x + y = n$, $n \in N$ on the circle $x^2 + y^2 = 4$ is k, then find $\dfrac{4k}{11}$.

35. The chord of contact of tangents from a point P to a circle passes through Q. If the lengths of tangents from P and Q to the circle are 5 units and $2\sqrt{6}$ units, then find the distance between P and Q.

36. If $A_0A_1A_2A_3A_4A_5$ is a regular hexagon inscribed in a unit circle with centre at the origin, then find the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4.

37. The value of α in $[0, \pi]$ so that the circle $x^2 + y^2 + 2(\sin\alpha)x + 2(\cos\alpha)y + \sin^2\alpha = 0$ always lies inside the square of unit side length is $\left[\dfrac{\pi}{\ell}, \dfrac{2\pi}{\ell}\right]$. Then find ℓ.

38. If the equation of the pair of straight lines parallel to x-axis and touching the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ is $y^2-\lambda y-21=0$, then find λ.

39. The chord of contact of tangents from a point A to a circle $x^2+y^2=a^2$ passes through a point B. If 3 and 4 are the lengths of tangents from A and B on the given circle, then find AB.

40. If the end points A and B of the diameter of a circle are such that their abscissas are the roots of the equation $x^2+ax-16=0$ and their ordinates are the roots of $x^2+24x-9=0$, If $a \in N$ and radius 'R' of the circle, with AB as diameter is an integer, then find $R - a$.

41. If the equations of four circles are $\left(x\pm\left(\sqrt{2}-1\right)\right)^2+\left(y\pm\left(\sqrt{2}-1\right)\right)^2=\left(\sqrt{2}-1\right)^2$, then find the radius of a circle touching all the four circles internally.

42. Two circles of radii 1 and 4 touch each other externally. Another circle of radius r touches both circles externally and also one direct common tangent of the two circles. Then, $\left[\dfrac{1}{r}\right]$ = (where [.] denotes greatest integer function).

43. If the maximum distance between the curves $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2+y^2-10x+4y+20=0$ is 3λ, then find λ.

44. The curve $x^2-y-\sqrt{5}\,x+1=0$ intersects x-axis at A and B. If a circle is drawn passing through A and B, then find the length of tangent drawn from origin to the circle.

45. If the locus of midpoints of the chords of the circle $x^2+y^2=64$, which subtends right angled at (10, 0) is $x^2+y^2-10x+k=0$, then find $\frac{k}{2}$.

46. Let x and y be the real numbers satisfying the equation $x^2-4x+y^2+3=0$. If the maximum and the minimum values of x^2+y^2 are M and m, respectively, then find the numerical value of $M-m$.

47. If the locus of centre of circle passing through the point of intersection of pair of straight lines $cxy-c^2x+y-c=0$ and $y=cx+1$ ('c' is the parameter) is $Lxy-My+N=0$, then find $L^3+M^3+N^3-1$.

ARCHIVE: JEE MAIN

1. **[JEE MAIN 2020: Jan Attempt]**
If a line, $y=mx+c$ is a tangent to the circle, $(x-3)^2+y^2=1$ and it is perpendicular to a line L_1, where L_1 is the tangent to the circle, $x^2+y^2=1$ at the point $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$; then :

(A) $c^2-6c+7=0$ (B) $c^2+7c+6=0$
(C) $c^2+6c+7=0$ (D) $c^2-7c+6=0$

2. **[JEE MAIN 2020: Jan Attempt]**
Let the tangents drawn from the origin to the circle, $x^2+y^2-8x-4y+16=0$ touch it at the points A and B. The $(AB)^2$ is equal to :

(A) $\frac{52}{5}$ (B) $\frac{32}{5}$
(C) $\frac{56}{5}$ (D) $\frac{64}{5}$

3. **[JEE MAIN 2020: Jan Attempt]**
A circle touches the y-axis at the point (0, 4) and passes through the point (2, 0). Which of the following lines is not a tangent to this circle?

(A) $3x+4y-6=0$ (B) $4x+3y-8=0$
(C) $3x-4y-24=0$ (D) $4x-3y+17=0$

4. **[JEE MAIN 2020: Jan Attempt]**
If the curves, $x^2-6x+y^2+8=0$ and $x^2-8y+y^2+16-k=0$, $(k>0)$ touch each other at a point, then the largest value of k is ________.

5. **[JEE Main 2019]**
If the tangent and the normal lines at the point $(\sqrt{3}, 1)$ to the circle $x^2+y^2=4$ and the x-axis form a triangle, then the area of this triangle (in square units) is:

(A) $\frac{4}{\sqrt{3}}$ (B) $\frac{1}{3}$
(C) $\frac{1}{\sqrt{3}}$ (D) $\frac{2}{\sqrt{3}}$

6. **[JEE Main 2019]**
The common tangent to the circles $x^2+y^2=4$ and $x^2+y^2+6x+8y-24=0$ also passes through the point:

(A) (–6, 4) (B) (4, –2)
(C) (–4, 6) (D) (6, – 2)

7. **[JEE Main 2019]**
If a tangent to the circle $x^2+y^2=1$ intersects the coordinate axes at distinct points P and Q, then the locus of the midpoint of PQ is:

(A) $x^2+y^2-2x^2y^2=0$ (B) $x^2+y^2-4x^2y^2=0$
(C) $x^2+y^2-16x^2y^2=0$ (D) $x^2+y^2-2xy=0$

8. **[JEE Main 2019]**
If the circles $x^2+y^2-16x-20y+164=r^2$ and $(x-4)^2+(y-7)^2=36$ intersect at two distinct points, then:

(A) $r>11$ (B) $r=11$
(C) $1<r<11$ (D) $0<r<1$

9. **[JEE Main 2019]**
If the circles $x^2+y^2+5Kx+2y+K=0$ and $2(x^2+y^2)+2Kx+3y-1=0$, $(K\in R)$, intersect at the points P and Q, then the line $4x+5y-K=0$ passes through P and Q, for:

(A) exactly two values of K
(B) no value of K
(C) exactly one value of K
(D) infinitely many values of K

10. **[JEE Main 2019]**
The line $x=y$ touches a circle at the point (1, 1). If the circle also passes through the point (1, –3), then its radius is:

(A) 3 (B) 2
(C) $2\sqrt{2}$ (D) $3\sqrt{2}$

11. **[JEE Main 2019]**
The locus of the centres of the circles, which touch the circle, $x^2+y^2=1$ externally, also touch the y-axis and lie in the Ist quadrant is:

(A) $x = \sqrt{1+2y}\,, y \geq 0$ (B) $y = \sqrt{1+2x}\,, x \geq 0$

(C) $y = \sqrt{1+4x}\,, x \geq 0$ (D) $x = \sqrt{1+4y}\,, y \geq 0$

12. [JEE Main 2019]

If a circle C passing through the point (4, 0) touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point (1, –1), then the radius of C is:

(A) 5 (B) $2\sqrt{5}$

(C) 4 (D) $\sqrt{57}$

13. [JEE Main 2019]

If the line $3x + 4y - 24 = 0$ intersects the x-axis at the point A and the y-axis at the point B, then the incentre of the ΔOAB, where O is the origin, is:

(A) (3, 4) (B) (2, 2)

(C) (4, 4) (D) (4, 3)

14. [JEE Main 2019]

A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the co-ordinate axes. Then the distance of the vertex of this square which is nearest to the origin is:

(A) $\sqrt{137}$ (B) 6

(C) $\sqrt{41}$ (D) 13

15. [JEE Main 2019]

Two circles with equal radii are intersecting at the points (0, 1) and (0, –1). If the tangent at the point (0, 1) to one of the circles passes through the centre of the other circle, then the distance between the centres of these circles is:

(A) $2\sqrt{2}$ (B) $\sqrt{2}$

(C) 2 (D) 1

16. [JEE Main 2019]

If the angle of intersection at a point where the two circles with radii 5 cm and 12 cm intersect is 90°, then the length (in cm) of their common chord is:

(A) $\frac{13}{5}$ (B) $\frac{60}{13}$

(C) $\frac{120}{13}$ (D) $\frac{13}{2}$

17. [JEE Main 2019]

If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval is:

(A) (23, 31) (B) (2, 17)

(C) (13, 23) (D) (12, 21)

18. [JEE Main 2019]

Let C_1 and C_2 be the centres of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 6x - 6y + 14 = 0$, respectively. If P and Q are the points of intersection of these circles, then the area (in sq. units) of the quadrilateral PC_1QC_2 is:

(A) 4 (B) 6

(C) 9 (D) 8

19. [JEE Main 2019]

If a circle of radius R passes through the origin O and intersects the co-ordinates axes at A and B, then the locus of the foot of perpendicular from O on AB is:

(A) $(x^2 + y^2)^2 = 4R^2x^2y^2$ (B) $(x^2 + y^2)(x + y) = R^2xy$

(C) $(x^2 + y^2)^2 = 4Rx^2y^2$ (D) $(x^2 + y^2)^3 = 4R^2x^2y^2$

20. [JEE Main 2019]

A circle touching the x-axis at (3, 0) and making an intercept of length 8 on the y-axis passes through the point:

(A) (1, 5) (B) (2, 3)

(C) (3, 5) (D) (3, 10)

21. [JEE Main 2019]

The straight line $x + 2y = 1$ meets the co-ordinate axes at A and B. A circle is drawn through A, B and the origin. Then the sum of perpendicular distances from A and B on the tangent to the circle at the origin is:

(A) $\frac{\sqrt{5}}{2}$ (B) $2\sqrt{5}$

(C) $\frac{\sqrt{5}}{4}$ (D) $4\sqrt{5}$

22. [JEE Main 2019]

Three circles of radii a, b and c $(a < b < c)$ touch each other externally. If they have x-axis as a common tangent, then:

(A) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$

(B) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

(C) a, b and c are in A.P.

(D) $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ are in A.P.

23. [JEE Main 2019]

A rectangle is inscribed in a circle with a diameter lying along the line $3y = x + 7$. If the two adjacent vertices of the rectangle are (–8, 5) and (6, 5), then the area of the rectangle (in sq. units) is:

(A) 72 (B) 84

(C) 98 (D) 56

24. [JEE Main 2019]

If the area of an equilateral triangle inscribed in the circle, $x^2 + y^2 + 10x + 12y + c = 0$ is $27\sqrt{3}$ sq. units, then c is:

(A) 13 (B) 20

(C) – 25 (D) 25

25. [JEE Main 2019]

A circle cuts a chord of length $4a$ on the x-axis and passes through a point on the y-axis, distant $2b$ from the origin. Then the locus of the centre of this circle is:

(A) a hyperbola (B) an ellipse

(C) a straight line (D) a parabola

26. [JEE Main 2019]

The sum of the squares of the length of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$. $n \in N$, where N is the set of all natural numbers is:

(A) 160 (B) 320

(C) 210 (D) 105

27. [JEE Main 2016]

If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$ is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is:

(A) 10 (B) $5\sqrt{2}$

(C) $5\sqrt{3}$ (D) 5

28. [JEE Main 2016]

The centres of those circles, which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on:

(A) a parabola

(B) a circle

(C) an ellipse which is not a circle

(D) a hyperbola

29. [JEE Main 2015]

The number of common tangents to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is:

(A) 3 (B) 4

(C) 1 (D) 2

30. [JEE- Main-2015]

Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a:

(A) circle of radius $\sqrt{2}$

(B) circle of radius $\sqrt{3}$

(C) straight line parallel to x-axis

(D) straight line parallel to y-axis

31. [JEE Main 2014]

Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at $(0, y)$, passing through origin and touching the circle C externally, then the radius of T is:

(A) $\frac{\sqrt{3}}{\sqrt{2}}$ (B) $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$ (D) $\frac{1}{4}$

32. [JEE Main 2013]

The circle passing through (1, – 2) and touching the x-axis at (3, 0) also passes through the point:

(A) (–5, 2) (B) (2, –5)

(C) (5, –2) (D) (–2, 5)

33. [AIEEE 2012]

The length of the diameter of the circle, which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is:

(A) $\frac{5}{3}$ (B) $\frac{10}{3}$

(C) $\frac{3}{5}$ (D) $\frac{6}{5}$

34. [AIEEE 2011]

The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if:

(A) $a = 2c$ (B) $|a| = 2c$

(C) $2|a| = c$ (D) $|a| = c$

35. [AIEEE 2010]

The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points, if

(A) $-85 < m < -35$ (B) $-35 < m < 15$

(C) $15 < m < 65$ (D) $35 < m < 85$

36. [AIEEE 2009]

Three distinct points A, B and C are given in the two-dimensional co-ordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then the circumcentre of the ΔABC is at the point:

(A) $\left(\frac{5}{2}, 0\right)$ (B) $\left(\frac{5}{3}, 0\right)$

(C) (0, 0) (D) $\left(\frac{5}{4}, 0\right)$

37. [AIEEE 2009]
If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and $(1, 1)$ for:

(A) all except one value of p
(B) all except two values of p
(C) exactly one value of p
(D) all values of p

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE Advanced 2019]
A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-co-ordinate $-\frac{3}{5}$, then which one of the following options is correct?

(A) $4 \le m < 6$ (B) $6 \le m < 8$
(C) $-3 \le m < -1$ (D) $2 \le m < 4$

2. [JEE 2012]
The locus of the midpoint of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is:

(A) $20(x^2 + y^2) - 36x + 45y = 0$
(B) $20(x^2 + y^2) + 36x - 45y = 0$
(C) $36(x^2 + y^2) - 20x + 45y = 0$
(D) $36(x^2 + y^2) + 20x - 45y = 0$

3. [JEE 2011]
The circle passing through the point $(-1, 0)$ and touching the y-axis at $(0, 2)$ also passes through the point:

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$
(C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) $(-4, 0)$

4. [JEE 2009]
Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the ΔPAB is:

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$
(B) $x^2 + y^2 - 4x - 10y + 19 = 0$
(C) $x^2 + y^2 - 2x + 6y - 29 = 0$
(D) $x^2 + y^2 - 6x - 4y + 19 = 0$

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. [JEE Advanced 2018]
Let T be the line passing through the points $P(-2, 7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distance points of E_1 and passing through the point $R(1, 1)$ be F_2. Let E_2 be the set of midpoints of the line segments in the set F_2. Then, which of the following statement(s) is/are TRUE?

(A) The point $(-2, 7)$ lies in E_1
(B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does not lie in E_2
(C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2
(D) The point $\left(0, \frac{3}{2}\right)$ does not lie in E_1

2. [JEE Advanced 2016]
Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E, then the locus of E passes through the point(s) is:

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$
(C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

3. [JEE Advanced 2014]
A circle S passes through the point $(0, 1)$ and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then:

(A) radius of S is 8 (B) radius of S is 7
(C) centre of S is $(-7, 1)$ (D) centre is S is $(-8, 1)$

4. [JEE Advanced 2013]
Circle(s) touching x-axis at a distance 3 units from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is/are:

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$
(B) $x^2 + y^2 - 6x + 7y + 9 = 0$
(C) $x^2 + y^2 - 6x - 8y + 9 = 0$
(D) $x^2 + y^2 - 6x - 7y + 9 = 0$

Reasoning Based Questions

This section contains Reasoning type questions, each having four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

1. [JEE 2008]
Consider, $L_1: 2x + 3y + p - 3 = 0$ and $L_2: 2x + 3y + p + 3 = 0$, where p is a real number and C: $x^2 + y^2 + 6x - 10y + 30 = 0$.

Statement 1: If line L_1 is a chord of circle C, then the line L_2 is not always a diameter of circle C.

and

Statement 2: If line L_1 is a diameter of circle C, then the line L_2 is not a chord of circle C.

(A) Statement 1 is True, Statement 2 is True and Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True and Statement 2 is **NOT** a correct explanation for Statement 1.
(C) Statement 1 is True and Statement 2 is False.
(D) Statement 1 is False and Statement 2 is True.

2. [JEE 2007]
Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$.

Statement 1: The tangents are mutually perpendicular.

because

Statement 2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

(A) Statement 1 is True, Statement 2 is True and Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True and Statement 2 is **NOT** a correct explanation for Statement 1.
(C) Statement 1 is True and Statement 2 is False.
(D) Statement 1 is False and Statement 2 is True.

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

[JEE Advanced 2018]

Let S be the circle in the xy plane defined by the equation $x^2 + y^2 = 4$.

1. Let E_1E_2 and F_1F_2 be the chords of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slope –1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents to S at F_1 and F_2 meet at F_3 and tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3, and G_3, lie on the curve:
(A) $x + y = 4$
(B) $(x - 4)^2 + (y - 4)^2 = 16$
(C) $(x - 4)(y - 4) = 4$
(D) $xy = 4$

2. Let P be a point on the circle S with both co-ordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve:
(A) $(x + y)^2 = 3xy$
(B) $x^{2/3} + y^{2/3} = 2^{4/3}$
(C) $x^2 + y^2 = 2xy$
(D) $x^2 + y^2 = x^2y^2$

Comprehension 2

[JEE 2012]

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$.

3. A common tangent of the two circles is:
(A) $x = 4$
(B) $y = 2$
(C) $x + \sqrt{3}\,y = -1$
(D) $x + 2\sqrt{2}\,y = 6$

4. A possible equation of L is:
(A) $x - \sqrt{3}\,y = 1$
(B) $x + \sqrt{3}\,y = 1$
(C) $x - \sqrt{3}\,y = -1$
(D) $x + \sqrt{3}\,y = 5$

Comprehension 3

[JEE 2008]

A circle C of radius 1 unit is inscribed in an equilateral ΔPQR. The points of contact of C with the sides PQ, QR and RP are D, E and F, respectively. The line PQ is given by the equation $\sqrt{3}\,x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Further, it is given that the origin and the centre of C are on the same side of the line PQ.

5. The equation of circle C is:

(A) $(x-2\sqrt{3})^2+(y-1)^2=1$

(B) $(x-2\sqrt{3})^2+(y+\frac{1}{2})^2=1$

(C) $(x-\sqrt{3})^2+(y+1)^2=1$

(D) $(x-\sqrt{3})^2+(y-1)^2=1$

6. Points E and F are given by:

(A) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),(\sqrt{3},0)$ (B) $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right),(\sqrt{3},0)$

(C) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ (D) $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

7. Equations of the sides RP and RQ are:

(A) $y=\frac{2}{\sqrt{3}}x+1,\ y=-\frac{2}{\sqrt{3}}x-1$

(B) $y=\frac{1}{\sqrt{3}}x, y=0$

(C) $y=\frac{\sqrt{3}}{2}x+1,\ y=-\frac{\sqrt{3}}{2}x-1$

(D) $y=\sqrt{3}x, y=0$

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [JEE Advanced 2019]
Let the point B be the reflection of the point $A(2, 3)$, with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B, respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is:

2. [JEE 2017]
For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the co-ordinate axes have exactly three common points?

3. [JEE 2011]
The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S=\left\{\left(2,\frac{3}{4}\right),\left(\frac{5}{2},\frac{3}{4}\right)\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{1}{8},\frac{1}{4}\right)\right\},$$

then the number of point(s) in S lying inside the smaller part is

4. [IIT JEE 2009]
The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is:

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. D	**2.** C	**3.** A	**4.** C	**5.** D	**6.** B	**7.** D	**8.** C	**9.** A	**10.** D
11. D	**12.** C	**13.** A	**14.** B	**15.** A	**16.** A	**17.** C	**18.** A	**19.** D	**20.** B
21. D	**22.** C	**23.** C	**24.** C	**25.** C	**26.** B	**27.** A	**28.** C	**29.** C	**30.** D
31. C	**32.** D	**33.** A	**34.** A	**35.** B	**36.** A	**37.** B	**38.** D	**39.** D	**40.** A
41. B	**42.** A	**43.** A	**44.** A	**45.** C	**46.** D	**47.** A	**48.** D	**49.** B	**50.** C
51. B	**52.** A	**53.** C	**54.** B	**55.** D	**56.** A	**57.** B	**58.** C	**59.** B	**60.** A
61. B	**62.** B	**63.** B	**64.** C	**65.** B	**66.** B	**67.** D	**68.** A	**69.** C	**70.** B
71. C	**72.** B	**73.** D	**74.** D	**75.** B	**76.** C	**77.** C	**78.** A	**79.** C	**80.** A
81. A	**82.** B	**83.** C	**84.** A	**85.** B	**86.** D	**87.** C	**88.** C	**89.** B	**90.** A
91. A	**92.** C	**93.** B	**94.** A	**95.** C	**96.** A	**97.** D	**98.** C	**99.** A	**100.** C
101. D	**102.** D	**103.** B	**104.** A	**105.** A	**106.** C	**107.** B	**108.** A	**109.** C	**110.** D

Multiple Correct Choice Type Questions

1. B, D	**2.** A, B, D	**3.** A, B, C, D	**4.** A, C, D	**5.** A, B, C
6. A, B, D	**7.** A, C	**8.** A, C, D	**9.** A, B, C, D	**10.** A, B, C, D
11. A, C, D	**12.** A, B, C, D	**13.** A, C	**14.** A, C	**15.** A, B
16. B, C, D	**17.** A, C, D	**18.** A, B	**19.** A, B	**20.** A, C, D
21. A, B, C, D	**22.** B, C	**23.** A, C	**24.** B, C	**25.** A, B
26. A, B, C	**27.** A, B, C	**28.** A, D	**29.** A, C	**30.** A, D
31. B, C	**32.** A, B, C, D	**33.** A, B, C	**34.** A, B, C, D	**35.** A, C
36. A, C	**37.** A, C	**38.** A, C, D	**39.** A, C	**40.** A, B, C
41. A, B, C	**42.** A, C, D	**43.** B, C		

Linked Comprehension Type Questions

1. A	**2.** D	**3.** A	**4.** A	**5.** C	**6.** C	**7.** C	**8.** B	**9.** A	**10.** C
11. B	**12.** B	**13.** C	**14.** C	**15.** B	**16.** B	**17.** A	**18.** D	**19.** D	**20.** C
21. C	**22.** D	**23.** A	**24.** C	**25.** A	**26.** D	**27.** A	**28.** A	**29.** B	**30.** A
31. C	**32.** A	**33.** A	**34.** B	**35.** B	**36.** B	**37.** B	**38.** B	**39.** C	**40.** B
41. B	**42.** C	**43.** A	**44.** B	**45.** A	**46.** B	**47.** B	**48.** B	**49.** A	**50.** B
51. D	**52.** A	**53.** B	**54.** D	**55.** C	**56.** A	**57.** A	**58.** C	**59.** A	**60.** C
61. D	**62.** A	**63.** B	**64.** A	**65.** B	**66.** D	**67.** B	**68.** C		

Matrix Match/Column Match Type Questions

1. A → (p)	B → (s)	C → (q)	D → (r)
2. A → (r)	B → (s)	C → (q)	D → (p)
3. A → (p)	B → (s)	C → (q)	D → (r)

4. A → (s)	B → (q)	C → (p)	D → (r)
5. A → (p)	B → (r)	C → (s)	D → (q)
6. A → (s)	B → (r)	C → (q)	D → (p, q, s)
7. A → (r)	B → (p)	C → (q)	D → (s)
8. A → (r)	B → (p)	C → (p)	D → (s)
9. A → (q)	B → (q)	C → (s)	D → (p, r)
10. A → (q)	B → (r)	C → (q, s)	D → (p)
11. A → (r)	B → (q)	C → (p, t)	D → (q, s)
12. A → (r)	B → (s)	C → (p)	D → (q)
13. A → (p)	B → (q)	C → (r)	D → (q)
14. A → (r)	B → (q)	C → (q)	D → (p)
15. A → (r)	B → (s)	C → (q)	D → (p)

Integer/Numerical Answer Type Questions

1. 1	**2.** 6	**3.** 6	**4.** 8	**5.** 1
6. 1	**7.** 5	**8.** 4	**9.** 1	**10.** 2
11. 1	**12.** 5	**13.** 4	**14.** 1	**15.** 1
16. 2	**17.** 1	**18.** 3	**19.** 4	**20.** 4
21. 2	**22.** 5	**23.** 2	**24.** 8	**25.** 5
26. 5	**27.** 2	**28.** 2	**29.** 3	**30.** 3
31. 1	**32.** 1	**33.** 4	**34.** 8	**35.** 7
36. 3	**37.** 3	**38.** 4	**39.** 5	**40.** 1
41. 1	**42.** 2	**43.** 3	**44.** 1	**45.** 9
46. 8	**47.** 9			

ARCHIVE: JEE MAIN

1. C	**2.** D	**3.** B	**4.** 36	**5.** D	**6.** D	**7.** B	**8.** C	**9.** B	**10.** C
11. B	**12.** A	**13.** B	**14.** C	**15.** C	**16.** C	**17.** D	**18.** A	**19.** D	**20.** D
21. A	**22.** A	**23.** B	**24.** D	**25.** D	**26.** C	**27.** C	**28.** A	**29.** A	**30.** A
31. D	**32.** C	**33.** B	**34.** D	**35.** B	**36.** D	**37.** A			

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. D	**2.** A	**3.** D	**4.** B

Multiple Correct Choice Type Problems

1. B, D	**2.** A, C	**3.** B, C	**4.** A, C

Reasoning Based Questions

1. C	**2.** A

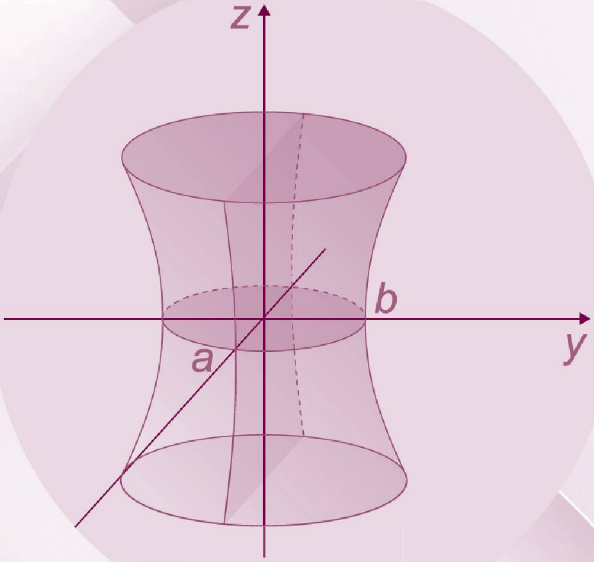
Linked Comprehension Type Questions

1. A	**2.** D	**3.** D	**4.** A	**5.** D	**6.** A	**7.** D

Integer/Numerical Answer Type Questions

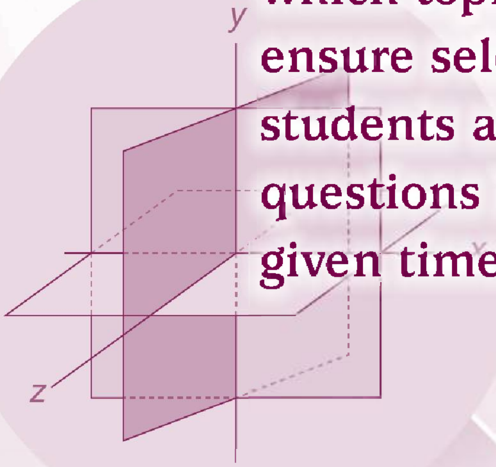
1. 10	**2.** 2	**3.** 2	**4.** 8

From the Author's Desk



Dear Learners,

In Parabola chapter, the most important thing to concentrate on is its parameters and their relative position. The concepts of the same should be very much clear while attempting the related questions. Tangent, chord of contact, focal chord, normal and its properties are main key concepts from exam point of view. Parabola is related to many other chapters like Circle/Ellipse/Hyperbola. PYQs of any exam are best to identify which topics to practice more so as to ensure selection with good rank, hence students are advised to practice adequate questions based on various concepts in given time frame. Good luck!!



CHAPTER 4

Parabola

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Conic Section definition
(b) Parabola and its parameters
(c) Focal chord, tangent
(d) Normal and its properties
(e) Miscellaneous problems

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

CONIC SECTION

Parabola belongs to the family of conic section. Hence, let us first understand the definition of **conic section**. Conic section is the intersection of right circular cone by a plane in different possibilities.

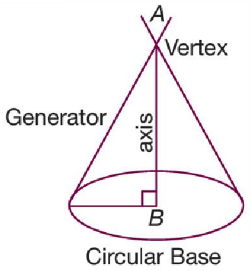
When a right circular cone is cut by a plane in various ways, different types of sections known as conic sections are obtained.

Let us start with, how the conic sections are obtained, by slicing a right circular cone.

A right circular cone is as shown in the figure.

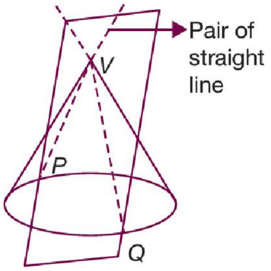
It has circular base and line perpendicular from vertex to base is known as axis of cone.

Generator line is slant height of cone.

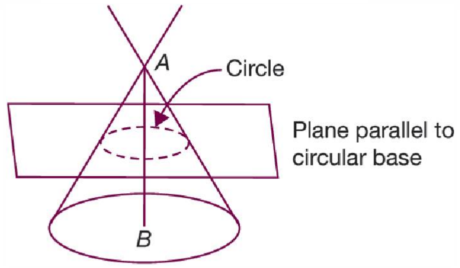


Section of Right Circular Cone by Different Planes

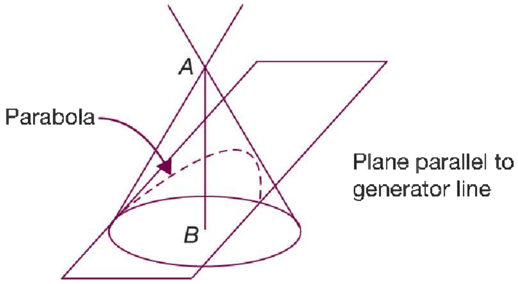
(i) Section of a right circular cone by a plane passing through its vertex is **a pair of straight lines** passing through the vertex, as shown in the figure.



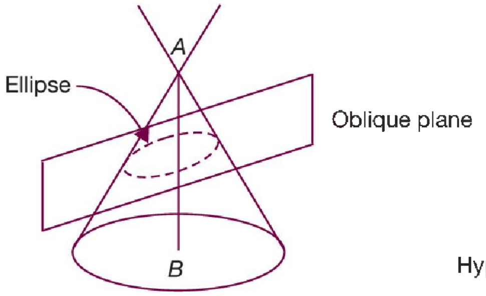
(ii) Section of a right circular cone by a plane parallel to its base is **a Circle,** as shown in the figure.



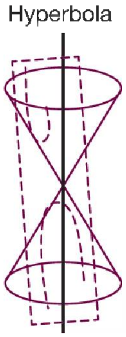
(iii) Section of a right circular cone by a plane parallel to a generator of the cone is **a Parabola,** as shown in the figure.



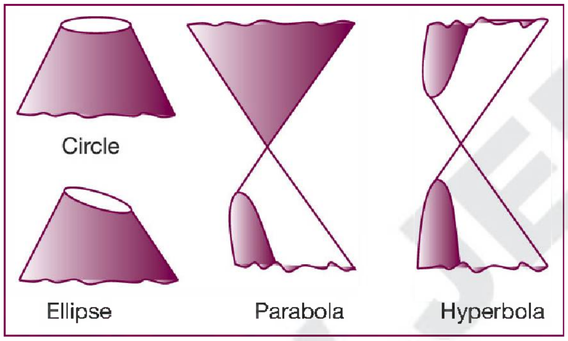
(iv) Section of a right circular cone by a plane, which is neither parallel to base nor parallel to generator, is an **Ellipse**, as shown in the figure.


A
Ellipse
Oblique plane
B

(v) Section of a double right circular cone by a plane, which is either parallel to neither generator of the cone nor perpendicular or parallel to the axis of the cone, is a **Hyperbola,** as shown in the figure.

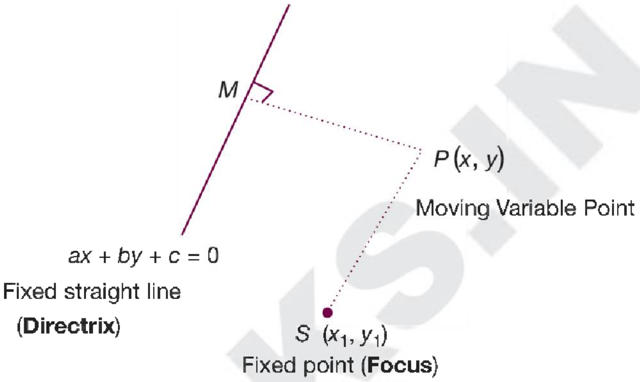


3D View


Circle
Ellipse
Parabola
Hyperbola

EQUATION OF CONIC SECTION

The locus of point, which moves in a plane such that its ratio of distance from a fixed point to the perpendicular distance from a fixed line is constant, is known as conic section.


M
P (x, y)
Moving Variable Point
ax + by + c = 0
Fixed straight line
(Directrix)
S (x1, y1)
Fixed point (Focus)

The fixed point is called the **focus** of the conic and this fixed line is called the **directrix** of the conic. Also, the constant ratio is called the **eccentricity** of the conic, which is denoted by e.

$$\frac{PS}{PM} = e \text{ (eccentricity)}$$

$$(PS)^2 = e^2(PM)^2$$

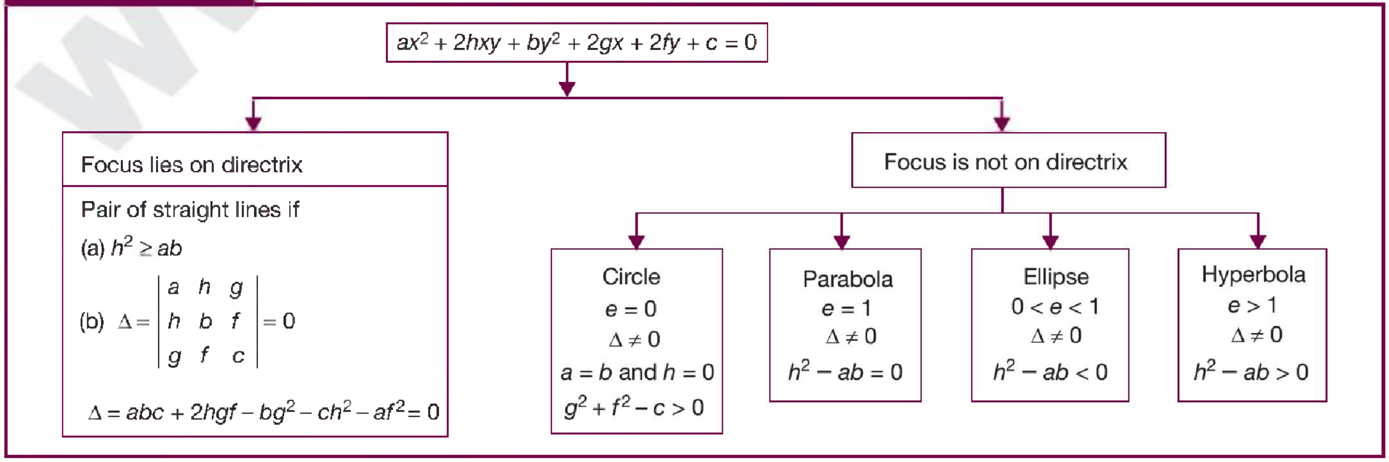
The equation of conic section is given as,

$$\boxed{(x-x_1)^2+(y-y_1)^2 = e^2\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|^2}$$

It is degree two equation in x and y. Let us consider a general two degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which we have also discussed in previous chapters. Now let understand when it represents various conic sections under different conditions.

Working Rule


ax² + 2hxy + by² + 2gx + 2fy + c = 0
Focus lies on directrix
Pair of straight lines if
(a) h² ≥ ab
(b) Δ = | a h g ; h b f ; g f c | = 0
Δ = abc + 2hgf – bg² – ch² – af² = 0
Focus is not on directrix
Circle
e = 0
Δ ≠ 0
a = b and h = 0
g² + f² – c > 0
Parabola
e = 1
Δ ≠ 0
h² – ab = 0
Ellipse
0 < e < 1
Δ ≠ 0
h² – ab < 0
Hyperbola
e > 1
Δ ≠ 0
h² – ab > 0

Let us understand a few terminologies, which are frequently used in conic section.

Axis

(i) A line, in which conic is symmetric.
(ii) A line perpendicular to directrix and passes through focus.

Vertex

Point of intersection of conic and axis, is/are known as vertices of conic sections.

Centre

The point which bisects every chord of the conic passing through it is called the centre of the conic section.

Double Ordinate

Chords of conic, which are perpendicular to axis is known as double ordinate.

Latus Rectum

A double ordinate passing through focus is known as latus rectum.

ILLUSTRATION 1

Identify the curve, $x^2+y^2-2x-2y+2=(2x+y-3)^2$.

SOLUTION

$$(x-1)^2+(y-1)^2=\left(\sqrt{5}\right)^2\left(\frac{2x+y-3}{\sqrt{5}}\right)^2$$

Focus $S(1, 1)$, $e=\sqrt{5}$, directrix $2x+y-3=0$

Here, the focus lies on directrix and hence, it is a pair of straight lines.

Alternate

$$x^2+y^2-2x-2y+2=(2x+y-3)^2$$

$$\Rightarrow \quad 3x^2+4xy-10x-4y+7=0;\ h^2-ab=4-0>0$$

$$\Delta=\begin{vmatrix}3 & 2 & -5\\ 2 & 0 & -2\\ -5 & -2 & 7\end{vmatrix}=20+20-12-28=0,$$

which represents a pair of straight lines.

ILLUSTRATION 2

Identify the locus of $P(x, y)$, for the following.

(i) $(5x-1)^2+(5y-2)^2=\left(\dfrac{3x+4y+2}{5}\right)^2$

(ii) $(5x-1)^2+(5y-2)^2=(3x+4y+2)^2$

SOLUTION

(i) $(5x-1)^2+(5y-2)^2=\left(\dfrac{3x+4y+2}{5}\right)^2$

The given curve is

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=\left(\frac{1}{5}\right)^2\left[\frac{3x+4y+2}{5}\right]^2$$

$$\Rightarrow \sqrt{\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2}=\frac{1}{5}\left|\frac{3x+4y+2}{5}\right|$$

$PS=ePM$

Focus $S\left(\dfrac{1}{5}, \dfrac{2}{5}\right)$ and directrix $3x+4y+2=0$

Focus does not lie on directrix.

$e=\dfrac{1}{5}$

$\therefore$ It is an ellipse.

(ii) $(5x-1)^2+(5y-2)^2=(3x+4y+2)^2$

The given curve is

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=\left(\frac{3x+4y+2}{5}\right)^2$$

$$\Rightarrow \sqrt{\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2}=\left|\frac{3x+4y+2}{5}\right|$$

$\Rightarrow PS=ePM$

Focus $S\left(\dfrac{1}{5}, \dfrac{2}{5}\right)$ and directrix $3x+4y+2=0$

Focus does not lie on directrix.

$e=1$

$\therefore$ It is a parabola.

ILLUSTRATION 3

Identify the locus of $P(x, y)$, for the following.

(i) $(5x-1)^2+(5y-2)^2=25(3x+4y+2)^2$

(ii) $(5x-1)^2+(5y-2)^2=\left(3x+4y-\dfrac{11}{5}\right)^2$

SOLUTION

(i) $(5x-1)^2+(5y-2)^2=25(3x+4y+2)^2$

The given curve is

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=25\left(\frac{3x+4y+2}{5}\right)^2$$

$$\Rightarrow \sqrt{\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2}=5\left|\frac{3x+4y+2}{5}\right|$$

$\Rightarrow PS=ePM$

Focus $S\left(\frac{1}{5},\frac{2}{5}\right)$

Focus does not lie on directrix $3x+4y+2=0$.

$e=5>1$

∴ It is hyperbola

(ii) $(5x-1)^2+(5y-2)^2=\left(3x+4y-\frac{11}{5}\right)^2$

The given curve is

$$\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=1\left[\frac{3x+4y-\frac{11}{5}}{5}\right]^2$$

$$\Rightarrow \sqrt{\left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2}=\left|\frac{3x+4y-\frac{11}{5}}{5}\right|$$

$\Rightarrow PS=ePM$

Focus $S\left(\frac{1}{5},\frac{2}{5}\right)$ lie on directrix $3x+4y-\frac{11}{5}=0$

∴ It is a pair of straight line.

Alternate

$$25x^2+25y^2+5-10-20y$$

$$=9x^2+16y^2+24xy-\frac{66}{5}x-\frac{88}{5}y+\frac{121}{25}$$

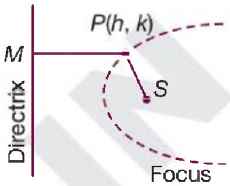
$$\Rightarrow \quad 16x^2+9y^2-24xy+\frac{16}{5}x-\frac{12}{5}y+\frac{4}{25}=0$$

$$\Rightarrow \quad (4x-3y)^2+\frac{4}{5}(4x-3y)+\frac{4}{25}=0$$

$$\Rightarrow \quad (4x-3y+\frac{2}{5})^2=0$$

∴ It is a pair of coincident straight lines.

PARABOLA



A parabola is the locus of a point, which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

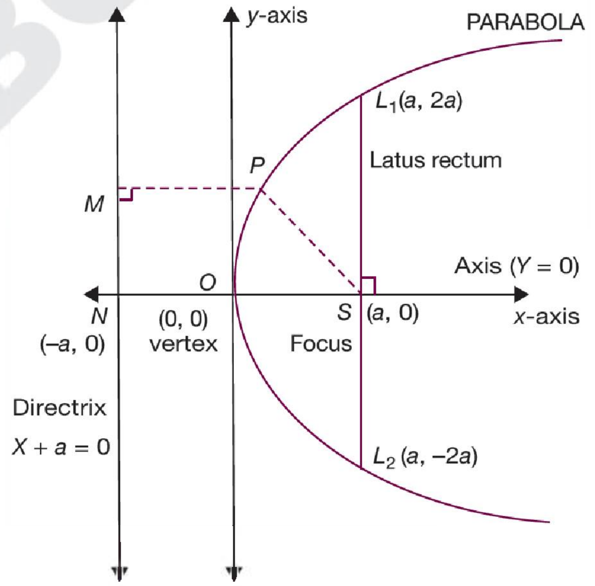
From the figure, $PS=PM$.

For parabola, eccentricity, $e=1$.

In simple words, **every point on parabola maintains equal distance from a point and a line**.

Standard Parabola

If we take vertex as the origin, axis as x-axis and the distance between vertex and focus as 'a' then, the equation of the parabola in the simplest form will be $\boldsymbol{Y^2=4aX}$ where $a>0$.



Parameters of Parabola

Focus (Fixed Point)

$S(a,0)\ X=a,\ Y=0$

Directrix (Fixed Line)

Equation of directrix, $X+a=0$.

Axis

(i) A line about which the parabola is symmetric.

(ii) A line perpendicular to directrix and passes through focus.

Equation of axis, $Y=0$.

Vertex

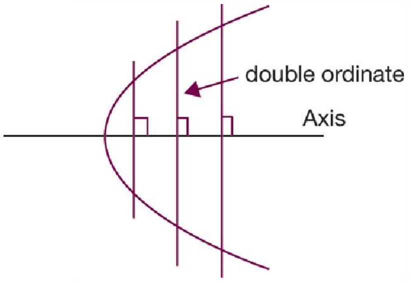
Point of intersection of conic (parabola) and axis is $X = Y = 0$.
Vertex is the midpoint of point N and focus S.

Centre

It does not exist for parabola.

Double Ordinate

Chords of parabola, which are perpendicular to axis is called double ordinate.



Latus Rectum

A double ordinate passing through focus is called latus rectum.
Extremities (end points) of latus rectum L_1, L_2 $(a, \pm 2a)$
$X = a$ and $Y = \pm 2a$
Length of latus rectum $= 4a$
$= 2$ (Distance between focus and directrix)
$= 4$ (Distance between focus and vertex)

ILLUSTRATION 4

Find vertex, focus, axis, directrix and latus rectum of the parabola, $y^2 - 4y - 4x - 8 = 0$.

SOLUTION

The given equation is $(y - 2)^2 = 4x + 12$.

$\Rightarrow \quad (y - 2)^2 = 4(x + 3)$

$\Rightarrow \quad Y^2 = 4aX$, where $Y = y - 2$, $X = x + 3$ and $a = 1$

Vertex is $X = 0$, $Y = 0$

$\Rightarrow \quad x + 3 = 0, y - 2 = 0$

$\therefore$ Vertex is $(-3, 2)$.

Latus rectum $= 4a = 4$.

Focus is $X = a$, $Y = 0$

$\Rightarrow \quad x + 3 = 1, y - 2 = 0$

$\therefore$ Focus is $(-2, 2)$.

Axis of the parabola is $Y = 0$

$\Rightarrow \quad y - 2 = 0$

Directrix of the parabola is

$$X = -a \text{ or } x + 3 = -1$$

$\Rightarrow \quad x + 4 = 0$

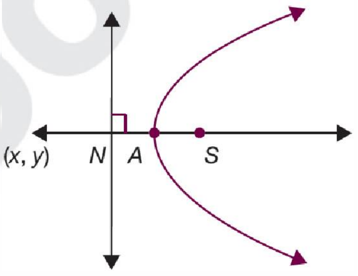
ILLUSTRATION 5

Find the focus, directrix, vertex, axis and latus rectum of the parabola, $y^2 - 8y + 2x + 8 = 0$.

SOLUTION

We can rewrite the given equation as,

$$(y - 4)^2 = -2(x - 4).$$



$\Rightarrow \quad Y^2 = 4aX$ where, $Y = y - 4$, $X = x - 4$ and $a = -\dfrac{1}{2}$

For vertex, $X = 0 \Rightarrow x - 4 = 0 \Rightarrow x = 4$

$Y = 0 \Rightarrow y - 4 = 0 \Rightarrow y = 4$

$\therefore$ The vertex is $(4, 4)$.

For focus, $X = a \Rightarrow x - 4 = -\dfrac{1}{2} \Rightarrow x = \dfrac{7}{2}$

$Y = 0 \Rightarrow y - 4 = 0 \Rightarrow y = 4$

$\therefore$ Focus is $\left(\dfrac{7}{2}, 4\right)$.

Directrix, $X + a = 0$

$\Rightarrow \quad x - 4 + \left(-\dfrac{1}{2}\right) = 0$ or $x = \dfrac{9}{2}$

Axis $Y = 0 \Rightarrow y - 4 = 0 \Rightarrow y = 4$,

Length of latus rectum $= |4a| = 2$

End points of latus rectum

$X = a \Rightarrow x - 4 = -\dfrac{1}{2} \Rightarrow x = \dfrac{7}{2}$

$Y = \pm 2a$

$$\Rightarrow \quad y-4=\pm 2\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad y=5 \text{ and } y=3$$

∴ The end points of the latus rectum are $\left(\frac{7}{2},3\right)$ and $\left(\frac{7}{2},5\right)$.

ILLUSTRATION 6

Find the equation of the parabola with its vertex at (3, 2) and its focus at (5, 2).

SOLUTION

Let the vertex be $A(3, 2)$ and focus is $S(5, 2)$.

Slope of $AS=\frac{2-2}{5-3}=0$ (which is parallel to x-axis).

Hence, the axis of parabola is parallel to x-axis.

The equation is of the form

$$(y-k)^2=4a(x-h) \quad \text{or} \quad (y-2)^2=4a(x-3)$$

As (h, k) is the vertex (3, 2).

a = Distance between the focus and the vertex

$$=\sqrt{(5-3)^2+(2-2)^2}=2$$

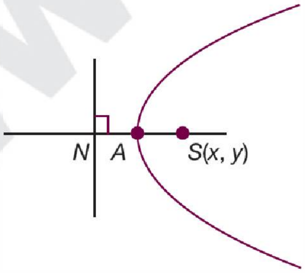
Hence, the required equation is $(y-2)^2=8(x-3)$ or $y^2-8x-4y+28=0$.

ILLUSTRATION 7

Find the equation of parabola, whose vertex is origin and directrix is $x+y=2$.

SOLUTION

Vertex and directrix are given, we need to get focus.



Foot of perpendicular from vertex (0, 0) to directrix = Point N

Using formula

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=-\frac{(ax_1+by_1+c)}{a^2+b^2}$$

$$\frac{x-0}{1}=\frac{y-0}{1}=-\frac{(0+0-2)}{1+1}=1$$

We get $N(1, 1)$

Vertex is midpoint of N and focus $S(x, y)$.

$$\left(\frac{x+1}{2},\frac{y+1}{2}\right)=(0,0)$$

We get $x=y=-1$ Focus $S(-1,-1)$

Equation of parabola

$$PS=PM$$

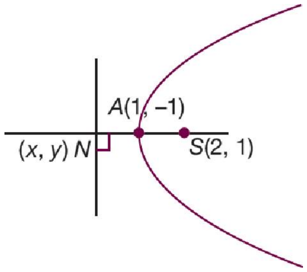
$$(x+1)^2+(y+1)^2=\left(\frac{x+y-2}{\sqrt{2}}\right)^2$$

ILLUSTRATION 8

Find the length of latus rectum of the parabola, whose focus is (2, 1) and vertex (1, –1). Also, find the equation of parabola.

SOLUTION

Here, vertex and focus are given. We need to find the directrix let us first get foot of perpendicular from focus on directrix = Point N



It is a relative sketch, not exact representation

Midpoint of N and S = Vertex A

$$\left(\frac{x+2}{2},\frac{y+1}{2}\right)\equiv(1,-1)$$

We get, $x=0$ and $y=-3$ ∴ point $N(0, -3)$

Directrix is perpendicular to axis (AS)

slope of directrix $=-\frac{1}{m_{AS}}=-\left(\frac{2-1}{1-(-1)}\right)=-\frac{1}{2}$

Equation of directrix is $y+3=-\frac{1}{2}(x-0)$

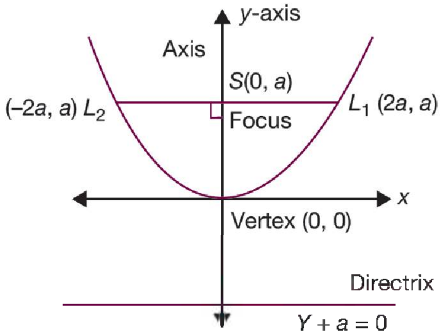
$$\Rightarrow \quad x+2y+6=0$$

Now, equation of parabola

$$PS = PM$$

$$\Rightarrow \quad (x-2)^2 + (y-1)^2 = \left(\frac{x+2y+6}{\sqrt{5}}\right)^2$$

Standard Parabola $X^2 = 4aY$



We know that, $X^2 = 4aY$

Here, in standard parabola, X is replaced by Y.

$\therefore$ In standard parabola takes the form, $Y^2 = 4aX$.

Vertex (0, 0) $\quad X = Y = 0$

Focus (0, a) $\quad X = 0, Y = a$

Equation of directrix is $Y + a = 0$

Equation of axis, $X = 0$

Length of latus rectum, $\ell_{LR} = 4a$

Extremities of latus rectum are $L_1(2a, a)$ and $L_2(-2a, a)$.

Problem Solving Technique

As we can see, if we replace X by Y in standard parabola $Y^2 = 4aX$ we get, $X^2 = 4aY$

Hence, we just need to interchange $X \leftrightarrow Y$, to get all the parameters accordingly.

ILLUSTRATION 9

Find the focus, directrix, vertex, axis, latus rectum of the parabola $y = -x^2 + 2x - 3$.

SOLUTION

Let us rewrite the given equation as $(x-1)^2 = -(y+2)$.

$$\Rightarrow \quad X^2 = 4aY$$

So, $X = x - 1$; $Y = y + 2$; $4a = -1$

Vertex $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$$Y = 0 \Rightarrow y + 2 = 0$$

$\Rightarrow \quad y = -2$ we get vertex (1, –2)

Axis $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

Directrix $Y + a = 0 \Rightarrow y + 2 - \dfrac{1}{4} = 0$

$$\Rightarrow \quad y + \frac{7}{4} = 0$$

Focus S, $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$$Y = a \Rightarrow y + 2 = -\frac{1}{4} \Rightarrow y = -\frac{9}{4}$$

$\therefore$ We get focus as $\left(1, -\dfrac{9}{4}\right)$

Length of latus rectum $= |4a| = 1$

Extremities of latus rectum $(\pm 2a, a)$

$$X = \pm 2a$$

$$\Rightarrow \quad x - 1 = \pm 2\left(-\frac{1}{4}\right) = \mp\frac{1}{2} \text{ or } x = \frac{3}{2}, \frac{1}{2}$$

$$Y = a \Rightarrow y + 2 = -\frac{1}{4} \Rightarrow y = -\frac{9}{4}$$

We get the extremities of latus rectum as $\left(\dfrac{3}{2}, -\dfrac{9}{4}\right)$ and $\left(\dfrac{1}{2}, -\dfrac{9}{4}\right)$.

ILLUSTRATION 10

Find the equation of the parabola, whose axis is parallel to the y-axis and which passes through the points (0, 4), (1, 9) and (–2, 6). Also, find its latus rectum.

SOLUTION

As the axis is parallel to the y-axis, let vertex is (α, β), the equation of the parabola will be of the form $(x - \alpha)^2 = 4a(y - \beta)$

Parabola passes through the points (0, 4), (1, 9) and (–2, 6).

So, $(0-\alpha)^2 = 4a(4-\beta) \Rightarrow \alpha^2 = 4a(4-\beta)$ ...(1)

and $(1-\alpha)^2 = 4a(9-\beta) \Rightarrow 1-2\alpha+\alpha^2 = 4a(9-\beta)$...(2)

and $(-2-\alpha)^2 = 4a(6-\beta) \Rightarrow 4+4\alpha+\alpha^2 = 4a(6-\beta)$ ...(3)

Difference of (2) and (1) $1-2\alpha = 20a$ ...(4)

Difference of (3) and (2) $3+6\alpha = -12a$

$$\therefore \quad \frac{1-2\alpha}{3+6\alpha} = \frac{20a}{-12a}$$

$$\Rightarrow \quad -3(1-2\alpha) = 5(3+6\alpha)$$

$$\Rightarrow \quad -3+6\alpha = 15+30\alpha$$

$$\Rightarrow \quad 24\alpha = -18$$

$\therefore$ We get $\alpha = -\frac{3}{4}$; $a = \frac{1}{8}$; $\beta = \frac{23}{8}$ by solving these relations.

$\therefore$ The equation of the parabola is

$$\left(x+\frac{3}{4}\right)^2 = 4\cdot\frac{1}{8}\cdot\left(y-\frac{23}{8}\right)$$

Its latus rectum is $4a = 4\times\frac{1}{8} = \frac{1}{2}$

Alternate

Let parabola is $y = ax^2 + bx + c$ (axis is parallel to y-axis)

Point (0, 4) lies on it

$$4 = 0+0+c \Rightarrow c = 4$$

(1, 9) lies on it

$$9 = a+b+c \Rightarrow a+b = 5$$

(–2, 6) lies on it

$$6 = 4a-2b+c \Rightarrow 2a-b = 1$$

We get $a = 2$, $b = 3$ and $c = 4$

Required parabola $y = 2x^2+3x+4$

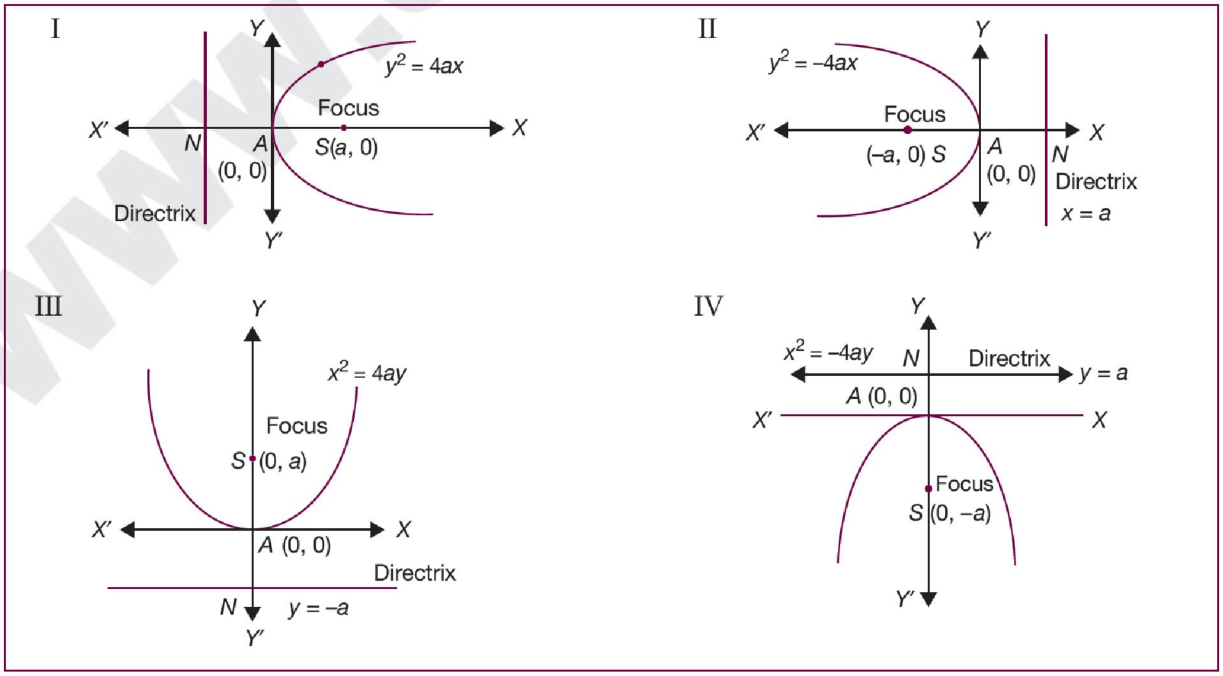
Or $\quad y = 2\left(x+\frac{3}{4}\right)^2 + \frac{23}{8}$

Or $\quad \left(x+\frac{3}{4}\right)^2 = \frac{1}{2}\left(y-\frac{23}{8}\right)$

Length of latus rectum $= \frac{1}{2}$

Standard Parabolas

Let $a > 0$

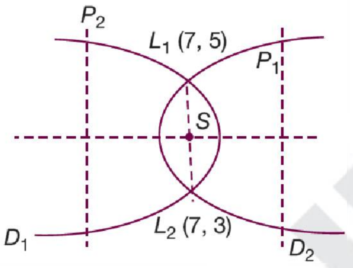

I
Y
$y^2 = 4ax$
Focus
X′
X
N
A
S(a, 0)
(0, 0)
Directrix
Y′
II
Y
$y^2 = -4ax$
Focus
X′
X
(–a, 0) S
A
N
(0, 0)
Directrix
x = a
Y′
III
Y
$x^2 = 4ay$
Focus
S (0, a)
X′
X
A (0, 0)
Directrix
N
y = –a
Y′
IV
Y
$x^2 = -4ay$
N
Directrix
y = a
A (0, 0)
X′
X
Focus
S (0, –a)
Y′

Parameters	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Co-ordinates of vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Co-ordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the latus rectum	$4a$	$4a$	$4a$	$4a$

ILLUSTRATION 11

Find the equation of the parabola, whose end points of latus rectum are (7, 5) and (7, 3), respectively.

SOLUTION



Line joining end points of latus rectums is parallel to y-axis is given by $x = 7$

Focus = Midpoint of end points of latus rectum = (7, 4)

Length of latus rectum = $4a = L_1L_2 = 2$

Directrix is parallel to L_1L_2 at distance $2a$

Directrix D_1: $x = 6$ and Directrix D_2: $x = 8$

Equation of parabola P_1

$$PS = PM$$

$$\Rightarrow \quad \sqrt{(x-7)^2 + (y-4)^2} = |x - 6|$$

$$\Rightarrow \quad (y-4)^2 = 2(x - 6.5)$$

Equation of parabola P_2 $PS = PM$

$$\sqrt{(x-7)^2 + (y-4)^2} = |x - 8|$$

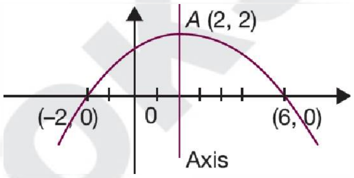
$$(y-4)^2 = -2(x - 7.5)$$

The required parabolas are $(y - 4)^2 = -2(x - 7.5)$; $(y - 4)^2 = 2(x - 6.5)$.

ILLUSTRATION 12

If the vertex of a parabola is (2, 2) and the co-ordinates of its two extremities of the latus rectum are (–2, 0) and (6, 0), then find the equation of the parabola.

SOLUTION



As shown in figure, we can clearly conclude that the length of latus rectum is 8.

Now, shifting the origin at A, the equation of parabola is $X^2 = -8Y$

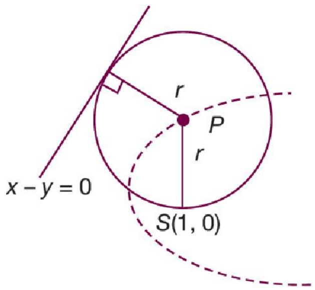
We get, $(x - 2)^2 = -8(y - 2)$

ILLUSTRATION 13

A variable circle always passes through (1, 0) and its tangent is $y = x$. Find the locus of centre of the variable circle. Also, find the directrix and the axis of symmetry and the length of latus rectum of the conic.

SOLUTION

Variable circle touches the line $x - y = 0$ and passes through the point $S(1, 0)$. In other words, centre is equidistant from fixed point $S(1, 0)$ and fixed line.



$\therefore$ It is a parabola.

$$PS = PM$$

$$(x-1)^2+(y-0)^2=\left|\frac{x-y}{\sqrt{2}}\right|^2$$

Focus (1, 0) and directrix $x-y=0$

Axis = perpendicular to directrix passing through focus is $x+y=1$.

$$l_{\text{latus rectum}}=2(\text{Distance between focus and directrix})$$

$$=\left(\frac{2}{\sqrt{2}}\right)=\sqrt{2}$$

Directrix $x-y=0$, axis $x+y=1$, latus rectum = $\sqrt{2}$

Test Your Concepts-I

(Solutions on page H.140)

1. Show that the equation $3x^2-18xy+27y^2+2x+14y-2=0$ represents a parabola.
2. If the line $x-1=0$ is the directrix of the parabola $y^2-kx+8=0$, then find the value of k.
3. Show that the curve described parametrically by $x=t^2-2t+2$, $y=t^2+2t+2$ represents a parabola
4. If the ends of latus rectum of parabola are (2, 6) and (6, 2) and the equation of the possible directrix is $x+y=\lambda_i$, where $i=1, 2$. Then find the value of A.M. of λ_1 and λ_2.
5. If one end of the diameter of a circle is (3, 4) and it touches the x-axis, then find the locus of other end of the diameter of the circle.
6. If the focus of a parabola is (–2, 1) and the directrix has the equation $x+y=3$, then find the vertex.
7. Find the length of the latus rectum of the parabola $169\{(x-1)^2+(y-3)^2\}=(5x-12y+17)^2$.

Answer Keys

2. –8, 4 **4.** 8 **5.** $(x-3)^2=16y$ **6.** (–1, 2) **7.** 28/13

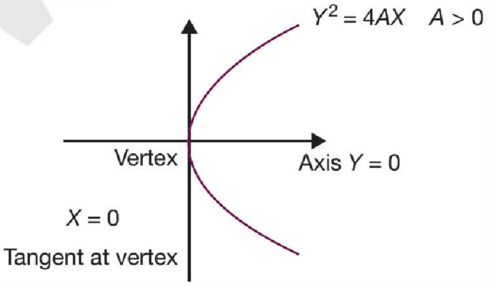
EQUATION OF PARABOLA USING AXIS AND TANGENT AT VERTEX

Let us observe the equation of standard parabola $Y^2=4aX$ closely and then we conclude that the parabola equation can be written as

(Equation of axis)^2 = ℓ_{LR} (Equation of tangent at vertex)

where, ℓ_{LR} = Length of latus rectum

$$Y^2=4AX$$



NOTE:

Both the equation of axis and tangent at vertex must be written in normalised form.

∴ The normalised form of straight line $ax+by+c=0$ is $\dfrac{ax+by+c}{\sqrt{a^2+b^2}}=0$.

ILLUSTRATION 14

Find the equation of parabola, whose tangent at vertex is $x-y+1=0$ and focus being (2, 0).

SOLUTION

Axis perpendicular to tangent at vertex and passes through focus (2, 0) is $y-0=-1(x-2)$.

$\Rightarrow \quad x+y-2=0$

Now, length of latus rectum = 4 (distance between focus and tangent at vertex)

$$= 4\left(\frac{2-0+1}{\sqrt{2}}\right) = 6\sqrt{2}$$

Equation of parabola

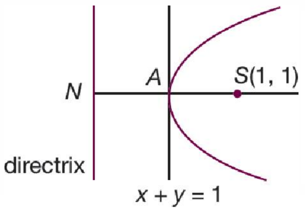
$$\left(\frac{x+y-2}{\sqrt{2}}\right)^2 = 6\sqrt{2}\left(\frac{x-y+1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad (x+y-2)^2 = 12\,(x-y+1)$$

ILLUSTRATION 15

Find the equation of the parabola, whose focus is (1, 1) and the tangent at the vertex is $x + y = 1$. Also, find its latus rectum.

SOLUTION



It is a relative sketch, not exact representation

The directrix is parallel to the tangent at the vertex A.

$\therefore$ The directrix will be of the form $x + y = \lambda$ ...(1)

Now, A is the foot of the perpendicular from S (1, 1) to the line $x + y = 1$.

Let $A = (\alpha, \beta)$, then $\alpha + \beta = 1$ ...(2)

and $\frac{\beta - 1}{\alpha - 1}(-1) = -1$, i.e., $\alpha = \beta$ ...(3)

Solving (2), (3), we get $\alpha = \beta = \frac{1}{2}$.

So, $A\left(\frac{1}{2}, \frac{1}{2}\right)$

Let, $N(x_1, y_1)$. As $NA = AS$, A is the midpoint of NS.

$$\therefore \quad \frac{x_1+1}{2} = \frac{1}{2}, \frac{y_1+1}{2} = \frac{1}{2}$$

$x_1 = 0, y_1 = 0$ So, $N = (0, 0)$

As N is on the directrix, (0, 0) satisfies (1).

Hence, $\lambda = 0$.

The equation of the directrix is $x + y = 0$.

Using definition of parabola, the equation of the parabola is $(x-1)^2 + (y-1)^2 = \left(\frac{x+y}{\sqrt{1^2+1^2}}\right)^2$

$$\Rightarrow \quad 2\{(x-1)^2 + (y-1)^2\} = (x+y)^2$$

$$\Rightarrow \quad 2(x^2 + y^2 - 2x - 2y + 2) = x^2 + y^2 + 2xy$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 4x - 4y + 4 = 0$$

$$\Rightarrow \quad (x-y)^2 = 4\,(x+y-1)$$

$$\therefore \text{ The latus rectum} = 4AS = 4\sqrt{\left(\frac{1}{2}-1\right)^2 + \left(\frac{1}{2}-1\right)^2}$$

$$= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

Alternate

Axis is perpendicular to the tangent at vertex and passes through focus (1, 1).

Then, the equation of axis is $x - y = 0$.

Length of latus rectum ℓ_{LR} = 4 (distance between focus and tangent at vertex)

$$= 4\left|\frac{1+1-1}{\sqrt{2}}\right| = 2\sqrt{2}$$

Equation of parabola

$$\left(\frac{x-y}{\sqrt{2}}\right)^2 = 2\sqrt{2}\left(\frac{x+y-1}{\sqrt{2}}\right)$$

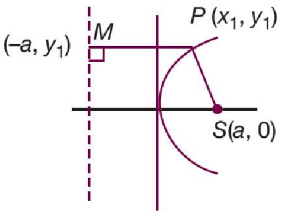
$$\Rightarrow \quad (x-y)^2 = 4(x+y-1)$$

NOTE:

Two parabolas are said to be equal, if the lengths of their latus rectum are equal.

FOCAL DISTANCE

Distance of a point $P(x_1, y_1)$ on parabola from focus is known as focal distance.



Focal distance = $PS = PM$ (Definition of parabola)

$$= x_1 + a$$

PARAMETRIC EQUATION OF PARABOLA

Parabola $y^2 = 4ax$.

Parametric equation $x = at^2, y = 2at$, where $t \in R$, which helps assuming any variable point $P(at^2, 2at)$ on parabola. In short, the term $P(t)$ is used to write point $P(at^2, 2at)$.

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Any point $P(x, y)$	$P(at^2, 2at)$	$P(-at^2, 2at)$	$P(2at, at^2)$	$P(2at, -at^2)$
Focal distance (PS)	$x + a$	$x - a$	$y + a$	$y - a$

If the points $P(t_1)$ and $Q(t_2)$ are given, it means the points are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.

ILLUSTRATION 16

Find the parametric equation of the parabola,

$$(x - 1)^2 = -12(y - 2).$$

SOLUTION

$$X^2 = -4AY;\ X = x - 1;\ 4a = 12 \Rightarrow a = 3;\ Y = y - 2$$

$$X = 2at \Rightarrow x - 1 = 6t \Rightarrow x = 1 + 6t$$

$$Y = -at^2 \Rightarrow y - 2 = -3t^2 \Rightarrow y = 2 - 3t^2$$

Hence parametric equations

$$x = 1 + 6t,\ y = 2 - 3t^2$$

ILLUSTRATION 17

Find the distance between the points $P(1)$ and $Q(-2)$ of the parabola $y^2 = 4x$.

SOLUTION

$$P(1) = P(at^2, 2at) = P(1, 2) \qquad \text{(Using } t = 1, a = 1)$$

$$Q(-2) = Q(4, -4) \qquad \text{(Using } t = -2 \text{ and } a = 1)$$

Now, distance $PQ = \sqrt{3^2 + 6^2} = 3\sqrt{5}$ units

ILLUSTRATION 18

Show that the curve described parametrically by $x = t^2 + t + 1$ and $y = t^2 - t + 1$ represents a parabola. Also find length of latus rectum.

SOLUTION

$$x = t^2 + t + 1 \qquad \ldots(1)$$

$$y = t^2 - t + 1 \qquad \ldots(2)$$

Subtract (2) from (1)

$$x - y = 2t$$

Thus, $x = t^2 + t + 1$

$$\Rightarrow \quad x = \left(\frac{x-y}{2}\right)^2 + \left(\frac{x-y}{2}\right) + 1$$

$$\Rightarrow \quad 4x = (x - y)^2 + 2x - 2y + 4$$

$$\Rightarrow \quad (x - y)^2 = 2(x + y - 2)$$

$$\Rightarrow \quad x^2 + y^2 - 2xy = 2x + 2y - 4$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 2x - 2y + 4 = 0$$

Now, $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2$

$$= 4 - 2 - 1 - 1 - 4 = -4$$

$\therefore \quad \Delta \neq 0$ and $h^2 - ab = 1 - 1 = 0$

So, it is the equation of a parabola.

Rewriting the equation of parabola

$$(x - y)^2 = 2x + 2y - 4$$

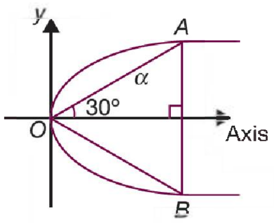
$$\left(\frac{x-y}{\sqrt{2}}\right)^2 = \sqrt{2}\left(\frac{x+y-2}{\sqrt{2}}\right)$$

Length of latus rectum $= \sqrt{2}$

ILLUSTRATION 19

Find the side of an equilateral triangle with an angular points being origin, such that the triangle is inscribed in the parabola $y^2 = 4ax$.

SOLUTION



About axis, parabola is symmetric equilateral triangle should also be symmetric about x-axis.
Let side of equilateral triangle is α point A($\alpha\cos 30°$, $\alpha\sin 30°$) it lies on parabola $y^2 = 4ax$.

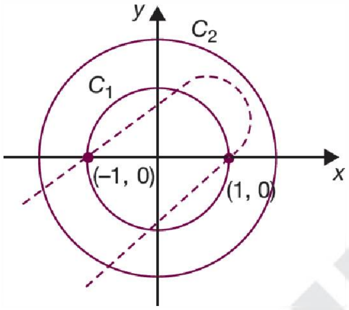
$$\Rightarrow \quad \left(\frac{\alpha}{2}\right)^2 = 4a\left(\frac{\alpha\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad \alpha = 8\sqrt{3}\,a$$

ILLUSTRATION 20

Let C_1: $x^2 + y^2 = 1$ and C_2: $x^2 + y^2 = 4$ are two concentric circles. If a parabola passes through the end points of diameter along the x-axis of C_1 and whose directrix is any tangent of the circle C_2, then find the locus of the focus of the parabola.

SOLUTION



Let focus is (h, k) and any tangent on C_2 is

$x\cos\theta + y\sin\theta = 2$ (directrix of parabola)

Equation of parabola $PS = PM$

$$(x-h)^2 + (y-k)^2 = \left|\frac{x\cos\theta + y\sin\theta - 2}{1}\right|^2$$

it passes through points (1, 0) and (–1, 0)

$$(1-h)^2 + k^2 = (\cos\theta - 2)^2$$

and $\quad (-1-h)^2 + k^2 = (-\cos\theta - 2)^2$

After taking difference, we get $-4h = -8\cos\theta$

$$\Rightarrow \quad \cos\theta = \frac{h}{2}$$

Required locus is $(1-h)^2 + k^2 = \left(\frac{h}{2} - 2\right)^2$

$$\Rightarrow \quad 1 + h^2 - 2h + k^2 = \frac{h^2}{4} + 4 - 2h$$

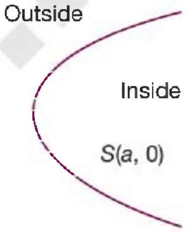
$$\Rightarrow \quad \frac{3h^2}{4} + k^2 = 3$$

$$\Rightarrow \quad \frac{h^2}{4} + \frac{k^2}{3} = 1$$

By substituting $h = x$ and $k = y$

Hence, locus is $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1 \Rightarrow 3x^2 + 4y^2 = 12$

POSITION OF A POINT RELATIVE TO PARABOLA



Parabola S: $y^2 - 4ax = 0$

For point (x_1, y_1), we are defining a term $S_1 : y_1^2 - 4ax_1$

As we know that focus $S(a, 0)$ always lies inside the parabola, so $S_1 : y_1^2 - 4ax_1 = 0 - 4a^2 < 0$

It means, we can conclude that for all points inside the parabola, the expression S_1 should be negative and for points outside the parabola, S_1 should be positive.

Parabola Equation	Expression S_1	Inside Point	On the Parabola	Outside Point
S: $y^2 = 4ax$	$S_1 : y_1^2 - 4ax_1$	$S_1 < 0$	$S_1 = 0$	$S_1 > 0$
S: $y^2 = -4ax$	$S_1 : y_1^2 + 4ax_1$	$S_1 < 0$	$S_1 = 0$	$S_1 > 0$
S: $x^2 = 4ay$	$S_1 : x_1^2 - 4ay_1$	$S_1 < 0$	$S_1 = 0$	$S_1 > 0$
S: $x^2 = -4ay$	$S_1 : x_1^2 + 4ay_1$	$S_1 < 0$	$S_1 = 0$	$S_1 > 0$

Remark

If parabola is not written in standard form, then as we know focus lies inside the parabola we should observe the location of given point (x_1, y_1) accordingly.

ILLUSTRATION 21

Find the position of the point (–2, 2) with respect to the parabola $y^2 - 4y + 9x + 13 = 0$.

SOLUTION

Let the point (h, k) be (–2, 2).

We have $S_1 = k^2 - 4k + 9h + 13$

$$= (2)^2 - 4(2) + 9(-2) + 13 = 4 - 8 - 18 + 13$$

$$= -9 < 0$$

Hence, $S_1 < 0$

∴ The point (–2, 2) lies inside the parabola.

ILLUSTRATION 22

For what values of α, the point (α, α) lies inside, on or outside the parabola $(y-2)^2 = 4(x-3)$.

SOLUTION

$$S_1 = (\alpha-2)^2 - 4(\alpha-3)$$
$$= \alpha^2 - 8\alpha + 16 = (\alpha-4)^2$$

For inside point (α, α), $S_1 < 0 \Rightarrow (\alpha-4)^2 < 0$, which is impossible.

$\therefore \quad \alpha \in \phi$

For on the parabola, $S_1 = 0 \Rightarrow (\alpha-4)^2 = 0$

$\therefore \quad \alpha = 4$

For outside parabola, $S_1 = 0 \Rightarrow (\alpha-4)^2 > 0$

$\therefore \quad \alpha = R - \{4\}$

ILLUSTRATION 23

Find the set of values of 'α' in the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, for which the point $(\sin\alpha, \cos\alpha)$ does not lie outside the parabola $2y^2 + x - 2 = 0$.

SOLUTION

$$S_1 = 2\cos^2\alpha + \sin\alpha - 2 \le 0$$

$\Rightarrow \quad \sin\alpha - 2\sin^2\alpha \le 0$

$\Rightarrow \quad \sin\alpha\,(1 - 2\sin\alpha) \le 0$

It gives $\sin\alpha \le 0$ or $\sin\alpha \ge \frac{1}{2}$

We get, $\alpha \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$

ILLUSTRATION 24

For what real values of a, the point $(-2a, a+1)$ will be an interior point of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$.

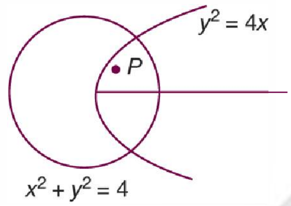
SOLUTION

The point $P(-2a, a+1)$ will be an interior point of the circle $x^2 + y^2 - 4 = 0$.

$\therefore \quad (-2a)^2 + (a+1)^2 - 4 < 0,$

$\Rightarrow \quad 5a^2 + 2a - 3 < 0 \Rightarrow (5a-3)(a+1) < 0$

We get, $-1 < a < \frac{3}{5}$ …(1)



The point $P(-2a, a+1)$ is an interior point of the parabola $y^2 - 4x = 0$.

Then, $(a+1)^2 - 4(-2a) < 0 \Rightarrow a^2 + 10a + 1 < 0.$

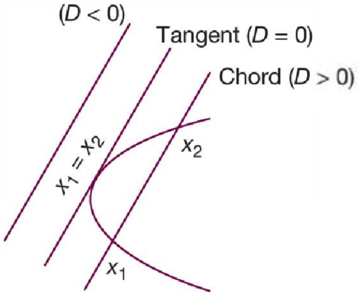
$\Rightarrow \quad (a - (-5 + 2\sqrt{6}))\,(a + 5 + 2\sqrt{6}) < 0$

We get, $-5 - 2\sqrt{6} < a < -5 + 2\sqrt{6}$ …(2)

The set of values that satisfies Equations (1) and (2) is $-1 < a < -5 + 2\sqrt{6}$.

POSITION OF STRAIGHT LINE RELATIVE TO PARABOLA

Let the parabola be $y^2 = 4ax$ and the given line be $y = mx + c$.



Solving them together, we get, $(mx + c)^2 = 4ax$

$\Rightarrow \quad m^2x^2 + 2x(mc - 2a) + c^2 = 0$ …(1)

This equation is quadratic in x, which shows that any straight line can cut the parabola at maximum two points.

Let us calculate discriminant of this quadratic equation in x.

$$D = 4(mc-2a)^2 - 4m^2c^2$$

Position of Line	Condition on D
Chord	$D > 0$
Tangent	$D = 0$
No intersection	$D < 0$

CONDITION OF TANGENCY

If the straight line touches the parabola, then the equation in x has equal roots.
Discriminant, $D = 0$

$$\Rightarrow \quad 4(mc-2a)^2 - 4m^2c^2 = 0$$

$$\Rightarrow \quad -4amc + 4a^2 = 0$$

$$\Rightarrow \quad c = \frac{a}{m}, m \neq 0$$

So, the line $y = mx + c$ touches the parabola $y^2 = 4ax$, if $c = \frac{a}{m}$ (Which is condition of tangency).

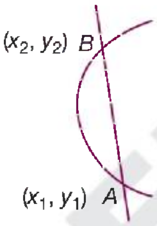
LENGTH OF CHORD

Chord $y = mx + c$

Parabola $y^2 = 4ax$

Length of chord (AB)

$$= \frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$$



PROOF:

By solving the given chord with parabola, we get,

$$y^2 = 4a\left(\frac{y-c}{m}\right) \text{ or } my^2 - 4ay + 4ac = 0.$$

Let it has roots y_1 and y_2.

Then, $y_1 + y_2 = \frac{4a}{m}$ and $y_1y_2 = \frac{4ac}{m}$

$$AB = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

$$= \sqrt{\left(\frac{y_1^2 - y_2^2}{4a}\right)^2 + (y_1-y_2)^2}$$

$$= |y_1 - y_2| \sqrt{\frac{(y_1+y_2)^2}{16a^2} + 1}$$

$$= \sqrt{(y_1+y_2)^2 - 4y_1y_2}\sqrt{\frac{\left(\frac{4a}{m}\right)^2}{16a^2} + 1}$$

$$= \sqrt{\frac{16a^2}{m^2} - \frac{16ac}{m}}\sqrt{\frac{1}{m^2} + 1}$$

We get, length of chord $(AB) = \frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$ units

FOCAL CHORD

A chord passing through focus is known as focal chord of parabola.

(i) If the extremities of focal chord are $P(t_1)$ and $Q(t_2)$, then $t_1t_2 = -1$

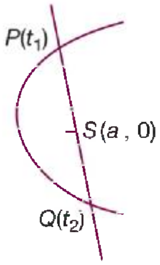
PROOF:

Let $y^2 = 4ax$ be a parabola, if PQ be a chord.

If $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to be a focal chord PQ must pass through the focus $S(a, 0)$.

$\therefore$ Q, S and P are collinear.

$\therefore$ Slope of PS = Slope of QS



$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2}$$

$$\Rightarrow \quad \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

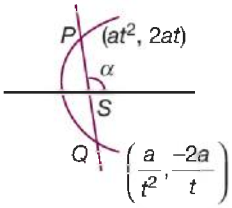
$$\Rightarrow \quad t_2 - t_1 \neq 0 \quad \therefore \quad t_1t_2 + 1 = 0 \text{ or } t_1t_2 = -1 \text{ or } t_2 = -\frac{1}{t_1}$$

which is a required relation.

(ii) Length of focal chord = PQ = $a\left(t + \frac{1}{t}\right)^2$

PROOF:

Extremities of focal chord are $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.



Now, focal distance $PS = a + at^2$

and $QS = a + \frac{a}{t^2}$

Now, focal chord is $PQ = PS + QS = a + at^2 + a + \frac{a}{t^2}$

$$= a\left(t + \frac{1}{t}\right)^2$$

Here, slope of focal chord is $m_{PS} = \frac{2at}{at^2 - a} = \tan\alpha$

$$\Rightarrow \quad t - \frac{1}{t} = 2\cot\alpha$$

$$\text{Length of focal chord} = a\left(t+\frac{1}{t}\right)^2 = a\left(\left(t-\frac{1}{t}\right)^2+4\right)$$

$$= a(4\cot^2\alpha+4) = 4a\operatorname{cosec}^2\alpha$$

(iii) Length of focal chord = 4a cosec$^2\alpha$ where, α is inclination angle of focal chord with x-axis.

(iv) Harmonic Mean of segment of focal chords

$= \frac{1}{2}$ length of latus rectum

or $\frac{2(PS)(QS)}{PS+QS} = \frac{1}{2}$ (Length of latus rectum)

PROOF:

We know that the value of segment of focal chord PS and QS as $PS = a+at^2$ and $QS = a+\frac{a}{t^2}$

$$\text{Now, } \frac{2(PS)(QS)}{PS+QS} = \frac{2(a+at^2)\left(a+\frac{a}{t^2}\right)}{a+at^2+a+\frac{a}{t^2}} = 2a$$

$$= \frac{1}{2}\text{Length of latus rectum}$$

ILLUSTRATION 25

Find the latus rectum of a parabola, whose focal chord PSQ is such that $SP = 3$ and $SQ = 2$.

SOLUTION

Semi-latus rectum is the H.M. between two focal segments.

$$\frac{1}{2}\ell_{LR} = \frac{2(SP)(SQ)}{SP+SQ}$$

$$\Rightarrow \quad \ell_{LR} = \frac{4(3)(2)}{3+2} = \frac{24}{5}$$

ILLUSTRATION 26

If P_1Q_1 and P_2Q_2 are two focal chords of the parabola $y^2 = 4ax$, then prove that the chords P_1P_2 and Q_1Q_2 intersect on the directrix of the parabola.

SOLUTION

Let co-ordinates of P_1 and P_2 be $\left(at_1^2, 2at_1\right)$ and $\left(at_2^2, 2at_2\right)$, then co-ordinates of Q_1 and Q_2 are $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ and $\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$, respectively.

($\because$ P_1Q_1 and P_2Q_2 are focal chords)

$\therefore$ Equation of P_1P_2 is $y(t_1+t_2) = 2(x+at_1t_2)$ ...(1)

Replacing t_1 by $-\frac{1}{t_1}$ and $t_2 = \frac{-1}{t_2}$, then equation of Q_1Q_2 is $y\left(-\frac{1}{t_1}-\frac{1}{t_2}\right) = 2\cdot\left\{x+a\left(-\frac{1}{t_1}\right)\left(-\frac{1}{t_2}\right)\right\}$

$\Rightarrow \quad -y(t_1+t_2) = 2(xt_1t_2+a)$ ...(2)

Adding (1) and (2), we get,

$$0 = 2(a+t_1t_2)(x+a) \qquad \because 2(1+t_1t_2) \neq 0$$

$\therefore \quad x+a = 0$ (Directrix)

ILLUSTRATION 27

If the extremities of a focal chord of a parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then compute $x_1x_2 + y_1y_2$.

SOLUTION

Let $x_1 = at_1^2;\ x_2 = at_2^2$

$\Rightarrow \quad x_1x_2 = a^2t_1^2t_2^2$

Let $y_1 = 2at_1;\ y_2 = 2at_2$

$\Rightarrow \quad y_1y_2 = 4a^2t_1t_2$

Using $t_1t_2 = -1$ as a focal chord, we get on addition

$$x_1x_2 + y_1y_2 = -3a^2.$$

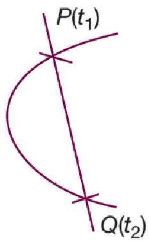
ILLUSTRATION 28

If $t_1t_2 = k$, then prove that chord joining point t_1 and point t_2 always passes through a fixed point.

SOLUTION

Let the points be $P(t_1)$ and $Q(t_2)$.

Equation of PQ is



$$y-2at_1 = \frac{2}{(t_1+t_2)}(x-at_1^2)$$

$\Rightarrow \quad y(t_1+t_2) - 2at_1(t_1+t_2) = 2x - 2at_1^2$

$\Rightarrow \quad y(t_1+t_2) - 2(x+ak) = 0$

$\Rightarrow \quad L_1 + \lambda L_1 = 0$, which is a family of straight line

For fixed point $L_1 = 0$ and $L_2 = 0$;

$y = 0$ and $x + ak = 0$

∴ It always passes through a fixed point $(-ak, 0)$.

If $k = -1$; then $t_1 t_2 = -1$ fixed point $(a, 0)$ = focus. It is a true result.

ILLUSTRATION 29

If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B and $P\left(\sqrt{3}, 0\right)$, then find the value of $PA.PB$

SOLUTION

Given line is $y = \sqrt{3}x - 3$, then $\theta = 60^\circ$ $\quad P(\sqrt{3}, 0)$

Rewriting the equation in parametric form, we get,

$$\frac{x - \sqrt{3}}{\cos 60^\circ} = \frac{y - 0}{\sin 60^\circ} = r$$

Any point on the line is $\left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$

Substituting in $y^2 = x + 2$ as line cuts parabola at points A and B, we get, $\frac{3r^2}{4} = \sqrt{3} + \frac{r}{2} + 2$

$$\Rightarrow \quad 3r^2 = 4\sqrt{3} + 2r + 8$$

$$\Rightarrow \quad 3r^2 - 2r - 8 - 4\sqrt{3} = 0$$

Let, $PA = r_1$ and $PB = r_2$

We get, $PA.PB = r_1 r_2 = \frac{4\sqrt{3} + 8}{3}$

ILLUSTRATION 30

If the line $y = x + 2$ does not intersect any member of family of parabolas $y^2 = ax$, $(a \in R^+)$ at two distinct point, then find the maximum value of latus rectum of parabola.

SOLUTION

Given parabola is $y^2 = ax$.

Given line is $y = x + 2$. Let us solve both curves.

$$(x + 2)^2 - ax = 0$$

$$\Rightarrow \quad x^2 + x(4 - a) + 4 = 0$$

They do not intersect at two distinct points, then $D \le 0$.

$$\Rightarrow \quad (4 - a)^2 - 16 \le 0$$

$$\Rightarrow \quad a^2 - 8a \le 0$$

$$\Rightarrow \quad a(a - 8) \le 0 \qquad (\because a \in R^+)$$

We get $a \le 8$

Length of latus rectum $= a_{\max} = 8$.

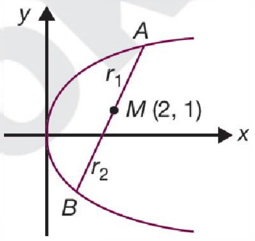
ILLUSTRATION 31

Find the length of the chord of the parabola $y^2 = x$, which is bisected at the point (2, 1).

SOLUTION

Use parametric through (2, 1)

$$\text{Equation of } AB = \frac{x - 2}{\cos\theta} = \frac{y - 1}{\sin\theta} = r,$$



Point $(2 + r\cos\theta, 1 + r\sin\theta)$ lies on parabola $y^2 = x$, then $(1 + r\sin\theta)^2 = 2 + r\cos\theta$

$\Rightarrow$ $r^2\sin^2\theta + r(2\sin\theta - \cos\theta) - 1 = 0$ has roots r_1 and r_2, for $A(r_1 > 0)$, $B(r_2 < 0)$

Here, $r_1 + r_2 = 0$

We get, $\tan\theta = \frac{1}{2}$

Now, compute $AB = |r_1 - r_2| = \sqrt{(r_1 + r_2)^2 - 4r_1 r_2}$

$$= \sqrt{0 + 4\operatorname{cosec}^2\theta} = \sqrt{4(1 + 4)} = \sqrt{20} = 2\sqrt{5}$$

Alternate

Let the points are $A\left(t_1^2, t_1\right)$, $B\left(t_2^2, t_2\right)$ on parabola $y^2 = x$

Midpoint of AB is (2, 1).

$$t_1^2 + t_2^2 = 4,\ t_1 + t_2 = 2 \text{ on square } t_1^2 + t_2^2 + 2t_1 t_2 = 4$$

$\Rightarrow t_1 t_2 = 0$

$$\ell = AB = \sqrt{\left(t_1^2 - t_2^2\right)^2 + (t_1 - t_2)^2}$$

$$= |t_1 - t_2|\sqrt{(t_1 + t_2)^2 + 1}$$

$$= \sqrt{(t_1 + t_2)^2 - 4t_1 t_2}\sqrt{(t_1 + t_2)^2 + 1}$$

$$= \sqrt{4 - 0}\sqrt{4 + 1} = 2\sqrt{5}$$

Test Your Concepts-II

(Solutions on page H.140)

1. Find the locus of trisection point of any double ordinate of $y^2 = 4ax$.
2. If the line $4y - 3x - 8 = 0$ cuts the parabola $x^2 + y - 4 = 0$ at A and B, then find $PA.PB$, where $P(0, 2)$.
3. PQ is the latus rectum of the parabola $y^2 = 4x$. O is the vertex. A rectangle $ABCD$ has two vertices namely, A and B, on the arc POQ and C and D on the segment PQ. If the perimeter of the rectangle is $\frac{11}{2}$ units, then find its area.
4. $y = \sqrt{3}x + \lambda$ is drawn through focus S of the parabola $y^2 = 8x + 16$. If two intersection points of the given line and the parabola are A and B such that perpendicular bisector of AB intersect the x-axis at P, then find the length of PS.
5. If the focal chord to $y^2 = 16x$ is a tangent to $(x-6)^2 + y^2 = 2$, then find the slope of focal chord.
6. If AFB is a focal chord of the parabola $y^2 = 4ax$ and $AF = 4$, $FB = 5$, then find the latus rectum of the parabola.
7. Find the number of point(s) (x, y) (where x and y both are perfect squares of integers) on the parabola $y^2 = px$, such that p being a prime number.
8. If the point $(1, 2)$ is one extremity of focal chord of parabola $y^2 = 4x$, then find the length of this focal chord.
9. Find the number of focal chords of length $\frac{4}{7}$ units in the parabola $7y^2 = 8x$.

Answer Keys

1. $9y^2 = 4ax$
2. 25/8
3. 3/2
4. 16/3
5. ± 1
6. 80/9
7. 1
8. 4
9. Zero

EQUATION OF TANGENT

(i) Tangent in Point Form

If (x_1, y_1) be a point on the parabola $y^2 = 4ax$, then the equation of the tangent at P is

$$T = 0 \text{ or } yy_1 = 2a(x + x_1).$$

(ii) Tangent in Slope Form

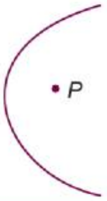
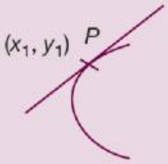
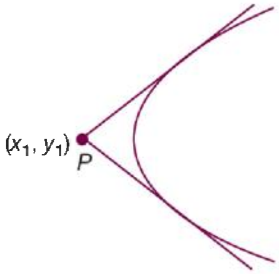
$y = mx + \frac{a}{m}, m \neq 0$ will always be a tangent to the parabola $y^2 = 4ax$ at point of contact $\left(\frac{a}{m^2}, \frac{2a}{m}\right), (m \neq 0)$.

(iii) Tangents in Parametric Form

Equation of tangent at point $P\left(at^2, 2at\right)$ on the parabola $y^2 = 4ax$ is given by $yt = x + at^2$.

Working Rule

- For parabola $y^2 = 4ax$, equation of tangent $y = mx + \frac{a}{m}$ at point $P\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
- For parabola $x^2 = 4ay$, equation of tangent $x = my + \frac{a}{m}$ at $P\left(\frac{2a}{m}, \frac{a}{m^2}\right)$
(Interchange $x \Leftrightarrow y$ in result from parabola $y^2 = 4ax$)
- For parabola $(y - k)^2 = 4a(x - h)$,
equation of tangent $(y - k) = m(x - h) + \frac{a}{m}$
- For parabola $(x - h)^2 = 4a(y - k)$,
equation of tangent $(x - h) = m(y - k) + \frac{a}{m}$

Location of Point	No. of Tangent	Equation of Tangent
(i) Inside ($S_1 < 0$)  	0	Does not exist
(ii) ($S_1 = 0$) Point is on the parabola  	1	**Ist Method (Preferred)** Equation of tangent at point (x_1, y_1) on parabola $y^2 = 4\,ax$ $T = 0$ $yy_1 - 2a(x + x_1) = 0$ **IInd Method** Let tangent $(y - y_1) = m(x - x_1)$ Solve it with equation of parabola Now, by making $D = 0$ on either in quadratic in x or y, we get m.
(iii) Outside ($S_1 > 0$)  	2	**Ist Method (Preferred)** Let, tangent $y - y_1 = m(x - x_1)$ Solve it with equation of parabola Now, by making $D = 0$ on either in quadratic in x or y, we get two values of m_1 and m_2. **IInd Method** Equation of pair of tangents $SS_1 = T^2$ Here, $S : y^2 - 4ax$ $S_1 \colon y_1^2 - 4ax_1$ $T \colon yy_1 - 2a(x + x_1)$

Remark

If general two-degree equation

$$S\colon ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

For point $P(x_1, y_1)$, we have

$$S_1\colon ax_1{}^2 + 2hx_1y_1 + by_1{}^2 + 2gx_1 + 2fy_1 + c = 0$$

For $T = 0$, we need to replace,

$$x^2 \to xx_1,\ y^2 \to yy_1,\ xy \to \frac{1}{2}(xy_1 + x_1y),$$

$$x \to \frac{1}{2}(x + x_1), y \to \frac{1}{2}(y + y_1), c \to c$$

$$T = axx_1 + byy_1 + h(xy_1 + x_1y) + g(x + x_1) + f(y + y_1) + c = 0$$

ILLUSTRATION 32

For the parabola $y^2 = 9x$, write the equation of tangent from the following points.

(i) Point $P(1, 3)$

(ii) Point $Q(4, 10)$

SOLUTION

(i) Point $P(1, 3)$

First, we have to check the location of point $P(1, 3)$.

$S_1 = 9 - 9 = 0$; it lies on parabola

Now, equation of tangent $T = 0$

$$yy_1 = \frac{9}{2}(x + x_1) \Rightarrow 3y = \frac{9}{2}(x + 1)$$

$$\Rightarrow 2y = 3x + 3$$

(ii) Point $Q(4, 10)$

Location of given point (4, 10)

$S_1 = 100 - 9(4) = 64 > 0$, which is outside the parabola.

Let the tangent is $y = mx + \frac{9}{4m}$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

It passes through point (4, 10) and then we get,

$$10 = 4m + \frac{9}{4m}$$

$$\Rightarrow 16m^2 - 40m + 9 = 0$$

$$\Rightarrow 16m^2 - 4m - 36m + 9 = 0$$

$$\Rightarrow 4m(4m - 1) - 9(4m - 1) = 0$$

$$\Rightarrow (4m - 1)(4m - 9) = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

Equation of tangents is $y = \frac{x}{4} + 9$.

$$\Rightarrow x - 4y + 36 = 0 \text{ at } (36, 18)$$

and $y = \frac{9x}{4} + 1$

$$\Rightarrow 9x - 4y + 4 = 0 \text{ at } \left(\frac{4}{9}, 2\right)$$

ILLUSTRATION 33

Prove that the straight line $\ell x + my + n = 0$ touches the parabola $y^2 = 4ax$, if $\ell n = am^2$.

SOLUTION

The given line is $\ell x + my + n = 0$.

$$\Rightarrow \quad y = \frac{-\ell}{m}x - \frac{n}{m}$$

Comparing this line with

$$y = Mx + c \qquad \ldots(1)$$

$$\therefore \quad M = -\frac{\ell}{m} \text{ and } c = \frac{-n}{m}$$

The line (1) will touch the parabola $y^2 = 4ax$, if

$$c = \frac{a}{M} \Rightarrow cM = a$$

$$\Rightarrow \quad \left(-\frac{n}{m}\right)\left(-\frac{\ell}{m}\right) = a \Rightarrow \ell n = am^2$$

ILLUSTRATION 34

Identify the relative position of parabola $x^2 = 64y$ and straight line $x - y + 1 = 0$.

SOLUTION

By solving both curves, we get, $x^2 = 64(x + 1)$

$$\Rightarrow \quad x^2 - 64x - 64 = 0$$

$$D = (-64)^2 - 4(1)(-64) > 0$$

Hence, it is a chord of parabola.

ILLUSTRATION 35

Find the length of the chord intercepted by the line $x + y = 5$ on the parabola $y = x^2 + 3x$.

SOLUTION

Let solve both curves $y = x^2 + 3x = 5 - x$

$$\Rightarrow \quad x^2 + 4x - 5 = 0$$

$$\Rightarrow \quad (x + 5)(x - 1) = 0$$

We get, $x = -5$ and $y = 10$

$\therefore$ Point $A(-5, 10)$,

$x = 1, y = 4$

$\therefore$ Point $B(1, 4)$

Length of chord $AB = \sqrt{36 + 36} = 6\sqrt{2}$ units.

ILLUSTRATION 36

Find the equation of tangent to parabola $y^2 = 8x$, which is inclined at an angle 45° to line $y = 3x + 5$.

SOLUTION

Let the slope of tangent be m.

Angle between tangent and given line (slope 3) is 45°.

$$\tan 45^\circ = \left|\frac{m - 3}{1 + 3m}\right| = 1$$

$$\Rightarrow \quad m - 3 = \pm(1 + 3m)$$

We get, $m = -2$ and $m = \frac{1}{2}$

Now, the required tangent is $y = mx + \frac{a}{m}$

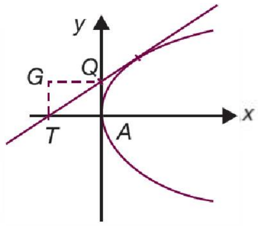
$$y = -2x - 1 \Rightarrow 2x + y + 1 = 0$$

and $y = \frac{x}{2} + 4 \Rightarrow x - 2y + 8 = 0$

ILLUSTRATION 37

A tangent drawn at any point on the parabola $y^2 = 4ax$ intersects the co-ordinate axes at T and Q, respectively. If the parallelogram $TAQG$ is completed, then find the locus of G, where A is the vertex of the parabola.

SOLUTION



Let, tangent $y = mx + \frac{a}{m}$

Tangent cuts co-ordinate axes at points $T\left(\frac{-a}{m^2}, 0\right)$ and $Q\left(0, \frac{a}{m}\right)$

$TAQG$ is rectangle $G(h, k) = \left(\frac{-a}{m^2}, \frac{a}{m}\right)$

We get, $h = -\frac{a}{m^2}$ and $k = \frac{a}{m}$

$$h = \left(\frac{-a}{\left(a/k\right)^2}\right) = -\frac{k^2}{a} \text{ or } k^2 + ah = 0$$

Locus of point (h, k), we get $y^2 + ax = 0$.

ILLUSTRATION 38

Find the value of λ, if $y = 2x + \lambda$ touches the parabolas.

(i) $y = x - x^2$

(ii) $(y - 1)^2 = 4(x - 2)$

SOLUTION

(i) Given parabola $y = x - x^2$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 = -\left(y - \frac{1}{4}\right)$$

Tangent $\left(x - \frac{1}{2}\right) = m\left(y - \frac{1}{4}\right) - \frac{1}{4m}$

Given line (tangent) $y = 2x + \lambda$, we get $m = \frac{1}{2}$

Now, equation of tangent $x - \frac{1}{2} = \frac{1}{2}\left(y - \frac{1}{4}\right) - \frac{1}{2}$

$$\Rightarrow 2x - 1 = y - \frac{1}{4} - 1$$

$$\Rightarrow y = 2x + \frac{1}{4} \qquad \therefore \lambda = \frac{1}{4}$$

(ii) Given parabola $(y - 1)^2 = 4(x - 2)$

Equation of tangent $y - 1 = m(x - 2) + \frac{1}{m}$

Given tangent is $y = 2x + \lambda$ and we can get $m = 2$

$$y - 1 = 2(x - 2) + \frac{1}{2}$$

$$y = 2x - \frac{5}{2}$$

$\therefore$ We get, $\lambda = -\frac{5}{2}$

ILLUSTRATION 39

A tangent is drawn to the parabola $y^2 = 4x$ at the point 'P', whose abscissa lies in the interval [1, 4], then find the maximum possible area of the triangle formed by the tangent at 'P', ordinate of the point 'P' and the x axis.

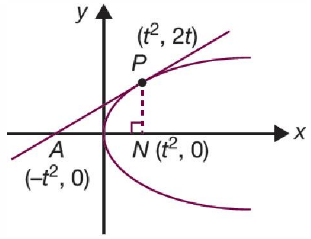
SOLUTION

Tangent $ty = x + t^2$, $\tan\theta = \frac{1}{t}$

Required area

$$S = \frac{1}{2}(AN)(PN) = \frac{1}{2}(2t^2)(2t)$$

$$S = 2t^3 = 2(t^2)^{\frac{3}{2}}$$



i.e., $t^2 \in [1, 4]$ and S_{max} occurs, when $t^2 = 4$

$\Rightarrow$ $S_{max} = 16$

ILLUSTRATION 40

From an external point P, a pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 and θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then find the locus of P.

SOLUTION

Equation of tangent is $y = mx + \frac{1}{m}$

It passes though point $P(h, k)$, we get $m^2h - mk + 1 = 0$

$$m_1 + m_2 = \frac{k}{h}\ ;\ m_1 m_2 = \frac{1}{h}$$

Given, $\theta_1 + \theta_2 = \frac{\pi}{4}$

$$\Rightarrow \quad \frac{m_1 + m_2}{1 - m_1 m_2} = 1$$

$$\Rightarrow \quad \frac{k}{h} = 1 - \frac{1}{h} \Rightarrow y = x - 1$$

ILLUSTRATION 41

Find the common tangent to the parabola $y^2 = 8x$ and the curve $xy = -1$.

SOLUTION

Let the equation of tangent to the parabola be $y^2 = 8x$.

$y = mx + \frac{2}{m}$, which is also tangent to $xy = -1$

Now, solving tangent and curve $xy = -1$, we get $x\left(mx + \frac{2}{m}\right) = -1$

$$\Rightarrow \quad mx^2 + \frac{2x}{m} + 1 = 0$$

Now, making discriminant $D = 0$

$$\Rightarrow \quad \frac{4}{m^2} - 4m = 0 \Rightarrow m = 1$$

∴ Required common tangent is $y = x + 2$.

ILLUSTRATION 42

Find the locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other.

SOLUTION

Equation of tangent is $y = mx + \frac{a}{m}$, which passes through (h, k), then $m^2h - km + a = 0$.

As given, let the slopes are m and $2m$.

We get, $3m = \frac{k}{h}$; $2m^2 = \frac{a}{h}$

$$(2)\left(\frac{k}{3h}\right)^2 = \frac{a}{h} \text{ or } 2k^2 = 9ah$$

$$\Rightarrow \quad y^2 = \frac{9}{2}ax$$

ILLUSTRATION 43

Find the locus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that (i) $m_1 + m_2 = m_0$ (constant) (ii) $\theta_1 + \theta_2 = \theta_0$ (constant), where θ_1 and θ_2 are inclination angle of tangents made with positive x-axis.

SOLUTION

Let, tangent $y = mx + \frac{a}{m}$

It passes through $P(h, k)$

$$k = mh + \frac{a}{m}$$

$\Rightarrow \quad m^2h - mk + a = 0$ has roots m_1 and m_2

$$m_1 + m_2 = \frac{k}{h}\ ;\ m_1 m_2 = \frac{a}{h}$$

(i) $m_1 + m_2 = m_0$ (constant)

$$m_1 + m_2 = m_0 = \frac{k}{h}$$

$\Rightarrow k = m_0 h$

Required locus, $y = m_0 x$

(ii) $\theta_1 + \theta_2 = \theta_0$ (constant)

$$\tan(\theta_1 + \theta_2) = \tan\theta_0$$

$$\Rightarrow \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \tan\theta_0$$

$$\Rightarrow m_1 + m_2 = \tan\theta_0(1 - m_1 m_2)$$

$$\Rightarrow \frac{k}{h} = \tan\theta_0\left(1 - \frac{a}{h}\right)$$

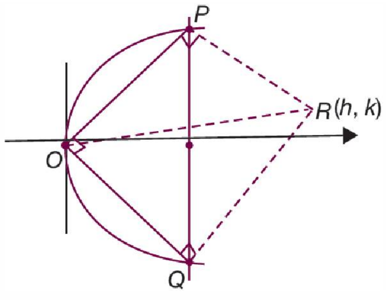
$$k = \tan\theta_0\,(h - a)$$

∴ Required locus, $y = \tan\theta_0(x - a)$

ILLUSTRATION 44

Two perpendicular chords OP and OQ are drawn from the vertex of the parabola $y^2 = 4ax$. If the rectangle $OPRQ$ is completed, then find the locus of R.

SOLUTION



Let a line through origin is $y = mx$. (line OP)

Parabola $y^2 = 4ax$

$$y^2 = 4\frac{ay}{m}$$

On solving we get,

$$y = 0;\ y = \frac{4a}{m}$$

Point of intersection are $O(0, 0)$ and $P\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$

$OP \perp OQ$, now the slope of OQ is $-\frac{1}{m}$

So point $P\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$ and $Q(4am^2, -4am)$, let point $R(h, k)$

Midpoint of OR = Midpoint of PQ

$$\left(\frac{h+0}{2}, \frac{k+0}{2}\right) = \left(\frac{\frac{4a}{m^2}+4am^2}{2}, \frac{\frac{4a}{m}-4am}{2}\right)$$

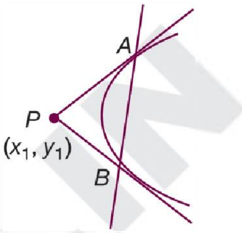
We get, $h = 4a\left(m^2 + \frac{1}{m^2}\right)$ and $k = 4a\left(\frac{1}{m} - m\right)$

On squaring it, we get $k^2 = 16a^2\left[\frac{1}{m^2} + m^2 - 2\right]$

$$k^2 = 4a(h - 8a)$$

$\therefore$ Required locus is $y^2 = 4a(x - 8a)$.

CHORD OF CONTACT



Chord of contact

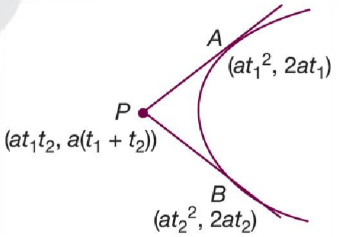
From a point $P(x_1, y_1)$, two tangents namely, PA and PB can be drawn to the parabola. The chord AB joining the points of contact A and B of the tangents from P is called the chord of contact of point $P(x_1, y_1)$ with respect to the given parabola.

For parabola, $y^2 = 4ax$

Working Rule

Equation of chord of contact is given by $T = 0$ from external point (x_1, y_1), where T: $yy_1 - 2a(x + x_1) = 0$.

* Point of Intersection of tangents at t_1 and t_2 is $(at_1t_2, a(t_1 + t_2))$



PROOF:

Equations of the tangents at the points $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ are

$yt_1 = x + at_1^2$ and $yt_2 = x + at_2^2$, respectively.

Take difference $y(t_1 - t_2) = a(t_1^2 - t_2^2)$

$\Rightarrow \quad y = a(t_1 + t_2) \qquad (\because\ t_1 \neq t_2)$

$\Rightarrow \quad yt_1 = x + at_1^2$

$\Rightarrow \quad a\,t_1(t_1 + t_2) = x + at_1^2$

We get, $x = at_1t_2$

$\therefore \quad P\{at_1t_2, a(t_1 + t_2)\}$

$\therefore$ Tangent at point $A(t_1)$ and $B(t_2)$ on parabola meet at point $P(at_1t_2, a(t_1 + t_2))$.

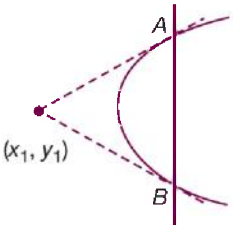
ILLUSTRATION 45

Find the point of intersection of tangents drawn at points, where straight line $x - y - 1 = 0$ cuts parabola $y^2 = 8x$.

SOLUTION

Let the given line $x - y - 1 = 0$ cuts parabola at A and B. Tangents at A and B meet at point $P(x_1, y_1)$.

From point $P(x_1, y_1)$



Equation of chord of contact $T = 0$

$\Rightarrow \quad yy_1 = 4(x + x_1)$

$\Rightarrow \quad 4x - yy_1 + 4x_1 = 0 \qquad \ldots(1)$

Given chord is $x - y - 1 = 0$

Both represent same straight line. On comparison

$$\frac{4}{1} = \frac{-y_1}{-1} = \frac{4x_1}{-1}$$

$\Rightarrow \quad y_1 = 4;\ x_1 = -1$

We get, $P(x_1, y_1)$ as $(-1, 4)$.

ILLUSTRATION 46

From a variable point on the straight line $x - 2y + 1 = 0$, a pair of tangents are drawn to parabola $y^2 = 8x$. Prove that chord of contact always passes through fixed point.

SOLUTION

Let $P(x_1, y_1)$ lies on given line then $x_1 - 2y_1 + 1 = 0$

Equation of chord of contact $T = 0$; $yy_1 = 4(x + x_1)$

$\Rightarrow \quad yy_1 = 4(x + 2y_1 - 1)$

$\Rightarrow \quad y_1(y - 8) - 4(x - 1) = 0$

$L_1 + \lambda L_2 = 0$; it is a family of line

For fixed point, $L_1 = 0 \Rightarrow y = 8$

$L_2 = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$\therefore$ The fixed point is $(1, 8)$.

ILLUSTRATION 47

Find locus of point, whose chord of contact with respect to parabola $y^2 = 4bx$ is a tangent to another parabola $y^2 = 4ax$.

SOLUTION

From point $P(h, k)$, chord of contact $T = 0$ of first parabola.

$\Rightarrow \quad yk = 2b\,(x + h)$

$\Rightarrow \quad y = \frac{2b}{k}x + \frac{2bh}{k}$, which is tangent to $y^2 = 4ax$

So, $\quad c = \frac{a}{m}$

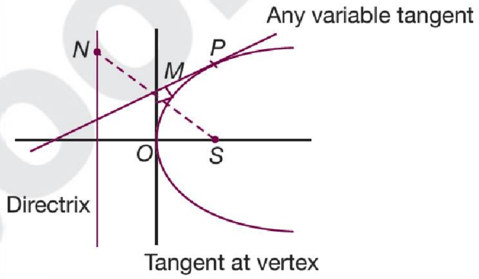
$$\frac{2bh}{k} = \frac{ak}{2b} \Rightarrow k^2 = \frac{4b^2h}{a}$$

$\therefore$ Locus of point $P\ (h, k)$ is $y^2 = \frac{4b^2}{a}x \qquad$ (Parabola)

ILLUSTRATION 48

Prove that foot of perpendicular drawn from focus on any tangent lies on tangent at vertex and image of focus with respect to any tangent lies on directrix.

SOLUTION



Equation of tangent at point $P(t)$ is $yt = x + at^2$.

$\Rightarrow \quad x - yt + at^2 = 0$

Point M: Foot of perpendicular from focus $S(a, 0)$ on variable tangent

$$\frac{x-a}{1} = \frac{y-0}{-t} = -\frac{(a - 0 + at^2)}{1+t^2} = -a$$

$M(0, at)$, which lies on tangent at vertex $(x = 0)$.

Point N: Image of focus $S(a, 0)$ relative to variable tangent

$$\frac{x-a}{1} = \frac{y-0}{-t} = \frac{-2(a - 0 + at^2)}{1+t^2} = -2a$$

$\therefore \quad N(-a, 2at)$, which lies on directrix $(x + a = 0)$.

Working Rule

- $*$ Foot of perpendicular (x, y) from point (x_1, y_1) on line $ax + by + c = 0$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

- $*$ Image of point (x_1, y_1) relative to straight line $ax + by + c = 0$ is (x, y)

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

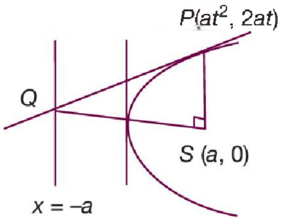
ILLUSTRATION 49

Prove that segment of tangent intercepted between point of contact and directrix subtend right angle at focus.

SOLUTION

Equation of tangent at point $P(t)$

$$yt = x + at^2$$



It intersects directrix $x + a = 0$ at point $Q\left(-a, a\left(t - \frac{1}{t}\right)\right)$

Now, product of slopes of PS and QS

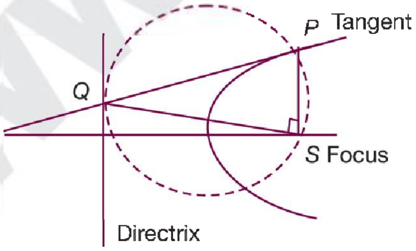
$$= m_{PS}\, m_{QS} = \left(\frac{2at - 0}{at^2 - a}\right)\left(\frac{a\left(t - \frac{1}{t}\right) - 0}{-a - a}\right) = -1$$

Hence, $\angle QSP = 90°$

ILLUSTRATION 50

By taking segment of any tangent intercepted between point of contact and directrix as a diameter, a variable circle is drawn, which always passes through a fixed point (focus).

SOLUTION



$P(t)(at^2, 2at)$

Tangent at P

$yt = x + at^2$ it cuts directrix $(x + a = 0)$ at point Q

then $Q\left(-a, a\left(t - \frac{1}{t}\right)\right)$

Equation of circle PQ as diameter

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad (x - at^2)(x + a) + (y - 2at)\left(y - a\left(t - \frac{1}{t}\right)\right) = 0$$

$$\Rightarrow \quad x^2 + y^2 + (a - at^2)x - a^2t^2 - y\left[2at + at - \frac{a}{t}\right] + 2a^2t\left(t - \frac{1}{t}\right) = 0$$

It cuts y-axis, $y = 0$, $x^2 + (a - at^2)x + a^2t^2 - 2a^2 = 0$

We get $x = a$, $x = at^2 - 2a$

As can see $(a, 0)$ is fixed point = Focus.

Conceptual Note(s)

- Foot of perpendicular drawn from focus on any tangent lies on tangent at vertex and image of focus with respect to any tangent lies on directrix.
- Segment of tangent intercepted between point of contact and directrix subtend right angle at focus.
- A variable circle whose diameter is intercept of tangent between point of contact and directrix, always passes through focus of parabola.

ILLUSTRATION 51

If focus of parabola is (1, 2) and feet of perpendicular on any two tangents to this parabola from focus are respectively (3, 4) and (4, 6), then find the vertex of the parabola.

SOLUTION

Feet of perpendicular on any tangent, lies on tangent at vertex have line joining (3, 4) and (4, 6) is tangent at vertex.

$$y - 4 = 2(x - 3)$$

$$\Rightarrow \quad 2x - y - 2 = 0$$

Vertex = Foot of perpendicular from focus (1, 2) on tangent at vertex

$$\frac{x - 1}{2} = \frac{y - 2}{-1} = -\frac{(2 - 2 - 2)}{4 + 1} = \frac{2}{5}$$

$\therefore$ The vertex of the parabola is $\left(\frac{9}{5}, \frac{8}{5}\right)$.

ILLUSTRATION 52

Let (2, 3) be the focus of a parabola and $x + y = 0$ and $x - y = 0$ be its two tangents, then find equation of its directrix.

SOLUTION

Mirror image of focus in the tangent of parabola lie on its directrix,

Image of $S(2, 3)$ w.r.t tangent $x + y = 0$ is $A(-3, -2)$

Image of $S(2, 3)$ w.r.t tangent $x - y = 0$ is $B(3, 2)$ directrix is AB $2x - 3y = 0$

ILLUSTRATION 53

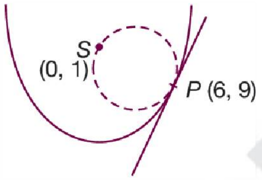
Find the equation of circle, which passes through the focus of the parabola $x^2 = 4y$ and touches it at the point (6, 9).

SOLUTION

Tangent at (6, 9) to parabola $x^2 = 4y$

$$(x)(6) = 2(y + 9)$$

$$\Rightarrow \quad y - 3x + 9 = 0$$



Equation of circle, $S + \lambda L = 0$

$$(x - 6)^2 + (y - 9)^2 + \lambda(y - 3x + 9) = 0$$

It passes through focus (0, 1).

$$36 + 64 + \lambda(1 - 0 + 9) = 0$$

$$\Rightarrow \quad \lambda = -10$$

$\therefore$ The required circle is $x^2 + 36 - 12x + y^2 + 81 - 18y - 10y + 30x - 90 = 0$ or $x^2 + y^2 + 18x - 28y + 27 = 0$.

ILLUSTRATION 54

Find the equation of tangent perpendicular to the line $x - 2y = 2$ and to the parabola $x^2 = 4y$. Also, find the point of contact.

SOLUTION

Given, parabola $x^2 = 4y$; $a = 1$

Required tangent is $x = my + \frac{1}{m}$ at $\left(\frac{2a}{m}, \frac{a}{m^2}\right)$

It is perpendicular to line $x - 2y = 2$.

$$\left(\frac{1}{m}\right)\left(\frac{1}{2}\right) = -1$$

$$\Rightarrow \quad m = -\frac{1}{2}$$

Required tangent, $x = -\frac{y}{2} - 2$

$$\Rightarrow \quad 2x + y + 4 = 0 \text{ at } (-4, 4)$$

ILLUSTRATION 55

If tangents at any point P of the parabola $x^2 = 4ay$ intersects to the parabola $y^2 = 4bx$ at points Q and R, then find the locus of midpoint of QR.

SOLUTION

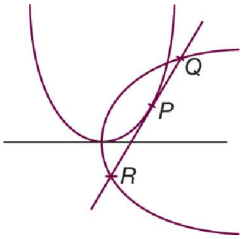
Let the midpoint of QR be $M(h, k)$.

Equation of chord QR $\quad T = S_1$

$$yk - 2b(x + h) = k^2 - 4bh$$

$$\Rightarrow \quad yk - 2bx + 2bh - k^2 = 0$$

$$\Rightarrow \quad x = \left(\frac{k}{2b}\right)y + \frac{2bh - k^2}{2b} \qquad \ldots(1)$$



Tangent to $x^2 = 4ay$ is $x = my + \frac{a}{m}$ $\qquad \ldots(2)$

Line (1) and (2) represents same straight line, on comparison, we get

$$m = \frac{k}{2b} \text{ and } \frac{a}{m} = \frac{2bh - k^2}{2b}$$

$$\Rightarrow \quad \frac{(a)(2b)}{k} = \frac{2bh - k^2}{2b}$$

$$\Rightarrow \quad k(2bh - k^2) = 4b^2a$$

Locus of $M(h, k)$ is $y(2bx - y^2) = 4ab^2$

ILLUSTRATION 56

If tangents at P and Q points of the parabola intersect at point R, then show that SP, SR and SQ are in G.P.

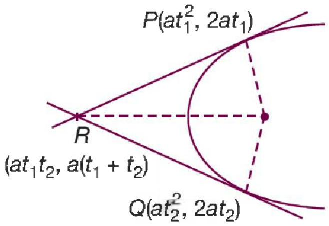
SOLUTION

Let tangents at $P(at_1^2, 2at_1^2)$

and $Q(at_2^2, 2at_2^2)$ meet at $R(at_1t_2, a(t_1+t_2))$

$$SP = a + at_1^2$$

$$SQ = a + at_2^2$$



$$SR = \sqrt{(a - at_1t_2)^2 + (a(t_1+t_2))^2}$$

$$= \sqrt{a^2 + a^2t_1^2t_2^2 - 2a^2t_1t_2 + a^2t_1^2 + a^2t_2^2 + 2a^2t_1t_2}$$

$$= \sqrt{a^2(1 + t_1^2 + t_2^2 + t_1^2t_2^2)}$$

$$= a\sqrt{(1+t_1^2)(1+t_2^2)}$$

$$SP \cdot SQ = a(1 + t_1^2) \cdot a(1 + t_2^2)$$

$$= a^2(1 + t_1^2)(1 + t_2^2)$$

$$= SR^2$$

$\therefore$ SP, SR and SQ are in G.P.

ILLUSTRATION 57

Find the locus of point of intersection of tangents to $y^2 = 4ax$, which meet at an angle α.

SOLUTION

Let the point of intersection of tangents is $P(x_1, y_1)$. Equation of pair of tangents

$$SS_1 = T^2$$

$$\Rightarrow \quad (y^2 - 4ax)(y_1^2 - 4ax_1) = (yy_1 - 2a(x + x_1))^2$$

$$= y^2y_1^2 + 4a^2(x + x_1)^2 - 4ay_1y(x + x_1)$$

$$\Rightarrow \quad 4a^2x^2 + y^2(4ax_1) - 4ay_1xy + \ell x + my + n = 0$$

If angle between tangents is α,

then $\tan\alpha = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\dfrac{2\sqrt{4a^2y_1^2 - 4a^2(4ax_1)}}{4a^2 + 4ax_1}\right|$

$$4a(x_1 + a)\tan\alpha = 4a\sqrt{y_1^2 - 4ax_1}$$

$$(x_1 + a)^2\tan^2\alpha = (y_1^2 - 4ax_1)$$

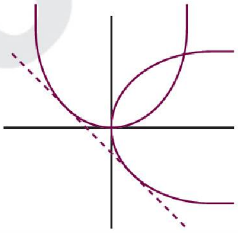
$\therefore$ Locus of required point $P(x_1, y_1)$ is $(y^2 - 4ax) = (x + a)^2\tan^2\alpha$.

ILLUSTRATION 58

Find the equation of the common tangent to $y^2 = 4ax$ and $x^2 = 4by$.

SOLUTION

Equation of tangent to $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$ ...(1)



If it is a tangent to $x^2 = 4by$, then solving them

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$\therefore$ Discriminant $= 0$ for tangential condition.

Hence, $b^2m^2 + \dfrac{ab}{m} = 0$ or $m^3 = -\dfrac{a}{b}$ or $m = -\dfrac{a^{1/3}}{b^{1/3}}$

Required common tangent is $y = -\dfrac{a^{1/3}}{b^{1/3}}x - a^{2/3}b^{1/3}$

$$\Rightarrow \quad a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$$

ILLUSTRATION 59

Find the common tangents to $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$.

SOLUTION

Any tangent to

$$y^2 = 8ax \text{ is } y = mx + \frac{2a}{m} \quad \text{...(1)}$$

If it is a tangent to $x^2 + y^2 = 2a^2$, then the condition is that perpendicular distance of (tangent) from centre (0, 0) is equal to radius $\sqrt{2}a$.

$$\left|\frac{0+0+\dfrac{2a}{m}}{\sqrt{1+m^2}}\right|=\sqrt{2}\,a$$

$$\Rightarrow \quad m^4+m^2-2=0$$

$$\Rightarrow \quad (m^2+2)(m^2-1)=0$$

Hence, $m = \pm 1$

$\therefore$ Common tangents are $y = \pm (x + 2a)$.

ILLUSTRATION 60

Prove that two straight lines, one a tangent to the parabola $y^2 = 4a(x + a)$ and the other to the parabola $y^2 = 4a'(x + a')$, are at right angles to one another and they meet on the straight line $x + a + a' = 0$. Show also that this straight line is a common chord of the two parabolas.

SOLUTION

Given, parabola $y^2 = 4a(x + a)$ …(1)

$$y^2 = 4a'(x + a') \qquad \text{…(2)}$$

Equation to the tangents of (1) and (2) are

$$y = m(x + a) + \frac{a}{m}$$

and $y = m'(x + a') + \dfrac{a'}{m'}$

$$mm' = -1$$

For Point of intersection of these tangents, take difference of tangents

$$(m - m')x + am - m'a' + \frac{a}{m} - \frac{a'}{m'} = 0$$

$$\Rightarrow \quad (m - m')x + am - m'a' + \frac{(am' - a'm)}{-1} = 0$$

$$\Rightarrow \quad (m - m')x + a(m - m') + a'(m - m') = 0$$

$$\Rightarrow \quad x + a + a' = 0$$

To get common chord, subtract equation of parabolas, We get, $x + a + a' = 0$

Hence proved.

ILLUSTRATION 61

If the circle $x^2 + y^2 = 5$ meets the parabola $y^2 = 4x$ at P and Q, then find the length of common chord PQ.

SOLUTION

The given circle $x^2 + y^2 = 5$ meets the parabola $y^2 = 4x$ at P and Q.
On solving these curves we get, $P(1, 2)$ and $Q(1, -2)$, which are coinciding with extremities of latus rectum, then the length of common chord is PQ = latus rectum = 4 units.

ILLUSTRATION 62

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the parabola $y^2 = 4ax$. If the circle with chord AB, as a diameter touches the parabola, then compute $|y_1 - y_2|$.

SOLUTION

Equation of the circle is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Solving with $y^2 = 4ax$, we get,

$$16a^2(y-y_1)(y-y_2) + (y^2-y_1^2)(y^2-y_2^2) = 0$$

$$\Rightarrow \quad (y-y_1)(y-y_2)\left[16a^2 + (y+y_1)(y+y_2)\right] = 0$$

$$\Rightarrow \quad (y+y_1)(y+y_2) + 16a^2 = 0$$

$$\Rightarrow \quad y^2 + (y_1+y_2)y + y_1y_2 + 16a^2 = 0\,.$$

The roots of the equation are equal, if

$$(y_1+y_2)^2 = 4y_1y_2 + 64a^2 \Rightarrow |y_1-y_2| = 8a\,.$$

ILLUSTRATION 63

Two parabolas have the same focus. If their directrices are the x-axis and the y-axis respectively, then find the slope of their common chord.

SOLUTION

Let the focus be $S(a, b)$. The equations of parabola are $(x - a)^2 + (y - b)^2 = x^2$ and $(x - a)^2 + (y - b)^2 = y^2$.
Using definition of parabola $PS = PM$. Common chord $s_1 - s_2 = 0$ is given by $x^2 - y^2 = 0$

$$\Rightarrow \quad y = \pm x$$

Test Your Concepts-III

(Solutions on page H.141)

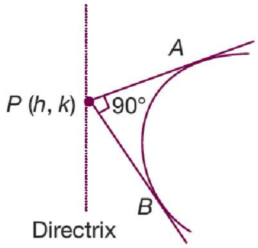
1. TP and TQ are tangents to the parabola, $y^2 = 4ax$ at P and Q. If the chord PQ passes through the fixed point $(-a, b)$, then find the locus of T.
2. Prove that chord of contact of any two tangents drawn from each point from the line $x + 2y = 4$ to the parabola $y^2 = 4x$ passes through a fixed point. Also, find the fixed point.
3. If m_1, m_2 are slopes of tangents drawn from (1, 4) to the parabola $y^2 = 4x$, then find $|m_1 - m_2|$.
4. Find the points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$.
5. A tangent is drawn to the parabola $y^2 = 4(x + 2)$ and cut the co-ordinate axis at A and B. If rectangle $OAPB$ is completed, then find the locus of P.
6. If the distances of two points P and Q from the focus of a parabola $y^2 = 4x$ are 4 and 9, respectively, then find the distance of the point of intersection of tangents at P and Q from the focus.
7. Two tangents are drawn from point (1, 4) to the parabola $y^2 = 4x$. If the angle between these tangents is $\frac{\pi}{k}$, then find k.
8. Find the locus of point of intersection of tangents to the parabola $y^2 = 4ax$, such that the angle between them being always 45°.
9. If $a_1x + by + c = 0$ and $a_2x + by + c = 0$ are two tangents to $y^2 = 8a(x - 2a)$, then find the condition.
10. If $2p^2 - 3q^2 + 4pq - p = 0$ and a variable line $px + qy = 1$ always touches a parabola, whose axis is parallel to x-axis, then find the equation of the parabola.
11. If $y = 2x - 3$ is a tangent to the parabola $y^2 = 4a\left(x - \frac{1}{3}\right)$, then find a.
12. Find the slope of tangent to $x^2 = 4y$ from $(-1, -1)$.
13. If perpendiculars are drawn on any tangent to a parabola $y^2 = 4ax$ from the points $(a \pm k, 0)$ on the axis, then find the difference of their squares.
14. If a parabola having focus at (2, 3) touches both the axes, then find the equation of its directrix.
15. Find the equation of common tangent to $x^2 = 32y$ and $y^2 = 32x$.

Answer Keys

1. $by = 2a(x - a)$
2. $(-4, -4)$
3. $2\sqrt{3}$
4. (1, 2), (4, 4)
5. $x^2 + y^2(x + 2) = 0$
6. 6
7. 3
8. $x^2 - y^2 + 6ax + a^2 = 0$
9. $a_1a_2 + b^2 = 0$
10. $(y - 4)^2 = 12(x - 2)$
11. $-14/3$
12. $\frac{-1 \pm \sqrt{5}}{2}$
13. $4ak$
14. $3x + 2y = 0$
15. $x + y + 8 = 0$

DIRECTOR CIRCLE

Locus of a point from which mutually perpendicular tangents can be drawn to the given parabola (conic) is called director circle.



For parabola, $y^2 = 4ax$

Equation of tangent, $y = mx + \frac{a}{m}$

It passes through point (h, k), then $k = mh + \frac{a}{m}$

Or $m^2h - km + a = 0$

Let it has two roots m_1 and m_2 corresponding to two tangents.

Product of slopes $= m_1m_2 = -1$

$\Rightarrow \quad \frac{a}{h} = -1$

Or $\quad h + a = 0$

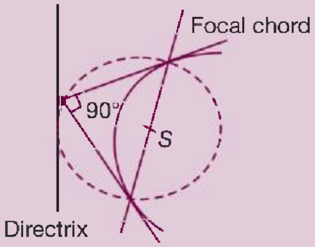
For locus of point, $P(h, k)$.

$\therefore \quad x + a = 0$ is the directrix of parabola.

We can conclude that **director circle of parabola is directrix.**

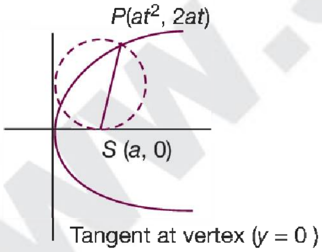
Conceptual Note(s)

- From any variable point on directrix, every chord of contact always passes through focus.
- Tangents drawn at extremities of any focal chord meet at the directrix, at right angle.
- By taking any focal chord at diameter, a circle is drawn, which touches the directrix.



- Circle whose diameter is focal distance always touches tangent at vertex.

PROOF:



Equation of circle with PS as diameter is

$$(x - a)(x - at^2) + (y - 0)(y - 2at) = 0$$

$$\Rightarrow x^2 + y^2 - a(1 + t^2)x - 2aty + a^2t^2 = 0$$

$$f^2 = a^2t^2 = c$$

$\therefore$ Circle touches the y-axis (tangent at vertex).

ILLUSTRATION 64

Find the locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$.

SOLUTION

Given equation reduces to $Y^2 = 6X$, where $X = x + 1$ and $y + 2 = Y$.

Required locus is directrix $X + A = 0$

$$\Rightarrow \quad x + 1 + \frac{3}{2} = 0$$

$$\Rightarrow \quad 2x + 5 = 0$$

POLE AND POLAR OF PARABOLA

(i) Equation of the polar of the point $P\ (x_1, y_1)$ with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a\ (x + x_1)$ $(T = 0)$.

(ii) The pole of the line $\ell x + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{\ell}, -\frac{2am}{\ell}\right)$.

NOTES:

- The polar of the focus of the parabola is the directrix.
- When the point (x_1, y_1) lies outside the parabola, the equation to its polar is the same as the equation to the chord of contact of tangent drawn from (x_1, y_1).
- When (x_1, y_1) is on the parabola, then polar is the same as the tangent at the point.
- If the polar of a point P passes through the point Q, then the polar of Q goes through P, then P and Q are called **conjugate points**.
- Two straight lines are said to be conjugated to each other with respect to a parabola, when the pole of one lies on the other.

ILLUSTRATION 65

Find the locus of a pole, if it's polar with respect to the circle $x^2 + y^2 + 2ax = 0$ touches the parabola $x^2 = 4by$.

SOLUTION

Let the pole be (x_1, y_1).

Equation of polar $T = 0$ or $xx_1 + yy_1 + a(x + x_1) = 0$

$$\Rightarrow \quad (x_1 + a)x + yy_1 + ax_1 = 0 \quad \ldots(1)$$

Tangent to parabola $x^2 = 4by$ is $x = my + \frac{b}{m}$

$$\Rightarrow \quad x - my - \frac{b}{m} = 0 \qquad \ldots(2)$$

Equations (1) and (2) lie on the same straight line.

$$\frac{x_1 + a}{1} = \frac{y_1}{-m} = \frac{ax_1}{-\frac{b}{m}}$$

We get, $m = \dfrac{-y_1}{x_1 + a} = -\dfrac{b(x_1 + a)}{ax_1}$

$$\Rightarrow \quad ax_1y_1 = b(x_1 + a)^2$$

$\therefore$ Locus of $P(x_1, y_1)$ as $b(x + a)^2 = axy$

ILLUSTRATION 66

Show that the locus of the poles of the tangents to the parabola $y^2 = 4ax$ with respect to the parabola $y^2 = 4bx$, is the parabola $ay^2 = 4b^2x$.

SOLUTION

Any tangent to the parabola

$$y^2 = 4ax \text{ is } ty = x + at^2 \qquad \ldots(1)$$

Let (α, β) be the pole of (1) with respect to the parabola $y^2 = 4bx$.

Then, Equation (1) is the polar of (α, β) with respect to $y^2 = 4bx$

$\therefore$ Equation (1) and $y\beta = 2b(x + \alpha)$ are identical.

So, by comparing these, we get,

$$\frac{t}{\beta} = \frac{1}{2b} = \frac{at^2}{2b\alpha}$$

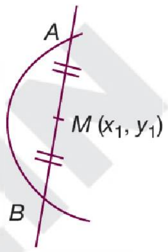
$$\therefore \quad t = \frac{\beta}{2b},\ t^2 = \frac{\alpha}{a}$$

$$\therefore \quad \left(\frac{\beta}{2b}\right)^2 = \frac{\alpha}{a}$$

$$\Rightarrow \quad a\beta^2 = 4b^2\alpha$$

$\therefore$ The equation of the required locus of the poles is $ay^2 = 4b^2x$.

CHORD WITH GIVEN MIDDLE POINT

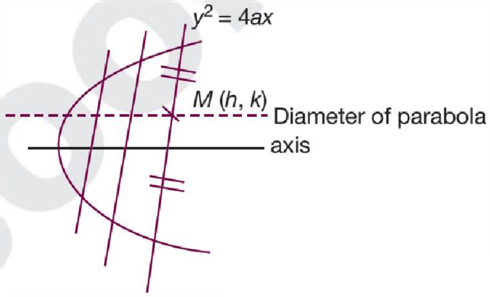


The equation of the chord of the parabola S, whose midpoint is M (x_1, y_1) is $T = S_1$,

where, $S: y^2 - 4ax$, S_1: $y_1^2 - 4ax_1$ and T: $yy_1 - 2a(x + x_1)$.

DIAMETER OF PARABOLA

Locus of midpoint of a set of parallel chords is known as diameter of that curve.



A set of parallel chords of slope m are drawn on parabola $y^2 = 4ax$

Let midpoint of chord is $M(h, k)$ then equation of chord is

$$T = S_1$$

$$\Rightarrow \quad yk - 2a(x + h) = k^2 - 4ah$$

$$\Rightarrow \quad y = \frac{2a}{k}x + k - \frac{2ah}{k}$$

Now $\dfrac{2a}{k} = m$ (slope of given set of parallel chord)

$$\Rightarrow \quad k = \frac{2a}{m}$$

Locus $h = y$,

$$y = \frac{2a}{m} \text{ Diameter of parabola, } y^2 = 4ax$$

$\therefore$ Diameter of parabola is parallel to the axis of parabola.

ILLUSTRATION 67

If the equation of the system of parallel chords of the parabola $y^2 = \frac{2}{3}x$ is given by $y + 2x + \lambda = 0$, then find the equation of the corresponding diameter.

SOLUTION

$$\text{Equation of diameter } y = \frac{2a}{m} = \frac{1/3}{-2}$$

$$\Rightarrow \quad y = -\frac{1}{6}$$

ILLUSTRATION 68

The general equation to a system of parallel chords of the parabola $y^2 = \frac{25}{7}x$ is $4x - y + k = 0$. What is the equation to the corresponding diameter?

SOLUTION

Let PQ be a chord of the system, whose equation is

$$4x - y + k = 0 \qquad \dots(1)$$

Here, k is a parameter.

Let $M(\alpha, \beta)$ be the midpoint of PQ. The locus of M is the required diameter.

The equation of the parabola is

$$y^2 = \frac{25}{7}x \qquad \dots(2)$$

Solving (1) and (2), we get, $4 \times \frac{7y^2}{25} - y + k = 0$

Or $\quad 28y^2 - 25y + 25k = 0$

Let its roots be y_1, y_2

Then, $\quad \beta = \frac{y_1 + y_2}{2} = \frac{25}{2.28} = \frac{25}{56}$

$\therefore$ The equation of the locus of $M(\alpha, \beta)$ is $y = \frac{25}{56}$

Alternate

Equation of diameter $y = \frac{2a}{m}$ or $y = \frac{25/14}{4} = \frac{25}{56}$

ILLUSTRATION 69

Find the locus of midpoint of chords of parabola $y^2 = 8x$, which passes through (p, q).

SOLUTION

Let (h, k) is midpoint, then equation of chord is $T = S_1$

Or $\quad yk - 4(x + h) = k^2 - 8h$

It passes through the point (p, q),

we get $qk - 4(p + h) = k^2 - 8h$

Or $\quad qk - 4p = k^2 - 4h$

The required locus is $h = x$ and $k = y$

$\therefore$ The locus of midpoint of the chords of parabola is $y^2 - 4x - qy + 4p = 0$.

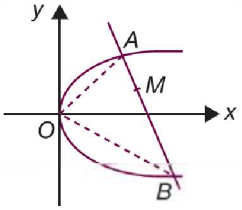
ILLUSTRATION 70

Find the locus of midpoint of the chord of the parabola $y^2 = 4ax$, which subtends right angle at the vertex of parabola.

SOLUTION

Let the midpoint of chord is $M(x_1, y_1)$.

Equation of chord $T = S_1$



$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

Or $\quad yy_1 - 2ax = y_1^2 - 2ax_1$

Homogenisation of parabola $y^2 = 4ax$ with chord.

$$y^2 = 4ax\left(\frac{yy_1 - 2ax}{y_1^2 - 2ax_1}\right)$$

$$\Rightarrow \quad 8a^2x^2 + (y_1^2 - 2ax_1)y^2 - 4ay_1xy = 0$$

It is the combined equation of OA and OB

We know that, $\angle AOB = 90°$,

Then coefficient of x^2 + coefficient of $y^2 = 0$

$$8a^2 + y_1^2 - 2ax_1 = 0$$

Required locus $\; x_1 = x$ and $y_1 = y$

$$y^2 - 2ax + 8a^2 = 0$$

$$\Rightarrow \quad y^2 = 2a(x - 4a)$$

ILLUSTRATION 71

Find the midpoint of the chord $x + y = 2$ of the parabola $y^2 = 4x$.

SOLUTION

Let the midpoint of chord is $M(x_1, y_1)$.

Equation of chord, $T = S_1$

$$yy_1 - 2(x + x_1) = y_1^2 - 4x_1$$

⇒ $yy_1 - 2x = y_1^2 - 2x_1$

⇒ $2x - yy_1 + y_1^2 - 2x_1 = 0$

Given chord is $x + y - 2 = 0$, both are same straight line

On comparison, we get, $\frac{2}{1} = -\frac{y_1}{1} = \frac{y_1^2 - 2x_1}{-2}$

$$y_1 = -2; -4 = y_1^2 - 2x_1$$

⇒ $2x_1 = 8$ or $x_1 = 4$

∴ Midpoint of the chord is (4, – 2).

ILLUSTRATION 72

Through the vertex 'O' of a parabola $y^2 = 4x$, chords OP and OQ are drawn at right angle to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point. Also, find the locus of midpoint of PQ.

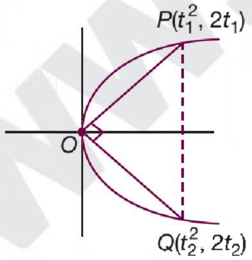
SOLUTION

Let point $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ are on parabola

OP and OQ are perpendicular.

$$m_{OP} \cdot m_{OQ} = -1$$

⇒ $\left(\frac{2t_1}{t_1^2}\right)\left(\frac{2t_2}{t_2^2}\right) = -1 \Rightarrow t_1t_2 = -4$



Equation of PQ

$$y - 2t_1 = \frac{2(t_2 - t_1)}{t_2^2 - t_1^2}(x - t_1^2) = \frac{2}{t_1 + t_2}(x - t_1^2)$$

⇒ $y(t_1 + t_2) - 2t_1^2 - 2t_1t_2 = 2x - 2t_1^2$

⇒ $y(t_1 + t_2) - 2x + 8 = 0$

⇒ $y(t_1 + t_2) - 2(x - 4) = 0$ or $\lambda L_1 + L_2 = 0$

Fixed point, $L_1 = 0 \Rightarrow y = 0$

$$L_2 = 0 \Rightarrow x - 4 = 0 \text{ or } x = 4$$

Fixed point is (4, 0).

Locus of midpoint $M(h, k)$ of PQ

$$h = \frac{t_1^2 + t_2^2}{2}; k = t_1 + t_2$$

⇒ $2h = (t_1 + t_2)^2 - 2t_1t_2$ $(\because t_1t_2 = -4)$

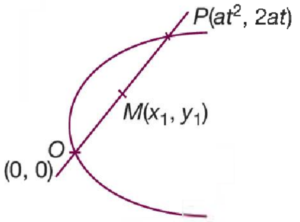
⇒ $2h = k^2 + 8 \Rightarrow k^2 = 2(h - 4)$

∴ Locus of $M(h, k)$ is $y^2 = 2(x - 4)$.

ILLUSTRATION 73

Find the locus of all the midpoints of all the chords of parabola $y^2 = 4ax$, which are drawn through the vertex.

SOLUTION



Let the midpoint of chord OP be $M(x_1, y_1)$.

We get, $x_1 = \frac{0 + at^2}{2} = \frac{at^2}{2}$ and $y_1 = \frac{0 + 2at}{2} = at$

⇒ $2x_1 = a\left(\frac{y_1}{a}\right)^2$

⇒ $y_1^2 = 2ax_1$

∴ Locus of $M(x_1, y_1)$ is $y^2 = 2ax$.

ILLUSTRATION 74

Find the equation of chord, whose ordinate of midpoint is 1 and the length of chord being 5 units to $y^2 = 4x$.

SOLUTION

Let the midpoint $(x_1, y_1) = (x_1, 1)$

Equation of chord $T = S_1$

⇒ $yy_1 - 2(x + x_1) = y_1^2 - 4x_1$ $(\because y_1 = 1)$

⇒ $y - 2(x + x_1) = 1 - 4x_1$

⇒ $y - 2x = 1 - 2x_1$

Length of chord $= 5 = \frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$, where $m = 2, a = 1$ and $c = 1 - 2x_1$

$\Rightarrow \quad 25 = 1(1 + 4)(1 - 2(1 - 2x_1))$

$\Rightarrow \quad 4x_1 = 6$

$\Rightarrow \quad x_1 = \frac{3}{2}$

$\therefore$ Equation of chord $y - 2x = 1 - 2\left(\frac{3}{2}\right) = -2$

$\Rightarrow \quad y = 2x - 2.$

Test Your Concepts-IV

(Solutions on page H.143)

1. Two parabolas $y^2 = 4a(x - \ell_1)$ and $x^2 = 4a(y - \ell_2)$ always touch one another, where the quantities l_1 and l_2 are both variables. Find the locus of their point of contact.
2. The tangents PA and PB are drawn from a point P on $y^2 = 4x$ to the circle $(x+3)^2 + (y-2)^2 = 1$.

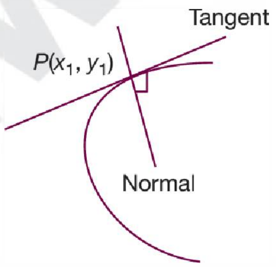
   Find the number of points of intersections of director circle of the circumcircle of ΔPAB and the directrix of the parabola.
3. If the given circle $x^2 + y^2 + 2px = 0$, $p \in R$ touches the parabola $y^2 = 4x$ externally, then find p.
4. If 'P' be (1, 0) and Q be any point on the parabola $y^2 = 8x$, then find the locus of midpoint of PQ.
5. Tangents at points P and Q to the parabola $y^2 = 4ax$ intersect at T, where $ST = 2\sqrt{10}$ and S being focus. If product of length of perpendiculars from focus to these tangents is $8\sqrt{10}$, then find the length of latus rectum of parabola.
6. Find the locus of point of intersection of two tangents to the parabola $y^2 = 4x$ such that their chord of contact subtends a right angle at the vertex.

Answer Keys

1. $xy = 4a^2$ **2.** Zero **3.** $p > 0$ **4.** $y^2 - 4x + 2 = 0$ **5.** 16
6. $x + 4 = 0$

NORMAL OF PARABOLA

A straight line perpendicular to tangent and passing through point of contact is known as normal of that curve.



Let the parabola be $y^2 = 4ax$

Equation of tangent at point $P(x_1, y_1)$ is $T = 0$

Or $\quad yy_1 = 2a(x + x_1)$

Slope of tangent $m_T = \frac{2a}{y_1}$

Tangent and normal are perpendicular lines.

Hence, slope of normal is $m_N = \frac{-y_1}{2a} = m$ (let)

Remark(s)

- Equation of normal in Cartesian form

  $y - y_1 = \frac{-y_1}{2a}(x - x_1)$ at point $P(x_1, y_1)$.
- Equation of normal in parametric form

  $y + xt = 2at + at^3$ at point $P(at^2, 2at)$.
- Equation of normal in slope form

  $y = mx - 2am - am^3$ at point $P(am^2, -2am)$.

Let us make a chart for equation of tangent and normal for parabola $y^2 = 4ax$ at mentioned point of contact for comparison purpose.

Equation of Tangent	Equation of Normal
(i) Cartesian form $T = 0$ or $yy_1 = 2a(x + x_1)$; Point (x_1, y_1)	(i) Cartesian form $(y - y_1) = \frac{-y_1}{2a}(x - x_1)$; Point (x_1, y_1)
(ii) Parametric form $yt = x + at^2$; Point $(at^2, 2at)$	(ii) Parametric form $y + xt = 2at + at^3$; Point $(at^2, 2at)$
(iii) Slope form $y = mx + \frac{a}{m}$; Point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	(iii) Slope form $y = mx - 2am - am^3$; Point $(am^2, -2am)$

Point of intersection of normal at point t_1 and t_2 of parabola

$$y^2 = 4ax \text{ is } \left(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2)\right).$$

ILLUSTRATION 75

If $x + y = k$ is normal to $y^2 = 12x$, then find k.

SOLUTION

The slope, $y = mx + c$ is normal to the parabola, if $y^2 = 4ax$

$$c = -2am - am^3.$$

From given condition, $y^2 = 12x$

$\Rightarrow \quad y^2 = (4)(3)(x)$

$\Rightarrow \quad a = 3$ and $x + y = k$

$\Rightarrow \quad y = (-1)x + k$

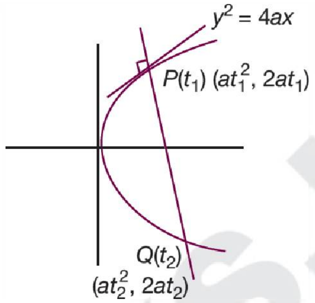
$\Rightarrow \quad m = -1$ and $c = k$

$\therefore \quad c = k = -2(3)(-1) - 3(-1)^3 = 9$

Remark

- If normal at point $P(t_1)$ cuts parabola at point $Q(t_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$

PROOF:



Slope of PQ = Slope of normal at $P(t_1)$

$$m_{PQ} = m_{\text{normal}}$$

$$\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = -t_1 \Rightarrow \frac{2}{t_1 + t_2} = -t_1 \Rightarrow t_2 = -t_1 - \frac{-2}{t_1}$$

ILLUSTRATION 76

If normal at point $P(1, 2)$ on parabola $y^2 = 4x$ cuts parabola at point Q, then find the length of normal chord.

SOLUTION

Normal at point $P(1, 2) = (at_1^2, 2at_1)$, we get $t_1 = 1$

Now, from result $t_2 = -t_1 - \frac{2}{t_1}$ we get, $t_2 = -3$.

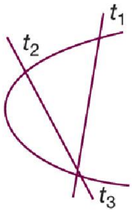
Then, point Q $(at_2^2, 2at_2) = (9, -6)$

Now, length of chord $PQ = \sqrt{8^2 + 8^2} = 8\sqrt{2}$ units

ILLUSTRATION 77

If normal drawn at point $P(t_1)$ and point $Q(t_2)$ meet parabola at point $R(t_3)$, then
Prove that (i) $t_1t_2 = 2$ (ii) $t_1 + t_2 + t_3 = 0$.

SOLUTION



Normal at $P(t_1)$ meet on parabola at point $R(t_3)$, then

$$t_3 = -t_1 - \frac{2}{t_1} \qquad \ldots(1)$$

Normal at $Q(t_2)$ meet on parabola at point $R(t_3)$, then

$$t_3 = -t_2 - \frac{2}{t_2} = -t_1 - \frac{2}{t_1} \qquad \text{using (1)}$$

$$\Rightarrow \quad t_2 - t_1 - \frac{2}{t_1} + \frac{2}{t_2} = 0 \Rightarrow (t_2 - t_1)\left(1 - \frac{2}{t_1 t_2}\right) = 0$$

$$\Rightarrow \quad t_2 \neq t_1;\ t_1 t_2 = 2$$

Now, $t_3 = -t_1 - \dfrac{2}{t_1} = -t_1 - t_2$

∴ $t_1 + t_2 + t_3 = 0$ Hence proved

ILLUSTRATION 78

Two tangents are drawn from a point to $y^2 = 4ax$. If these tangents are normals to $x^2 = 4by$, then prove that $a^2 > 8b^2$.

SOLUTION

Tangent to

$$y^2 = 4ax \text{ is } y = mx + \frac{a}{m} \qquad \ldots(1)$$

Any point on $x^2 = 4by$ is $A\ (2bt, bt^2)$ and normal at A is

$$y - bt^2 = -\frac{1}{t}(x - 2bt)$$

Or $\quad \dfrac{x}{t} + y = 2b + bt^2 \qquad \ldots(2)$

As equation (1) and (2) represent same line, compare

$$1 = \frac{m}{-1/t} = \frac{a/m}{2b + bt^2} \Rightarrow tm = -1$$

and $\quad 2bm^2 - am + b = 0$

For two tangents, m should be real and distinct discriminant > 0 or $a^2 - 8b^2 > 0 \Rightarrow a^2 > 8b^2$. Hence proved

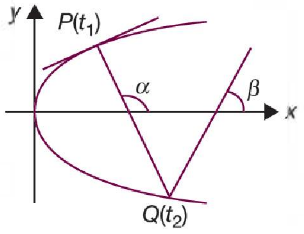
ILLUSTRATION 79

If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angles α and β, respectively with the x-axis, then find $\tan\alpha\,(\tan\alpha + \tan\beta)$.

SOLUTION

Slope of normal at $t_1 = \tan\alpha = -t_1$

Slope of normal at $t_2 = \tan\beta = -t_2$



As normal at t_1 meet parabola at t_2, then $t_2 = -t_1 - \dfrac{2}{t_1}$

Or $\quad t_1 t_2 + t_1^2 = -2$

$\Rightarrow \quad \tan\alpha \tan\beta + \tan^2\alpha = -2$

ILLUSTRATION 80

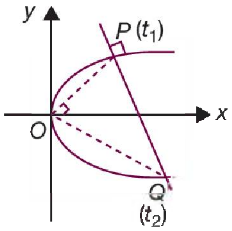
If a normal chord of the parabola $y^2 = 4x$ subtending a right angle at the vertex makes an acute angle θ with the x axis, then find θ.

SOLUTION

Equation of normal at point $P\ (t_1)$,

we get $y + t_1 x = 2at_1 + at_1^{\ 3}$, it meets parabola again at point $Q(t_2)$.

Then, $\quad t_2 = -t_1 - \dfrac{2}{t_1} \Rightarrow t_1 t_2 = -t_1^2 - 2 \qquad \ldots(1)$



Normal chord subtends right angle at vertex, which means $\quad m_{OP} \cdot m_{OQ} = -1$ or $\dfrac{4}{t_1 t_2} = -1 \qquad \ldots(2)$

Solving (1) and (2), we get, $t_1 = \sqrt{2}$ or $-\sqrt{2}$

Slope of normal is $\tan\theta = \sqrt{2}$

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$

ILLUSTRATION 81

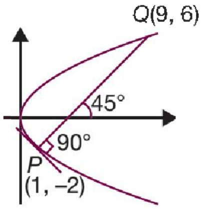
Find the length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\dfrac{\pi}{4}$ with the x-axis.

SOLUTION

Normal: $y + tx = 2t + t^3$ and slope of the normal is 1. Hence, $-t = 1$

$\Rightarrow \quad t = -1$

∴ Co-ordinates of P are $(1, -2)$.



Hence, parameter at $Q = t_2 = -t_1 - \dfrac{2}{t_1} = 1 + 2 = 3$

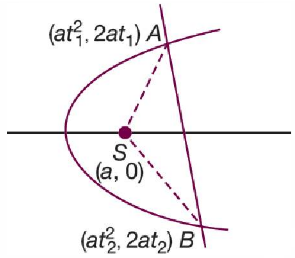
$\therefore$ Co-ordinates at Q are (9, 6).

$\therefore$ $(PQ) = \sqrt{64+64} = 8\sqrt{2}$

ILLUSTRATION 82

Prove that normal chord to $y^2 = 4ax$ at point, whose ordinate is equal to abscissa subtends 90° at focus.

SOLUTION



Let normal be $y = mx - 2am - am^3$ at $A(am^2, -2am)$

$am^2 = -2am$

$\Rightarrow$ $m = 0$ (Only one point of intersection)

$m = -2$; $A(4a, 4a) = \left(at_1^2, 2at_1\right)$

$t_1 = 2$

$t_2 = -t_1 - \dfrac{2}{t_1} = -3$

Other end $B\left(at_2^2, 2at_2\right) = (9a, -6a)$

$$m_{AS}\, m_{BS} = \left(\frac{4a-0}{4a-a}\right)\left(\frac{-6a-0}{9a-a}\right) = \frac{4}{3}\left(\frac{-6}{8}\right) = -1$$

$\angle ASB = 90°$

ILLUSTRATION 83

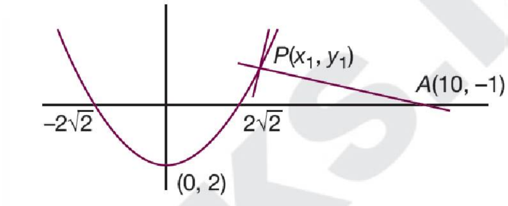
Find the equation for the line that passes through (10, –1) and is perpendicular to parabola $y = \dfrac{x^2}{4} - 2$.

SOLUTION

Parabola $4y = x^2 - 8$

Tangent at $P(x_1, y_1)$ is $2(y + y_1) = xx_1 - 8$

$$m_T = \frac{x_1}{2}$$



$\therefore$ Slope of normal $= -\dfrac{2}{x_1}$ and

slope of normal $= \dfrac{y_1+1}{x_1-10}$

$\therefore$ $\dfrac{y_1+1}{x_1-10} = -\dfrac{2}{x_1}$

$\Rightarrow$ $x_1y_1 + x_1 = -2x_1 + 20$

$\Rightarrow$ $x_1y_1 + 3x_1 = 20$

Substituting $y_1 = \dfrac{x_1^2 - 8}{4}$ (From the given equation)

$$x_1\left(\frac{x_1^2-8}{4} + 3\right) = 20$$

$\Rightarrow$ $x_1^3 + 4x_1 - 80 = 0$

$\Rightarrow$ $x_1^2(x_1 - 4) + 4x(x_1 - 4) + 20(x_1 - 4) = 0$

$\Rightarrow$ $(x_1 - 4)(x_1^2 + 4x_1 + 20) = 0$

Hence, $x_1 = 4$; $y_1 = 2$

$\therefore$ $P = (4, 2)$

Equation of PA is $y + 1 = -\dfrac{1}{2}(x - 10)$

$\Rightarrow$ $2y + 2 = -x + 10$

$\Rightarrow$ $x + 2y - 8 = 0$

ILLUSTRATION 84

If normal to $y^2 = 4ax$ at point t_1 cuts parabola at t_2, then prove that $|t_2| \geq 2\sqrt{2}$.

SOLUTION

$$t_2 = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow \quad t_2^2 = t_1^2 + \frac{4}{t_1^2} + 4$$

$$\because \quad AM \ge GM$$

$$t_1^2 + \frac{4}{t_1^2} \ge 2\sqrt{t_1^2 \cdot \frac{4}{t_1^2}} = 4;\ t_2^2 \ge 4 + 4$$

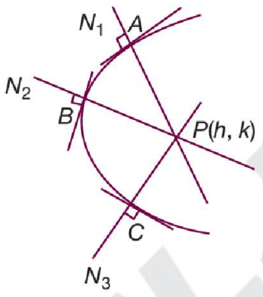
$$\therefore \quad t_2^2 \ge 8 \Rightarrow |t_2| \ge 2\sqrt{2}$$

NORMAL FROM EXTERNAL POINT (*h*, *k*)

Equation of normal $y = mx - 2am - am^3$, point $(am^2, -2am)$ on parabola $y^2 = 4ax$ and this normal passes through a point $P(h, k)$.

$$k = mh - 2am - am^3$$

$am^3 + (2a - h)\,m + k = 0$, it is a cubic in m.



Let us assume that from point $P(h, k)$, three normal are drawn to parabola and their slopes are m_1, m_2, m_3.

Normal drawn at point $A(am_1^2, -2am_1), B(am_2^2, -2am_2)$, $C(am_3^2 - 2am_3)$ are concurrent at point $P(h, k)$. So, these points A, B and C are known as **co-normal points**.

From the equation, $am^3 + 0.m^2 + (2a - h)\,m + k = 0$

We get the relation as $m_1 + m_2 + m_3 = 0$,

$$\Sigma m_1 m_2 = \frac{2a-h}{a} \text{ and } m_1 m_2 m_3 = \frac{-k}{a}$$

Conclusion

If three normals can be drawn from point (h, k) to parabola $y^2 = 4ax$, then

(i) Sum of slopes of normal is zero, it means, $m_1 + m_2 + m_3 = 0$.

(ii) Sum of ordinates of co-normal points is zero, which means $y_1 + y_2 + y_3 = -2a(m_1 + m_2 + m_3) = 0$.

(iii) Centroid of triangle formed by co-normal points lies on x-axis (axis of parabola).

$$\left(\frac{x_1+x_2+x_3}{3}, \overset{0}{\cancel{\frac{y_1+y_2+y_3}{3}}}\right) = \left(\frac{x_1+x_2+x_3}{3}, 0\right)$$

Condition for Three Normals from External Point (*h*, *k*)

From point (h, k), three normals are drawn to parabola $y^2 = 4ax\ (a > 0)$, if

(i) $h > 2a$

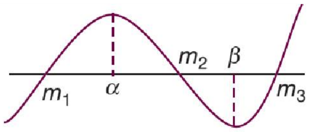
(ii) $27ak^2 < 4(h - 2a)^3$

PROOF:

For the given parabola, let the equation of normal is $y = mx - 2am - am^3$.

It passes through the point $P(h, k)$.

$$am^3 + (2a - h)m + k = 0$$



Let $f(m) = am^3 + (2a - h)m + k$ ...(1)

$$f'(m) = 3am^2 + (2a - h) = 0$$

$$\Rightarrow \quad m = \pm\sqrt{\frac{h-2a}{3a}}$$

For local extrema, $h > 2a$ must be a condition.

$$\text{Let, } \alpha = -\sqrt{\frac{h-2a}{3a}},\ \beta = \sqrt{\frac{h-2a}{3a}}$$

$$\Rightarrow \quad \beta = -\alpha$$

Now, for having three real roots of m, $f(\alpha)f(\beta) < 0$

Substituting the value of α and β in Equation (1), we get,

$$f(\alpha)\,f(-\alpha) < 0$$

$$\text{or} \quad (a\alpha^3 + (2a - h)\alpha + k)(-a\alpha^3 - (2a - h)\alpha + k) < 0$$

$$\Rightarrow \quad k^2 - (a\alpha^3 + (2a - h)\alpha)^2 < 0$$

$$\Rightarrow\ k^2-\left(\frac{h-2a}{3a}\right)\left[\frac{h-2a}{3}+2a-h\right]^2<0$$

$$\Rightarrow\ k^2-\frac{4}{27a}(h-2a)^3<0$$

We get, $27ak^2<4(h-2a)^3$

Suppose, if we want to count number of normal drawn from focus $S(a, 0)$, we can see that first condition $h>2a$ is not satisfied.

So, **only one normal can be drawn from the focus**.

ILLUSTRATION 85

Find the locus of point from which three normals can be drawn to parabola, if two normals are as follows.

(i) Mutually perpendicular

(ii) Equally inclined to x-axis

SOLUTION

(i) Equation of normal is $y=mx-2am-am^3$

It passes through the point (h, k).

$$am^3+(2a-h)\,m+k=0 \qquad \ldots(1)$$

$$m_1m_2=-1;\ m_1m_2m_3=-\frac{k}{a}$$

$\Rightarrow\ m_3=\dfrac{k}{a}$, which is a root of equation and hence, it should satisfy the relation.

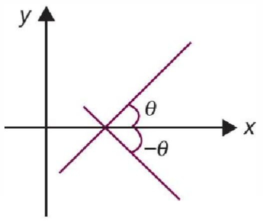
We get, $a\left(\dfrac{k}{a}\right)^3+(2a-h)\,\dfrac{k}{a}+k=0$

Locus $y\left(\dfrac{y^2}{a^2}+\dfrac{(2a-x)}{a}+1\right)=0$

$\therefore\ y=0$ or $y^2+a(2a-x)+a^2=0$

(ii) $m_1=\tan\theta;\ m_2=-\tan\theta$

$m_1+m_2=0$



$\therefore\ m_1+m_2+m_3=0$

$m_3=0$ it is root of equation, then it satisfies Equation (1).

We get, $k=0$

Hence, the locus of point (h, k) is $y=0$.

ILLUSTRATION 86

Prove that the circle passing through co-normal points always passes through the vertex of parabola.

SOLUTION

Let the equation of normal is $y=mx-2am-am^3$.

It passes through the point (h, k), which satisfies the equation of normal.

$am^3+(2a-h)\,m+k=0$, let m_1, m_2 and m_3 are roots

$$m_1+m_2+m_3=0 \qquad \ldots(1)$$

Let the circle be $x^2+y^2+2gx+2fy+c=0$.

Point of contact of normal is $(am^2, -2am)$, which lies on circle, then $a^2m^4+4a^2m^2+2gam^2-4afm+c=0$.

$a^2m^4+0.m^3+4a^2m^2+2gam^2-4afm+c=0$ has roots m_1, m_2, m_3 and m_4

$m_1+m_2+m_3+m_4=0$ (Using Equation (1), $m_1+m_2+m_3=0$)

We get $m_4=0$

Fourth point $(am_4^2, -2am_4)=(0, 0)$

$\therefore$ It is a vertex of parabola.

ILLUSTRATION 87

If three distinct and real normals can be drawn to $y^2=8x$ from the point $(a, 0)$, then find the condition.

SOLUTION

Equation of normal in terms of m is $y=mx-4m-2m^3$.

If it passes through $(a, 0)$, then $am-4m-2m^3=0$.

$$\Rightarrow\quad m\left(a-4-2m^2\right)=0$$

$$\Rightarrow\quad m=0,\ m^2=\frac{a-4}{2}$$

For three distinct normal, $a-4>0\ \Rightarrow a>4$

ILLUSTRATION 88

Find the locus of the centre of a circle, which orthogonally cuts the parabola $y^2=4ax$ at $(1, 2)$.

SOLUTION

As both are orthogonal, tangent to parabola becomes normal of circle.

The centre of such circle must lies on the tangent at (1, 2) of the parabola $y^2 = 4x$, which is $y = x + 1$.

Hence, it is the desired locus of centre of circle.

ILLUSTRATION 89

Find the common normal of the parabola $y^2 = 4x$ and $(y - 2)^2 = 4(x - 1)$.

SOLUTION

Equation of normal

$y = mx - 2am - am^3$ of parabola $y^2 = 4x$, $a = 1$

Equation of normal

$y - 2 = m(x - 1) - 2m - m^3$ of second parabola

$y = mx - 3m - m^3 + 2$

Both represent the same line.

$$\frac{1}{1} = \frac{-2m - m^3}{-3m - m^3 + 2} \Rightarrow m = 2$$

$\therefore$ Equation of normal is $y = 2x - 12$.

ILLUSTRATION 90

If the normal at P meets the axis of the parabola $y^2 = 4ax$ in G and S is the focus, then prove that $SG = SP$.

SOLUTION

Equation of the normal at $P(at^2, 2at)$ is $tx + y = 2at + at^3$.

Since it meets the axis, $y = 0$

$\Rightarrow \quad tx = 2at + at^3 \Rightarrow x = 2a + at^2$

$\therefore \quad G = (2a + at^2, 0)$, Focus $S = (a, 0)$

$$SG = a(1 + t^2)$$

$$SP = \sqrt{(at^2 - a)^2 + (2at - 0)^2} = \sqrt{(at^2 - a)^2 + 4a^2t^2}$$

$$= \sqrt{(at^2 + a)^2} = a(1 + t^2)$$

$\therefore \quad SG = SP$

ILLUSTRATION 91

If the normals from any point to the parabola $x^2 = 4y$ cuts the line $y = 2$ in points, whose abscissae are in A.P., then prove that the slopes of the tangents at the 3 co-normal points are in A.P.

SOLUTION

A point on $x^2 = 4y$ is $(2t, t^2)$ and required point be $P(x_1, y_1)$.

Equation of normal at $(2t, t^2)$ is $x + ty = 2t + t^3$ ...(1)

Given line equation is $y = 2$ ...(2)

Solving (1) and (2), we get, $x + t(2) = 2t + t^3 \Rightarrow x = t^3$

This passes through $P(x_1, y_1) \Rightarrow t^3 = x_1$ ...(3)

Let, $(2t_1, t_1^2)$ $(2t_2, t_2^2)$, $(2t_3, t_3^2)$ be the co-normal points from P.

$2t_1, 2t_2, 2t_3$ are in A.P.

$\Rightarrow \quad 4t_2 = 2(t_1 + t_3)$

$$\Rightarrow \quad t_2 = \frac{t_1 + t_3}{2}$$

Slope of tangent $= t$

$\therefore$ Slopes of the tangents t_1, t_2 and t_3 are in A.P.

ILLUSTRATION 92

Let L be an end of the latus rectum of $y^2 = 4x$. The normal at L meets the curve again at M. If the normal at M meets the curve again at N, then find the area of ΔLMN .

SOLUTION

Normal at t meets the curve again at t_1, then $t_1 = -t - \frac{2}{t}$.

Let one end of latus rectum is L (1, 2).

Here, $t = 1$, then $t_1 = -3$

So, point M is (9, –6).

If normal at point M(9, –6) cuts again curve at N, here

$$t_1 = -3, t_2 = 3 + \frac{2}{3} = \frac{11}{3}$$

We get point $N = \left(\frac{121}{9}, \frac{22}{3}\right)$

$$\text{Area of } \Delta LMN = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 9 & -6 & 1 \\ \frac{121}{9} & \frac{22}{3} & 1 \end{vmatrix} = \frac{640}{9} \text{ sq. units}$$

ILLUSTRATION 93

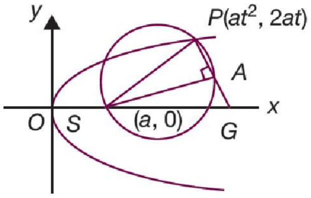
Prove that the length of the intercepts on the normal at the point $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{1+t^2}$ units.

SOLUTION

The equation of the normal PG at $P(at^2, 2at)$ to

$$y^2 = 4ax \text{ is } y + xt = 2at + at^3 \qquad \ldots(1)$$

Let the circle described on the line PS where S is the focus $(a, 0)$ as diameter cut PG at A. Join SA.



Then $\angle SAP = 90°$

$$\Rightarrow \quad PA = \sqrt{PS^2 - SA^2} \qquad \ldots(2)$$

Now, $PS = \sqrt{(at^2 - a)^2 + (2at - 0)^2}$

$$= a\sqrt{t^4 - 2t^2 + 1 + 4t^2}$$

$$= a(t^2 + 1)$$

Also, SA = length of perpendicular from S to normal(1)

$$= \frac{-at + 2at + at^3}{\sqrt{t^2+1}} = at\sqrt{t^2+1}$$

So, from Equation (2),

$$PA = \sqrt{a^2(t^2+1)^2 - a^2t^2(t^2+1)}$$

$$= a\sqrt{t^2+1}\sqrt{t^2+1-t^2} = a\sqrt{1+t^2}$$

ILLUSTRATION 94

Let (h, k) be the point on line $x = 3$. If three real distinct normal cannot be drawn from (h, k) to parabola $y^2 = 4x$, then find the condition.

SOLUTION

From (h, k), three real and distinct normal cannot be drawn if, $27ak^2 \geq 4(h-2a)^3$

$$27k^2 \geq 4(h-2)^3$$

Put, $h = 3$, we get, $\left(k - \dfrac{2}{3\sqrt{3}}\right)\left(k + \dfrac{2}{3\sqrt{3}}\right) \geq 0$

$$\Rightarrow \quad k \in \left(-\infty, \frac{-2}{3\sqrt{3}}\right] \cup \left[\frac{2}{3\sqrt{3}}, \infty\right)$$

ILLUSTRATION 95

Prove that two parabolas $y^2 = 4ax$ and $y^2 = 4c(x-b)$ cannot have a common normal, other than the axis, unless $\dfrac{b}{(a-c)} > 2$.

SOLUTION

Equation of normal to $y^2 = 4ax$ is

$$y - 2at = -t(x - at^2) \qquad \ldots(1)$$

Equation of normal to $y^2 = 4c(x-b)$

$$y - 2ct_1 = -t_1[(x-b) - ct_1^2] \qquad \ldots(2)$$

For common normal, compare Equations (1) and (2)

$$\frac{1}{1} = \frac{t}{t_1} = \frac{at^3 + 2at}{bt_1 + ct_1^3 + 2ct_1}$$

$$\Rightarrow \quad t = t_1 \text{ and } at^3 + 2at = bt_1 + ct_1^3 + 2ct_1$$

However, $t = 0$, which gives common normal as the axis of parabola.

In our case, $(a-c)t^2 = 2c - 2a + b$

Or $\quad t^2 = \dfrac{b}{a-c} - 2$

Normal cannot be axis

$$\Rightarrow \quad t \neq 0$$

$$t^2 > 0$$

$$\Rightarrow \quad \frac{b}{a-c} - 2 > 0$$

$$\Rightarrow \quad \frac{b}{a-c} > 2$$

Hence proved

ILLUSTRATION 96

Prove that three normals can be drawn from the point $(c, 0)$ to the parabola $y^2 = x$, if $c > \frac{1}{2}$ and then one of the normals is always the axis of the parabola. Also, find c for which the other two normals will be perpendicular to each other.

SOLUTION

For three normal from point (h, k) on parabola $y^2 = 4ax$ are possible drawn if

(i) $h > 2a$

(ii) $27ak^2 < 4(h - 2a)^3$

Here point $(c, 0)$; $h = c$; $k = 0$; $4a = 1 \Rightarrow a = \frac{1}{4}$

(i) $c > \frac{1}{2}$ (ii) $27\left(\frac{1}{4}\right) 0 < 4\left(c - \frac{1}{2}\right)^3 \quad \Rightarrow c > \frac{1}{2}$

Equation of normal, $y = mx - 2am - am^3$

It passes through $(c, 0)$

$$0 = mc - 2am - am^3$$

$$m(c - 2a - am^2) = 0$$

$$m = 0,\ y = 0\ (x\text{-axis})$$

$$c - 2a - am^2 = 0$$

$$c - \frac{1}{2} - \frac{1}{4}m^2 = 0$$

$$\Rightarrow m^2 - 4c + 2 = 0$$

$$m_1m_2 = -1 = 2 - 4c$$

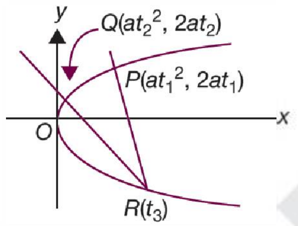
$$\Rightarrow 4c = 3 \Rightarrow c = \frac{3}{4}$$

ILLUSTRATION 97

P, Q are the points t_1, t_2 on the parabola $y^2 = 4ax$. The normals at P, Q meet on the parabola. Show that the midpoint of PQ lies on the parabola $y^2 = 2a(x + 2a)$.

SOLUTION

Let normals at $P(t_1)$ and $Q(t_2)$ meet at point $R(t_3)$.



$$t_3 = -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$$\Rightarrow \quad t_2 - t_1 - \frac{2(t_2 - t_1)}{t_1t_2} = 0 \Rightarrow \frac{(t_1t_2 - 2)(t_2 - t_1)}{t_1t_2} = 0$$

$$t_2 \neq t_1 \text{ and } t_1t_2 = 2$$

Now, midpoint of PQ is

$$h = \frac{at_1^2 + at_2^2}{2} = \frac{at_1^2 + \frac{4a}{t_1^2}}{2} \qquad (\because t_1t_2 = 2)$$

$$\text{Or} \quad h = \frac{a(t_1^4 + 4)}{2t_1^2}$$

$$k = \frac{2a(t_1 + t_2)}{2} = a\left(t_1 + \frac{2}{t_1}\right) = a\left(\frac{t_1^2 + 2}{t_1}\right)$$

(using $t_1t_2 = 2$)

Putting these values in $y^2 = 2a\,(x + 2a)$, we get,

$$\frac{a^2(t_1^2 + 2)^2}{t_1^2} = 2a\left[\frac{a(t_1^4 + 4)}{2t_1^2} + 2a\right]$$

$$\Rightarrow \quad \frac{a^2(t_1^2 + 2)^2}{t_1^2} = \frac{a^2}{t_1^2}(t_1^4 + 4 + 4t_1^2) = \frac{a^2}{t_1^2}(t_1^2 + 2)^2$$

Thus, L.H.S. = R.H.S.

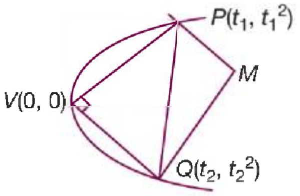
Hence, midpoint of PQ lies on $y^2 = 4a\,(x + 2a)$.

ILLUSTRATION 98

A variable chord PQ of the parabola $y = x^2$ subtends a right angle at the vertex. Find the locus of points of intersection of the normals at P and Q.

SOLUTION

The vertex V of the parabola is $(0, 0)$ and any point on $y = x^2$ has the co-ordinates (t, t^2).



So, let us take $P(t_1, t_1^2)$, $Q(t_2, t_2^2)$ and $\angle PVQ = 90°$

As 'm' of $VP = \dfrac{t_1^2 - 0}{t_1 - 0} = t_1$

And 'm' of $VQ = \dfrac{t_2^2 - 0}{t_2 - 0} = t_2$

$\therefore \quad VP \perp VQ$

$\therefore \quad t_1 t_2 = -1 \qquad \ldots(1)$

Equation of tangent at $P(t_1, t_1^2)$ is $\dfrac{1}{2}(y + y_1) = xx_1$

Slope of tangent $m_T = 2x_1 = 2t_1$

Slope of normal $m_N = -\dfrac{1}{2t_1}$

$\therefore$ The equation of the normal at $P(t_1, t_1^2)$ is

$$y - t_1^2 = \frac{-1}{2t_1}(x - t_1)$$

Or $\quad 2t_1 y + x = 2t_1^3 + t_1 \qquad \ldots(2)$

Similarly, the equation of the normal at $Q(t_2, t_2^2)$ is

$$2t_2 y + x = 2t_2^3 + t_2 \qquad \ldots(3)$$

Solving (2) and (3), we get point M

$$2y = 2(t_1^2 + t_1t_2 + t_2^2) + 1$$

$$x = -2t_1t_2(t_1 + t_2) = 2(t_1 + t_2)$$

For locus of point M

$$2y = 2\{(t_1 + t_2)^2 - t_1t_2\} + 1$$

$$= 2\left\{\left(\frac{1}{2}x\right)^2 + 1\right\} + 1$$

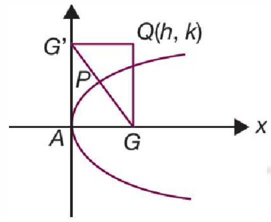
$\therefore$ The equation of the required locus is

$2y = \dfrac{x^2}{2} + 3$ or $x^2 = 2(2y - 3)$, which is a parabola.

ILLUSTRATION 99

The normal at any point 'P' meets the x-axis in G and the tangent at the vertex in G'. If A be the vertex and the rectangle $AGQG'$ be completed, prove that the equation to the locus of Q is $x^3 = 2ax^2 + ay^2$.

SOLUTION



Normal at $P(am^2, -2am)$ is $y = mx - 2am - am^3$

It meets the axis of parabola (x-axis) at

$$y = 0,\ x = 2a + am^2 \Rightarrow G(2a + am^2, 0)$$

and the tangent at the vertex (y-axis) at

$$x = 0,\ y = -2am - am^2 \Rightarrow G'(0, -2am - am^3)$$

$\therefore \quad h = 2a + am^2$ and $k = -2am - am^3$

$$m = \frac{-k}{h}$$

Eliminating m, we get, $h^3 = 2ah^2 + ak^2$.

Now, taking the locus of $Q(h, k)$, we get, $x^3 = 2ax^2 + ay^2$.

ILLUSTRATION 100

Show that the locus of the midpoints of normal chords of the parabola $y^2 = 4ax$ is $y^4 - 2a(x - 2a)y^2 + 8a^4 = 0$.

SOLUTION

If (x_1, y_1) is the midpoint of the chord of the parabola, then $T = S_1$

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$\Rightarrow \quad 2ax - yy_1 + y_1^2 - 2ax_1 = 0 \qquad \ldots(1)$

Equation of normal of the parabola at 't' is

$$y = -tx + 2at + at^3$$

$\Rightarrow \quad tx + y - 2at - at^3 = 0 \qquad \ldots(2)$

Equation (1) and (2) are identical, then

$$\frac{2a}{t} = \frac{-y_1}{1} = \frac{y_1^2 - 2ax_1}{-2at - at^3}$$

From first two relations,

$t = -\dfrac{2a}{y_1} \qquad \ldots(3)$

From (1) and (3) relations,

$$2a(-2a - at^2) = y_1^2 - 2ax_1 \qquad \ldots(4)$$

Substituting the value of t from (3) in (4), then

$$2a\left(-2a - \frac{4a^3}{y_1^2}\right) = y_1^2 - 2ax_1$$

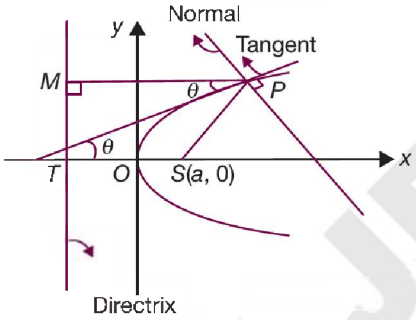
$$\Rightarrow \quad y_1^4 - 2a(x_1 - 2a)\, y_1^2 + 8a^4 = 0$$

Hence, locus of the midpoint of (x_1, y_1) is

$$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0.$$

HIGHLIGHTS OF PARABOLA

(i) The tangent and normal at any point of a parabola are the bisectors of the angles between the focal distance PS and the perpendicular on the directrix from the point PM.



PROOF:

Let $P(at^2, 2at)$ be any point of the parabola $y^2 = 4ax$.

Equation of tangent at 'P' is $ty = x + at^2$

$$\text{Slope of tangent} = \frac{1}{t}$$

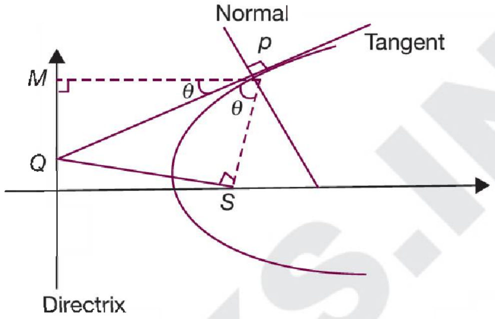
$$\text{Slope of } SP = \frac{2t}{t^2 - 1}$$

$$\therefore\ \tan\angle SPT = \frac{\left(\frac{2t}{t^2-1} - \frac{1}{t}\right)}{1 + \frac{2t}{t^2-1}\cdot\frac{1}{t}} = \frac{1}{t} = \tan\angle PTS$$

$$\therefore\ \angle SPT = \angle PTS$$

Hence, $\angle TPM = \angle SPT$

Alternate



Let us consider the triangles ΔPMQ and ΔPSQ

$PS = PM$ (Definition of parabola)

PQ is the common side of these triangles.

Segment of any tangent intercepted between point of contact and directrix subtends right angle at focus

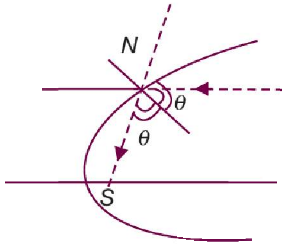
$$\angle PMQ = \angle PSQ = 90°$$

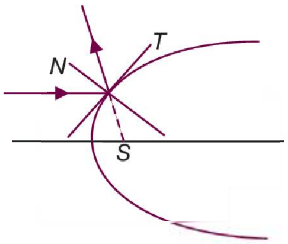
Now, we can conclude that both triangles are congruent $(\Delta PMQ \cong \Delta PSQ)$

$$\angle MPQ = \angle SPQ = \theta$$

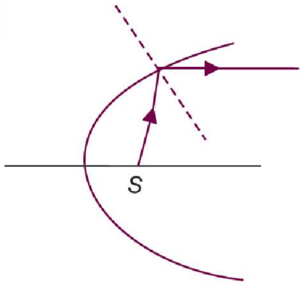
Angle bisectors of PS and PM ≡ Tangent and normal at point P

(ii) A ray travelling parallel to axis of parabola after getting reflected from parabolic surface passes through focus.



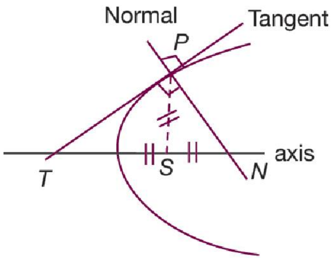


(iii) A ray originated from focus after reflection becomes parallel to the axis of parabola.

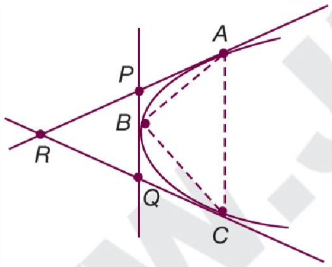


(iv) Focus is the circumcentre of triangle formed by tangent, normal at any point and axis of parabola

$SP = ST = SN$ (Focus S is circumcentre of ΔPTN)



(v) The area of triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.



PROOF:

Let the three points on the parabola be $A\left(at_1^2, 2at_1\right)$, $B\left(at_2^2, 2at_2\right)$ and $\left(at_3^2, 2at_3\right)$.

$$\text{The area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$= \left|a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$$

The intersections of the tangents at these points are $P(at_1t_2, a(t_1 + t_2))$, $Q(at_2t_3, a(t_2 + t_3))$, $R(at_1t_3, a(t_1 + t_3))$

The area of the ΔPQR formed is

$$= \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_1t_3 & a(t_1+t_3) & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$$

$\therefore$ area of $\Delta ABC = 2$(area of ΔPQR)

(vi) Orthocenter of triangle formed by tangents at co-normal points lies on directrix.

(vii) Circle circumscribing the triangle formed by co-normal points always passes through vertex of parabola.

(viii) Circle circumscribing the triangle formed by tangents always passes through focus.

ILLUSTRATION 101

The circumcircle of ΔABC is $x^2 + y^2 - 5x - 4y + 6 = 0$. If a parabola $(k + 1)y^2 = x$ have sides AB, BC and CA as tangents, then find the value of k.

SOLUTION

Circumcircle of ΔABC passes through the focus $\left(\frac{1}{4k+4}, 0\right)$ of the given parabola.

$\therefore$ Putting $y = 0$, in equation of circumcircle, we get

either $\frac{1}{4k+4} = 2$ or $\frac{1}{4k+4} = 3$

We get, $k = -\frac{7}{8}$ or $-\frac{11}{12}$

ILLUSTRATION 102

A ray of light bisects the chord $y = 2x - 1$ and $y = 2x - 3$ and it is incident on the parabola $y^2 = 4x$ (interior). Find the co-ordinates of the point, where the ray strikes the parabola for the following.

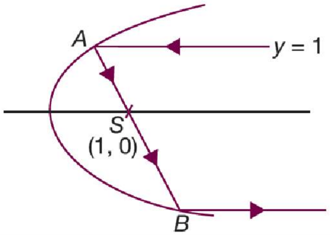
(i) First time
(ii) Second time
(iii) Third time

SOLUTION

Given parallel chord $y = 2x - 1$ and $y = 2x - 3$. A ray of light bisects them, which means ray of light is diameter of parabola.

$$y = \frac{2a}{m} = 1$$

Parabola $y^2 = 4x$ Ist strike at point $A\left(\frac{1}{4}, 1\right)$



Equation of AS $y - 0 = \frac{1-0}{\frac{1}{4}-1}(x-1) = -\frac{4}{3}(x-1)$

$\Rightarrow \quad 4x + 3y - 4 = 0$

For B, $y^2 = 4x = 4 - 3y \Rightarrow y^2 + 3y - 4 = 0$

$\Rightarrow \quad (y-1)(y+4) = 0$

$\Rightarrow \quad y = 1$ (Incident ray)

$y = -4;\ 16 = 4x \Rightarrow x = 4,$

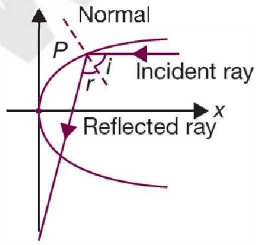
$B(4, -4)$ (IInd strike)

It becomes parallel to x-axis after IInd strike.
So, it never strikes third time

ILLUSTRATION 103

A ray of light is coming along the line $y = b$ from the positive direction of x-axis and strikes a concave mirror whose intersection with xy-plane is a parabola $y^2 = 4ax$. Find the equation of the reflected ray and show that it passes through the focus of the parabola. $(a > 0, b > 0)$.

SOLUTION



Incident ray line is $y = b$ (slope = 0)

P is a point on $y^2 = 4ax$ and $y = b$

$\Rightarrow \quad P$ is $\left(\frac{b^2}{4a}, b\right)$

$y^2 = 4ax$ tangent at (x_1, y_1)

$$yy_1 = 2a(x + x_1)$$

Or $\quad m_T = \frac{2a}{y_1}$ and $m_N = -\frac{y_1}{2a}$

Slope of normal at $P = -\frac{b}{2a}$

Let slope of reflected ray be m.

As incident and reflected rays are equally inclined to normal at P, we use

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\therefore \quad \left|\frac{0 - \left(-\frac{b}{2a}\right)}{1 + 0\left(-\frac{b}{2a}\right)}\right| = \left|\frac{m - \left(-\frac{b}{2a}\right)}{1 + m\left(-\frac{b}{2a}\right)}\right|$$

Or $\quad \frac{b}{2a} = \frac{2am + b}{2a - bm}$ and $\frac{b}{2a} = -\left(\frac{2am + b}{2a - bm}\right)$

Or $\quad m = 0$ and $2ab - b^2 m + 4a^2 m + 2ab = 0$

$\Rightarrow \quad m\,(4a^2 - b^2) + 4ab = 0$ or $m = \frac{4ab}{b^2 - 4a^2}$

$\therefore$ Equation of reflected ray is $y - b = \frac{4ab}{b^2 - 4a^2}\left(x - \frac{b^2}{4a}\right)$

This equation is satisfied by the co-ordinates of focus $(a, 0)$.

Test Your Concepts-V

(Solutions on page H.144)

1. If $y = 2x + \lambda$ is normal to parabola $y^2 = 8x$, then find λ.
2. If the parabolas $y^2 = 4Kx (K > 0)$ and $y^2 = 4(x - 1)$ do not have a common normal other than the axis of parabola, then find K.
3. If the normal at a point P on $y^2 = 4ax (a > 0)$ meet it again at Q in such a way that PQ is of minimum length. If 'O' is vertex, then show that ΔOPQ is a right angled triangle.
4. PN is the ordinate of any point P on $y^2 = 4x$. If the normal at P to the curve meets the axis at G, then find NG.
5. If a normal to a parabola $y^2 = 4ax$ make an angle ϕ with its axis, then it will cut the curve again at an angle θ, find it.
6. If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then show that $y_1 = y_2$.
7. If two normals to a parabola $y^2 = 4ax$ intersect at right angles, then the chord joining their feet passes through a fixed point. Find the point.
8. Normal at P to the parabola $(12x+5y+3)^2 = 52(5x-12y+1)$ meets the line $12x + 5y + 3 = 0$ at G, perpendicular GN is drawn to SP, S being focus, then find NP.
9. If $x^2+y^2-4x-6y+c=0$ is the circumcircle of the triangle formed by three tangents to the parabola, $y^2 = 4x - 4$, then find the radius of the circle.
10. From any point P on the parabola $y^2 = 4ax$, perpendicular PN is drawn on the axis meeting it at N. Normal at P meets the axis in G. For what value of 't', the point N divides SG internally in the ratio 1:3.
11. Find the ordinate of the centroid of the triangle formed by co-normal points on the parabola $y^2 = 4ax$.
12. Find the minimum distance between $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.

Answer Keys

1. -24
2. $K \in (1, 3/2)$
4. 2
5. $\tan^{-1}\left(\frac{\tan\phi}{2}\right)$
7. $(a, 0)$
8. 2
9. 3
10. $\pm\sqrt{\frac{5}{3}}$
11. Zero
12. $\sqrt{5}$

SOLVED PROBLEMS

PROBLEM 1

Identify the locus of the centre of the cirlce touching to the x-axis and the circle $x^2 + y^2 - 2y = 0$ externally.

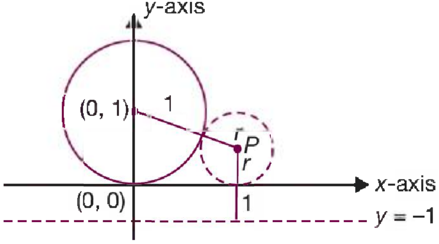
SOLUTION

Given circle can be (0, 1) $r = 1$

Centre of variable circle is $P(x, y)$.

Now, it is equidistant from point $S(0, 1)$ and line $y = -1$.

$\therefore$ Locus of P is parabola, focus (0, 1) and directrix $y = -1$.



$$PS = PM$$

$$\sqrt{(x-0)^2 + (y-1)^2} = |y+1|$$

$$x^2 + y^2 + 1 - 2y = y^2 + 1 + 2y$$

$$x^2 = 4y \text{ (Parabola)}$$

Alternate

Let centre of $P(x, y)$ and it touches given circle $x^2 + y^2 - 2y = 0$.

$$\text{Centre } C(0, 1)\ r = 1$$

$$PC = r + 1$$

Variable circle touches x-axis.

$$|y| = r$$

$$\therefore \quad PC = |y| + 1$$

$$(x-0)^2 + (y-1)^2 = (|y| + 1)^2$$

$$x^2 + y^2 + 1 - 2y = 1 + y^2 + 2|y|$$

We get $x^2 = 4y$

(Here $|y| = y$ as $y \geq 0$)

Required locus is $x^2 = 4y$.

PROBLEM 2

Let (x_r, y_r), where $r = 1, 2, 3, 4$ be the points of intersection of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Prove that $y_1 + y_2 + y_3 + y_4 = 0$.

SOLUTION

$$\text{Let} \quad x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(1)$$

$$y^2 = 4ax \qquad \ldots(2)$$

Solving (1) and (2), we get the co-ordinates of points of intersection.

From (2), $x = \dfrac{y^2}{4a}$. Putting in (1), we get,

$$\left(\frac{y^2}{4a}\right)^2 + y^2 + 2g\left(\frac{y^2}{4a}\right) + 2fy + c = 0$$

$$\text{Or} \quad \frac{1}{(4a)^2}y^4 + \left(1 + \frac{g}{2a}\right)y^2 + 2fy + c = 0$$

It has four roots.

Its roots are y_1, y_2, y_3 and y_4.

$$\text{Now, sum of roots} = -\frac{\text{Coefficient of } y^3}{\text{Coefficient of } y^4}$$

$$y_1 + y_2 + y_3 + y_4 = -\frac{0}{1/(4a)^2} = 0$$

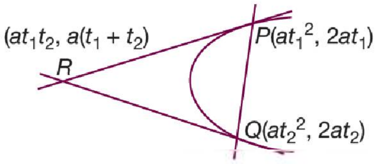
PROBLEM 3

The tangents to the parabola $y^2 = 4ax$ at $P(t_1)$ and $Q(t_2)$ intersect at R. Prove that the area of the ΔPQR is $\dfrac{1}{2}a^2(t_2 - t_1)^3$ sq. units.

SOLUTION

The tangent to the parabola $y^2 = 4ax$ at $P\ (at_1^2\ , 2at_1)$ is

$$yt_1 = x + at_1^2 \qquad \ldots(1)$$



$\therefore \quad R = \{at_1t_2, a(t_1 + t_2)\}$

$$\text{The area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2\begin{vmatrix} t_1^2 & 2t_1 & 1 \\ t_2^2 & 2t_2 & 1 \\ t_1t_2 & t_1+t_2 & 1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \frac{1}{2}a^2\begin{vmatrix} t_1^2 & 2t_1 & 1 \\ (t_2+t_1)(t_2-t_1) & 2(t_2-t_1) & 0 \\ t_1(t_2-t_1) & t_2-t_1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_2-t_1)^2\begin{vmatrix} t_2+t_1 & 2 \\ t_1 & 1 \end{vmatrix} = \frac{1}{2}a^2(t_2-t_1)^3$$

PROBLEM 4

Find the centre and radius of two circles that touch the parabola $75y^2 = 64(5x - 3)$ at $\left(\frac{6}{5}, \frac{8}{5}\right)$ and the x-axis.

SOLUTION

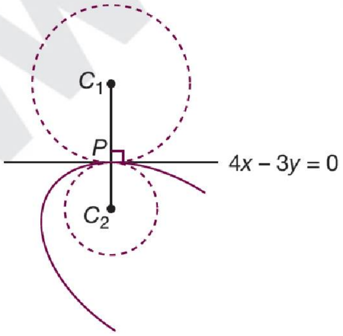
The parabola can be written as

$$75y^2 = 32(10x - 6)$$

Tangent at (x_1, y_1) is $75yy_1 = 32[5(x + x_1) - 6]$

Or $\quad 75y \cdot \frac{8}{5} = 32\left[5\left(x + \frac{6}{5}\right) - 6\right]$

Or $\quad 4x - 3y = 0$



Above is a common tangent to parabola and circle at $\left(\frac{6}{5}, \frac{8}{5}\right)$. The centre of the circle will be on normal at a distance ± r from the point of contact. Slope of tangent is $\frac{4}{3}$ and hence of normal will be $\frac{-3}{4} = \tan\theta$

$\therefore \quad \cos\theta = -\frac{4}{5}$, $\sin\theta = \frac{3}{5}$

Hence, the co-ordinates of centres are points C_1 and C_2 on the normal through $P\left(\frac{6}{5}, \frac{8}{5}\right)$ and at a distance r or $-r$ from P.

$$\frac{x - 6/5}{-4/5} = \frac{y - 8/5}{3/5} = r, -r$$

$$\therefore \quad C_1\left(\frac{-4r+6}{5}, \frac{3r+8}{5}\right), C_2\left(\frac{4r+6}{5}, \frac{-3r+8}{5}\right)$$

Since the circle touches x-axis, y co-ordinate of centre is radius.

$\therefore \quad \frac{3r+8}{5} = r \quad \Rightarrow \quad r = 4$ for C_1

Or $\quad \frac{-3r+8}{5} = r \quad \Rightarrow \quad r = 1$ for C_2

Hence, $C_1(-2, 4)$ and $C_2(2, 1)$, respectively.

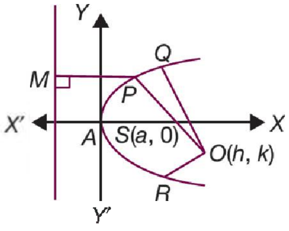
For smaller circle, $r = 1$ and $C_2(2, 1)$.

Larger circle $r = 4$, $C_1(-2, 4)$.

PROBLEM 5

If the normals at P, Q and R of the parabola $y^2 = 4ax$ meet in O and S be its focus, then prove that $SP \cdot SQ. SR = a(SO)^2$.

SOLUTION



Normal at t is $y + tx = 2at + at^3$

It passes through $O(h, k)$, then

$$at^3 + (2a - h)t - k = 0$$

If the co-ordinates of P, Q and R are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$, then we have relations

$$\sum t_1 = 0$$

$$\Sigma\, t_1t_2 = \frac{2a-h}{a}$$

$$t_1t_2t_3 = \frac{k}{a}$$

Then, $SP = PM = a + at_1^2$

Similarly, $SQ = a + at_2^2$ and

$$SR = a + at_3^2$$

$$\therefore \quad SP.\,SQ.\,SR = a^3\,(1+t_1^2)\,(1+t_2^2)\,(1+t_3^2)$$

$$= a^3\,\{1+\Sigma\, t_1^2 + \Sigma\, t_1^2\, t_2^2 + t_1^2\, t_2^2\, t_3^2\}$$

$$= a^3\{(1 + (\Sigma t_1)^2 - 2\Sigma t_1t_2 + (\Sigma t_1t_2)^2 - 2t_1t_2t_3\Sigma t_1$$

$$+ t_1^2\, t_2^2\, t_3^2\}$$

$$= a^3\left\{1+0-\frac{2(2a-h)}{a}+\frac{(2a-h)^2}{a^2}-0+\frac{k^2}{a^2}\right\}$$

$$= a(a^2 - 2a(2a-h) + (2a-h)^2 + k^2\}$$

$$= a\{h^2 + a^2 - 2ah + k^2\} = a\{(h-a)^2 + k^2\}$$

Hence, $SP \cdot SQ \cdot SR = a(SO)^2$

PROBLEM 6

The three normals from a point to the parabola $y^2 = 4ax$ cut the axes in points, whose distances from the vertex are in A.P., then find the locus of the point.

SOLUTION

Let $P(x_1, y_1)$ be any point.

Equation of any normal is $y = mx - 2am - am^3$

If it passes through P, then $y_1 = mx_1 - 2am - am^3$

$\Rightarrow \quad am^3 + (2a - x_1)m_1 + y_1 = 0$, which is cubic in m.

Let m_1, m_2 and m_3 be its roots.

Then, $m_1 + m_2 + m_3 = 0$, $m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-x_1}{a}$

Normal meets the axis $(y = 0)$, where $0 = mx - 2am - am^3$

$\Rightarrow \quad x = 2a + am^2$

$\therefore$ Distances of points from the vertex are $2a + am_1^2 \cdot 2a + am_2^2$, $2a + am_3^2$

If these are in A.P., then

$$2(2a + am_2^2) = (2a + am_1^2) + (2a + am_3^2)$$

$$\Rightarrow \quad 2m_2^2 = m_1^2 + m_3^2$$

$$\Rightarrow \quad 3m_2^2 = m_1^2 + m_2^2 + m_3^2$$

$$= (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)$$

$$= \frac{-2(2a-x_1)}{a}$$

$$\therefore \quad m_2^2 = \frac{2(x_1-2a)}{3a}$$

However, $y_1 = m_2(x_1 - 2a - am_2^2)$

$$\Rightarrow \quad y_1^2 = m_2^2\,(x_1 - 2a - am_2^2)^2$$

$$= \frac{2(x_1-2a)^3}{27a}$$

$\therefore$ Locus of P is $27ay^2 = 2(x - 2a)^3$.

PROBLEM 7

If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the feet of the three normals drawn from a point to the parabola $y^2 = 4ax$, then find $\dfrac{x_1-x_2}{y_3} + \dfrac{x_2-x_3}{y_1} + \dfrac{x_3-x_1}{y_2}$

SOLUTION

$y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$ for co-normal points $y_1 + y_2 + y_3 = 0$

$$\frac{x_1-x_2}{y_3} + \frac{x_2-x_3}{y_1} + \frac{x_3-x_1}{y_2}$$

$$= \frac{y_1^2-y_2^2}{4ay_3} + \frac{y_2^2-y_3^2}{4ay_1} + \frac{y_3^2-y_1^2}{4ay_2}$$

$$= \frac{1}{4a}\left[\frac{(y_1-y_2)(y_1+y_2)}{-(y_1+y_2)} + \frac{(y_2-y_3)(y_2+y_3)}{-(y_2+y_3)}\right.$$

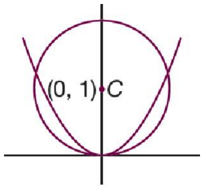
$$\left. + \frac{(y_3-y_1)(y_3+y_1)}{-4a(y_1+y_3)}\right]$$

$$= -\frac{1}{4a}\,(y_1 - y_2 + y_2 - y_3 + y_3 - y_1) = 0$$

PROBLEM 8

C is the centre of the circle with centre (0, 1) and radius unity. P is the parabola $y = ax^2$, then find the set of values of 'a', for which they meet at a point other than the origin.

SOLUTION



Put $x^2 = \frac{y}{a}$ in circle and we get $x^2 + (y-1)^2 = 1$.

Note that, for $a < 0$, they cannot intersect other than origin. Hence $a > 0$; $y = ax^2 > 0$

Solving with circle $\frac{y}{a} + y^2 - 2y = 0$

Hence, we get, $y = 0$ or $y = 2 - \frac{1}{a} > 0$

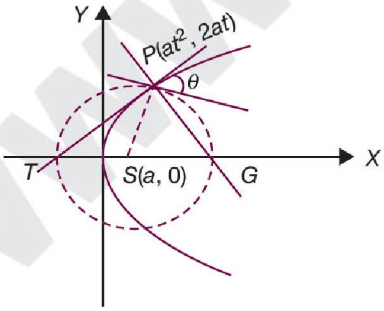
$$\Rightarrow \quad 2 > \frac{1}{a} \qquad (a > 0)$$

$$\therefore \quad a > \frac{1}{2}$$

PROBLEM 9

The tangent and normal at $P(t)$, for all real positive t, to the parabola $y^2 = 4ax$ meet the axis of the parabola in T and G respectively, then find the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle passing through the points P, T and G.

SOLUTION



Slope of tangent at P on parabola $= \frac{1}{t}(m_1)$

Slope of $PS = \frac{2at}{a(t^2-1)} = \frac{2t}{t^2-1}$

S is centre of circumcircle of ΔPTG.

Circle $(x-a)^2 + (y-0)^2 = r^2$

$$\Rightarrow \quad x^2 + y^2 - 2ax + a^2 - r^2 = 0$$

Tangent at $P(at^2, 2at)$

$$at^2x + 2aty - a(x + at^2) + a^2 - r^2 = 0$$

Slope of tangent at P on circle $= \frac{1-t^2}{2t}(m_2)$

Let θ is angle between both tangents

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$= \frac{\frac{1}{t} - \frac{1-t^2}{2t}}{1 + \frac{1-t^2}{2t^2}} = \frac{(2-1+t^2)2t^2}{2t(1+t^2)} = t$$

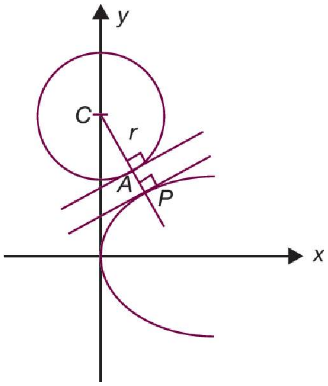
$$\theta = \tan^{-1}t$$

PROBLEM 10

Find the shortest distance between the curves, $y^2 = 4ax$ and $x^2 + y^2 - 24ay + 128a^2 = 0$.

SOLUTION

Distance should be measured along the common normal.



Normal of parabola at $P(at^2, 2at)$

$y + xt = 2at + at^3$ passes through the centre of circle C(0, 12a)

$$t^3 + 2t - 12 = 0 \Rightarrow t = 2$$

Point $P(4a, 4a)$

Shortest distance $PA = PC - r$

$$= \sqrt{16a^2 + 64a^2} - \sqrt{144a^2 - 128a^2}$$

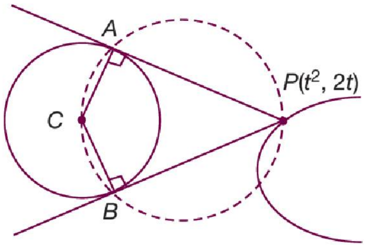
$$= 4\sqrt{5}\ a - 4a$$

$$= 4a\ (\sqrt{5} - 1)$$

PROBLEM 11

Tangents PA and PB are drawn to circle $(x + 3)^2 + (y - 2)^2 = 1$ from point P lying on parabola $y^2 = 4x$, then find the locus of circumcentre of ΔPAB.

SOLUTION



Point $P\left(t^2, 2t\right)$ on parabola and the centre of circle is $C(-3, 2)$.

Circumcentre of ΔPAB = Midpoint of PC

$$\text{Circumcentre of } \Delta PAB = \left(\frac{t^2 - 3}{2}, t + 1\right)$$

By eliminating 't',

we get the locus of circumcentre of ΔPAB as

$$2x + 3 = (y - 1)^2.$$

PROBLEM 12

Let $M = \left\{(x, y) \middle| y \geq x^2\right\}$ and $N = \left\{(x, y) \middle| x^2 + (y - a)^2 \leq 1\right\}$

The necessary and sufficient condition is $M \cap N = N$, then find condition on a.

SOLUTION

The desired condition is to have the circle N inside the convex region of the parabola, $y = x^2$.

Clearly, $a \geq 1$. Otherwise, circle will cross x-axis.

Also, for circle to be inside the parabola, the points of intersection must be less than or equal to 2.

Equation $y + (y - a)^2 = 1$ has one or less than one solution: $y^2 + (1 - 2a)y + a^2 - 1 = 0$

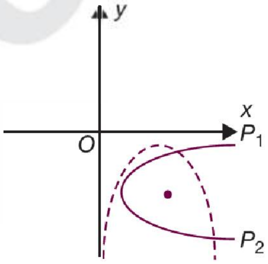
$$\Rightarrow \quad D \leq 0 \Rightarrow (2a - 1)^2 - 4\left(a^2 - 1\right) \leq 0$$

$$\Rightarrow \quad a \geq \frac{5}{4}$$

PROBLEM 13

Two parabolas have the same focus namely at the point (3, –28). If their directrices are the x- axis and y-axis, respectively and slope of common chord is 't', then find $|t|$.

SOLUTION



For parabola P_1, we have, $PS^2 = PM^2$ or $(x - 3)^2 + (y + 28)^2 = x^2$

For parabola P_2, we have, $PS^2 = PM^2$ or $(x - 3)^2 + (y + 28)^2 = y^2$

Equation of common chord $P_1 - P_2 = 0$

$$x^2 - y^2 = 0$$

$$(x + y)(x - y) = 0$$

$$m = \pm 1$$

We get, $y = \pm x$, which means $|t| = 1$

PROBLEM 14

In a square matrix A of order 3, $a_{ii} = m_i + i$, where $i = 1, 2, 3$ and m_i's are the slopes (in increasing order of their absolute value) of the 3 normals concurrent at the point (9, – 6) to the parabola $y^2 = 4x$. Rest all other entries of the matrix are one, then find the value of Det (A).

SOLUTION

Equation of normal to $y^2 = 4x \qquad (a = 1)$

$$y = mx - 2m - m^3$$

Passes through (9, – 6)

$$-6 = 9m - 2m - m^3$$

$$m^3 - 7m - 6 = 0$$

$$(m+1)(m+2)(m-3) = 0$$

$$m = -1 \text{ or } -2, 3$$

$$\therefore \quad m_1 = -1;\ m_2 = -2;\ m_3 = 3$$

$$\therefore \quad a_{11} = 1 + m_1 = 0$$

$$a_{22} = 2 + m_2 = 0$$

$a_{33} = 3 + m_3 = 6$, rest all elements are 1.

$$\therefore \quad \text{Det}(A) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 6 \end{vmatrix} = -4$$

PROBLEM 15

Find the co-ordinates of the focus of the parabola, $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

SOLUTION

Parabola $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$

For this parabola, x is a tangent at $P(a, 0)$.

y-axis a tangent at $Q(0, b)$.

$\therefore$ $O(0, 0)$ is the point of intersection perpendicular tangents.

$\therefore$ Directrix pass through this point.

Clearly, $\angle OSP = 90°$

Hence, circle on OP as diameter passing though S,

$$x^2 + y^2 - ax = 0 \qquad \ldots(1)$$

Similarly, $\angle OSQ = 90°$

Now circle OQ as diameter passing through S.

$$\therefore \quad x^2 + y^2 - by = 0 \qquad \ldots(2)$$

Point of intersecting above circle is focus.

Take difference of (1) and (2)

$$ax - by = 0$$

$$y = \frac{ax}{b} \Rightarrow x^2 = \frac{a^2x^2}{b^2} = ax \Rightarrow x\left(\frac{b^2 + a^2}{b^2}\right) = a$$

We get

$$x = \frac{ab^2}{a^2 + b^2},\ y = \frac{a^2b}{a^2 + b^2}$$

$$\text{Focus } S\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2b}{a^2 + b^2}\right).$$

PROBLEM 16

In a ΔABC, $P = \left(\frac{AB}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^3 + \ldots\ldots\infty$. Let $x^2 + y^2 = 1$ and $y^2 - 8x + 16 = 0$ be the equations of a circle and the parabola, respectively. If P is the shortest distance between the circle and the parabola, then find $\frac{AB}{AC}$.

SOLUTION

Normal of parabola at P

$$y + t(x-2) = 2at + at^3 \qquad (\because a = 2)$$

$$\Rightarrow \quad y + t(x-2) = 4t + 2t^3$$

It passes through centre (0, 0)

$$0 + 0 - 2t = 4t + 2t^3$$

$$\Rightarrow \quad 2t^3 + 6t = 0 \qquad t = 0 \text{ only}$$

$$P(2 + at^2, 2at) = (2, 0)$$

$$\text{Minimum distance} = pc - r = 1$$

$$P = 1 = \left(\frac{AB}{AC}\right)^2\left(\frac{1}{1 - \frac{AB}{AC}}\right) \Rightarrow \left(\frac{AB}{AC}\right)^2 + \left(\frac{AB}{AC}\right) - 1 = 0$$

We get, $\frac{AB}{AC} = \frac{\sqrt{5} - 1}{2}$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The Cartesian equation of the curve, whose parametric equations are $x = t^2 + 2t + 3$ and $y = t + 1$, is:

(A) $y = (x - 1)^2 + 2(y - 1) + 3$
(B) $x = (y - 1)^2 + 2(y - 1) + 5$
(C) $x = y^2 + 2$
(D) none of these

2. If $x + a = 0$ is the directrix of the parabola $y^2 = 2y + ax + 2$, then the value of a is:

(A) 2 (B) 3
(C) 4 (D) $\frac{4}{3}$

3. The length of latus rectum of the parabola $x = t^2 + t + 1$, $y = t^2 + 2t + 3$ is:

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$
(C) $\frac{1}{2\sqrt{2}}$ (D) $\frac{1}{8}$

4. If any point $P(x, y)$, which satisfies the relation $(5x - 1)^2 + (5y - 2)^2 = \lambda(3x - 4y - 1)^2$, represents a parabola, then:

(A) $\lambda = 1$ (B) $\lambda < 1$
(C) $\lambda > 1$ (D) $\lambda > 2$

5. The locus of the vertex of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$, where a is a parameter is:

(A) $xy = \frac{105}{64}$ (B) $xy = \frac{3}{4}$
(C) $xy = \frac{35}{16}$ (D) $xy = \frac{64}{105}$

6. The locus of centroid of triangle formed by a tangent to the parabola $y^2 = 36x$ with the co-ordinate axes is:

(A) $y^2 = -9x$ (B) $y^2 + 3x = 0$
(C) $y^2 = 3x$ (D) $y^2 = 9x$

7. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$, then:

(A) $d^2 + (2b + 3c)^2 = 0$ (B) $d^2 + (3b - 2c)^2 = 0$
(C) $d^2 + (2b - 3c)^2 = 0$ (D) $d^2 + (3b + 2c)^2 = 0$

8. If the points (2, 3) and (3, 2) on a parabola are equidistant from the focus, then the slope of its tangent at vertex is:

(A) parallel to $y = x$ (B) parallel to $y = -x$
(C) parallel to x-axis (D) parallel to y-axis

9. Equation of the circle of minimum radius, which touches both the parabolas $y = x^2 + 2x + 4$ and $x = y^2 + 2y + 4$ is:

(A) $4x^2 + 4y^2 - 11x - 11y - 31 = 0$
(B) $4x^2 + 4y^2 - 11x - 11y - 13 = 0$
(C) $4x^2 + 4y^2 - 11x - 11y + 11 = 0$
(D) $4x^2 + 4y^2 - 11x - 11y - 6 = 0$

10. The locus of the centre of the circle described on any focal chord of a parabola $y^2 = 4ax$ as diameter is:

(A) $x^2 = 2a\,(y - a)$ (B) $x^2 = -2a\,(y - a)$
(C) $y^2 = 2a\,(x - a)$ (D) $y^2 = -2a\,(x - a)$

11. Which of the following parametric equations does not represent a parabola?

(A) $x = t^2 + 2t + 1, y = 2t + 2$
(B) $x = a(t^2 - 2t + 1), y = 2at - 2a$
(C) $x = 3\sin^2 t, y = 6\sin t$
(D) $x = a\sin t, y = 2a\cos t$

12. If the area of the triangle inscribed in the parabola $y^2 = 4ax$ with one vertex at the vertex of the parabola and other two vertices at the extremities of a focal chord is $\frac{5a^2}{2}$, then the length of the focal chord is:

(A) $3a$ (B) $5a$
(C) $\frac{25a}{4}$ (D) None of these

13. If the tangents at the extremities of a focal chord of the parabola $x^2 = 4ay$ meet the tangent at the vertex at points, whose abscissa are x_1 and x_2, then x_1x_2 is:

(A) a^2 (B) $a^2 - 1$
(C) $a^2 + 1$ (D) $-a^2$

14. Area of a trapezium, whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through (1, 0) and having length $\frac{25}{4}$ units each is:

(A) $\frac{75}{4}$ sq. units
(B) $\frac{625}{16}$ sq. units
(C) $\frac{25}{4}$ sq. units
(D) $\frac{25}{8}$ sq. units

15. Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ maybe equal to:

(A) 2
(B) 4
(C) 5
(D) 3

16. A parabola $y = ax^2 + bx + c$ crosses x-axis at $(\alpha, 0)$ and $(\beta, 0)$ both right of origin. A circle passes through these two points. The length of tangent from origin to the circle is:

(A) $\sqrt{\frac{bc}{a}}$
(B) ac^2
(C) $\frac{b}{a}$
(D) $\sqrt{\frac{c}{a}}$

17. If the parabola $y = ax^2 + bx + c$ has vertex at (4, 2) and $a \in [1, 3]$, then the difference between the extreme values of abc is:

(A) 3600
(B) 144
(C) 3456
(D) cannot be found

18. The focus of the parabola $x^2 + y^2 + 2xy - 6x - 2y + 3 = 0$ is:

(A) (1, –1)
(B) (–1, 1)
(C) (3, 1)
(D) (1, 1)

19. Let the line $lx + my = 1$ intersects the parabola $y^2 = 4ax$ at points P and Q. The normals at P and Q meet at point R. If R lies on the parabola $y^2 = 4ax$, then a equals:

(A) $\frac{1}{l}$
(B) $\frac{1}{2l}$
(C) $\frac{-1}{l}$
(D) $\frac{-1}{2l}$

20. A circle is described, whose centre and diameter are vertex and three quarters of the latus rectum of the parabola $y^2 = 4ax$, respectively. The common chord of the circle and parabola divides the distance between the vertex and the focus in the ratio of:

(A) 1:1
(B) 1:2
(C) 1:3
(D) 3:1

21. Consider one side AB of a square $ABCD$ on the line $y = 2x - 17$ and the other two vertices C, D lies on the parabola $y = x^2$ and the maximum possible area of the square $ABCD$ (in square units) is:

(A) 160
(B) 1280
(C) 640
(D) 320

22. The focus of a parabola, whose axis is parallel to x-axis is $S(3, 1)$. The normal at $P(h, k)$ is $x + 2y - 10 = 0$ and M is the foot of the perpendicular from P to the directrix, then area of ΔSPM is:

(A) 10 sq. units
(B) 20 sq. units
(C) 25 sq. units
(D) 50 sq. units

23. Area of the triangle formed by the tangents at the points (4, 6), (10, 8) and (2, 4) on the parabola $y^2 - 2x = 8y - 20$, is (in sq. units):

(A) 4
(B) 2
(C) 1
(D) 8

24. Tangents are drawn to $y^2 = 4ax$ from a variable point P moving on $x + a = 0$, then the locus of foot of perpendicular drawn from P on the chord of contact of P is:

(A) $y = 0$
(B) $(x - a)^2 + y^2 = a^2$
(C) $(x - a)^2 + y^2 = 0$
(D) $y(x - a) = 0$

25. A line from (–1, 0) intersects the parabola $x^2 = 4y$ at A and B. Then the locus of centroid of ΔOAB is:

(A) $3x^2 - 2x = 4y$
(B) $3y^2 - 2y = 4x$
(C) $3x^2 + 2x = 4y$
(D) none of these

26. The line $x - b + \lambda y = 0$ cuts the parabola $y^2 = 4ax$ $(a > 0)$ at $P(t_1)$ and $Q(t_2)$. If $b \in [2a, 4a]$, then range of $t_1 t_2$, where $\lambda \in R$, is:

(A) [–4, – 2]
(B) [2, 4]
(C) [4, 16]
(D) [–16, – 4]

27. If BC is a latus rectum of parabola $y^2 = 4ax$ and A is the vertex. Then the minimum length of the projection of BC on a tangent drawn in the portion BAC is:

(A) $\sqrt{2}a$
(B) $2\sqrt{2}a$
(C) $2a$
(D) $3\sqrt{2}a$

28. Two distinct chords of the parabola $y^2 = 4ax$ passing through $(a, 2a)$ are bisected by the line $x + y = 1$. The length of the latus rectum of the parabola can be:

(A) 9
(B) 3
(C) 4
(D) 5

29. A line bisecting the ordinate PN of a point $P\left(at^2, 2at\right), t > 0$, on the parabola $y^2 = 4ax$ is drawn parallel to the axis to meet the curve at Q. If NQ meets the tangent at the vertex at the point T, then the co-ordinates of T are:

(A) $\left(0, \frac{4}{3}at\right)$ (B) $(0, 2at)$

(C) $\left(\frac{1}{4}at^2, at\right)$ (D) $(0, at)$

30. If two of the three feet of normals drawn from a point to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other, then:

(A) $\left(2, 2\sqrt{2}\right)$ (B) $(0, 0)$

(C) $\left(2, -2\sqrt{2}\right)$ (D) none of these

31. Let $A(0, 2)$, B and C are points on parabola $y^2 = x + 4$ and such that angle $CBA = \frac{\pi}{2}$, then the range of ordinate of C is:

(A) $(-\infty, 0) \cup (4, \infty)$ (B) $(-\infty, 0] \cup [4, \infty)$

(C) $[0, 4]$ (D) $(-\infty, 0) \cup [4, \infty)$

32. From a point 'A', common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and the parabola $y^2 = 4ax$. The area of the quadrilateral formed by the tangents, the chord of contact of the circle and the chord of contact of parabola is:

(A) $\frac{11a^2}{4}$ (B) $\frac{13a^2}{4}$

(C) $\frac{15a^2}{4}$ (D) $\frac{17a^2}{4}$

33. Consider the parabola $x^2 + 4y = 0$. Let, $p(a, b)$ be any fixed point inside the parabola and let 'S' be the focus of the parabola. Then the minimum value at $SQ + PQ$ as point Q moves on the parabola is:

(A) $|1 - a|$ (B) $|ab| + 1$

(C) $\sqrt{a^2 + b^2}$ (D) $1 - b$

34. The normals are drawn from $(2k, 0)$ to the curve $y^2 = 4x$. One normal is x-axis. Then the value of k for which other two normals are perpendicular to each other is:

(A) $\frac{3}{2}$ (B) 2

(C) $\frac{5}{2}$ (D) $\frac{7}{2}$

35. The co-ordinates of the point on the parabola $y = x^2 + 7x + 2$, which is nearest to the straight line $y = 3x - 3$ are:

(A) $(-2, -8)$ (B) $(1, 10)$

(C) $(2, 20)$ (D) $(-1, -4)$

36. The point P on the parabola $y^2 = 4ax$ for which $|PR - PQ|$ is maximum, where $R = (-a, 0)$, $Q = (0, a)$ is:

(A) $(a, 2a)$ (B) $(a, -2a)$

(C) $(4a, 4a)$ (D) $(4a, -4a)$

37. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is:

(A) $x^2 + 2y^2 - ax = 0$ (B) $2x^2 + y^2 - 2ax = 0$

(C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$

38. The locus of point of intersection of any tangent to the parabola $y^2 = 4a(x - 2)$ with a line perpendicular to it and passing through the focus, is:

(A) $x = 2$ (B) $y = 2$

(C) $x = a$ (D) $x = a + 2$

39. If the parabola $y = (a - b)x^2 + (b - c)x + (c - a)$ touches the x-axis, then the line $ax + by + c = 0$:

(A) always passes through a fixed point

(B) represents the family of parallel lines

(C) always perpendicular to x-axis

(D) always has negative slope

40. From any point P on the straight line $x = 1$, a tangent PQ is drawn to the parabola $y^2 - 8x + 24 = 0$, then the abscissae of N, where N is the foot of the perpendicular drawn from $A(5, 0)$ to PQ is:

(A) 1 (B) 2

(C) 3 (D) 4

41. If $P(-3, 2)$ is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is:

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$

(C) 2 (D) -2

42. The locus of the focus of the family of parabolas having directrix of slope m and touching the lines $x = a$ and $y = b$ is:

(A) $y + mx = am + b$ (B) $y + mx = am - b$

(C) $y - mx = am + b$ (D) $y - mx = am - b$

43. If at $x = 1$, $y = 2x$ the tangent to the parabola is $y = ax^2 + bx + c$, then respective possible values of a, b and c are:

(A) $\frac{1}{2}, 1, \frac{1}{2}$ (B) $1, \frac{1}{2}, \frac{1}{2}$

(C) $\frac{1}{2}, \frac{1}{2}, 1$ (D) $\frac{-1}{2}, 1, \frac{3}{2}$

44. A circle drawn on any focal chord of the parabola $y^2 = 4ax$ as diameter cuts the parabola and two points t and t' (other than extremity of a focal chord). Then the value of tt' is:

(A) 2 (B) 3
(C) 1 (D) 4

45. The length of the chord of the parabola $x^2 = 4y$ passing through the vertex and having slope as $\cot \alpha$ is:

(A) $4 \cos \alpha \operatorname{cosec}^2 \alpha$ (B) $4 \tan \alpha \sec \alpha$
(C) $4 \sin \alpha \sec^2 \alpha$ (D) none of these

46. Let S be the set of all possible values of the parameter 'a', for which the points of intersection of the parabolas $y^2 = 3ax$ and $y = \frac{1}{2}(x^2 + ax + 5)$ are concyclic. Then S contains interval:

(A) $(-\infty, 2)$ (B) $(-2, 0)$
(C) $(0, 2)$ (D) $(2, \infty)$

47. The equation of the line of the shortest distance between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 - 4x - 2y + 4 = 0$ is:

(A) $x + y = 3$ (B) $x - y = 3$
(C) $2x + y = 5$ (D) none of these

48. The ends of the latus rectum of the conic $x^2 + 10x - 16y + 25 = 0$ are:

(A) $(3, -4), (13, 4)$ (B) $(-3, -4), (13, -4)$
(C) $(3, 4), (-13, 4)$ (D) $(5, -8), (-5, 8)$

49. If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ and are the tangents of $y^2 = 4ax$, then:

(A) $m_1 + m_2 = 0$ (B) $m_1 m_2 = 1$
(C) $m_1 m_2 = -1$ (D) $m_1 + m_2 = 1$

50. Radius of the largest circle, which passes through the focus of the parabola $x^2 - 2x - 4y + 5 = 0$ and contained in it is:

(A) $\sqrt{2} + 1$ (B) $4\sqrt{3} + 1$
(C) $\sqrt{3} + 1$ (D) 4

51. The locus of the centre of the circle passing through the vertex and the midpoints of perpendicular chords from the vertex of the parabola $y^2 = 4ax$ is:

(A) $y^2 = 4a(x - 2a)$ (B) $y^2 = a(x - 2a)$
(C) $y^2 = 4a(x - a)$ (D) $(x - a)^2 + y^2 = a^2$

52. If focal distance of a point on the parabola $y = x^2 - 4$ is $\frac{25}{4}$ and points are of the form $\left(\pm\sqrt{a}, b\right)$, then $a + b$ equals:

(A) 8 (B) 4
(C) 2 (D) 0

53. Length of side of an equilateral triangle inscribed in a parabola $y^2 - 2x - 2y - 3 = 0$, whose one angular point is vertex of the parabola is:

(A) $2\sqrt{3}$ (B) $4\sqrt{3}$
(C) $-\sqrt{3}$ (D) $\sqrt{3}$

54. Tangents PA and PB are drawn to circle $(x + 3)^2 + (y - 2)^2 = 1$ from point P lying on $y^2 = 4x$, then the locus of circumcentre of ΔPAB is:

(A) $(y - 1)^2 = 2x - 3$ (B) $(y + 1)^2 = 2x + 3$
(C) $(y + 1)^2 = 2x - 3$ (D) $(y - 1)^2 = 2x + 3$

55. In the parabola, $y^2 - 2y + 8x - 23 = 0$, the length of double ordinate at a distance of 4 units from its vertex is:

(A) $4\sqrt{2}$ (B) $8\sqrt{2}$
(C) 6 (D) 4

56. Tangents are drawn from the point $(-1, 2)$ to the parabola $y^2 = 4x$. The length of the intercept made by the line $x = 2$ on these tangents is:

(A) 6 (B) $6\sqrt{2}$
(C) $2\sqrt{6}$ (D) none of these

57. Consider $y^2 = 8x$. If the normal at a point P on the parabola meets it again at a point Q, then the least distance of Q from the tangent at the vertex of the parabola is:

(A) 16 (B) 8
(C) 4 (D) 2

58. The ΔPQR of area A is inscribed in the parabola $y^2 = 4ax$ such that P lies at the vertex of the parabola and base QR is a focal chord. The numerical difference of the ordinates of the points Q and R is:

(A) $\frac{A}{2a}$ (B) $\frac{A}{a}$
(C) $\frac{2A}{a}$ (D) $\frac{4A}{a}$

59. Through the vertex O of the parabola $y^2 = 4ax$, two chords namely, OP and OQ are drawn and the circles on OP and OQ as diameter intersect at R. If θ_1, θ_2 and ϕ are the inclinations of the tangents at P and Q on the parabola and the line through O, R, respectively, then the value of $\cot\theta_1 + \cot\theta_2 \cot\theta_1$ is:

(A) $-2 \tan \phi$ (B) $-2 \tan(\pi - \phi)$
(C) 0 (D) $2 \cot \phi$

60. AB and AC are tangents to the parabola $y^2 = 4ax$. p_1, p_2 and p_3 are the perpendiculars from A, B and C, respectively on any tangent to the curve (other than the tangents at B and C), then p_1, p_2 and p_3 are in:

(A) A.P. (B) G.P.
(C) H.P. (D) none of these

61. A tangent to the parabola $x^2 + 4ay = 0$ at the point T cuts the parabola $x^2 = 4by$ at A and B. Then locus of the midpoint of AB is:

(A) $(b + 2a)x^2 = 4b^2y$ (B) $(b + 2a)x^2 = 4a^2y$
(C) $(a + 2b)y^2 = 4b^2x$ (D) $(a + 2b)x^2 = 4b^2y$

62. The number of normals that can be drawn through $(-1, 4)$ to the parabola $y^2 - 4x + 6y = 0$ are:

(A) 4 (B) 3
(C) 2 (D) 1

63. If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then:

(A) $x_1 = x_2$ (B) $x_1 = y_2$
(C) $y_1 = y_2$ (D) $x_2 = y_1$

64. The normals at three points P, Q and R of the parabola $y^2 = 4ax$ meet in (h, k). The centroid of $\Delta\, PQR$ lies on:

(A) $x = 0$ (B) $y = 0$
(C) $x = -a$ (D) $y = 0$

65. A parabola of latus rectum '$4a$' touches a fixed equal parabola, the axes of the two curves being parallel and the locus of the vertex of moving curve is parabola of latus rectum is k, then k:

(A) $2a$ (B) $4a$
(C) $8a$ (D) $16a$

66. The normals at two points P and Q of a parabola $y^2 = 4ax$ meet at (x_1, y_1) on the parabola. Then PQ^2:

(A) $(x_1 + 4a)(x_1 + 8a)$ (B) $(x_1 + 4a)(x_1 - 8a)$
(C) $(x_1 - 4a)(x_1 + 8a)$ (D) $(x_1 - 4a)(x_1 - 8a)$

67. If a normal subtends a right angle at the vertex of the parabola $y^2 = 4ax$, then its length is:

(A) $\sqrt{5}a$ (B) $3\sqrt{5}a$
(C) $6\sqrt{3}a$ (D) $7\sqrt{5}a$

68. Three normals with slopes m_1, m_2 and m_3 are drawn from any point P not on the axis of the parabola $y^2 = 4x$. If $m_1 . m_2 = a$, results in locus of P being a part of parabola, the value of 'a' equals:

(A) 2 (B) -2
(C) 4 (D) -4

69. The length of the normal chord drawn at one end of the latus rectum of $y^2 = 4ax$ is:

(A) $2\sqrt{2}a$ (B) $4\sqrt{2}a$
(C) $8\sqrt{2}a$ (D) $10\sqrt{2}a$

70. A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at B. If AB subtends a right angle at the vertex of the parabola, then the slope of AB is:

(A) $\sqrt{2}$ (B) 2
(C) $\sqrt{3}$ (D) 3

71. The normal at P cuts the axis of the parabola $y^2 = 4ax$ in G and S is the focus of the parabola. If ΔSPG is equilateral, then each side is of length:

(A) a (B) $2a$
(C) $3a$ (D) $4a$

72. If the normals at two points on the parabola $y^2 = 4ax$ intersect on the parabola, then the product of the abscissa is:

(A) $4a^2$ (B) $-4a^2$
(C) $2a$ (D) $4a^4$

73. If the normals at two points on the parabola intersect on the curve, then the product of the ordinates of the points is:

(A) $8a$ (B) $8a^2$
(C) $8a^3$ (D) $8a^4$

74. The equation of the locus of the point of intersection of two normals to the parabola $y^2 = 4ax$, which are perpendicular to each other is:

(A) $y^2 = a(x - 3a)$ (B) $y^2 = a(x + 3a)$
(C) $y^2 = a(x + 2a)$ (D) $y^2 = a(x - 2a)$

75. The line $x - y = 1$ intersects the parabola $y^2 = 4x$ at A and B. Normals at A and B intersect at C. If D is the point other that A and B at which CD is normal to the parabola then the co-ordinate of D are:

(A) $(4, 4)$ (B) $(4, -4)$
(C) $(1, 2)$ (D) $(16, -8)$

76. Locus of poles of chords of the parabola $y^2 = 4ax$ which subtends 45° at the vertex is $(x + 4a)^2 = \lambda(y^2 - 4ax)$, then λ equals:

(A) 1 (B) 2
(C) 3 (D) 4

77. The angle subtended at the focus by the normal chord of the point (λ, λ), $\lambda \neq 0$ on the parabola $y^2 = 4ax$ is:

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

78. A circle with its centre at the focus of the parabola $y^2 = 4ax$ and touching its directrix intersects the parabola at points A, B. Then the length of AB is equal to:

(A) $4a$
(B) $2a$
(C) a
(D) $7a$

79. Three normals are drawn to the curve $y^2 = x$ from a point $(c, 0)$. Out of three, one is always on x-axis. If two other normals are perpendicular to each other, then the value of c equals:

(A) $\frac{3}{4}$
(B) $\frac{1}{2}$
(C) $\frac{3}{2}$
(D) 2

80. If P is a point $(2, 4)$ on the parabola $y^2 = 8x$ and PQ is a focal chord, then the co-ordinate of the mirror image of Q with respect to tangent at P is given by:

(A) $(6, 4)$
(B) $(-6, 4)$
(C) $(2, 4)$
(D) $(6, 2)$

81. The locus of the midpoint of chord of the circle $x^2 + y^2 = 9$ such that segment intercepted by the chord on the curve $y^2 - 4x - 4y = 0$ subtends the right angle at the origin is:

(A) $x^2 + y^2 - 4x - 4y = 0$
(B) $x^2 + y^2 + 4x + 4y = 0$
(C) $x^2 + 4x + 4y - 9 = 0$
(D) none of these

82. For different values of k and λ, the two parabolas $y^2 = 16(x - k)$, $x^2 = 16(y - \lambda)$ always touch each other, then the locus of point of contact is:

(A) $x^2 + y^2 = 64$
(B) $xy = 8$
(C) $y^2 = 8x$
(D) $xy = 64$

83. TP and TQ are any two tangents of a parabola $y^2 = 4ax$ and T is the point of intersection of two tangents. If the tangent at a third point on the parabola meets the above two tangents at P' and Q'. Then $\frac{TP'}{TP} + \frac{TQ'}{TQ}$ equals:

(A) -1
(B) $\frac{1}{2}$
(C) $-\frac{1}{2}$
(D) 2

84. The locus of the orthocentre of the triangle formed by three tangents of the parabola $(4x - 3)^2 = -64(2y + 1)$ is:

(A) $y = \frac{-5}{2}$
(B) $y = 1$
(C) $x = \frac{7}{4}$
(D) $y = \frac{3}{2}$

85. The length of the latus rectum of a parabola is $4a$. A pair of perpendicular tangents is drawn to the parabola to meet the axis of the parabola at the points A and B. If S is the focus of the parabola, then $\frac{1}{|SA|} + \frac{1}{|SB|}$ equals:

(A) $\frac{2}{a}$
(B) $\frac{4}{a}$
(C) $\frac{1}{a}$
(D) $2a$

86. Length of the focal chord of the parabola $(y + 3)^2 = -8(x - 1)$, which lies at a distance 2 units from the vertex of the parabola is:

(A) 8
(B) $6\sqrt{2}$
(C) 9
(D) $5\sqrt{3}$

87. The slope of normal to the parabola $y = \frac{x^2}{4} - 2$ drawn through the point $(10, -1)$ is:

(A) -2
(B) $-\sqrt{3}$
(C) $\frac{-1}{2}$
(D) $\frac{-5}{3}$

88. m_1, m_2 and m_3 are the slopes of normal $(m_1 < m_2 < m_3)$ drawn through the point $(9, 6)$ to the parabola $y^2 = 4x$. $A = [a_{ij}]$ is a square matrix of order 3 such that $a_{ij} = 1$, if $i \neq j$ and $a_{ij} = m_i$, if $i = j$. Then det A is:

(A) 6
(B) -4
(C) -9
(D) 8

89. If parabola of latus rectum 'u' touches a fixed equal parabola, the axes of two curves being parallel, then the locus of the vertex of the moving curve is:

(A) a parabola of latus rectum '$2u$'
(B) a parabola of latus rectum 'u'
(C) an ellipse, whose major axis is '$2u$'
(D) an ellipse, whose minor axis is '$2u$'

90. A ray of light moving parallel to the x-axis gets reflected form a parabolic mirror, whose equation is $(y - 2)^2 = 4(x + 1)$. After reflection, the ray must pass through the point:

(A) $(0, 2)$
(B) $(2, 0)$
(C) $(0, -2)$
(D) $(-1, 2)$

91. A normal, whose inclination is 30°, to a parabola cuts it again at an angle of:

(A) $\tan^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (B) $\tan^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$

(C) $\tan^{-1}(2\sqrt{3})$ (D) $\tan^{-1}\left(\dfrac{1}{2\sqrt{3}}\right)$

92. The locus of vertices of family of parabolas, $y = ax^2 + 2a^2x + 1$ is $(a \neq 0)$ is a curve passing through:

(A) $(1, 0)$ (B) $(1, 1)$

(C) $(0, 1)$ (D) $(0, 0)$

93. Equation of circle of minimum radius, which touches both the parabolas $y = x^2 + 2x + 4$ and $x = y^2 + 2y + 4$ is:

(A) $2x^2 + 2y^2 - 11x - 11y - 13 = 0$

(B) $4x^2 + 4y^2 - 11x - 11y - 13 = 0$

(C) $3x^2 + 3y^2 - 11x - 11y - 13 = 0$

(D) $x^2 + y^2 - 11x - 11y - 13 = 0$

94. An equilateral ΔSAB is inscribed in the parabola $y^2 = 4ax$ having its focus at 'S'. If the chord AB lies to the left of S, then the length of the side of this triangle is:

(A) $3a(2 - \sqrt{3})$ units (B) $4a(2 - \sqrt{3})$ units

(C) $2a(2 - \sqrt{3})$ units (D) $8a(2 - \sqrt{3})$ units

95. The normals to the parabola $y^2 = 4ax$ at points Q and R meet the parabola again at P. If T is the intersection point of the tangents to the parabola at Q and R, then the locus of the centroid of ΔTQR, is:

(A) $y^2 = 3a(x + 2a)$ (B) $y^2 = a(2x + 3a)$

(C) $y^2 = a(3x + 2a)$ (D) $y^2 = 2a(2x + 3a)$

96. The normal at a point P of a parabola $y^2 = 4ax$ meets its axis in G and tangent at its vertex in H. If A is the vertex of the parabola and if the rectangle $AGQH$ is completed, then the equation to the locus of vertex Q is:

(A) $y^2(y - 2a) = ax^2$ (B) $y^2(y + 2a) = ax^2$

(C) $x^2(x - 2a) = ay^2$ (D) $x^2(x + 2a) = ay^2$

97. If the focus of the parabola $(y - \beta)^2 = 4(x - \alpha)$ always lies between the lines $x + y = 1$ and $x + y = 3$, then:

(A) $3 < \alpha + \beta < 4$ (B) $0 < \alpha + \beta < 3$

(C) $0 < \alpha + \beta < 2$ (D) $-2 < \alpha + \beta < 2$

98. Consider the two parabolas $y^2 = 4a(x - \alpha)$ and $x^2 = 4a(y - \beta)$, where 'a' is the given constant and α, β are variables. If α and β vary in such a way that these parabolas touch each other, then equation to the locus of point of contact is a:

(A) circle

(B) parabola

(C) ellipse

(D) rectangular hyperbola

99. The points on the axis of the parabola $3y^2 + 4y - 6x + 8 = 0$ from where 3 distinct normals can be drawn is given by:

(A) $\left(a, \dfrac{4}{3}\right); a > \dfrac{19}{9}$ (B) $\left(a, -\dfrac{2}{3}\right); a > \dfrac{19}{9}$

(C) $\left(a, -\dfrac{2}{3}\right); a > \dfrac{16}{9}$ (D) $\left(a, -\dfrac{2}{3}\right); a > \dfrac{7}{9}$

100. Tangents PA and PB are drawn to $y^2 = 4ax$. If m_{PA} and m_{PB} are the slopes of the tangents satisfying $(m_{PA})^2 + (m_{PB})^2k = 4$, then the locus of P is:

(A) $y^2 = 2x(2x + a)$ (B) $y^2 = 2x(2x - a)$

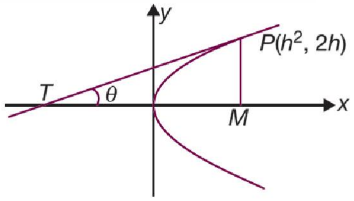
(C) $y^2 = x(x + a)$ (D) none of these

101. The common tangents to the circle $x^2 + y^2 = \dfrac{a^2}{2}$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola:

(A) $x^2 = 4ay$ (B) $x^2 = -4ay$

(C) $y^2 = -4ax$ (D) $y^2 = 4a(x + a)$

102. The triangle formed by the tangent to the parabola $y^2 = 4x$ at the point, whose abscissa lies in the interval $[a^2, 4a^2]$, then the ordinate and the x-axis has the greatest area equal to:



(A) $12a^3$ (B) $8a^3$

(C) $16a^3$ (D) none of these

103. Minimum distance between $y^2 - 4x - 8y + 40 = 0$ and $x^2 - 8x - 4y + 40 = 0$ is:

(A) 0 (B) $\sqrt{3}$

(C) $2\sqrt{2}$ (D) $\sqrt{2}$

104. If $(-2, 5)$ and $(3, 7)$ are the points of intersection of the tangent and normal at a point on a parabola with the axis of the parabola, then the focal distance of that point is:

(A) $\dfrac{\sqrt{29}}{2}$ (B) $\dfrac{5}{2}$

(C) $\sqrt{29}$ (D) $\dfrac{2}{5}$

105. PQ is any focal chord of the parabola $y^2 = 32x$. The length of PQ can never be less than:

(A) 8 units
(B) 16 units
(C) 32 units
(D) 48 units

106. Let $y^2 = 4ax$ be a parabola and PQ be a focal chord of parabola. Let T be the point of intersection of tangents at P and Q. Then:

(A) area of circumcircle of ΔPQT is $\left(\dfrac{\pi(PO)^2}{4}\right)$ sq. units
(B) orthocenter of ΔPQT will lie on tangent at vertex
(C) incenter of ΔPQT will be vertex of parabola
(D) incentre of ΔPQT will lie on directrix of parabola

107. Axis of a parabola is $y = x$ and vertex and focus are at a distance $\sqrt{2}$ units and $2\sqrt{2}$ units, respectively from the origin. Then equation of the parabola is:

(A) $(x-y)^2 = 8(x+y-2)$
(B) $(x+y)^2 = 2(x+y-2)$
(C) $(x-y)^2 = 4(x+y-2)$
(D) $(x+y)^2 = 2(x-y+2)$

108. Equation of the line passing through (α, β), cutting the parabola $y^2 = 4ax$ at two distinct points A and B such that AB subtends right angle at the origin is:

(A) $\beta x + (4a-\alpha)y - 4a\beta = 0$
(B) $2\beta x + (\alpha - 4a)y - 2a\beta = 0$
(C) $\beta x + (\alpha - 4a)y - 2a\beta = 0$
(D) none of these

109. The slope of the line, which belongs to family of these $(1+\lambda)x + (\lambda - 1)y + 2(1-\lambda) = 0$ and makes shortest intercept on $x^2 = 4y - 4$ is:

(A) $\dfrac{1}{2}$
(B) 1
(C) 0
(D) 2

110. If the tangents at two points (1, 2) and (3, 6) as a parabola intersect at the point (–1, 1), then the slope of the directrix of the parabola is:

(A) $\sqrt{2}$
(B) – 2
(C) – 1
(D) none of these

111. A variable chord PQ of the parabola $y = 4x^2$ subtends a right angle at the vertex. Then the locus of points of intersection of the tangents at P and Q is:

(A) $4y + 1 = 16x^2$
(B) $y + 4 = 0$
(C) $4y + 4 = 4x^2$
(D) $4y + 1 = 0$

112. Let A (9, 6), B(4, – 4) be two points on parabola $y^2 = 4x$ and $P(t^2, 2t)$, $t \in [-2, 3]$ be a variable point on it such that area of ΔPAB is maximum, then point P will be:

(A) (4, 4)
(B) $(3, -2\sqrt{3})$
(C) (4, 1)
(D) $\left(\dfrac{1}{4}, 1\right)$

113. The tangent and normal at the point P(4, 4) to the parabola, $y^2 = 4x$ intersect the x-axis at the points Q and R, respectively. Then the cirucmcentre of the ΔPQR is:

(A) (2, 0)
(B) (2, 1)
(C) (1, 0)
(D) (1, 2)

114. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola to the point (1, 2) is:

(A) $(x-1)^2 = 4(y+1)$
(B) $(x-1)^2 = 4(y+1)^2$
(C) $(x+1)^2 = 4(y-1)$
(D) $(x-1)^2 = 4(y-1)$

115. Let $3x - y - 8 = 0$ be the equation of tangent to a parabola at the point (7, 13). If the focus of the parabola is at (–1, – 1), then its directrix is:

(A) $x - 8y + 19 = 0$
(B) $8x + y + 19 = 0$
(C) $8x - y + 19 = 0$
(D) $x + 8y + 19 = 0$

116. The radius of largest circle, which passes through focus of parabola $y^2 = 4(x+y)$ and also contained in it is:

(A) 4
(B) 1
(C) 3
(D) 2

117. The locus of the midpoint of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix:

(A) $x = -a$
(B) $x = \dfrac{-a}{2}$
(C) $x = 0$
(D) $x = \dfrac{a}{2}$

118. The tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x+b)$ at Q and R, then the co-ordinates of the midpoint of QR are:

(A) $(x_1 - a, y_1 + b)$
(B) (x_1, y_1)
(C) $(x_1 + b, y_1 + a)$
(D) $(x_1 - b, y_1 - b)$

119. Equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is:

(A) $\sqrt{3}y = 3x + 1$
(B) $\sqrt{3}y = -(x+3)$
(C) $\sqrt{3}y = x + 3$
(D) $\sqrt{3}y = -(3x+1)$

120. If the normal chord at a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex, then the value of t is:

(A) 4
(B) $\sqrt{3}$
(C) $\sqrt{2}$
(D) 1

121. If the focus of a parabola divides a focal chord of the parabola in segments of length 3 and 2, then the length of the latus rectum of the parabola is:

(A) $\frac{3}{2}$ (B) $\frac{6}{5}$

(C) $\frac{12}{5}$ (D) $\frac{24}{5}$

122. A line L passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola in two distinct points. If 'm' be the slope of the line L, then:

(A) $m \in (-1, 1)$ (B) $m \in (-\infty, -1) \cup (1, \infty)$

(C) $m \in R$ (D) $m \in R - \{0\}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. The equation of a parabola is $25\left((x-2)^2 + (y+5)^2\right) = (3x + 4y - 1)^2$. For this parabola:

(A) tangent at vertex is $6x + 8y + 13 = 0$

(B) focus is (2, –5)

(C) directrix is $3x + 4y - 1 = 0$

(D) axis is $4x - 3y - 23 = 0$

2. The co-ordinates of a point on the parabola $y^2 = 8x$, whose focal distance is 4 units is:

(A) (2, 4) (B) (4, 2)

(C) (2, –4) (D) (4, –2)

3. The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that, $PR{:}QR = 1{:}\lambda$. If R is an interior point of the parabola $y^2 = 4x$, then:

(A) $\lambda \in (0, 1)$ (B) $\lambda \in \left(-\frac{3}{5}, 1\right)$

(C) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$ (D) $\lambda \in \left(-\frac{3}{5}, 0\right)$

4. If PQ and RS are normal chords of the parabola $y^2 = 8x$ and the points P, Q, R and S are concyclic, then:

(A) tangents at P and R meet on x-axis

(B) tangents at P and R meet on y-axis

(C) PR is parallel to y-axis

(D) PR is parallel to x-axis

5. Tangents are drawn from $(-2, 0)$ to $y^2 = 8x$. Radius of circle(s) that would touch these tangents and the corresponding chord of contact can be:

(A) $4\left(\sqrt{2} + 1\right)$ (B) $4\left(\sqrt{2} - 1\right)$

(C) $8\sqrt{2}$ (D) none of these

6. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through the vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, then the co-ordinates of the vertices of the square are:

(A) $(4a, 4a)$ (B) $(4a, -4a)$

(C) (0, 0) (D) $(8a, 0)$

7. $A\left(at_1^2, 2at_1\right)$, $B\left(at_2^2, 2at_2\right)$, $C\left(at_3^2, 2at_3\right)$ be three points on the parabola $y^2 = 4ax$. If the orthocentre of ΔABC is focus S of the parabola, then:

(A) $t_1t_2 + t_3t_2 + t_3t_1 = -5$

(B) $\frac{1}{t_1t_2} + \frac{1}{t_2t_3} + \frac{1}{t_3t_1} = -1$

(C) if $t_1 = 0$, then $t_2 + t_3 = 0$

(D) $(1 + t_1)(1 + t_2)(1 + t_3) = -4$

8. The normals at the points $P(t_1)$, $Q(t_2)$ on the parabola $y^2 = 4ax$ intersect at $Q(t_3)$ on the parabola. Then, which of the following is/are true?

(A) $|t_3| \geq \sqrt{2}$ (B) $|t_3| \geq 2\sqrt{2}$

(C) $t_1t_2 = 2$ (D) $t_3 = -t_1 - t_2$

9. Consider the parabola represented by the parametric equations $x = t^2 - 2t + 2$ and $y = t^2 + 2t + 2$. Then which of the following is/are true?

(A) Auxiliary circle of the parabola is $x + y = 4$

(B) Vertex of the parabola is (2, 2)

(C) Director circle of the parabola is $x + y = 6$

(D) Focus of the parabola is (3, 3)

10. Let PQ be a chord of the parabola $y^2 = 4x$. A circle is drawn with PQ as diameter, which passes through the vertex 'V' of the parabola. If area of ΔPVQ = 20 sq. units, then the co-ordinates of P are:

(A) (16, 8) (B) (16, –8)

(C) (–16, 8) (D) (–16, –8)

11. If $Ay^2 + By + Cx + D = 0$, ($ABC \neq 0$) be the equation of parabola, then:

(A) the length of latus rectum is $\left|\dfrac{C}{A}\right|$

(B) the axis of the parabola is a vertical line

(C) y-co-ordinate of the vertex is $-\dfrac{B}{2A}$

(D) x-co-ordinates of the vertex is $\left(\dfrac{B^2-4AD}{4AC}\right)$

12. The points on the axis of parabola $3x^2+4x-6y+8=0$, from which three distinct normals can be drawn to them, are:

(A) $\left(\dfrac{-2}{3}, 2\right)$
(B) $\left(\dfrac{-2}{3}, 3\right)$
(C) $\left(\dfrac{-2}{3}, 4\right)$
(D) $\left(\dfrac{-2}{3}, 1\right)$

13. Which of the following statements are true for the curve $9x^2-24xy+16y^2-20x-15y-60=0$?

(A) It represents a parabola
(B) Length of latus rectum of the curve is 2 units
(C) The equation of directrix of curve is $16x+12y+53=0$
(D) The equation of axis of the curve is $3x-4y-35=0$

14. If the equation of directrix of the parabola $x^2+4y-6x+k=0$ is $y+1=0$, then:

(A) $k=17$
(B) $k=-17$
(C) focus is $(3,-3)$
(D) vertex is $(3,-3)$

15. If a normal chord of $y^2=4ax$ subtends an angle $\dfrac{\pi}{2}$ at the vertex of the parabola, then its slope is:

(A) $\sqrt{2}$
(B) $-\sqrt{2}$
(C) 1
(D) -1

16. Equation of common tangent of $y=x^2$, $y=-x^2+4x-4$ is:

(A) $y=4(x-1)$
(B) $y=0$
(C) $y=-4(x-1)$
(D) $y=-30x-50$

17. Consider a circle with its centre lying on the focus of the parabola $y^2=2px$, such that it touches the directrix of the parabola. Then the point of intersection of the circle and the parabola is:

(A) $\left(\dfrac{p}{2}, p\right)$
(B) $\left(\dfrac{p}{2}, -p\right)$
(C) $\left(-\dfrac{p}{2}, p\right)$
(D) $\left(-\dfrac{p}{2}, -p\right)$

18. Let $y^2-5y+3x+k=0$ be a parabola, then:

(A) its latus rectum is least, when $k=1$
(B) its latus rectum is independent of k
(C) the line $y=2x+1$ will touch the parabola, if $k=\dfrac{73}{16}$
(D) $y=\dfrac{5}{2}$ is the only normal to the parabola, whose slope is zero.

19. If $f(x+y)=f(x).f(y)$ for all x and y and $f(1)=2$ and $a_r=f(r)$, for $r \in N$, then the co-ordinates of a point on the parabola $y^2=8x$, whose focal distance is 4 maybe:

(A) (a_1, a_2)
(B) $(a_1, -a_2)$
(C) (a_1, a_1)
(D) (a_2, a_2)

20. The line $x+y+2=0$ is a tangent to a parabola at point A, which intersects the directrix at B and tangent at vertex at C, respectively. If the focus of parabola is S $(2, 0)$, then:

(A) CS is perpendicular to AB
(B) $AC.BC=CS^2$
(C) $AC.BC=8$
(D) $AC=BC$

21. If $y^2=4ax$ be a parabola and (α, β) be a point, from where three normals are drawn to parabola, then:

(A) if two normals are coincidental, then $27a\beta^2=4(\alpha-2a)^3$
(B) if two normals are coincidental, then $4a\beta^2=27(\alpha-2a)^3$
(C) if these three normals cut the axis of the parabola at points, whose distances from vertex are in A.P., then $27a\beta^2=2(\alpha-2a)^3$
(D) if these three normals cut the axis of parabola points, whose distances from vertex are in A.P., then $2a\beta^2=27(\alpha-2a)^3$

22. The focus of parabola is (2, 3) and it is touching the co-ordinate axes. Then:

(A) equation of axis of parabola is $3x-2y=0$
(B) length of latus rectum of parabola is $\dfrac{24}{\sqrt{13}}$
(C) equation of tangent at vertex is $3x+2y-6=0$
(D) equation of directrix is $3x+2y=0$

23. If the equation of a conic is $y^2+2ax+2by+c=0$, then:

(A) it is an ellipse
(B) it is a parabola
(C) its latus rectum is a
(D) its latus rectum is $2a$

24. Let 'L' be the point $(t, 2)$ and 'M' be a point on the y-axis such that 'LM' has slope '$-t$'. Then the locus of the midpoint of 'LM', as 't' varies over all real values, is a parabola, whose:

(A) vertex is $(0, 2)$

(B) latus rectum is of length 2 units

(C) focus is $\left(0, \frac{17}{8}\right)$

(D) directrix is $8y - 15 = 0$

25. The straight line $y + x = 1$ touches the parabola, if:

(A) $x^2 + 4y = 0$

(B) $x^2 - x + y = 0$

(C) $4x^2 - 3x + y = 0$

(D) $x^2 - 2x + 2y = 0$

26. Let 'P' be a point dividing a chord of the parabola $y^2 = 4x$ having slope 2 in the ratio 1:2, then the locus of 'P' is:

(A) parabola

(B) ellipse

(C) circle or hyperbola

(D) a conic with latus rectum $\frac{4}{9}$

27. The line $2x - y = 1$ intersect the parabola $y^2 = 4x$ at the point A and B and the normal at A and B intersect each other at the point G. If a third normal to the parabola through G meets the parabola at C, then:

(A) sum of the abscissa and ordinate of the point C is -1

(B) the normal at C passes through the lower end of the latus rectum of the parabola

(C) centroid of the ΔABC lies at the focus of the parabola

(D) normal at C has the gradient 1

28. If two distinct chords of a parabola $y^2 = 4ax$ passing through the point $(a, 2a)$ are bisected by the line $x + y = 1$, then the length of latus rectum can be:

(A) 2

(B) 3

(C) $\frac{5}{2}$

(D) 4

29. If the parabola $y = (a - b)x^2 + (b - c)x + (c - a)$ touches the x-axis in the interval $(0, 2)$, then the line $ax + by + c = 0$:

(A) always passes through a fixed point

(B) represents the family of parallel lines

(C) data insufficient

(D) passes through the point $(-2, 1)$

30. Two parabolas with same axis, focus of each being exterior to the other and the latus rectum being $4a$ and $4b$ are given. The locus of the midpoints of the intercepts between the parabolas made on the lines parallel to the common axis is:

(A) straight line, if $a = b$

(B) parabola, if $a \neq b$

(C) parabola, for all a and b

(D) ellipse, if $b > a$

31. The equation of the common tangents to the parabolas $y = x^2$ and $y = -(x - 2)^2$ is:

(A) $y = 4(x - 1)$

(B) $y = 0$

(C) $y = -4(x - 1)$

(D) $y = -30x - 50$

32. If a parabola touches x-axis at $(4, 0)$ and y-axis at $(0, 2)$, then:

(A) equation of directrix is $2x + y = 0$

(B) its focus is $\left(\frac{4}{5}, \frac{8}{5}\right)$

(C) the x-intercept of the axis of parabola is $\frac{-12}{5}$

(D) its latus rectum is $\frac{32}{5}$

33. Consider the parabola, whose equation is $x^2 + 2xy + y^2 - 22x + 26y + 25 = 0$ then, which of the following are correct?

(A) The vertex is $(0, -1)$

(B) The length of the latus rectum is 24 units

(C) Equation of the axis is $x + y + 1 = 0$

(D) Two real and distinct tangents can be drawn from the point $(1, 2)$ to the parabola

34. If a line L touches the curves $y^2 = 4x$ and $x^2 = 4y$ at the points P and Q, respectively and O is the origin, then:

(A) equation of the line L is $x + y + 1 = 0$

(B) length $PQ = 3\sqrt{2}$

(C) area of the region POQ bounded by the two curves and the line L is $\frac{5}{6}$ sq. units

(D) if the normal at P on the curve $y^2 = 4x$ meets the same curve again at R, then R is $(9, 6)$

35. If $12x = k$ is directrix of $y^2 + 4y + 3x = 0$, then:

(A) equation of latus rectum is $12x - 7 = 0$

(B) value of k is 25

(C) equation of tangent at vertex $3x - 4 = 0$

(D) if PS and SQ and segment of focal chord, then $\frac{1}{PS} + \frac{1}{SQ}$ is equal to $\frac{4}{3}$

36. The chord $x - y = 1$ cuts the parabola $y^2 - 4x = 0$ at P and Q with P in IVth quadrant and Q in Ist quadrant. If normals at P and Q meet at R, then:

(A) slope of normal at P is $\sqrt{2} - 1$

(B) slope of tangent at Q is $\sqrt{2} - 1$

(C) slope of normal through R other than normals at P and Q is 2

(D) point of concurrency of normals is inside the parabola

37. If straight line $x + y - 2 = 0$ touches a parabola at (1, 1) with focus at origin, then:

(A) its directrix passes through the point (1, 3)

(B) number of such parabolas is 1

(C) number of such parabolas is 2

(D) its axis passes through the point (3, 3)

38. If $A\left\{(x,y) \mid y \geq x^2\right\}$ and $B = \left\{(x,y) \mid x^2 + (y-a)^2 \leq 1\right\}$ and $A \cap B = B$, then a can be:

(A) $\frac{5}{4}$ (B) $\frac{15}{8}$

(C) $-\frac{1}{4}$ (D) $\frac{29}{31}$

39. A tangent is drawn at any point (l, m), $l, m \neq 0$ on the parabola $y^2 = 4ax$ and the tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ (where, $a > 0$) such that all chords of contact pass through a fixed point (α, β), then:

(A) $\alpha l > 0, \beta m > 0$ (B) $\alpha l < 0, \beta m > 0$

(C) $2\left(\frac{l}{\alpha}\right)^2 + \left(\frac{m}{\beta}\right)^2 = 8$ (D) $4\left(\frac{l}{\alpha}\right) + \left(\frac{m}{\beta}\right)^2 = 0$

40. The curves $x^2 + y^2 + 6x - 24y + 72 = 0$ and $x^2 - y^2 + 6x + 16y - 46 = 0$ intersect at four points P, Q, R and S lying on a parabola. Let A be the focus of the parabola, then:

(A) $AP + AQ + AR + AS = 20$

(B) $AP + AQ + AR + AS = 40$

(C) vertex of the parabola is at (–3, 1)

(D) co-ordinates of A are (–3, 1)

41. The directrix of a parabola is $2x - y = 1$. The focus lies on y-axis. The locus of one end of the latus rectum is:

(A) $7x - y - 1 = 0$ (B) $2x - y - 1 = 0$

(C) $2x + y + 1 = 0$ (D) $3x + y + 1 = 0$

42. The parabola $y^2 = 4x$ and $x^2 + y^2 = 12$ intersect at P and Q. If the tangents at these points to the circle and the parabolas intersect at A and B, respectively, then:

(A) $PAQB$ is a rhombus

(B) Area of quadrilateral $PAQB$ is $16\sqrt{2}$ sq. units

(C) Angle between the tangents to the parabola at P, Q is $\tan^{-1}(2\sqrt{2})$

(D) P, A, B and Q are concyclic

43. The locus of the midpoint of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola, whose:

(A) latus rectum is half the latus rectum of the original parabola

(B) vertex is $\left(\frac{a}{2}, 0\right)$

(C) directrix is y-axis

(D) focus has the co-ordinates $(a, 0)$

44. P is a point on the parabola $y^2 = 4ax$ $(a > 0)$, whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter, then it intersects the x-axis at a point, whose co-ordinates are:

(A) $(-3a, 0)$ (B) $(-a, 0)$

(C) $(-2a, 0)$ (D) $(a, 0)$

45. The equation of the tangent to the parabola $y = (x - 3)^2$ parallel to the chord joining the points (3, 0) and (4, 1) is:

(A) $2x - 2y + 6 = 0$ (B) $2y - 2x + 6 = 0$

(C) $4y - 4x + 13 = 0$ (D) $4x - 4y = 13$

46. Locus of the centre of circles described on focal chords of $y^2 = 4x$ as diameter:

(A) is a parabola of latus rectum 2

(B) is a parabola with vertex (1, 0)

(C) always touches $y = 2x - \frac{7}{4}$

(D) is symmetrical about x-axis

47. A parabola touching both the co-ordinate axes lies in the Ist quadrant and has its focus at (3, 4). Then:

(A) equation of tangent at vertex is $4x + 3y = 12$

(B) equation of directrix is $3x + 4y = 0$

(C) latus rectum of the parabola is $\frac{48}{5}$

(D) equation of axis is $4x + 3y = 0$

48. The focus of a parabola, whose axis parallel to x-axis is $S(3, 1)$. The normal at $P(h, k)$ is $x + 2y - 10 = 0$ and M is the foot of the perpendicular from P to the directrix, then:

(A) $h + k = 5$

(B) area of ΔSPM is 10 sq. units

(C) length of latus rectum is 16 units

(D) vertex of parabola is (–1, 1)

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

$y = f(x)$ is a parabola of the form $y = x^2 + ax + 1$. Its tangent at the point of intersection of y-axis and the parabola, also touch the circle $x^2 + y^2 = r^2$. It is known that no point of the parabola is below x-axis. Then answer the following.

1. The radius of the circle, where 'a' attains its maximum value is:

(A) $\frac{1}{\sqrt{10}}$ (B) $\frac{1}{\sqrt{5}}$

(C) 1 (D) $\sqrt{5}$

2. The slope of the tangent, when the radius of the circle is maximum is:

(A) 0 (B) 1

(C) –1 (D) not defined

3. The minimum area bounded by the tangent and the co-ordinate axes is:

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$

(C) $\frac{1}{2}$ (D) 1

Comprehension 2

A parabola touches the bisectors of the angle obtained by the lines $x + 2y + 3 = 0$ and $2x + y + 3 = 0$ at the points (1, 1) and (0, –2).

4. The focus of the parabola is:

(A) (2, 3) (B) (–2, 3)

(C) $\left(\frac{2}{5}, \frac{7}{5}\right)$ (D) $\left(\frac{1}{5}, -\frac{7}{5}\right)$

5. The equation of the directrix is:

(A) $3x + y + 4 = 0$ (B) $x - 3y - 2 = 0$

(C) $8x - y + 7 = 0$ (D) none of these

6. The length of the latus rectum is:

(A) $\frac{16}{5^{\frac{3}{2}}}$ (B) $\frac{16\sqrt{2}}{5}$

(C) $\frac{16\sqrt{5}}{2^{\frac{3}{2}}}$ (D) $\frac{16\sqrt{2}}{5^{\frac{3}{2}}}$

Comprehension 3

Let S_1 be the parabola having equation $y^2 - 4ax = 0$. Through the vertex A of S_1, two chords namely, AP and AQ are drawn, which make an angle $\frac{\pi}{4}$ with one another.

7. If PQ be a focal chord of the curve S_1, then the area of the ΔAPQ equals:

(A) a^2 (B) $\frac{3}{2}a^2$

(C) $\frac{3}{4}a$ (D) $\frac{5}{4}a^2$

8. The line PQ always touches a curve S_2, whose equation is:

(A) $128a(x - 12a) = y^2$

(B) $(x - 12a)^2 - 8y^2 = 128a^2$

(C) $128y^2 + (x - 12a)^2 = 8a^2$

(D) $(x - 12a)^2 + 8y^2 = 128a^2$

9. If length of latus rectum of S_1 be l_1 and that of S_2 be l_2, then $\frac{l_2}{l_1}$ equals to:

(A) 32 (B) 4

(C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

Comprehension 4

$P_1 : y^2 = 4ax,\ P_2 : y^2 = -4ax,\ L : y = x$

Use the above data to answer the following questions.

10. Equation of tangent at the point on the parabola P_1, where the line L meets the parabola is:

(A) $x - 2y + 4a = 0$ (B) $x + 2y - 4a = 0$

(C) $x + 2y - 8a = 0$ (D) $x - 2y + 8a = 0$

11. Area of the rectangle formed by joining the ends of latus rectum of the two parabolas P_1 and P_2 is:

(A) $2a^2$ sq. units (B) $4a^2$ sq. units

(C) $8a^2$ sq. units (D) $16a^2$ sq. units

Comprehension 5

If the equation of parabola is $9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$, then answer the following questions.

12. Focus:

(A) $\left(\frac{43}{25}, 1\right)$ (B) $\left(\frac{-43}{25}, \frac{-1}{100}\right)$

(C) $\left(\frac{-43}{25}, \frac{-129}{50}\right)$ (D) $\left(\frac{-43}{25}, \frac{-129}{100}\right)$

13. Equation of directrix:

(A) $4x + 3y = 0$ (B) $16x + 12y + 53 = 0$

(C) $16x + 12y + 15 = 0$ (D) $4x + 3y + 1 = 0$

14. Equation of axis:

(A) $3x + 4y = 1$ (B) $3x - 4y = 0$

(C) $3x + 4y = 0$ (D) $4x + 3y = 0$

Comprehension 6

Point $A(8, 8)$ lies on the parabola $y^2 = 8x$. From point A, a focal chord is drawn, which cuts the given parabola at point B. Normals at point A and point B intersect at point C. From point C, a normal is drawn (other than the above two), which cuts the parabola at point D.

15. Equation of normal at point B is:

(A) $x + 2y = \frac{9}{2}$ (B) $x - 2y = \frac{9}{2}$

(C) $2y + x = \frac{9}{2}$ (D) $2y - x = \frac{9}{2}$

16. x-co-ordinate of point D is:

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$

(C) $\frac{7}{2}$ (D) $\frac{9}{2}$

17. Co-ordinate of point C is:

(A) $\left(\frac{9}{2}, 3\right)$ (B) $\left(\frac{17}{2}, 3\right)$

(C) $\left(\frac{21}{2}, 3\right)$ (D) $\left(\frac{13}{2}, 3\right)$

Comprehension 7

Let $P(h, k)$ be the midpoint of the chord AB of the parabola $y^2 = 4ax$ and then its equation will be $ky - 2ax + 2ah - k^2 = 0$. The locus of the mid-point of chords of the parabola which,

18. Subtend a constant angle α at the vertex is $(y^2 - 2ax + 8a^2)^2 \tan^2\alpha = \lambda a^2(4ax - y^2)$, where λ is:

(A) 4 (B) 8

(C) 16 (D) 32

19. Are such that the focal distances of their extremities are in the ratio 2:1 is $9(y^2 - 2ax)^2 = \lambda a^2 (2x - a)(4x + a)$, where λ is:

(A) 4 (B) 8

(C) 16 (D) 12

Comprehension 8

Normals at three points P, Q, R on the parabola $y^2 = 4ax$ meet at (α, β), then answer the following questions.

20. The centroid of the ΔPQR must be:

(A) $\left(\frac{\alpha - 2a}{3}, 0\right)$ (B) $\left(\frac{2\alpha - 4a}{3}, 0\right)$

(C) $\left(\frac{\alpha}{3}, \frac{\beta}{3}\right)$ (D) none of these

21. The orthocenter of the ΔPQR must be at:

(A) $\left(\alpha + 6a, \frac{\beta}{2}\right)$ (B) $\left(\alpha - 3a, \frac{\beta}{2}\right)$

(C) $\left(\alpha - 6a, \frac{-\beta}{2}\right)$ (D) none of these

22. The circumcenter of ΔPQR must be:

(A) $\left(\frac{\alpha + 2a}{2}, -\frac{\beta}{4}\right)$ (B) $\left(\frac{\alpha + 2a}{4}, \frac{\beta}{4}\right)$

(C) $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right)$ (D) none of these

Comprehension 9

Consider the parabola $y^2 = 4x$. Let $A\ (-1, 0)$ and $B\ (0, 1)$. F is the focus of the parabola. Then, answer the following questions.

23. If $P(a, b)$ is a point on the parabola such that $||PA| - |PB||$ is maximum, then $a + b$:

(A) 4 (B) $5\sqrt{2}$

(C) 3 (D) $4\sqrt{3}$

24. If $P(a, b)$ is a point on the parabola such that $||PA| - |PB||$ is minimum, then the value of $2a + b$ is:

(A) 4 (B) 3

(C) $4\sqrt{2}$ (D) $2\sqrt{3}$

Comprehension 10

Let S be a given fixed point (focus), 'L' is given fixed line and P is a movable point such that $\frac{SP}{PM} = e$, where $e = 1$. Then, locus of P is called a parabola.

25. A normal drawn at P cuts the parabola $y^2 = 4ax$ at Q and PQ subtends a right angle at the focus of the parabola, then its length is:

(A) $a\sqrt{5}$ (B) $5a\sqrt{5}$

(C) $6a\sqrt{3}$ (D) $7a\sqrt{5}$

26. If tangents at A and B on $y^2 = 4ax$ intersect on $x + 16a = 0$, then AB always pass through:

(A) $(-16a, 0)$ (B) $(16a, 0)$

(C) $(a, 0)$ (D) $(-a, 0)$

27. The equation of the smallest circle touching both the parabolas $y^2 = 4(x-2)$ and $x^2 = 4(y-2)$ is:

(A) $x^2 + y^2 - 10x - 10y + 6 = 0$

(B) $x^2 + y^2 - 5x - 5y + 12 = 0$

(C) $x^2 + y^2 - 2x - 2y - 1 = 0$

(D) $x^2 + y^2 - x - y - 3 = 0$

Comprehension 11

A parabola is drawn through two given points $A(1,0)$ and $B(-1,0)$ such that its directrix always touches the circle $x^2 + y^2 = 4$.

28. The equation of directrix is of the form:

(A) $x \cos\alpha + y \sin\alpha = 1$

(B) $x \cos\alpha + y \sin\alpha = 2$

(C) $x \cos\alpha + y \sin\alpha = 3$

(D) $x \tan\alpha + y \sec\alpha = 2\ x \tan\alpha + y \sec\alpha = 2$

29. The locus of focus of the parabola is:

(A) $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (B) $\frac{x^2}{4} + \frac{y^2}{5} = 1$

(C) $\frac{x^2}{3} + \frac{y^2}{4} = 1$ (D) $\frac{x^2}{5} + \frac{y^2}{4} = 1$

30. The maximum possible length of semi-latus rectum is:

(A) $2 + \sqrt{3}$ (B) $3 + \sqrt{3}$

(C) $4 + \sqrt{3}$ (D) $1 + \sqrt{3}$

Comprehension 12

Let, C: $y = x^2 - 3$, D: $y = kx^2$, L_1: $x = a$, L_2: $x = 1$, $(a \neq 0)$.

31. If the parabolas C and D intersect at a point A on the line L_1, then the tangent line L at A to the parabola D is:

(A) $2(a^2 - 3)x - ay + a^3 - 3a = 0$

(B) $2(a^2 - 3)x - ay + a^3 + 3a = 0$

(C) $(a^2 - 3)x - 2ay + 2a^3 + 6a = 0$

(D) $2(a^2 - 3)x - ay - a^3 + 3a = 0$

32. If the line L meets the parabola C at a point B on the line L_2, other than A, then 'a' is equal to:

(A) -3 (B) -2

(C) 2 (D) 30

33. If $a > 0$, the angle subtended by the chord AB at the vertex of the parabola C is:

(A) $\tan^{-1}\left(\frac{5}{7}\right)$ (B) $\tan^{-1}\left(\frac{1}{2}\right)$

(C) $\tan^{-1}(2)$ (D) $\tan^{-1}\left(\frac{1}{8}\right)$

Comprehension 13

The equation of normal to a parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$. From this, conclude that three normals real or imaginary can be drawn from point 'p'.

34. The locus of 'p' such that two normals make complementary angles with axis of parabola is:

(A) $y^2 = a(x + a)$ (B) $y^2 = a(x - a)$

(C) $y^2 = a(y - a)$ (D) none of these

35. The locus of 'p', if one normal is bisector of other two is:

(A) $27ay^2 = (2x - a)(x - 5a)^2$

(B) $y^2 = (x - 2a)(x - 5a)^2$

(C) $x^2 = (y - 2a)(y - 5a)^2$

(D) none of these

36. The locus of a point 'p', if the sum of the angles made by the normals with the axis is a constant is:

(A) a straight line (B) a parabola

(C) a circle (D) an ellipse

Comprehension 14

The chord AC of the parabola $y^2 = 4ax$ subtends an angle of 90° at points B and D on the parabola. If A, B, C and D are represented by t_1, t_2, t_3 and t_4, then:

37. Value of $\left|\frac{t_2 + t_4}{t_1 + t_3}\right|$ is:

(A) 0 (B) 1

(C) 2 (D) 4

38. Minimum value of $|t_1 - t_3|$ is:

(A) 0 (B) 1

(C) 2 (D) 4

39. The y-co-ordinate of the midpoint of the points of intersection of the tangents at A, C and B, D is:

(A) 0 (B) 1

(C) 2 (D) 4

Comprehension 15

Any point on the parabola $y^2 = 4ax$ can be considered as $x = at^2$, $y = 2at$ (where t is a parameter). Equation of tangent at $(at^2, 2at)$ is given by $ty = x + at^2$, if t_1, t_2 are parameters corresponding to two points on the parabola $y^2 = 4ax$, point of intersection of tangents at these point is given by $[at_1t_2, a(t_1 + t_2)]$. Now, answer the following questions.

40. Locus of point of intersection of tangents to a parabola $y^2 = 4ax$, whose chord of contact subtends an angle $\frac{\pi}{3}$ at the vertex, is:

(A) $3x^2 + 4y^2 + 40ax + 48a^2 = 0$
(B) $3x^2 - 4y^2 + 40ax + 48a^2 = 0$
(C) $4x^2 + 3y^2 + 40ax + 48a^2 = 0$
(D) $4x^2 - 3y^2 + 40ax + 48a^2 = 0$

41. TP and TQ are any two tangents to a parabola and the tangent at a third point R cuts them in P' and Q', then $\frac{TP'}{TP} + \frac{TQ'}{TQ}$ is:

(A) 1 (B) 2
(C) $\frac{1}{4}$ (D) $\frac{1}{2}$

42. Locus of point of intersection of tangents drawn at extremities of normal chord of the parabola $y^2 = 4x$ is:

(A) $\frac{4}{x^2} + y + 2 = 0$ (B) $\frac{4}{y^2} + y + 2 = 0$
(C) $\frac{4}{y^2} + x + 2 = 0$ (D) $\frac{4}{x^2} + y + 1 = 0$

Comprehension 16

The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $y - y_1 = \frac{-y_1}{2a}(x - x_1)$.

The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + tx = 2at + at^3$.

The equation of normal of slope m to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ at the point $(am^2, -2am)$.

43. If the normal at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then t^2 is:

(A) 1 (B) 2
(C) 3 (D) 4

44. The equations of the normals at the ends of the latus rectum of the parabola $y^2 = 4ax$ is:

(A) $x - y - 3a = 0$ (B) $x + y - 3a = 0$
(C) $x + y + 3a = 0$ (D) $x - y + 3a = 0$

45. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of the ordinates of P and Q is:

(A) $9a^2$ (B) $10a^2$
(C) $8a^2$ (D) a^2

Comprehension 17

If a parabola touches both the axes and its focus is (3, 4), then answer the following.

46. Equation of axis of this parabola is:

(A) $3x - 4y + 7 = 0$ (B) $3x - 4y = 0$
(C) $4x - 3y = 0$ (D) none of these

47. Length of its latus rectum is:

(A) $\frac{24}{5}$ (B) $\frac{48}{5}$
(C) $\frac{13}{5}$ (D) 48

48. The point where it touches the y-axis is:

(A) $\left(0, \frac{25}{4}\right)$ (B) $\left(0, \frac{24}{5}\right)$
(C) $\left(0, \frac{25}{3}\right)$ (D) $\left(0, \frac{4}{3}\right)$

Comprehension 18

If a source of light is placed at the fixed point of a parabola and if the parabola is a reflecting surface, then the ray will bounce back in a line parallel to the axis of the parabola.
On the basis of above information, answer the following questions.

49. A ray of light is coming along the line $y = 2$ from the positive direction of x-axis and strikes a concave mirror, whose intersection with the xy plane is a parabola $y^2 = 8x$, then the equation of the reflected ray is:

(A) $2x + 5y = 4$ (B) $3x + 2y = 6$
(C) $4x + 3y = 8$ (D) $5x + 4y = 10$

50. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror, whose equation is $y^2 + 10y - 4x + 17 = 0$. After reflection, the ray must pass through the point:

(A) (–2, –5) (B) (–1, –5)
(C) (–3, –5) (D) (–4, –5)

51. Two rays of light coming along the lines $y = 1$ and $y = -2$ form the positive direction of x-axis and strikes a concave mirror, whose intersection with the xy-plane is a parabola $y^2 = x$ at A and B, respectively. The reflected rays pass through a fixed point C, then the area of ΔABC is:

(A) $\frac{21}{8}$ sq. units (B) $\frac{19}{2}$ sq. units

(C) $\frac{17}{2}$ sq. units (D) $\frac{15}{2}$ sq. units

Comprehension 19

Tangent is drawn to the parabola $y^2 = 4x$ at the point P, which is the upper end of the latus rectum.

52. Image of the parabola $y^2 = 4x$ in the tangent line at the point P is:

(A) $(x + 4)^2 = 16y$ (B) $(x + 2)^2 = 8(y - 2)$

(C) $(x + 1)^2 = 4(y - 1)$ (D) $(x - 2)^2 = 2(y - 2)$

53. Radius of the circle touching the parabola $y^2 = 4x$ at the point P and passing through its focus is:

(A) 1 (B) $\sqrt{2}$

(C) $\sqrt{3}$ (D) 2

54. Area enclosed by parabola, the tangent at 'P' to the parabola and x-axis is:

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$

(C) $\frac{14}{3}$ (D) 4

Comprehension 20

If a circle is drawn of focal distance of any point P, lying on a parabola, as diameter and it touches the tangent at vertex of the parabola at the point, where the tangent to the parabola at point P meets the circle. Now, considering a parabola with focus at $S(-1, -1)$. Let $3x - y - 8 = 0$ be the equation of tangent to the parabola at point $P(7, 13)$.

55. The foot of perpendicular from focus upon the tangent to the parabola is:

(A) $\left(\frac{13}{5}, \frac{1}{5}\right)$ (B) $(2, -2)$

(C) $(-2, 2)$ (D) $\left(\frac{1}{5}, \frac{13}{5}\right)$

56. Slope of the normal to the circle through the point found in the previous question is:

(A) 4 (B) 8

(C) -4 (D) -8

57. Equation of tangent at vertex to the parabola is:

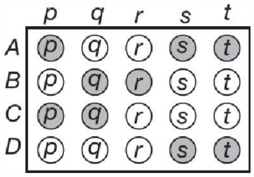
(A) $x - 8y + 14 = 0$ (B) $8x - y + 14 = 0$

(C) $x + 8y + 14 = 0$ (D) $8x + y + 14 = 0$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. If AB is a chord of the parabola $y^2 = 4x$ such that the normals at A and B intersect at the point $C(9, 6)$, then match the following.

COLUMN-I	COLUMN-II
(A) The length of AB	(p) 20
(B) The area of ΔABC	(q) $\frac{4}{\sqrt{13}}$
(C) The distance from vertex of parabola to the chord AB	(r) $\sqrt{13}$
(D) The area bounded by the co-ordinate axes and the line through AB	(s) $\frac{4}{3}$

(Continued)

2. Consider the curves $y^2 = 4|x|$, $x^2 = 4|y|$ and match the following.

COLUMN-I	COLUMN-II
(A) Number of distinct points of intersection of the two curves	(p) 4
(B) Radius of the largest circle inscribed in one loop formed by the two curves	(q) 8
(C) $x^2 + y^2 = 8$ intersects the given curves at n points, then n	(r) 5
(D) $x^2 + y^2 = 64$ intersects the curves at n points, then n	(s) $2\sqrt{10} - 4\sqrt{2}$
	(t) 10

3. Consider the parabola $(x-1)^2 + (y-2)^2 = \dfrac{(12x - 5y + 3)^2}{169}$ and match the following.

COLUMN-I	COLUMN-II
(A) Locus of point of intersection of perpendicular tangent	(p) $12x - 5y - 2 = 0$
(B) Locus of foot of perpendicular from focus upon any tangent	(q) $5x + 12y - 29 = 0$
(C) Line along which minimum length of focal chord occurs	(r) $12x - 5y + 3 = 0$
(D) Line about which parabola is symmetrical	(s) $24x - 10y + 1 = 0$

4. Consider the parabola $y^2 = 12x$ and match the following.

COLUMN-I	COLUMN-II
(A) Tangent and normal at the extremities of the latus rectum intersect the x-axis at T and G, respectively. The co-ordinates of midpoint of T and G are	(p) (0, 0)
(B) Variable chords of the parabola passing through a fixed point K on the axis, such that sum of the reciprocals of two parts of the chord through K, is a constant. Co-ordinates of K are	(q) (3, 0)
(C) All variable chords of the parabola subtending a right angle at the origin are concurrent at the point	(r) (6, 0)
(D) AB and CD are the chords of a parabola, which intersect at a point E on the axis. The radical axis of the two circles described on AB and CD as diameter always pass through the point	(s) (12, 0)

(*Continued*)

5. Match the following.

COLUMN-I	COLUMN-II
(A) If two distinct chords of a parabola $y^2 = 4ax$ passing through the point $(a, 2a)$ are bisected by the line $x + y = 1$, then the length of the latus rectum can be	(p) -1
(B) The parabola $y = x^2 - 5x + 4$ cuts the x-axis at P and Q. A circle is drawn through P and Q, so that the origin lies outside it. The length of tangent to the circle from the origin is equal to	(q) 0
(C) If $y + b = m_1(x + a)$ and $y + b = m_2(x + a)$ are two tangents to $y^2 = 4ax$, then $m_1 m_2$ is equal to	(r) 1
(D) If the point $(h, -1)$ is exterior to both the parabolas $y^2 = \|x\|$, then the integral part of h can be equal to	(s) 2

6. Match the following.

COLUMN-I	COLUMN-II
(A) $y = -2x + 12$ is a normal to the parabola $y^2 = 4x$ at the point, whose distance from the focus of the parabola is	(p) 2
(B) Length of the latus rectum of a parabola, whose focus is (2, 0) and directrix $3x + 4y + 4 = 0$ is	(q) 3
(C) If $2x + 3y = \alpha$, $x - y = \beta$ and $kx + 15y = \gamma$ are the three concurrent normals of parabola $y^2 = \lambda x$, then the value of k is	(r) 4
(D) If two distinct chords of a parabola $y^2 = 4ax$, passing through $(a, 2a)$ are bisected on the line $x + y = 1$, then length of the latus rectum can be	(s) 5

7. Match the statements/expressions in **COLUMN-I** with the open intervals in **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The x-co-ordinate of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$, from which all the three normals to the parabola are real is	(p) 4
(B) The x-co-ordinate of points on the axis of the parabola $4y^2 - 32x + 4y + 65 = 0$, from which all the three normals to the parabola are real is	(q) 5
(C) The x-co-ordinate of points on the axis of the parabola $4y^2 - 16x - 4y + 41 = 0$, from which all the three normals to the parabola are real is	(r) 6
(D) The x-co-ordinate of the point on the axis of the parabola, from which all the three normals drawn to the parabola $y^2 - 2y - 8x + 17 = 0$ are real is	(s) 7
	(t) 8

8. A ΔABC is inscribed in the parabola $y^2 = 4x$ with A as vertex and the orthocenter of the triangle as the focus of the parabola. Match the following.

COLUMN-I	COLUMN-II
(A) The distance of the centroid of the triangle from the vertex A is not more than	(p) 1
(B) The distance of the circumcenter of the triangle from the vertex A is more than	(q) 2
(C) The distance of the incenter of the triangle from the vertex A is not less than	(r) 3
(D) Distance between the incenter and the circumcenter of the triangle is less than	(s) 4
	(t) 5

9. Match the following.

COLUMN-I	COLUMN-II
(A) Radius of the largest circle, which passes through the focus of the parabola $y^2 = 4x$ and contained in it is	(p) 16
(B) If the shortest distance between parabola $y^2 = 4x$ and $y^2 = 2x - 6$ is d, then d^2 is	(q) 5
(C) The H.M. of the segments of a focal chord of a parabola $y^2 = 12x$ is	(r) 6
(D) If tangents drawn from P to the parabola $y^2 = 16x$ meet at A and B are perpendicular, then the least value of AB is	(s) 4

10. Match the following.

COLUMN-I	COLUMN-II
(A) If the normal chord at a point on the parabola $y^2 = 4x$ subtends a right angle at the vertex, then t^2 is	(p) 4

(Continued)

COLUMN-I	COLUMN-II
(B) The area of the triangle inscribed in the curve $y^2 = 4x$, the parameter of co-ordinates, whose vertices are 1, 2 and 4 is	(q) 2
(C) The number of distinct normal possible from $\left(\frac{11}{4}, \frac{1}{4}\right)$ to the parabola $y^2 = 4x$ is	(r) 3
(D) If the normal at $(a, 2a)$ on $y^2 = 4ax$ meets the curve again at $(at^2, 2at)$, then the value of $\lvert t - 1 \rvert$ is	(s) 6

11. Match the following.

COLUMN-I	COLUMN-II
(A) The locus of point of intersection of the perpendicular tangents of parabola $(y - 3)^2 = -4(x - 1)$ is	(p) $4x - 4y + 3 = 0$
(B) The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$, then the Cartesian form of its directrix is	(q) $x - 2 = 0$
(C) The directrix and focus of a parabola are $x + 2y + 10 = 0$ and (2, 1) respectively, then the equation of the tangent at the vertex is	(r) $x = 0$
(D) The lines which is/are tangent to the parabola $y^2 = 3x$	(s) $x + 2y + 3 = 0$

12. Match the following.

COLUMN-I	COLUMN-II
(A) If the circle with centre at (0, 5) touches the parabola $x^2 = 4y$, then radius of the circle is	(p) 3
(B) The length of the chord of the parabola $y^2 = x$, which is bisected at (2, 1) is	(q) 4
(C) The $x - 2y + 1 = 0$ is a tangent to the parabola $3y^2 = kx$, then k equal to	(r) 2
(D) If AFB is a focal chord of the parabola $y^2 = 4ax$ and $AF = 3$, $FB = 6$, then a is equal to	(s) $2\sqrt{5}$

(Continued)

13. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Normal of parabola $y^2 = 4x$ at P and Q meets at $R(x_2, 0)$ and tangent at P and Q meets at $T(x_1, 0)$, if $x_2 = 3$, then the area of quadrilateral $PTQR$ is	(p) 3
(B) The length of latus rectum plus the length of tangent PT will be	(q) 6
(C) The quadrilateral $PTQR$ can be inscribed in a circle, then the value of circumference $/4\pi$ will be	(r) 1
(D) The number of normals that can be drawn to the parabola from $R\ (x_2, 0)$ is $\sum t_1 = \frac{2}{c}, \sum t_1 t_2 = -\frac{20}{c^2}$ $\sum t_1 t_2 t_3 = \frac{4}{c}, t_1 t_2 t_3 t_4 = 1$	(s) 8

14. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If the normal at $R(t)$ on $y^2 = 4ax$ subtends a right angle at the focus, then t is	(p) 1
(B) The points of contact of the tangents drawn from (2, –3) to the parabola $y^2 = 4x$ are P and Q. The normals at P and Q intersect at $P(x_1, y_1)$, then $\frac{2x_1}{y_1}$ is	(q) 2

(Continued)

COLUMN-I	COLUMN-II
(C) PQ is a focal chord of $y^2 = 4ax$. The normals drawn at P and Q meet the parabola at P' and Q', then $\frac{P'Q'}{PQ}$ is	(r) 3

COLUMN-I	COLUMN-II
(D) If the normal chord PQ of $y^2 = 4ax$ drawn at P subtends a right angle at the vertex, then $\frac{SQ}{SP}$ is	(s) 4

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If the line $lx + my + n = 0$ touches $y^2 = 4x$ and $x^2 = 4y$ simultaneously, then find least value of $l + 2m + 3n$, where $l, m, n \in N$.

2. If the parabolas $y^2 = \lambda x$ and $25([x - 3]^2 + [y + 2]^2) = (3x - 4y - 2)^2$ are equal, then find λ.

3. If PQ is a focal chord of $y^2 = 4ax$ and PP' and QQ' are normal chords at P and Q, respectively, then find $\frac{P'Q'}{PQ}$.

4. The locus of point from which 3 normals drawn to the curve $y^2 + 2y - 4x + 5 = 0$ are such that two of them are perpendicular is $\left(y - \frac{\mu}{2}\right)^2 - (x - \lambda) = 0$, then find the value of $\lambda - 2\mu$.

5. If the line $y = mx + a$ meets the parabola $x^2 = 4ay$ in two points, whose abscissae are x_1 and x_2 such that $x_1 + x_2 = 0$, then find m.

6. If the angle between the 2 tangents drawn from the point (1, 4) to the parabola $y^2 = 12x$ is $\tan^{-1}(k)$, then find $2k$.

7. The straight line $3x - 4y + c = 0$ intersects the parabola $x^2 = 4y$ at the points A and B, if the tangents to the parabola at A and B are perpendicular. Find c.

8. If M is the foot of the perpendicular from a point P on a parabola, $y^2 = 8(x - 3)$ to its directrix and S is the focus of the parabola and SPM is an equilateral triangle, then find the length of each side of the triangle.

9. The normal at the point $P(ap^2, 2ap)$ meets the parabola $y^2 = 4ax$ again at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then find p^2.

10. From a point 'T', a tangent is drawn to touch the parabola $y^2 = 16x$ at $P(16,16)$. If 'S' be the focus of the parabola and $\angle TPS = \cot^{-1} k$, then find k.

11. Find the radius of the largest circle, which passes through the focus of the parabola $y^2 = 4x$ and contained in it.

12. Two perpendicular chords AB and AC are drawn through the vertex A of the parabola, $y^2 = 4ax$. If the locus of circumcentre of the ΔABC is $y^2 = 2ax - Ka^2$, then find K.

13. The tangent at the point $P(1, 1)$ on the curve $y = x^2 + bx - b$ meets the positive x and y axes at A and B, respectively and O is the origin. If b_1, b_2 and b_3 are the values of b for which the area of the ΔOAB is 2 sq. units with $b_1 < b_2 < b_3$, then find $|b_1| + |b_2 + b_3|$.

14. If 'm' is the slope of a common tangent of the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$, then find m^2.

15. A circle with its centre at the focus of the parabola $y^2 = 4ax$ and touching its directrix intersects the parabola at points $A$and B. If its length AB is equal to ka, then find k.

16. If parabola with focus $\left(\frac{2}{5}, \frac{4}{5}\right)$ touches x and y axes at A and B, respectively, then find the area of ΔOAB, where 'O' is the origin.

17. Consider the parabola $y^2 = 4x$. Let, P and Q be any two points $(4, -4)$ and $(9, 6)$ on the parabola. Let R be a moving point on the arc of the parabola between P and Q. If the maximum area of ΔRPQ is 'S', then find $(4S)^{\frac{1}{3}}$.

18. Tangents are drawn from the points on the parabola $y^2 = -8(x + 4)$ to the parabola $y^2 = 4x$. If the locus of midpoint of chord of contact is again a parabola, with the length of latus rectum as λ, then find 5λ.

19. Tangent and normal at the ends A and C of focal chord AC of a parabola $y^2 = 4x$ intersect at B and D, then find the minimum area of $ABCD$.

20. A parabola is drawn through two given points $A(1,0)$ and $B(-1,0)$ such that its directrix always touches the circle $x^2 + y^2 = 4$. If the maximum possible length of semi-latus rectum is 'k', then find $[k]$ (where [.] denotes greatest integer function).

21. A line passing through $(21,30)$ and normal to the curve $y = 2\sqrt{x}$. If m is slope of the normal, then find $m + 6$.

22. Length of the latus rectum of the parabola $\sqrt{x} + \sqrt{y} = 1$ is $\sqrt{k}$ then k is

23. If AFB is a focal chord of the parabola $y^2 = 4ax$ and $AF = 3, FB = 6$, then find the latus rectum of the parabola.

24. If $2x + 3y = \alpha, x - y = \beta$ and $kx + 15y = r$ are 3 concurrent normals of parabola $y^2 = \lambda x$, then find the value of k.

25. The locus of the midpoint of the portion of the normal to the parabola $y^2 = 16x$ intercepted between the curve and the axis is another parabola, whose latus rectum is ______.

26. If the shortest distance between the parabolas $y^2 = x - 1$ and $x^2 = y - 1$ is $\frac{3\sqrt{2}}{k}$, then find the numerical value of k.

27. Through the vertex 'O' of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. If the locus of midpoint of PQ is a parabola with latus rectum λ, then find λ.

28. If the parabola $41y = 7x^2 + 2x + 1$ and its origin is shifted to $(-2009, -2010)$, then find $\left[\left|\frac{\text{Coefficient of } y}{\text{Coefficient of } x^2}\right|\right]$ (where [x] denotes greatest integer function).

29. If (α, β) be the foot of normal drawn from the point (6, 2) on the parabola $y^2 = 4x$, then find $\beta(\alpha - 4)$.

30. The normal at any point $P(t^2, 2t)$ on the parabola $y^2 = 4x$ meets the curve again at Q and the area of ΔPOQ, with O being the origin is $\frac{k}{|t|}(1 + t^2)(2 + t^2)$. Find k.

31. The line $x + y + 2 = 0$ is a tangent to a parabola at point A, which intersects the directrix at B and tangent at vertex at C, respectively. If the focus of parabola is S (2, 0), then find $AC.BC$.

32. Find the number of distinct normals, which can be drawn from the point (2, 8) to the parabola $y^2 = 6x$.

33. For a parabola $y^2 - 2y - 4x + 9 = 0$, the tangent at some point B is $3y = x + 10$, where the normal at some other point K is $27y - 9x + 10 = 0$. Let α and β are the segments of the chord BK that is cut by the axes of the parabola. Find the number of integral values of 'a', for which the equation $3x^2 - (\alpha + \beta)x + \left(a^2 - 5a - \frac{353}{27}\right)\alpha\beta = 0$ has its real and distinct roots.

34. Two equal parabolas with vertical axis of symmetry p_1 and p_2 have their vertices at (0, 4) and (6, 0), respectively. If they touch each other at a point, then find the sum of abscissa and ordinate of point of contact.

35. A variable parabola is drawn through (–5, 0) and (5, 0), such that its directrix always touches the circle $x^2 + y^2 = 36$. If the locus of the focus of the parabola is $\frac{x^2}{\lambda^2} + \frac{y^2}{11} = 1$, then find the value of λ.

36. One side of a square is along $y = 2x - 17$. The other two vertices lie on $x^2 = y$. There are two such squares. If L and l are the sides of larger and smaller squares, then find $\frac{L}{l}$.

ARCHIVE: JEE MAIN

1. **[JEE MAIN 2020: Jan Attempt]**
Let a line $y = mx$ $(m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x-axis at the point Q. If area $(\Delta OPQ) = 4$ sq. units, then m is equal to ______ .

2. **[JEE MAIN 2020: Jan Attempt]**
The locus of a point which divides the line segment joining the point (0, –1) and a point on the parabola $x^2 = 4y$, internally in the ratio 1 : 2, is :

(A) $9x^2 - 3y = 2$
(B) $9x^2 - 12y = 8$
(C) $x^2 - 3y = 2$
(D) $4x^2 - 3y = 2$

3. **[JEE MAIN 2020: Jan Attempt]**

If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to :

(A) 128 (B) –64

(C) –128 (D) –32

4. **[JEE MAIN 2020: Jan Attempt]**

If one end of a focal chord AB of the parabola $y^2 = 8x$ is at $A\left(\frac{1}{2}, -2\right)$ then the equation of the tangent to it at B is:

(A) $2x - y - 24 = 0$ (B) $x + 2y + 8 = 0$

(C) $2x + y - 24 = 0$ (D) $x - 2y + 8 = 0$

5. **[JEE MAIN 2019]**

The shortest distance between the line $y = x$ and the curve $y^2 = x - 2$ is:

(A) $\frac{7}{4\sqrt{2}}$ (B) 2

(C) $\frac{7}{8}$ (D) $\frac{11}{4\sqrt{2}}$

6. **[JEE MAIN 2019]**

The tangent to the parabola $y^2 = 4x$ at the point, where it intersects the circle $x^2 + y^2 = 5$ in the Ist quadrant, passes through the point:

(A) $\left(-\frac{1}{3}, \frac{4}{3}\right)$ (B) $\left(\frac{3}{4}, \frac{7}{4}\right)$

(C) $\left(\frac{1}{4}, \frac{3}{4}\right)$ (D) $\left(-\frac{1}{4}, \frac{1}{2}\right)$

7. **[JEE MAIN 2019]**

The area (in sq. units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the x-axis is:

(A) $4\pi(2 - \sqrt{2})$ (B) $8\pi(3 - 2\sqrt{2})$

(C) $4\pi(3 + \sqrt{2})$ (D) $8\pi(2 - \sqrt{2})$

8. **[JEE MAIN 2019]**

If one end of a focal chord of the parabola, $y^2 = 16x$ is at (1, 4), then the length of this focal chord is:

(A) 24 (B) 20

(C) 25 (D) 22

9. **[JEE MAIN 2019]**

If the line $ax + y = c$, touches both the curves $x^2 + y^2 = 1$ and $y^2 = 4\sqrt{2}x$, then $|c|$ is:

(A) $\frac{1}{\sqrt{2}}$ (B) 2

(C) $\sqrt{2}$ (D) $\frac{1}{2}$

10. **[JEE MAIN 2019]**

If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which of the following is a valid choice for the ordered triad (a, b, c)?

(A) (1, 1, 3) (B) (1, 1, 0)

(C) $\left(\frac{1}{2}, 2, 0\right)$ (D) $\left(\frac{1}{2}, 2, 3\right)$

11. **[JEE MAIN 2019]**

The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}y + 4\sqrt{2} = 0$ is:

(A) $8\sqrt{2}$ (B) $6\sqrt{3}$

(C) $3\sqrt{2}$ (D) $2\sqrt{11}$

12. **[JEE MAIN 2019]**

A circle cuts a chord of length $4a$ units on the x-axis and passes through a point on the y-axis, distant $2b$ from the origin. Then, the locus of the centre of this circle is:

(A) an ellipse (B) a parabola

(C) a hyperbola (D) a straight line

13. **[JEE MAIN 2019]**

If the area of the triangle, whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and y-axis, is 250 sq. units, then the value of 'a' is:

(A) $5\sqrt{5}$ (B) $(10)^{\frac{2}{3}}$

(C) $5(2)^{\frac{1}{3}}$ (D) 5

14. **[JEE MAIN 2019]**

Let $A(4, -4)$ and $B(9, 6)$ be two points on the parabola, $y^2 = 4x$ and let C be any point on the arc AOB of this parabola, where O is the vertex of this parabola, such that the area of ΔACB is maximum. Then this maximum area (in sq. units) is:

(A) $\frac{625}{4}$ (B) $\frac{125}{4}$

(C) $\frac{75}{2}$ (D) $\frac{125}{2}$

15. **[JEE MAIN 2019]**

The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive directions of x-axis, is:

(A) $x = y \cot\theta - 2\tan\theta$

(B) $y = x \tan\theta + 2\cot\theta$

(C) $x = y \cot\theta + 2\tan\theta$

(D) $y = x \tan\theta - 2\cot\theta$

16. [JEE MAIN 2019]

The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point:

(A) $\left(\frac{1}{4}, \frac{7}{2}\right)$ (B) $\left(-\frac{1}{8}, 7\right)$

(C) $\left(\frac{7}{2}, \frac{1}{4}\right)$ (D) $\left(\frac{1}{8}, -7\right)$

17. [JEE MAIN 2019]

The tangents to the curve $y = (x - 2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point:

(A) $\left(\frac{5}{2}, -1\right)$ (B) $\left(-\frac{5}{2}, -1\right)$

(C) $\left(\frac{5}{2}, 1\right)$ (D) $\left(-\frac{5}{2}, 1\right)$

18. [JEE MAIN 2019]

The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}$. $(x > 0)$, is:

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{5}{4}$

(C) $\frac{3}{2}$ (D) $\frac{\sqrt{5}}{2}$

19. [JEE MAIN 2019]

Axis of a parabola lies along x-axis. If its vertex and focus are at distances 2 and 4, respectively from the origin, on the positive x-axis, then which of the following points does not lie on it?

(A) (8, 6) (B) (4, – 4)

(C) $(6, 4\sqrt{2})$ (D) $(5, 2\sqrt{6})$

20. [JEE MAIN 2018]

Tangent and normal are drawn at $P(16, 16)$ on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then the value of $\tan\theta$ is:

(A) 2 (B) 3

(C) $\frac{4}{3}$ (D) $\frac{1}{2}$

21. [JEE MAIN 2018]

If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is:

(A) 185 (B) 85

(C) 95 (D) 195

22. [JEE MAIN 2017]

The radius of a circle, having minimum area, which touches the curve, $y = 4 - x^2$ and the line, $y = |x|$ is:

(A) $4\left(\sqrt{2}+1\right)$ (B) $2\left(\sqrt{2}+1\right)$

(C) $2\left(\sqrt{2}-1\right)$ (D) $4\left(\sqrt{2}-1\right)$

23. [JEE (Main)-2016]

Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is:

(A) $x^2 + y^2 - 4x + 9y + 18 = 0$

(B) $x^2 + y^2 - 4x + 8y + 12 = 0$

(C) $x^2 + y^2 - x + 4y - 12 = 0$

(D) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

24. [JEE (Main)-2015]

Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1:3, then the locus of P is:

(A) $y^2 = 2x$ (B) $x^2 = 2y$

(C) $x^2 = y$ (D) $y^2 = x$

25. [JEE (Main)-2014]

The slope of the line touching both the parabolas, $y^2 = 4x$ and $x^2 = -32y$ is:

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$

(C) $\frac{1}{8}$ (D) $\frac{2}{3}$

26. [JEE (Main)-2013]

Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement-I: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement-II: If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

(A) Statement-I is true, Statement-II is true and Statement-II is a correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true and Statement-II is not a correct explanation for Statement-I.

(C) Statement-I is true and Statement-II is false.

(D) Statement-I is false and Statement-II is true.

27. **[AIEEE-2010]**

If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus is:

(A) $x = 1$ (B) $2x + 1 = 0$

(C) $x = -1$ (D) $2x - 1 = 0$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. **[JEE Advanced-2017]**

If a chord, which is not a tangent of the parabola $y^2 = 16x$, has the equation $2x + y = p$ and midpoint (h, k), then which of the following is/are the possible value(s) of p, h and k?

(A) $p = -2, h = 2, k = -4$ (B) $p = -1, h = 1, k = -3$

(C) $p = 2, h = 3, k = -4$ (D) $p = 5, h = 4, k = -3$

2. **[JEE (Advanced) 2014]**

The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P and Q and the parabola at the points R and S. Then the area of the quadrilateral $PQRS$ is:

(A) 3 (B) 6

(C) 9 (D) 15

3. **[JEE (Advanced) 2011]**

Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio 1:3. Then the locus of P is:

(A) $x^2 = y$ (B) $y^2 = 2x$

(C) $y^2 = x$ (D) $x^2 = 2y$

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. **[JEE Advanced-2016]**

The circle C_1: $x^2 + y^2 = 3$ with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the Ist quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii as $2\sqrt{3}$ and centres Q_2 and Q_3, respectively and if Q_2 and Q_3 lie on the y-axis, then:

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) area of the ΔOR_2R_3 is $6\sqrt{2}$ sq. units

(D) area of the ΔPQ_2Q_3 is $4\sqrt{2}$ sq. units

2. **[JEE Advanced-2016]**

Let P be the point on the parabola $y^2 = 4x$, which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then:

(A) $SP = 2\sqrt{5}$

(B) $SQ:QP = \left(\sqrt{5}+1\right):2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$

3. **[JEE Advanced 2015]**

Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the Ist quadrant and the area the ΔOPQ is $3\sqrt{2}$ sq. units, then which of the following is/are the co-ordinates of P?

(A) $\left(4, 2\sqrt{2}\right)$ (B) $\left(9, 3\sqrt{2}\right)$

(C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $\left(1, \sqrt{2}\right)$

4. **[JEE (Advanced) 2011]**

Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is given by:

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$

(C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

5. **[JEE (Advanced) 2010]**

Let A and B be two distinct point on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be:

(A) $\frac{-1}{r}$ (B) $\frac{1}{r}$

(C) $\frac{2}{r}$ (D) $\frac{-2}{r}$

6. **[JEE (Advanced) 2009]**

The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its; axis at points T and N, respectively. The locus of the centroid of the ΔPTN is a parabola, whose:

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is $x = 0$

(C) latus rectum is $\frac{2a}{3}$ (D) focus is $(a, 0)$

Reasoning Based Questions

This section contains Reasoning type questions, each having four choices (A), (B), (C) and (D) out of which ONLY ONE is correct. Each question contains STATEMENT 1 and STATEMENT 2. You have to mark your answer as

Bubble (A) If both statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

Bubble (B) If both statements are TRUE but STATEMENT 2 is not the correct explanation of STATEMENT 1.

Bubble (C) If STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

Bubble (D) If STATEMENT 1 is FALSE but STATEMENT 2 is TRUE.

1. [JEE (Advanced) 2007]

Statement-1: The curve $y = \frac{-x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$.

Statement-2: A parabola is symmetric about its axis.

(A) Statement-1 is true, Statement-2 is true and Statement-2 is correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is true and Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is true and Statement-2 is false.

(D) Statement-1 is false and Statement-2 is true.

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension I

Let the circles C_1: $x^2 + y^2 = 9$ and C_2: $(x - 3)^2 + (y - 4)^2 = 16$ intersect at the points X and Y such that another circle C_3: $(x - h)^2 + (y - k)^2 = r^2$ satisfies the following conditions.

(i) Centre of C_3 is collinear with the centres of C_1 and C_2.

(ii) C_1 and C_2 both lie inside C_3.

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expressions given in the **Column-I**, whose values are given in **Column-II**. Match **Column-I** with **Column-II**.

COLUMN-I	COLUMN-II
(I) $2h + k$	(P) 6
(II) $\frac{\text{Length of } ZW}{\text{Length of } XY}$	(Q) $\sqrt{6}$
(III) $\frac{\text{Area of } \Delta MZN}{\text{Area of } \Delta ZMW}$	(R) $\frac{5}{4}$
(IV) α	(S) $\frac{21}{5}$
	(T) $2\sqrt{6}$
	(U) $\frac{10}{3}$

1. [JEE Advanced 2019]

Which of the following is the only **CORRECT** combination?

(A) (I) – (U) (B) (I) – (S)

(C) (II) – (T) (D) (II) – (Q)

2. [JEE Advanced 2019]

Which of the following is the only **INCORRECT** combination?

(A) (IV) – (U) (B) (IV) – (S)

(C) (I) – (P) (D) (III) – (R)

Comprehension 2

Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$.

3. [JEE(Advanced) 2014]

The value of r is:

(A) $-\frac{1}{t}$ (B) $\frac{t^2 + 1}{t}$

(C) $\frac{1}{t}$ (D) $\frac{t^2 - 1}{t}$

4. [JEE(Advanced) 2014]
If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point, whose ordinate is:

(A) $\dfrac{(t^2+1)^2}{2t^3}$ (B) $\dfrac{a(t^2+1)^2}{2t^3}$

(C) $\dfrac{a(t^2+1)^2}{t^3}$ (D) $\dfrac{a(t^2+2)^2}{t^3}$

Comprehension 3

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangent to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, such that $a > 0$.

5. [JEE (Advanced) 2013]
Length of the chord PQ is:

(A) $7a$ (B) $5a$
(C) $2a$ (D) $3a$

6. [JEE (Advanced) 2013]
If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta$ is:

(A) $\dfrac{2}{3}\sqrt{7}$ (B) $\dfrac{-2}{3}\sqrt{7}$

(C) $\dfrac{2}{3}\sqrt{5}$ (D) $\dfrac{-2}{3}\sqrt{5}$

Comprehension 4

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the Ist and the IVth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

7. [JEE (Advanced) 2007]
The ratio of the areas of the triangles PQS and PQR is:

(A) $1:\sqrt{2}$ (B) 1:2
(C) 1:4 (D) 1:8

8. [JEE (Advanced) 2007]
The radius of the circumcircle of the ΔPRS is:

(A) 5 (B) $3\sqrt{3}$
(C) $3\sqrt{2}$ (D) $2\sqrt{3}$

9. [JEE (Advanced) 2007]
The radius of the incircle of the ΔPQR is:

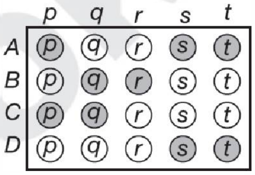
(A) 4 (B) 3

(C) $\dfrac{8}{3}$ (D) 2

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. [JEE(Advanced) 2013]
A line $L: y = mx + 3$ meets y-axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x, 0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the ΔEFG has a local maximum.

Match **COLUMN-I** with **COLUMN-II** and select the correct answer using the code given below in the options.

COLUMN-I	COLUMN-II
(P) m	(1) $\dfrac{1}{2}$
(Q) Maximum area of ΔEFG	(2) 4
(R) y_0	(3) 2
(S) y_1	(4) 1

Codes:

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	1	4
(D)	1	3	4	2

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. **[JEE Advanced 2015]**
If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is:

2. **[JEE Advanced 2015]**
Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is:

3. **[JEE (Advanced) 2012]**
Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola, then find the area of the ΔPQS.

4. **[JEE (Advanced) 2011]**
Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum then $\frac{\Delta_1}{\Delta_2}$ is

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** A	**3.** C	**4.** A	**5.** A	**6.** B	**7.** A	**8.** B	**9.** B	**10.** C
11. D	**12.** C	**13.** D	**14.** A	**15.** C	**16.** D	**17.** C	**18.** D	**19.** D	**20.** A
21. B	**22.** A	**23.** B	**24.** C	**25.** C	**26.** A	**27.** B	**28.** B	**29.** A	**30.** B
31. B	**32.** C	**33.** D	**34.** A	**35.** A	**36.** A	**37.** A	**38.** A	**39.** A	**40.** C
41. A	**42.** A	**43.** A	**44.** B	**45.** A	**46.** D	**47.** A	**48.** C	**49.** C	**50.** D
51. B	**52.** A	**53.** B	**54.** D	**55.** B	**56.** B	**57.** A	**58.** C	**59.** A	**60.** B
61. D	**62.** D	**63.** C	**64.** B	**65.** C	**66.** B	**67.** C	**68.** A	**69.** B	**70.** A
71. D	**72.** A	**73.** B	**74.** A	**75.** B	**76.** D	**77.** C	**78.** A	**79.** A	**80.** B
81. A	**82.** D	**83.** A	**84.** D	**85.** C	**86.** A	**87.** C	**88.** D	**89.** A	**90.** A
91. D	**92.** C	**93.** B	**94.** B	**95.** C	**96.** C	**97.** C	**98.** D	**99.** B	**100.** A
101. C	**102.** C	**103.** D	**104.** A	**105.** C	**106.** A	**107.** A	**108.** A	**109.** C	**110.** C
111. D	**112.** D	**113.** C	**114.** C	**115.** D	**116.** A	**117.** C	**108.** B	**109.** C	**120.** C
121. D	**122.** D								

Multiple Correct Choice Type Questions

1. A, B, C, D	**2.** A, C	**3.** A, B, C, D	**4.** A, C	**5.** A, B
6. A, B, C, D	**7.** A, B, C	**8.** B, C, D	**9.** A, B, D	**10.** A, B
11. A, C, D	**12.** B, C	**13.** A, B, C	**14.** A, C	**15.** A, B
16. A, B	**17.** A, B	**18.** B, C, D	**19.** A, B	**20.** A, B, C
21. A, C	**22.** B, C, D	**23.** B, D	**24.** A, C, D	**25.** A, B, C
26. A, D	**27.** A, B, C, D	**28.** A, B, C	**29.** A, D	**30.** A, B
31. A, B	**32.** A, B, C	**33.** A, C, D	**34.** A, B, C, D	**35.** A, B, C, D
36. A, B, C, D	**37.** A, B, D	**38.** A, B	**39.** B, D	**40.** B, C
41. A, D	**42.** A, B, C	**43.** A, B, C, D	**44.** A, D	**45.** C, D
46. A, B, C, D	**47.** A, C	**48.** A, B, C, D		

Linked Comprehension Type Questions

1. B	**2.** A	**3.** A	**4.** D	**5.** A	**6.** D	**7.** B	**8.** D	**9.** C	**10.** A
11. C	**12.** D	**13.** B	**14.** B	**15.** B	**16.** D	**17.** C	**18.** C	**19.** A	**20.** B
21. C	**22.** A	**23.** C	**24.** A	**25.** B	**26.** B	**27.** B	**28.** B	**29.** A	**30.** A
31. D	**32.** B	**33.** B	**34.** B	**35.** A	**36.** A	**37.** B	**38.** D	**39.** A	**40.** B
41. A	**42.** C	**43.** B	**44.** B	**45.** C	**46.** A	**47.** B	**48.** A	**49.** C	**50.** B
51. A	**52.** C	**53.** B	**54.** A	**55.** B	**56.** B	**57.** C			

Matrix Match/Column Match Type Questions

1. A → (r)	B → (p)	C → (q)	D → (s)
2. A → (r)	B → (s)	C → (q)	D → (q)
3. A → (r)	B → (s)	C → (p)	D → (q)
4. A → (q)	B → (q)	C → (s)	D → (p)
5. A → (r, s)	B → (s)	C → (p)	D → (p, q)
6. A → (s)	B → (r)	C → (s)	D → (p, q)
7. A → (p, q, r, s, t)	B → (s, t)	C → (q, r, s, t)	D → (s, t)

8. A → (s, t) B → (p, q, r, s) C → (p, q, r) D → (q, r, s, t)
9. A → (s) B → (q) C → (r) D → (p)
10. A → (q) B → (s) C → (q) D → (p)
11. A → (q) B → (r) C → (s) D → (p, r, s)
12. A → (q) B → (s) C → (p) D → (r)
13. A → (s) B → (q) C → (r) D → (r)
14. A → (q) B → (r) C → (r) D → (r)

Integer/Numerical Answer Type Questions

1. 6 **2.** 6 **3.** 3 **4.** 8 **5.** 0
6. 1 **7.** 4 **8.** 8 **9.** 2 **10.** 2
11. 4 **12.** 8 **13.** 5 **14.** 1 **15.** 4
16. 1 **17.** 5 **18.** 8 **19.** 8 **20.** 3
21. 1 **22.** 2 **23.** 8 **24.** 5 **25.** 4
26. 4 **27.** 2 **28.** 5 **29.** 4 **30.** 2
31. 8 **32.** 1 **33.** 8 **34.** 5 **35.** 6
36. 4

ARCHIVE: JEE MAIN

1. 0.50 **2.** B **3.** C **4.** D **5.** A **6.** B **7.** B **8.** C **9.** C
10. A or B or C or D **11.** B **12.** B **13.** D **14.** B **15.** C **16.** D **17.** A **18.** D
19. A **20.** A **21.** C **22.** Bonus **23.** B **24.** B **25.** A **26.** B **27.** C

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. C **2.** D **3.** C

Multiple Correct Choice Type Problems

1. A, B, C **2.** A, C, D **3.** A, D **4.** A, B, D **5.** C, D
6. A, D

Reasoning Based Questions

1. A

Linked Comprehension Type Questions

1. D **2.** B **3.** D **4.** B **5.** B **6.** D **7.** B **8.** B **9.** D

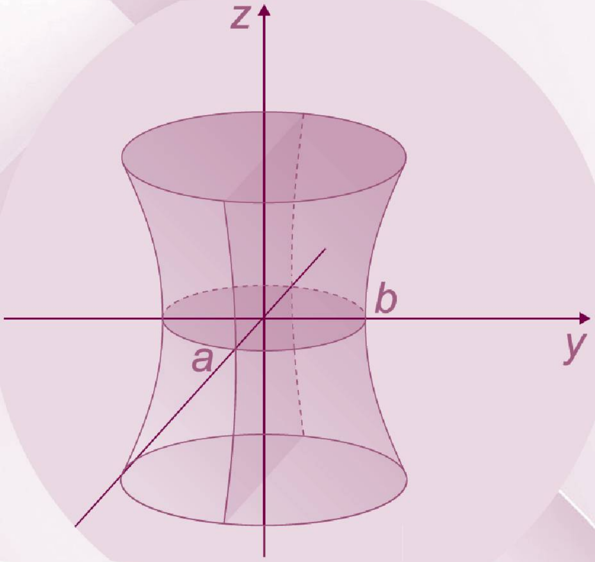
Matrix Match/Column Match Type Questions

1. A

Integer/Numerical Answer Type Questions

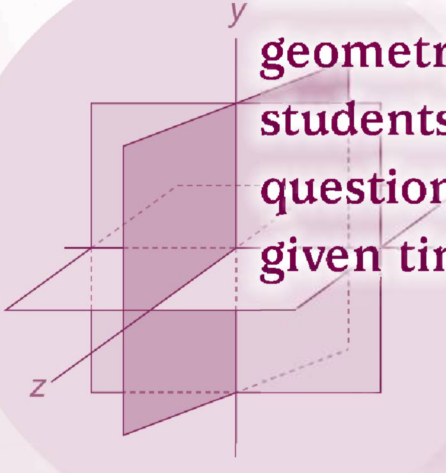
1. 2 **2.** 4 **3.** 4 **4.** 2

From the Author's Desk



“Dear Learners,

In Ellipse chapter, one should have a clear understanding of its parameters and their relative position while attempting questions. Problems based on topics, such as calculation of parameters of ellipse, auxiliary circle, tangent, chord of contact, focal chord, normal and its properties are the key concepts from examination point of view. To increase difficulty level, often ellipse may be merged with other chapters. Overall, to handle calculation part in coordinate geometry requires a lot of practice so students are advised to practice adequate questions based on various concepts in given time frame. Best of luck!!”



CHAPTER 5

Ellipse

WWW.JEE KS.IN

Learning Objectives

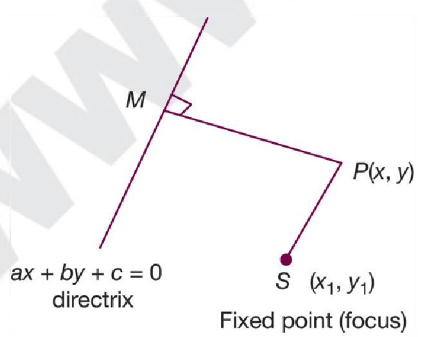
After reading this chapter, you will be able to understand concepts and problems based on:

(a) Definition of Ellipse and their parameters
(b) Tangent, chord, chord of contact
(c) Director circle, auxiliary circle
(d) Normal and properties of Ellipse
(e) Miscellaneous problems mixed with circle and parabola.

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

An ellipse is the locus of a point in a plane, which moves in the plane in such a way that the ratio of its distance from a fixed point (**focus**) to its distance from a fixed straight line (**directrix**) is always constant (**eccentricity**) and it is always less than unity.



The constant ratio is generally denoted by e and it is known as the eccentricity of the ellipse.

From the figure, $\dfrac{PS}{PM} = e$

$$\Rightarrow \quad PS = e\,PM$$

As $0 < e < 1$, then $PS < PM$, which tells that focus is closer to ellipse than directrix.

If $S(x_1, y_1)$ is the focus, $ax + by + c = 0$ is the directrix and $P(x, y)$ is any point on the ellipse, then by definition, $PS = e\,PM \Rightarrow PS^2 = e^2PM^2$

$$\Rightarrow \quad (x - x_1)^2 + (y - y_1)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}$$

$\therefore\ (a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = e^2\,(ax + by + c)^2$ is the equation of second degree form.

Remark(s)

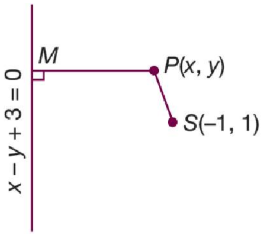
Any equation of second degree in x and y as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents equation of an ellipse, if

(i) $h^2 - ab < 0;\ 0 < e < 1$
(ii) $\Delta \neq 0$; where $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2$
(iii) Focus does not lie on directrix
(iv) Focus is closer to ellipse than directrix

ILLUSTRATION 1

Find the equation of an ellipse, whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$ and the directrix is $x - y + 3 = 0$.

SOLUTION



x − y + 3 = 0
M
P(x, y)
S(−1, 1)

Let $P(x, y)$ be any point on the ellipse, whose focus is $S(-1, 1)$ and the directrix is $x-y+3=0$. Draw PM, perpendicular from $P(x, y)$ on the directrix, $x - y + 3 = 0$. Then, by definition, $SP = ePM$.

$$\Rightarrow \quad (SP)^2 = e^2(PM)^2$$

$$\Rightarrow \quad (x+1)^2 + (y-1)^2 = \frac{1}{4}\left\{\frac{x-y+3}{\sqrt{2}}\right\}^2$$

$$\Rightarrow \quad 8(x^2+y^2+2x-2y+2) = x^2+y^2+9-2xy+6x-6y$$

$\therefore$ $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$ is the required equation of the ellipse.

ILLUSTRATION 2

If directrix $x - y + 2 = 0$, focus $(1, 2)$ and $e = \frac{1}{2}$, then find the equation of ellipse.

SOLUTION

From the definition of ellipse, $PS = ePM$.

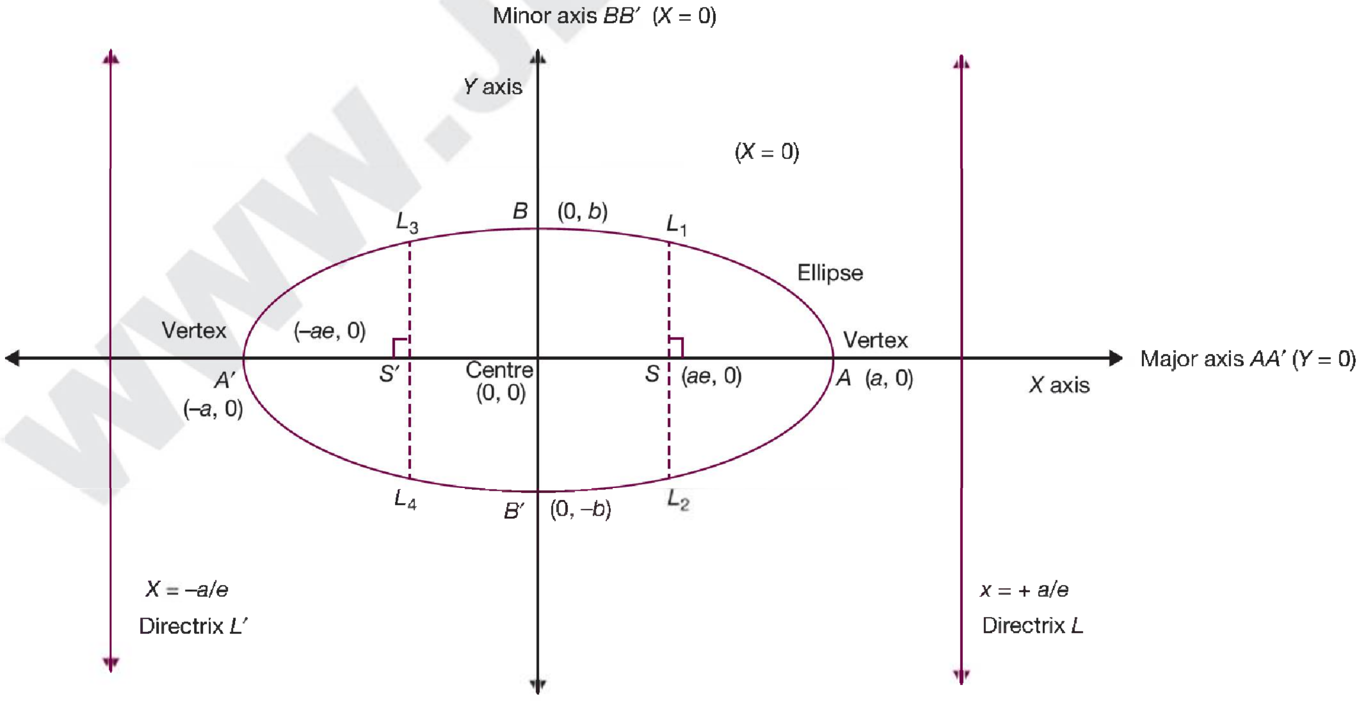
$$\Rightarrow \quad (x-1)^2 + (y-2)^2 = \frac{1}{4}\left|\frac{x-y+2}{\sqrt{2}}\right|^2$$

$\therefore$ $8((x-1)^2 + (y-2)^2) = (x - y + 2)^2$ is the required equation of ellipse.

STANDARD ELLIPSE

The standard ellipse is of the form, $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$

Let's analyze ellipse when $a > b$



Minor axis BB′ (X = 0)
Y axis
(X = 0)
L3
B
(0, b)
L1
Ellipse
Vertex
(−ae, 0)
Vertex
A′
(−a, 0)
S′
Centre
(0, 0)
S
(ae, 0)
A
(a, 0)
X axis
Major axis AA′ (Y = 0)
L4
B′
(0, −b)
L2
X = −a/e
Directrix L′
x = + a/e
Directrix L

Principal Axes

Major Axis

- Axis whose length is greater
- Both foci lie on it
- AA' major axis (if $a > b$)
- $\ell_{\text{major axis}} = AA' = 2a$ (if $a > b$)
- Equation $Y = 0$ (if $a > b$)

Minor Axis

- Axis with smaller length
- Parallel to directrix and passing through centre
- BB' Minor axis (if $a > b$)
- $\ell_{\text{minor axis}} = BB' = 2b$ (if $a > b$)
- Equation $X = 0$ (if $a > b$)

Working Rule

Parameters of Standard Ellipse ($a > b$)

Focus: $S, S'(\pm ae, 0)$ $X = \pm ae, Y = 0$

Directrix: $X = \frac{a}{e}$ and $X = \frac{-a}{e}$

Focus: $S(ae, 0)$ and $S'(-ae, 0)$ have their corresponding directrix as $L\left(X = \frac{a}{e}\right)$ and $L'\left(X = \frac{-a}{e}\right)$, respectively

Eccentricity (e): $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \left(\frac{\ell_{\text{minor axis}}}{\ell_{\text{major axis}}}\right)^2}$

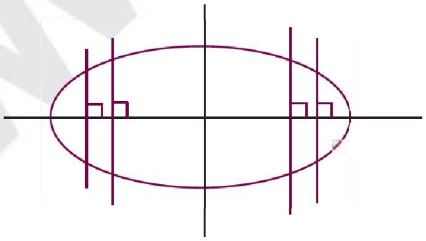
Centre: The center of a conic section is a point, which bisects every chord passing through it

$(0, 0)$ or $X = 0, Y = 0$.

Vertex: Point of intersection of conic section and axis (major axis) are called as vertices of ellipse.

$A(\pm a, 0)$ $X = \pm a, Y = 0$

Double ordinate: Any chord which is perpendicular to major axis is known as double ordinate.



Latus rectum: A double ordinate, which passes through the foci is known as latus rectum.

Extremities of latus rectum L_1, L_2, L_3, L_4 are $X = \pm ae$ and $Y = \pm\frac{b^2}{a}$

Length of latus rectum:

$$\ell_{L.R} = \frac{2b^2}{a} = \frac{(\ell_{\text{minor axis}})^2}{(\ell_{\text{major axis}})}$$

$= 2e$ (Distance between focus and corresponding directrix)

ILLUSTRATION 3

For the ellipse $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$, find centre, vertex, focus, directrix, principal axes and latus rectum.

SOLUTION

Given ellipse is $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$

where $X = x - 3$, $Y = y - 2$

$a = 5, b = 4$ and $a > b$

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

Centre $X = 0 \Rightarrow x - 3 = 0 \Rightarrow x = 3$

$Y = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

We get centre as $(3, 2)$.

Vertex $X = \pm a \Rightarrow x - 3 = \pm 5 \Rightarrow x = 8, -2$

$Y = 0 \Rightarrow y - 2 = 0$

We get vertices as $(8, 2)$ and $(-2, 2)$.

Focus $X = \pm ae \Rightarrow x - 3 = \pm 3 \Rightarrow x = 6, 0$

$Y = 0 \Rightarrow y = 2$

We get focus as $S(6, 2)$ $S'(0, 2)$

Directrix $X = \pm \dfrac{a}{e} \Rightarrow x - 3 = \pm 5 \cdot \dfrac{5}{3} \Rightarrow x = 3 \pm \dfrac{25}{3}$

Directrices $x = -\dfrac{16}{3}, x = \dfrac{34}{3}$

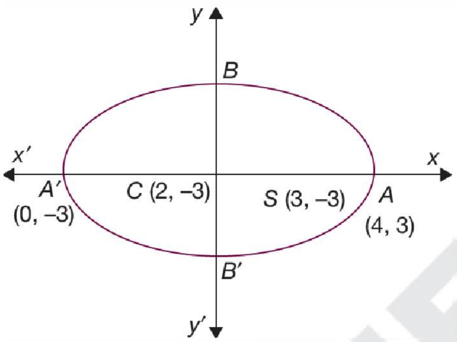
Length of latus rectum $\ell_{LR} = \dfrac{2b^2}{a} = \dfrac{32}{5}$ units

ILLUSTRATION 4

Find the equation of the ellipse, whose axes are parallel to the co-ordinate axes, having its centre at the point (2, –3) one focus at (3, –3) and one vertex at (4, –3).

SOLUTION

Let 2a and 2b be the major and minor axes of the ellipse.



Then its equation is

$$\frac{(x-2)^2}{a^2} + \frac{(y+3)^2}{b^2} = 1 \qquad \ldots(1)$$

We have centre, focus and vertex are on line parallel to x-axis, which means $a > b$.

$CA = a$ (Semi-major axis)

= Distance between centre and vertex

$$\Rightarrow \quad \sqrt{(4-2)^2 + (-3+3)^2} = a$$

$$\Rightarrow \quad a = 2 \qquad \ldots(2)$$

Since the distance between the focus and centre of an ellipse is equal to ae.

$$\therefore \quad CS = ae$$

$$\Rightarrow \quad \sqrt{(2-3)^2 + (-3+3)^2} = ae$$

$$\Rightarrow \quad ae = 1 \qquad \ldots(3)$$

From (2) and (3), we get, $e = \dfrac{1}{2}$

Now, $b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad b^2 = 4\left(1 - \frac{1}{4}\right) = 3$$

Substituting the values of a and b in (1), we get,

$$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{3} = 1$$

ILLUSTRATION 5

Find the eccentricity, foci and the length of the latus rectum of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

SOLUTION

The given equation of the ellipse is $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

$$\Rightarrow \quad x^2 - 2x + 4y^2 + 8y = -1$$

$$\Rightarrow \quad (x^2 - 2x + 1) + 4(y^2 + 2y + 1) = -1 + 1 + 4$$

$$\Rightarrow \quad (x-1)^2 + 4(y+1)^2 = 4$$

$$\Rightarrow \quad \frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{1^2} = 1 \qquad \ldots(1)$$

Equation (1) reduces to $\dfrac{X^2}{2^2} + \dfrac{Y^2}{1^2} = 1$

$$\Rightarrow \quad a^2 = 2^2 \text{ and } b^2 = 1$$

$$\Rightarrow \quad a = 2 \text{ and } b = 1$$

Let e be the eccentricity of the ellipse.

then, $b^2 = a^2(1 - e^2)$.

$$\Rightarrow \quad 1 = 4(1 - e^2)$$

$$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$$

The co-ordinates of foci are $(X = \pm ae, Y = 0)$

$$\Rightarrow \quad x - 1 = \pm 2 \times \frac{\sqrt{3}}{2} \text{ and } y + 1 = 0, \text{ i.e., } (x = 1 \pm \sqrt{3}, y = -1)$$

$\therefore$ Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{2(1)^2}{2} = 1$ unit

ILLUSTRATION 6

Find eccentricity of ellipse, whose minor axis is double of latus rectum.

SOLUTION

Length of minor axis = 2 (Length of latus rectum)

$$2b = 2\left(\frac{2b^2}{a}\right) \Rightarrow a = 2b$$

Now, $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

ILLUSTRATION 7

Find eccentricity of an ellipse, whose minor axis subtends 90° at focus.

SOLUTION

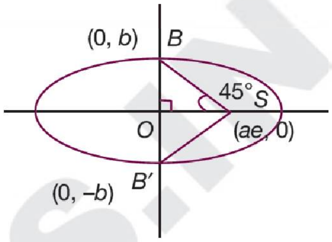
$\angle BSB' = 90°$ and $\angle BSO = 45°$

$OB = OS$

$\Rightarrow \quad b = ae$

$\Rightarrow \quad \frac{b}{a} = e = \sqrt{1-e^2}$

$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$

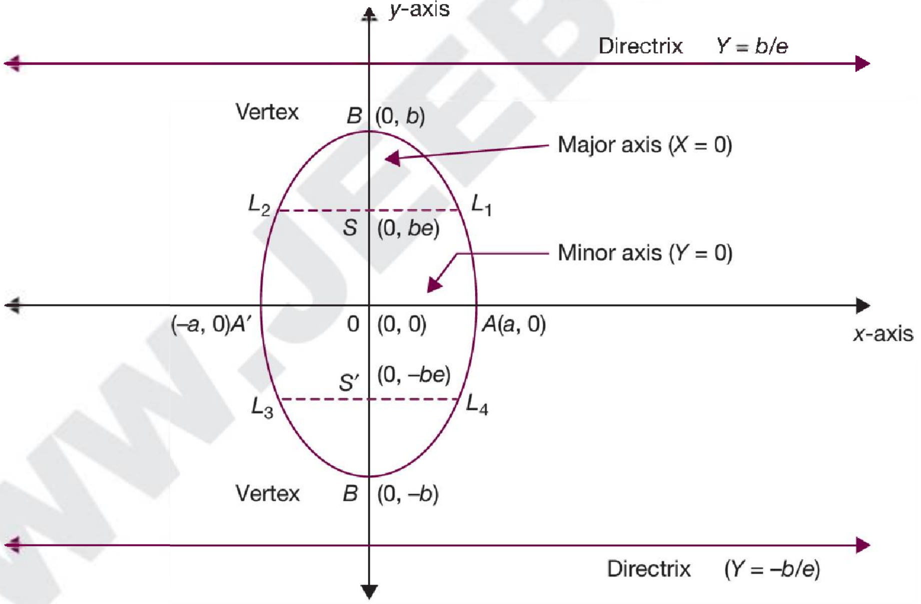

(0, b)
B
45°
S
O
(ae, 0)
(0, –b)
B′

Standard Ellipse when (*a* < *b*)

Equation of ellipse $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, if $a < b$

Problem Solving Technique

Interchange $X \leftrightarrow Y$ and $a \leftrightarrow b$ to get all parameters from standard Ellipse ($a > b$)


y-axis
Directrix
Y = b/e
Vertex
B
(0, b)
Major axis (X = 0)
L2
L1
S
(0, be)
Minor axis (Y = 0)
(–a, 0)A′
0
(0, 0)
A(a, 0)
x-axis
S′
(0, –be)
L3
L4
Vertex
B
(0, –b)
Directrix
(Y = –b/e)

Major axis BB'

$X = 0;\ \ell_{\text{major axis}} = 2b$

Minor axis AA'

$Y = 0;\ \ell_{\text{minor axis}} = 2a$

$e = \sqrt{1-\frac{a^2}{b^2}}$

Focus $(0, \pm be)$ $X = 0$, $Y = \pm be$

Directrix $\left(Y = \pm\frac{b}{e}\right)$

Vertex BB'

$X = 0,\ Y = \pm b$

Centre

$X = Y = 0$

Length of **latus rectum** $= \frac{2a^2}{b}$

Extremities of latus rectum L_1, L_2, L_3, L_4 are $\left(\pm\frac{a^2}{b}, \pm be\right)$

Comparison Chart for Parameters for Ellipse

Ellipse / Parameters	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a<b)$
Centre	$(0, 0)$	$(0, 0)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Equation of major axis	$y=0$	$x=0$
Length of minor axis	$2b$	$2a$
Equation of minor axis	$x=0$	$y=0$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Distance between directrices	$\frac{2a}{e}$	$\frac{2b}{e}$
Tangents at the vertices	$x=a, x=-a$	$y=b, y=-b$

ILLUSTRATION 8

Consider the ellipse $4x^2+y^2-8x+2y+1=0$. Find centre, vertex, focus, directrix, principal axes and latus rectum.

SOLUTION

Let us rewrite the given ellipse equation as

$$(x-1)^2+\frac{(y+1)^2}{4}=1$$

or $\frac{X^2}{1}+\frac{Y^2}{2^2}=1$ where, $X=x-1$ and $Y=y+1$

$a=1; b=2 \quad (a<b)$

$$e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$$

Focus $X=0 \Rightarrow x-1=0 \Rightarrow x=1$

$$Y=\pm be \Rightarrow y+1=\pm(2)\frac{\sqrt{3}}{2} \Rightarrow y=-1\pm\sqrt{3}$$

$\therefore$ Focus is $S(1, -1+\sqrt{3})$ $S'(1, -1-\sqrt{3})$

Centre $X=0 \Rightarrow x-1=0 \Rightarrow x=1$

$Y=0 \Rightarrow y+1=0 \Rightarrow y=-1$

$\therefore$ Centre is $C(1, -1)$

Vertex $X=0 \Rightarrow x-1=0 \Rightarrow x=1$

$Y=\pm b \Rightarrow y+1=\pm 2 \Rightarrow y=1, -3$

$\therefore$ Vertices are $(1, 1), (1, -3)$

Principal axes

Minor axis: $Y=0 \Rightarrow y=-1$

Major axis: $X=0 \Rightarrow x=1$

Directrix $Y=\pm\frac{b}{e} \Rightarrow y+1=\pm\frac{4}{\sqrt{3}}$

$\therefore$ Directrix is $y=-1+\frac{4}{\sqrt{3}}$ and $y=-1-\frac{4}{\sqrt{3}}$

$\therefore$ Length of latus rectum $=\frac{2a^2}{b}=\frac{2(1)}{2}=1$ unit

ILLUSTRATION 9

For an ellipse, focii $(\pm 4, 0)$ and $e=\frac{1}{3}$. Find equation of ellipse and length of the latus rectum.

SOLUTION

As we can observe, both focus lie on x-axis and centre $(0, 0)$.

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a>b)$

$$ae=4;\ a=12,\ e=\frac{1}{3}=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{b^2}{144}}$$

$\Rightarrow \quad b = 8\sqrt{2}$

Now, equation of ellipse is $\dfrac{x^2}{144}+\dfrac{y^2}{128}=1$

ILLUSTRATION 10

Find the equation of ellipse, whose centre is (1, 2) and one focus is at (6, 2), which also passes through (4, 6).

SOLUTION

Given, centre (1, 2) and focus (6, 2).

Line joining centre and focus is $y = 2$ (parallel to x-axis), then we can assume equation of ellipse as

$$\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1, \text{ where } a > b$$

It passes through point (4, 6) $\dfrac{9}{a^2}+\dfrac{16}{b^2}=1$ …(1)

Distance between centre and focus = ae = 5 units

$\Rightarrow \quad a^2e^2 = 25$

$\Rightarrow \quad a^2 - b^2 = 25$ …(2)

Solving (1) and (2), we get, $a^2 = 45$ and $b^2 = 20$

$\therefore$ Equation of ellipse is $\dfrac{(x-1)^2}{45}+\dfrac{(y-2)^2}{20}=1$

ILLUSTRATION 11

Find the equation of ellipse, whose foci are (2, 3) and (–2, 3) and whose semi-minor axis is $\sqrt{5}$.

SOLUTION

Both foci are on line $y = 3$ (parallel to x-axis) ($\because a > b$)

Distance between foci = $2ae$ = 4 units

$\Rightarrow \quad ae = 2 \Rightarrow a^2e^2 = 4 \Rightarrow a^2 - b^2 = 4, \qquad$ where $b = \sqrt{5}$

$\Rightarrow \quad a^2 = 9 \Rightarrow a = 3$ and $b = \sqrt{5}$

Centre of ellipse = Midpoint of foci (2, 3) and (–2, 3)

= (0, 3)

$\therefore$ Equation of ellipse is $\dfrac{(x-0)^2}{9}+\dfrac{(y-3)^2}{5}=1$

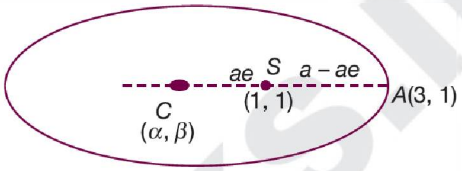
ILLUSTRATION 12

Find the equation of the ellipse, whose one vertex is (3, 1) and nearest focus is (1, 1) and $e = \dfrac{2}{3}$.

SOLUTION

Let the centre be (α, β).

Focus divides line joining centre C and vertex A in the ratio $ae : a–ae$ or $e : 1 - e$ or $\dfrac{2}{3}:\dfrac{1}{3}$ or 2:1.



$$(1, 1) = \left(\frac{6+\alpha}{3}, \frac{2+\beta}{3}\right)$$

$\Rightarrow \quad \alpha = -3$ and $\beta = 1$

We get centre (– 3, 1)

$SA = a(1-e) = 2 \Rightarrow a = 6$

$e = \dfrac{2}{3} = \sqrt{1-\dfrac{b^2}{a^2}}$

$\Rightarrow \quad \dfrac{4}{9} = 1 - \dfrac{b^2}{a^2}$

$\Rightarrow \quad b^2 = \dfrac{5}{9}a^2 = \dfrac{5}{9}(36) = 20$

Equation of ellipse is $\dfrac{(x+3)^2}{36}+\dfrac{(y-1)^2}{20}=1$

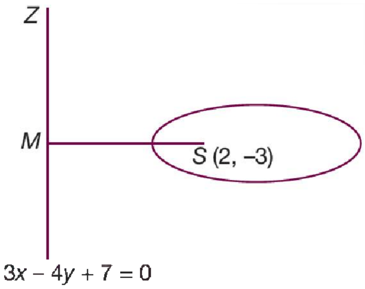
ILLUSTRATION 13

Find the length of the major axis of the ellipse $(5x-10)^2+(5y+15)^2 = \dfrac{(3x-4y+7)^2}{4}$

SOLUTION

The given ellipse is

$$(5x-10)^2+(5y+15)^2 = \frac{(3x-4y+7)^2}{4}$$



$$\Rightarrow\ (x-2)^2+(y+3)^2=\frac{1}{4}\left(\frac{3x-4y+7}{5}\right)^2$$

$$\therefore\quad e^2=\frac{1}{4}\quad\Rightarrow\quad e=\frac{1}{2}$$

Focus is (2, –3) and directrix is $3x-4y+7=0$

Now, distance between focus and directrix,

$$SM=\frac{a}{e}-ae$$

$$\text{Also, } SM=\left|\frac{6+12+7}{\sqrt{9+16}}\right|=5$$

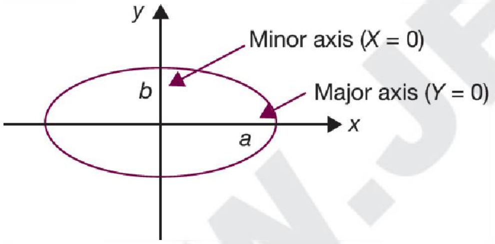
$$\therefore\quad 2a-\frac{a}{2}=5\quad\Rightarrow\quad a=\frac{10}{3}$$

$$\therefore\quad \text{Length of major axis }=2a=\frac{20}{3}\text{ units}$$

EQUATION OF ELLIPSE USING PRINCIPAL AXES

We should observe equation of ellipse closely

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\text{ where } a>b$$



Minor axis ($x=0$), length of minor axis $=2b$ units

Minor axis ($y=0$), length of major axis $=2a$ units

We can rewrite the equation of ellipse as,

$$\left(\frac{\text{Equation of major axis}}{\ell_{\text{semi minor axis}}}\right)^2+\left(\frac{\text{Equation of minor axis}}{\ell_{\text{semi major axis}}}\right)^2=1$$

NOTE:

Equation of principal axes is written in normalised form.

General form of straight line is $ax+by+c=0$ and the normalised form is $\frac{ax+by+c}{\sqrt{a^2+b^2}}=0$.

ILLUSTRATION 14

If principal axes of ellipse are of length 6, $2\sqrt{6}$ and equation are $3x-y+1=0$ and $x+3y-3=0$, respectively then find the equation of ellipse. Also, find eccentricity, centre and foci of ellipse.

SOLUTION

$\because$ $6>2\sqrt{6}$, which means $2a=6$; $2b=2\sqrt{6}$, we get $a=3$, $b=\sqrt{6}$

Now, we can conclude that $3x-y+1=0$ is major axis, as it has greater length.

and $x+3y-3=0$ is minor axis.

$$\text{Equation of ellipse}\left(\frac{\frac{3x-y+1}{\sqrt{9+1}}}{\sqrt{6}}\right)^2+\left(\frac{\frac{x+3y-3}{\sqrt{1+9}}}{3}\right)^2=1$$

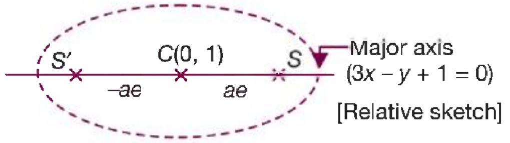
$$\frac{(3x-y+1)^2}{60}+\frac{(x+3y-3)^2}{90}=1$$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\frac{1}{\sqrt{3}}$$

We know that centre is point of intersection of given axes.

Now, on solving, we get centre as (0, 1).

Writing equation of major axis in parametric form, we get,



$$\tan\theta=3,\quad \cos\theta=\frac{1}{\sqrt{10}},\quad \sin\theta=\frac{3}{\sqrt{10}}$$

$$\frac{x-0}{\cos\theta}=\frac{y-1}{\sin\theta}=r$$

$$\frac{x-0}{\frac{1}{\sqrt{10}}}=\frac{y-1}{\frac{3}{\sqrt{10}}}=\pm ae=\pm\sqrt{3}$$

$$\text{Focii}\left(\pm\sqrt{\frac{3}{10}},\pm 3\sqrt{\frac{3}{10}}+1\right)$$

ILLUSTRATION 15

Identify major axis, minor axis and eccentricity of ellipse $\frac{(x-y+1)^2}{3}+\frac{(x+y-3)^2}{5}=1$.

SOLUTION

Given ellipse equation is (rewriting in normalized form)

$$\frac{\left(\frac{x-y+1}{\sqrt{2}}\right)^2}{\left(\sqrt{3}/\sqrt{2}\right)^2}+\frac{\left(\frac{x+y-3}{\sqrt{2}}\right)^2}{\left(\sqrt{5}/\sqrt{2}\right)^2}=1$$

Clearly, we can see that $\sqrt{\frac{5}{2}} > \sqrt{\frac{3}{2}}$

$$\therefore \quad a=\sqrt{\frac{5}{2}} \text{ and } b=\sqrt{\frac{3}{2}}$$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{3}{5}}=\sqrt{\frac{2}{5}}$$

$\therefore \quad x-y+1=0$ (Equation of major axis)

$x+y-3=0$ (Equation of minor axis)

Centre = Point of intersection of principal axes

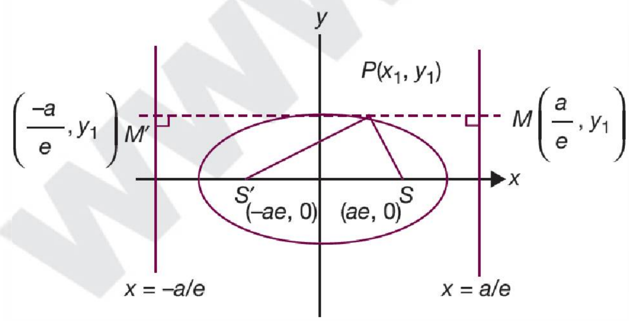
= (1, 2)

NOTES:

- Two ellipses are said to be equal, if their lengths of latus rectums are equal.
- Two ellipses are said to be similar, if their eccentricities are equal.

FOCAL DISTANCE

Distance of any point P on ellipse from focus is known as focal distance of point P.



Let the point $P(x_1, y_1)$ lies on ellipse, which has two focus.

Hence, we should get two focal distances.

From the definition of ellipse, we get,

$$PS=e(PM)=e\left(\frac{a}{e}-x_1\right)=a-ex_1$$

$$PS'=e(PM')=e\left(\frac{a}{e}+x_1\right)=a+ex_1$$

* Focal distance of point $P(x_1, y_1)$ are $a+ex_1$ and $a-ex_1$ if we add focal distances, we get

$$\ell(SP)+\ell(S'P)=2a=\text{Constant}=\ell_{\text{major axis}}$$

(By definition of ellipse)

Triangle rule says that sum of any two sides is greater than the third side.

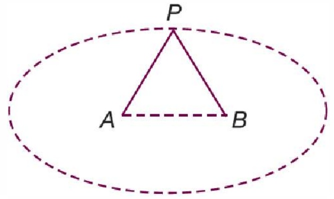
$PS + PS' > SS'$

$\Rightarrow \quad 2a > 2ae$

$\Rightarrow \quad 0 < e < 1$ for ellipse

Ellipse (Definition)

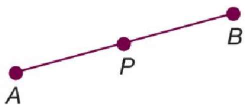
Locus of a point moving in a plane, whose sum of distances from two fixed point remains constant, these fixed points are focus.



$PA + PB = k$ (Constant)

(i) If $k > AB$, then locus of P is an ellipse.

(ii) If $k = AB$, then locus of P is a line segment joining A and B.



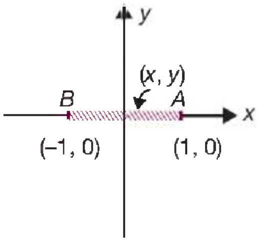
(iii) If $k < AB$, then no real locus of P is possible.

ILLUSTRATION 16

Identify the locus $P(x, y)$, which satisfies

$$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2}=2$$

SOLUTION



It is the sum of distances of variable point (x, y) from $A(1, 0)$ and $B(-1, 0)$ is 2 and $AB = 2$. Hence, point (x, y) lies on AB; it is x-axis.

Alternate

$$\sqrt{(x+1)^2+y^2} = 2-\sqrt{(x-1)^2+y^2}$$

$$\Rightarrow \quad (x+1)^2+y^2 = 4+(x-1)^2+y^2-4\sqrt{(x-1)^2+y^2}$$

$$\Rightarrow \quad \sqrt{(x-1)^2+y^2} = 1-x$$

On squaring, we get, $y = 0$ (x-axis).

ILLUSTRATION 17

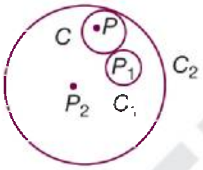
A circle C_1 lies completely inside a circle C_2, if a variable circle touches C_1 externally and C_2 internally, then identify the locus of centre of variable circle.

SOLUTION

Let, the centres of circles C_1, C_2, C_2 are P, P_1, P_2, respectively.

Circles C and C_1 touch externally, then $PP_1 = r_1 + r$

Circles C and C_2 touch internally, then $PP_2 = r_2 - r$



$$PP_1 + PP_2 = r_1 + r_2 = \text{Constant}$$

Hence, locus of P is an ellipse.

ILLUSTRATION 18

Find the sum of the distances of any point on the ellipse $3x^2+4y^2=12$ from its directrix.

SOLUTION

Given ellipse is

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad \dots(1)$$

We know that sum of distances of any point on ellipse from directrices $= \frac{2a}{e}$ units

Here, $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$

$\therefore$ Required sum $= \frac{2a}{e} = \frac{2\times 2}{(1/2)} = 2\times 2\times 2 = 8$

ILLUSTRATION 19

The perimeter of a triangle is 20 units and the points (–2, –3) and (–2, 3) are two of the vertices of it. Then find the locus of third vertex.

SOLUTION

Both the focii (–2, 3) and (–2, –3) lie on line $x = -2$

$\therefore \quad a < b$

$PS + PS' + SS' = 20$, where S and S' are foci, $SS' = 2be = 6$, then $PS + PS' = 14 = 2b$, $b = 7$.

Centre is the midpoint of S and S'

$\therefore$ Centre is $(-2, 0)$, $e = \frac{3}{7}$, $a = \sqrt{40}$

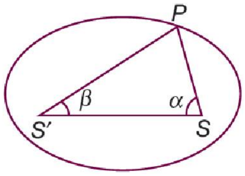
Locus of third vertex is ellipse $\frac{(x+2)^2}{40}+\frac{y^2}{49}=1$.

ILLUSTRATION 20

If S and S' are focii of an ellipse and P is any point on ellipse, then prove that

$$\tan\left(\frac{\angle PSS'}{2}\right)\tan\left(\frac{\angle PS'S}{2}\right) = \frac{1-e}{1+e}$$

SOLUTION



Let P is on ellipse.

Focal distances $PS = a - ex_1$

Or $\quad PS' = a + ex_1$

Distance between focii $SS' = 2ae$

Now, semi-perimeter $s = \frac{1}{2}(2a+2ae) = a + ae$ units

Using formula, $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$

$$\tan\left(\frac{\angle PSS'}{2}\right)\tan\left(\frac{\angle PS'S}{2}\right) = \frac{\Delta}{s(s-PS')}\cdot\frac{\Delta}{s(s-PS)}$$

$$= \frac{s(s-PS)(s-PS')(s-SS')}{s^2(s-PS')(s-PS)} = 1-\frac{SS'}{s}$$

$$= 1-\frac{2ae}{a+ae} = \frac{1-e}{1+e}$$

Hence proved.

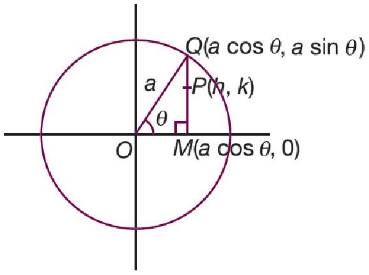
ILLUSTRATION 21

From point Q, which lies on circle $x^2+y^2=a^2$, a perpendicular QM on x-axis is drawn on x-axis. Find the locus of point P, which divide QM internally in the ratio 1:2.

SOLUTION

Let Q be the point on circle as $(a\cos\theta, a\sin\theta)$.

Foot of perpendicular on x-axis is $M(a\cos\theta, 0)$


Q(a cos θ, a sin θ)
a
P(h, k)
θ
O
M(a cos θ, 0)

Let $P(h, k)$ divides QM internally in the ratio 1:2.

$$P(h, k) = \left(a \cos\theta, \frac{2a}{3} \sin\theta\right)$$

We get, $h = a \cos\theta$ and $k = \dfrac{2a}{3}\sin\theta$

$$\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2}{\left(\frac{2a}{3}\right)^2} = 1$$

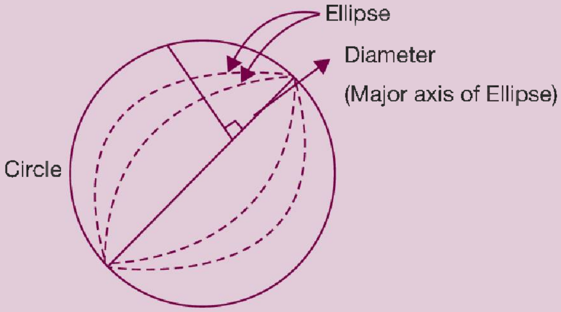
Hence, the locus is $P(h, k)$.

$$\frac{x^2}{a^2} + \frac{y^2}{(2a/3)^2} = 1$$

Or $\dfrac{x^2}{a^2} + \dfrac{9y^2}{4a^2} = 1$ (Ellipse)

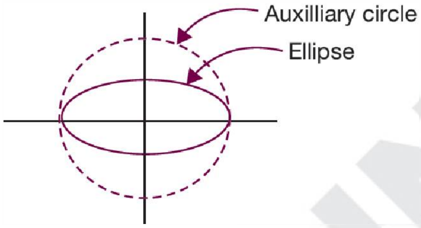
Conceptual Note(s)

From a point on a circle, a perpendicular is drawn on a diameter of circle, then the locus of any variable point, which divides this perpendicular in a fixed ratio internally, is an ellipse with given diameter of circle as major axis.

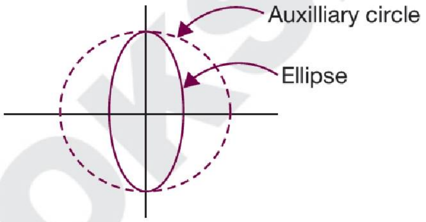

Ellipse
Diameter
(Major axis of Ellipse)
Circle

AUXILIARY CIRCLE

- A circle, whose diameter is major axis of an ellipse is known as auxiliary circle of ellipse.
- Auxiliary circle and ellipse are concentric.


Auxilliary circle
Ellipse

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a > b)$; Auxiliary circle $x^2 + y^2 = a^2$

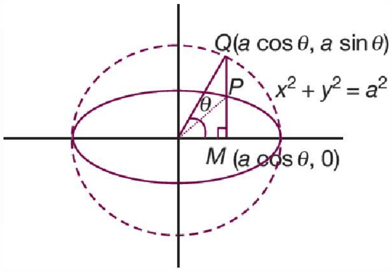

Auxilliary circle
Ellipse

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a < b)$; Auxiliary circle $x^2 + y^2 = b^2$

PARAMETRIC EQUATION OF ELLIPSE

Let us consider an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$$


Q(a cos θ, a sin θ)
P
x² + y² = a²
θ
M (a cos θ, 0)

A point Q is assumed on auxiliary circle and let its co-ordinates are $Q(a \cos\theta, a \sin\theta)$.

Now, a perpendicular QM is drawn to x-axis and the equation of QM is $x = a \cos\theta$

It cuts the ellipse at point P (x, y) and it lies on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Therefore, $\dfrac{a^2 \cos^2\theta}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \quad y^2 = b^2 (1 - \cos^2\theta) = b^2 \sin^2\theta$$

$$\Rightarrow \quad y = b \sin\theta$$

Hence, we get co-ordinate of point P on ellipse as $P(a\cos\theta, b\sin\theta) = p(\theta)$

FREEBOOKS.IN

Here, θ = **Eccentric angle** = Angle made by OQ with positive x-axis, where Q is corresponding point on auxiliary circle.

Remark(s)

(i) $P(\theta) = (a\cos\theta, b\sin\theta)$ and $Q(\theta) = (a\cos\theta, a\sin\theta)$ are known as the Corresponding points on the ellipse and the auxiliary circle, respectively.

(ii) We also observe that ratio of altitudes from auxiliary circle and ellipse on diameter is

$$\frac{\ell(PM)}{\ell(QM)} = \frac{b}{a} = \frac{\ell_{\text{minor axis}}}{\ell_{\text{major axis}}}$$

(iii) Equation of Ellipse with Centre (α, β) and axis being parallel to co-ordinate axes

Cartesian form is $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$

Parametric form is $x = \alpha + a\cos\theta$ and $y = \beta + b\sin\theta$

ILLUSTRATION 22

Consider the ellipse $x^2 + 3y^2 = 6$ and a point P on it in the I$^{\text{st}}$ quadrant at a distance of 2 units from the centre. Find the eccentric angle of P.

SOLUTION

Let point $P(\theta) = (\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on ellipse

$$\frac{x^2}{6} + \frac{y^2}{2} = 1$$

$$OP = 2 \Rightarrow (\sqrt{6}\cos\theta)^2 + (\sqrt{2}\sin\theta)^2 = 4$$

$$\Rightarrow \quad 4\cos^2\theta + 2 = 4 \Rightarrow \cos^2\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{1}{\sqrt{2}} \qquad \left(\text{as } \theta \in \left(0, \frac{\pi}{2}\right)\right)$$

$$\cos\theta = +\frac{1}{\sqrt{2}}, \text{ we get } \theta = \frac{\pi}{4}$$

ILLUSTRATION 23

Find the area of the triangle formed by the points on the ellipse $25x^2 + 16y^2 = 400$, whose eccentric angles are $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$.

SOLUTION

The ellipse is

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \qquad \ldots(1)$$

$\therefore$ The points on the ellipse are $\left(4\cos\frac{\pi}{2}, 5\sin\frac{\pi}{2}\right)$, $(4\cos\pi, 5\sin\pi)$ and $\left(4\cos\frac{3\pi}{2}, 5\sin\frac{3\pi}{2}\right)$, i.e., (0, 5), (–4, 0), (0, –5), respectively.

$\therefore$ Area of triangle formed by these points

$$= \frac{1}{2}\begin{vmatrix} 0 & 5 & 1 \\ -4 & 0 & 1 \\ 0 & -5 & 1 \end{vmatrix} = \frac{1}{2}\{-5(-4) + (20)\}$$

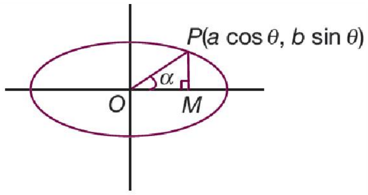
$$= 20 \text{ sq. units}$$

ILLUSTRATION 24

Find the distance between point P, which lies on ellipse and its centre, if line joining origin and P makes α angle with the x-axis.

SOLUTION

Let on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ point P is $(a\cos\theta, b\sin\theta)$



$$\tan\alpha = \frac{PM}{OM} = \frac{b\sin\theta}{a\cos\theta} = \frac{b}{a}\tan\theta$$

$$\Rightarrow \quad \tan\theta = \frac{a}{b}\tan\alpha$$

Distance between origin O and point P

$$OP = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

$$= \cos\theta\sqrt{a^2 + b^2\tan^2\theta}$$

$$= \frac{\sqrt{a^2 + b^2\tan^2\theta}}{\sqrt{1+\tan^2\theta}} \qquad \left(\text{Put } \tan\theta = \frac{a}{b}\tan\alpha\right)$$

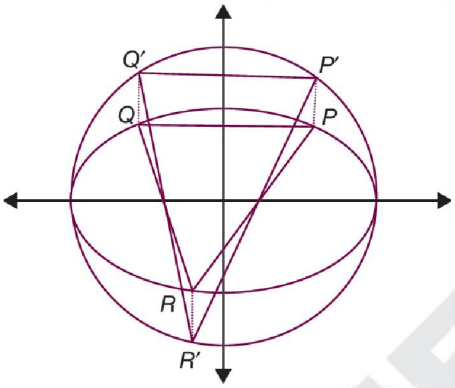
We get, $OP = \dfrac{ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$

ILLUSTRATION 25

Show that the area of a triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the Ist triangle.

SOLUTION

Let $P(a\cos\theta_1, b\sin\theta_1)$, $Q(a\cos\theta_2, b\sin\theta_2)$ and $R(a\cos\theta_3, b\sin\theta_3)$ be the vertices of the triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The points on the auxiliary circle corresponding to these points are $P'(a\cos\theta_1, a\sin\theta_1)$, $Q'(a\cos\theta_2, a\sin\theta_2)$ and $R'(a\cos\theta_3, a\sin\theta_3)$.



$$\therefore \quad \Delta_1 = \text{Area of } \Delta PQR$$

$$= \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1 \\ a\cos\theta_2 & b\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix}$$

$$\therefore \quad = \frac{1}{2}ab\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$$

$$\Delta_2 = \text{Area of } \Delta P'Q'R'$$

$$= \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & a\sin\theta_1 & 1 \\ a\cos\theta_2 & a\sin\theta_2 & 1 \\ a\cos\theta_3 & a\sin\theta_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$$

$$\therefore \quad \text{Clearly, } \frac{\Delta_1}{\Delta_2} = \frac{b}{a} = \text{constant}$$

ILLUSTRATION 26

If P is any point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$ with foci S and S', then prove that locus of the incentre of the $\Delta PSS'$ is

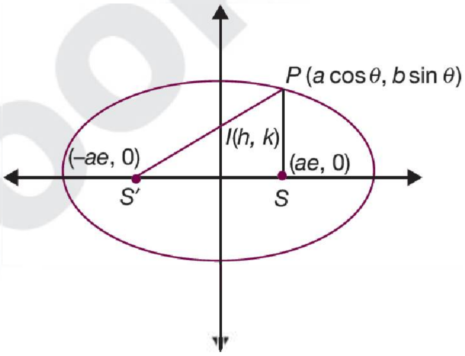
$$(1-e)x^2 + (1+e)y^2 = a^2e^2(1-e).$$

SOLUTION

Let P be $(a\cos\theta, b\sin\theta)$

$$SS' = 2ae,\ PS = a(1 - e\cos\theta),\ PS' = a(1 + e\cos\theta)$$

Let (h, k) be the incentre,



$$h = \frac{2ae(a\cos\theta) - ae\,a(1-e\cos\theta) + aea(1+e\cos\theta)}{2a(1+e)}$$

$$\Rightarrow \quad h = ae\cos\theta \qquad \ldots(1)$$

Similarly,

$$k = \frac{2ae(b\sin\theta) - 0\times a(1-e\cos\theta) + 0\times a(1+e\cos\theta)}{2a(1+e)}$$

$$\Rightarrow \quad k = \frac{be}{1+e}\sin\theta \qquad \ldots(2)$$

Eliminating θ from (1) and (2), we get,

$$\frac{h^2}{a^2e^2} + \frac{k^2(1+e)^2}{b^2e^2} = 1$$

$$\Rightarrow \quad b^2h^2 + a^2(1+e)^2k^2 = a^2b^2e^2$$

$$\because \quad b^2 = a^2(1-e^2)$$

$$\Rightarrow \quad (1-e^2)h^2 + (1+e)^2k^2 = a^2(1-e^2)e^2$$

$$\Rightarrow \quad (1-e)h^2 + (1+e)k^2 = a^2e^2(1-e)$$

$$\therefore \quad \text{Required locus is } (1-e)x^2 + (1+e)y^2 = a^2e^2(1-e)$$

Test Your Concepts-I

(Solutions on page H.195)

1. If $e = \frac{\sqrt{3}}{2}$, then length of latus rectum = k (length of major axis). Find k.
2. Find range of abscissae of the points, which lie on the ellipse $9x^2 + 25y^2 - 18x - 100y - 116 = 0$.
3. If the major axis is 'n' ($n > 1$) times the minor axis of the ellipse, then find the eccentricity.
4. If (5, 12) and (24, 7) are foci of an ellipse passing through origin, then find the eccentricity of the ellipse.
5. If there are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose distance from its centre is same and is equal to $\sqrt{\frac{a^2+2b^2}{2}}$, then find the eccentricity of the ellipse.
6. Find the centre of the ellipse, $4(x-2y+1)^2 + 9(2x+y+2)^2 = 25$.
7. If an ellipse passing through the origin has the foci (3, 4) (6, 8), then find the length of minor axis.
8. If e is the eccentricity of the conic $\sqrt{x^2+y^2} + \sqrt{(x+3)^2+(y-4)^2} = 10$, then find the eccentricity.
9. Suppose F_1 and F_2 are the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. P is a point on ellipse such that $PF_1 : PF_2 = 2 : 1$, then find the area of the ΔPF_1F_2.
10. Co-ordinates of the vertices B and C are $(2, 0)$ and $(8, 0)$, respectively. The vertex 'A' is varying in such a way that $4\tan\frac{B}{2}\tan\frac{C}{2} = 1$. If the locus of '$A$' is an ellipse, then find the length of its semi-major axis.
11. Find the sum of the distances of a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$.
12. In an ellipse, if the lines joining a focus to the extremities of the minor axis make an equilateral triangle with the minor axis, then find the eccentricity of the ellipse.

Answer Keys

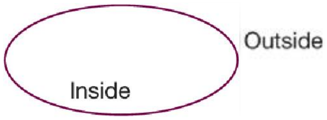
1. $\frac{1}{4}$
2. $x \in [-4, 6]$
3. $\frac{\sqrt{n^2-1}}{n}$
4. $\frac{\sqrt{386}}{38}$
5. $\frac{1}{\sqrt{2}}$
6. (–1, 0)
7. $10\sqrt{2}$
8. 0.5
9. 4 sq. units
10. 5
11. 8
12. $\frac{\sqrt{3}}{2}$

POSITION OF POINT RELATIVE TO ELLIPSE

Let us consider an ellipse, $S : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be any point.

We are defining an expression, $S_1 : \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$



As we know that centre (0, 0) always lies inside the ellipse, then the expression $S_1 : 0 + 0 - 1 = -1 < 0$

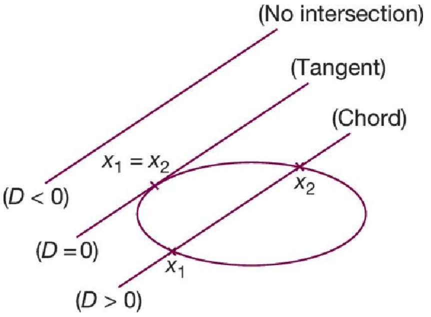
So, for all insider points, S_1 should be negative and for outsider points, S_1 should be positive.

Let point $P(x_1, y_1)$, the location is tabulated below.

Condition	Location of point
$S_1 < 0$	Inside
$S_1 = 0$	On the ellipse
$S_1 > 0$	Outside

POSITION OF LINE RELATIVE TO ELLIPSE

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the given line be $y = mx + c$.



Solving the line and ellipse we get,

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

Or $(a^2m^2 + b^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$

The above equation is a quadratic in x.

$\therefore$ Discriminant, $D = B^2 - 4AC$

$$= 4m^2c^2a^4 - 4a^2(a^2m^2 + b^2)(c^2 - b^2)$$

$$= 4a^2[m^2c^2a^2 - (a^2m^2c^2 - a^2b^2m^2 + b^2c^2 - b^4)]$$

$$= 4a^2[b^2[b^2 + a^2m^2 - c^2]] = 4a^2b^2(a^2m^2 + b^2 - c^2)$$

Working Rule

For chord,	$D > 0$	(When straight line cuts ellipse at two distinct points)
For tangent,	$D = 0$	(When straight line touches ellipse at one point)
For no intersection	$D < 0$	(When straight line never cut ellipse)

Condition for tangency, that line $y = mx + c$ touches standard ellipse $D = 0 \Rightarrow c^2 = a^2m^2 + b^2$

CHORD OF ELLIPSE

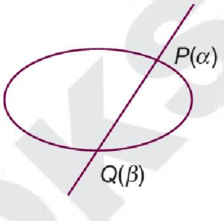
Equation of chord joining the points $P(\alpha)$ and $Q(\beta)$ is given by

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

where, α, β are eccentric angles.

PROOF:

Let, the points $P(\alpha)$ $(a\cos\alpha, b\sin\alpha)$ and $Q(\beta)$ $(a\cos\beta, b\sin\beta)$ lie on ellipse.



Now, equation of PQ will be as,

$$y - b\sin\alpha = \frac{b(\sin\alpha - \sin\beta)}{a(\cos\alpha - \cos\beta)}(x - a\cos\alpha)$$

$$= \frac{2b\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2a\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)}(x - a\cos\alpha)$$

$$\Rightarrow y - b\sin\alpha = \frac{-b\cos\left(\frac{\alpha+\beta}{2}\right)}{a\sin\left(\frac{\alpha+\beta}{2}\right)}(x - a\cos\alpha)$$

$$\Rightarrow a\sin\left(\frac{\alpha+\beta}{2}\right)y - ab\sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right) + b\cos\left(\frac{\alpha+\beta}{2}\right)x = ab\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow b\cos\left(\frac{\alpha+\beta}{2}\right)x + a\sin\left(\frac{\alpha+\beta}{2}\right)y$$

$$= ab\left[\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= ab\cos\left[\alpha - \left(\frac{\alpha+\beta}{2}\right)\right]$$

We get, $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

ILLUSTRATION 27

If $P(\alpha)$ and $Q(\beta)$ are the extremities of a focal chord of an ellipse with eccentricity e, then find $\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2}$

SOLUTION

Equation of chord PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

As we know that in ellipse there are two foci, so we should get two results.

It passes through focus $S(ae, 0)$.

Then, $e\cos\left(\frac{\alpha+\beta}{2}\right)+0=\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\Rightarrow \quad e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

$$=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$\Rightarrow \quad \cos\frac{\alpha}{2}\cos\frac{\beta}{2}(e-1)=(1+e)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

It passes through focus $S(-ae, 0)$.

Then, $-e\cos\left(\frac{\alpha+\beta}{2}\right)+0=\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\Rightarrow \quad -e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

$$=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$\Rightarrow \quad \cos\frac{\alpha}{2}\cos\frac{\beta}{2}(e+1)=(e-1)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e+1}{e-1}$$

Clearly, we get, $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$ or $\frac{e+1}{e-1}$

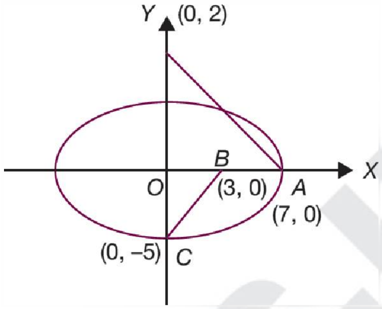
ILLUSTRATION 28

Find the eccentricity of the ellipse, which meets the line $\frac{x}{7}+\frac{y}{2}=1$ on the x-axis and the line $\frac{x}{3}-\frac{y}{5}=1$ on the y-axis and whose axes lie along the co-ordinate axes.

SOLUTION

Let the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(1)$$



As it passes through $A(7, 0)$ and $C(0, -5)$,

thus $\frac{49}{a^2}=1$ and $\frac{25}{b^2}=1$

$\therefore \quad a=7, b=5$

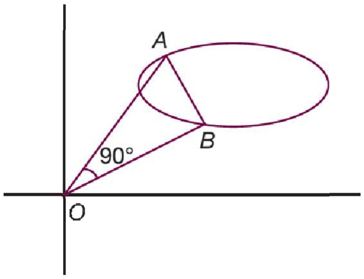
$$\Rightarrow \quad e=\sqrt{1-\frac{25}{49}}=\sqrt{\frac{24}{49}}=\frac{2\sqrt{6}}{7}$$

ILLUSTRATION 29

Prove that a chord of the ellipse $16x^2+9y^2-32x-36y=0$, which subtends a right angle at the origin always passes through a fixed point. Also find the co-ordinates of that point.

SOLUTION

Equation of the ellipse is $16x^2+9y^2-32x-36y=0$.
Let the equation of the chord be $ax+by=1$.



By homogenising the equation of the ellipse with respect to the equation of the chord, we will get the equation of two lines joining the origin and the point of intersection of the chord and the ellipse.

$$16x^2+9y^2-32x(ax+by)-36y(ax+by)=0$$

i.e., $\quad x^2(16-32a)+y^2(9-36b)-(32b+36a)xy=0$

Since the two lines are perpendicular to each other, coefficient of x^2 + coefficient of $y^2=0$

$$16-32a+9-36b=0$$

$$32a+36b=25, \text{ i.e., } \frac{32}{25}a+\frac{36}{25}b=1$$

Comparing the above condition with the equation of the chord, we get, $ax + by = 1$

It is clear that the chord passes through a fixed point, whose co-ordinates are $\left(\frac{32}{25}, \frac{36}{25}\right)$.

TANGENT OF ELLIPSE

Consider the ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $L : y = mx + c$

Solving the line and ellipse we get, $\frac{x^2}{a^2} + \left(\frac{mx + c}{b^2}\right)^2 = 1$

Or $(a^2m^2 + b^2)\, x^2 + 2\, mca^2\, x + a^2\,(c^2 - b^2) = 0$

The above equation is quadratic in x.

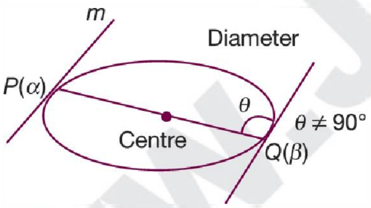
$\therefore$ Discriminant $D = B^2 - 4AC = 4m^2\, c^2a^4 - 4a^2\,(a^2\, m^2 + b^2)\,(c^2 - b^2)$

$= -b^2\, \{c^2 - (a^2m^2 + b^2)\}$

$= b^2\, \{(a^2m^2 + b^2) - c^2\} = 0$

We get, $c^2 = a^2m^2 + b^2$, which is a condition for tangency.

For tangency condition of an ellipse, $a > b$ or $a < b$ does not matter



If $P(\alpha)$ and $Q(\beta)$ are extremities of diameter, then equation of diameter is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

It passes through centre (0, 0), then we get

$$\cos\left(\frac{\alpha-\beta}{2}\right) = 0$$

$$\Rightarrow \quad \frac{\alpha-\beta}{2} = \pm\frac{\pi}{2}$$

$$\Rightarrow \quad |\alpha - \beta| = \pi$$

Remark(s)

(i) Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Tangent $y = mx \pm \sqrt{a^2m^2 + b^2}$ at point $\left(\mp\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$, where $c = \sqrt{a^2m^2 + b^2}$

Tangent T_1 : $y = mx + \sqrt{a^2m^2 + b^2}$ at point of contact $P\left(-\frac{a^2m}{c}, +\frac{b^2}{c}\right)$

Tangent T_2 : $y = mx - \sqrt{a^2m^2 + b^2}$ at point of contact $Q\left(\frac{a^2m}{c}, -\frac{b^2}{c}\right)$

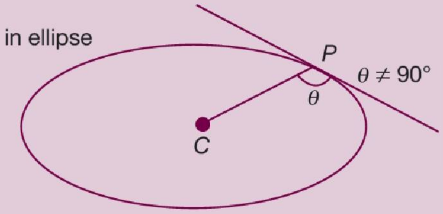
(ii) For a given slope, two parallel tangents can be drawn to an ellipse.

(iii) Chord joining points of contact of parallel tangents passes through the centre. Hence, it is a diameter.

(iv) In other words, tangents drawn at extremities of any diameter of an ellipse are parallel.

(v) Absolute difference of eccentric angle of extremities of any diameter of an ellipse is π.

(vi) We know that in a circle, perpendicular from centre meets any tangent at right angle. However, it is not always true for an ellipse.



(vii) For ellipse, perpendicular from centre only meets tangents at extremities of principal axes at right angle. In other cases, it is not right angle.

Table 5.1 Location of point and tangent

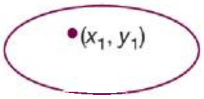
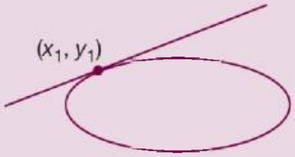
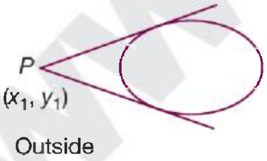
Location of Point (x_1, y_1) $S_1: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	**No. of Tangent**	**Equation of Tangent**
$S_1 < 0$ Point lies inside the ellipse. 	0	Does not exist
$S_1 = 0$ Point lies on the ellipse. 	1	**Ist Method (Preferred)** Equation of tangent, $T = 0$ **(i) Point form** Tangent $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ at point (x_1, y_1) **(ii) Parametric form** Tangent $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ at point $(a\cos\theta, b\sin\theta)$
		IInd Method **Slope form** Tangent $y = mx \pm \sqrt{a^2m^2 + b^2}$ at point $\left(\mp\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$
		IIIrd Method Let equation of tangent is $y - y_1 = m(x - x_1)$ By solving it with equation of ellipse, we get, quadratic in x or y. Now, make discriminant $D = 0$.
$S_1 > 0$ Point lies outside the ellipse. 	2	**Ist Method (Preferred)** Equation of tangent is $y - y_1 = m(x - x_1)$ In case of standard ellipse, use $c^2 = a^2m^2 + b^2$ Else for non-standard form of ellipse, solve tangent with equation of ellipse and get quadratic in x or y. Now, make discriminant $D = 0$
		IInd Method Equation of pair of tangents drawn from external point (x_1, y_1) is $SS_1 = T^2$ Where, $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$; $S_1: \frac{x_1^2}{a^2} + \frac{y_2^2}{b^2} - 1$; $T: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

ILLUSTRATION 30

Find equation of tangent to ellipse $3x^2 + 4y^2 = 12$, which are perpendicular to the line $y + 2x = 4$.

SOLUTION

Slope of the given line is – 2.

Then, slope of tangent, $m_T = \dfrac{1}{2}$

$$c = \sqrt{a^2m^2 + b^2} = \sqrt{1+3} = 2$$

Now, equations of tangents are

$$y = \frac{x}{2} - 2 \text{ at point} \left(1, \frac{-3}{2}\right) \text{ and}$$

$$y = \frac{x}{2} + 2 \text{ at point} \left(-1, \frac{3}{2}\right)$$

ILLUSTRATION 31

If $\sqrt{3}bx + ay = 2ab$ touches the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then find eccentric angle θ.

SOLUTION

Equation of tangent at point 'θ' to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ …(1)

Comparing (1) with $\left(\sqrt{3}\right)bx + ay = 2ab$ …(2)

we get, $\dfrac{\left(\dfrac{\cos\theta}{a}\right)}{\sqrt{3}\,b} = \dfrac{\left(\dfrac{\sin\theta}{b}\right)}{a} = \dfrac{1}{2ab}$

$\Rightarrow \quad \cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = \dfrac{\pi}{6}$

ILLUSTRATION 32

Find the distance of the center of the ellipse $x^2 + 2y^2 - 2 = 0$ to those tangents of the ellipse, which are equally inclined to both the axes.

SOLUTION

Given ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1 \qquad \text{…(1)}$$

Since tangents are equally inclined to both axes, $m = \pm 1$.

For tangency $c^2 = a^2m^2 + b^2 = 3$

$\Rightarrow \quad c = \pm\sqrt{3}$

$\therefore \quad$ Equations of tangents are

$$y = \pm x \pm \sqrt{3} \qquad \text{…(2)}$$

Thus, from (2), distance of centre (0, 0) $= \left|\dfrac{\pm\sqrt{3}}{\sqrt{2}}\right| = \sqrt{\dfrac{3}{2}}$ units

ILLUSTRATION 33

Find the equation(s) of the tangent(s) to the ellipse $9(x-1)^2 + 4y^2 = 36$ parallel to the latus rectum.

SOLUTION

Given ellipse is

$$\frac{(x-1)^2}{4} + \frac{(y-0)^2}{9} = 1 \qquad \text{…(1)}$$

Cleary, its major axis is parallel to y-axis and hence latus rectum will be parallel to x-axis.

$\therefore \quad$ Its slope = 0

$\Rightarrow \quad$ Equation of tangent to (1) is $y = 0 \cdot x + \sqrt{4 \times 0 + 9}$

$\Rightarrow \quad y = \pm 3$

ILLUSTRATION 34

Find the angle between pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2).

SOLUTION

Any tangent to the ellipse

$$\frac{x^2}{\frac{5}{3}} + \frac{y^2}{\frac{5}{2}} = 1 \qquad \text{…(1)}$$

is given by

$$y = mx + \sqrt{\frac{5}{3}m^2 + \frac{5}{2}} \qquad \text{…(2)}$$

As it passes through (1, 2),

$$2 = m + \sqrt{\frac{5}{3}m^2 + \frac{5}{2}}$$

$\Rightarrow \quad m^2 - 4m + 4 = \dfrac{5}{3}m^2 + \dfrac{5}{2}$

$\Rightarrow \quad 4m^2 + 24m - 9 = 0$

$$\therefore \quad m_1 + m_2 = -6 \text{ and } m_1 m_2 = -\frac{9}{4}$$

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\sqrt{36+9}}{1-\frac{9}{4}}\right| = \frac{12}{\sqrt{5}}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$$

ILLUSTRATION 35

If a tangent of slope 'm' at a point of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through $(2a, 0)$ and if 'e' denotes the eccentricity of the ellipse, then find the condition in e and m.

SOLUTION

Equation of tangent with slope m is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

It passes through $(2a, 0)$

$$\Rightarrow \quad 0 = 2am + \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad 3a^2m^2 - b^2 = 0$$

$$\Rightarrow \quad 3a^2m^2 - a^2(1-e^2) = 0$$

$$\Rightarrow \quad 3m^2 + e^2 = 1$$

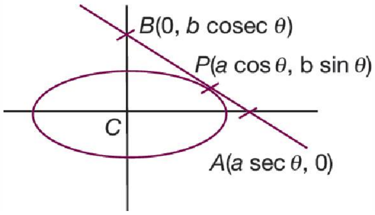
ILLUSTRATION 36

Tangent to point P in Ist quadrant to a ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts co-ordinates axes at A and B. If P divides AB in 3:1 internally, then find the equation of tangent at P.

SOLUTION

Tangent at $P(\theta)$ on ellipse

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$



Tangent cuts co-ordinate axis at points $A(a \sec\theta, 0)$ and $B(0, b \operatorname{cosec}\theta)$.

$P(\theta)$ divides AB in 3:1,

then $P(a\cos\theta, b\sin\theta)$

$$= \left(\frac{a\sec\theta}{4}, \frac{3b\operatorname{cosec}\theta}{4}\right)$$

We get, $\cos\theta = \frac{1}{4}\sec\theta$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{4}$$

$\Rightarrow \quad \theta = 60°$ as P lies in Ist quadrant

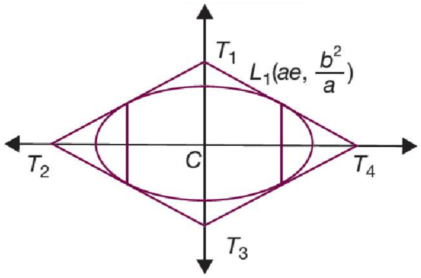
Now, equation of tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1$

ILLUSTRATION 37

Find the area of parallelogram formed by tangents at the extremities of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

SOLUTION

Equation of tangent at $L_1(ae, \frac{b^2}{a})$ is $T = 0$



$$\Rightarrow \quad \frac{x}{a^2}(ae) + \frac{y}{b^2}\left(\frac{b^2}{a}\right) = 1$$

$$\Rightarrow \quad \frac{xe}{a} + \frac{y}{a} = 1$$

Now, tangent is cutting co-ordinate axes at points $T_4\left(\frac{a}{e}, 0\right)$ and $T_1(0, a)$

Area of parallelogram = 4 (Area of triangle formed by tangent and co-ordinate axes ΔCT_1T_4)

$$= 4\left(\frac{1}{2}.\frac{a}{e}.a\right) = \frac{2a^2}{e} = \frac{2a^3}{\sqrt{a^2-b^2}} \text{ sq. units}$$

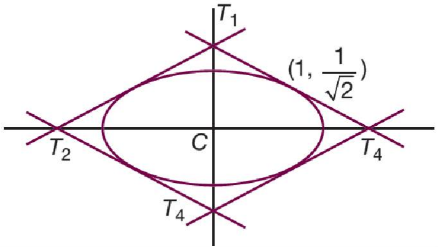
ILLUSTRATION 38

Find area of quadrilateral formed by tangents drawn at the ends of latus rectum of ellipse $x^2 + 2y^2 = 2$.

SOLUTION

Equation of ellipse is $\frac{x^2}{2} + \frac{y^2}{1} = 1$

$$e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \text{ latus rectum } \left(1, \frac{1}{\sqrt{2}}\right)$$



Equation of tangent is $\frac{x}{2} + \frac{y}{\sqrt{2}} = 1$

With tangents, we can derive the point of intersection with co-ordinate axes as $T_1(0, \sqrt{2})$ and $T_4(2, 0)$.

$$\text{Area of parallelogram} = 4(\Delta CT_1T_4) = 4\left(\frac{1}{2}.\sqrt{2}.2\right)$$

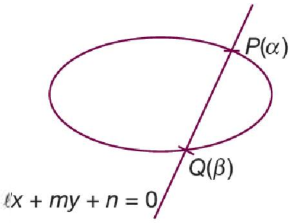
$$= 4\sqrt{2} \text{ sq. units}$$

ILLUSTRATION 39

If line $\ell x + my + n = 0$ intersects ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at two points, whose eccentric angle differ by $\frac{\pi}{2}$, then prove that $a^2\ell^2 + b^2m^2 = 2n^2$.

SOLUTION

Let the chord PQ cuts ellipse at the points $P(\alpha)$ and $Q(\beta)$.



Given, $\alpha - \beta = \frac{\pi}{2}$

$\Rightarrow \quad \alpha = 90° + \beta$

Equation of chord PQ

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\cos(45° + \beta) + \frac{y}{b}\sin(45° + \beta) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{x}{a}\left(\frac{\cos\beta - \sin\beta}{\sqrt{2}}\right) + \frac{y}{b}\left(\frac{\cos\beta + \sin\beta}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{x}{a}(\cos\beta - \sin\beta) + \frac{y}{b}(\cos\beta + \sin\beta) = 1$$

Given line $\ell x + my = -n$ also represents the same chord. On comparison, we get,

$$\frac{\cos\beta - \sin\beta}{a\ell} = \frac{\cos\beta + \sin\beta}{bm} = -\frac{1}{n}$$

$$\text{Or} \quad \cos\beta - \sin\beta = -\frac{a\ell}{n}; \cos\beta + \sin\beta = -\frac{bm}{n}$$

Square and add them, we get,

$$\cos^2\beta + \sin^2\beta + \cos^2\beta + \sin^2\beta = \frac{a^2\ell^2}{n^2} + \frac{b^2m^2}{n^2} = 2$$

$$\Rightarrow \quad a^2\ell^2 + b^2m^2 = 2n^2$$

Hence proved.

ILLUSTRATION 40

Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangent at each of them makes equal angles with the axes. Also prove that the length of the perpendicular from the centre on either of these tangent is $\sqrt{\frac{a^2 + b^2}{2}}$.

SOLUTION

Tangent at point $(a\cos\theta, b\sin\theta)$,

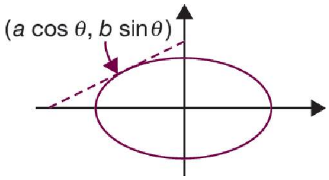
we get $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$$\text{Slope} = -\frac{b\cos\theta}{a\sin\theta} = \pm\tan 45° = \pm 1$$

$$\text{Or} \quad \cot\theta = \pm\frac{a}{b}$$

$$\text{Or} \quad \cos\theta = \pm\frac{a}{\sqrt{a^2 + b^2}}$$

$$\text{Or} \quad \sin\theta = \pm\frac{b}{\sqrt{a^2 + b^2}}$$



$\therefore$ Co-ordinates of required points are

$$\left[\pm\frac{a^2}{\sqrt{a^2 + b^2}}, \pm\frac{b^2}{\sqrt{a^2 + b^2}}\right]$$

∴ Length of perpendicular from (0, 0) on any tangent

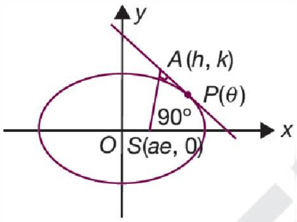
$$= \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} = \frac{1}{\sqrt{\frac{1}{(a^2+b^2)}+\frac{1}{(a^2+b^2)}}}$$

$$= \sqrt{\frac{a^2+b^2}{2}}$$

ILLUSTRATION 41

Find the locus of the foot of the perpendicular drawn from either focus upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

SOLUTION



Given ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(1)$$

Equation of any tangent to ellipse (1) maybe taken as

$$y-mx=\sqrt{a^2m^2+b^2} \qquad \ldots(2)$$

Equation of the line perpendicular to line (2) from the focus (± ae, 0) will be $y-0=-\frac{1}{m}(x\pm ae)$

Or $\quad my+x=\pm ae \qquad \ldots(3)$

In order to find the locus of A, the point of intersection of (2) and (3), we will have to eliminate m.

Squaring (2) and (3) and then adding we get,

$$(1+m^2)\,y^2+(1+m^2)\,x^2=a^2m^2+b^2+a^2e^2$$

$$=a^2m^2+a^2\ (\because b^2=a^2\,[1-e^2])$$

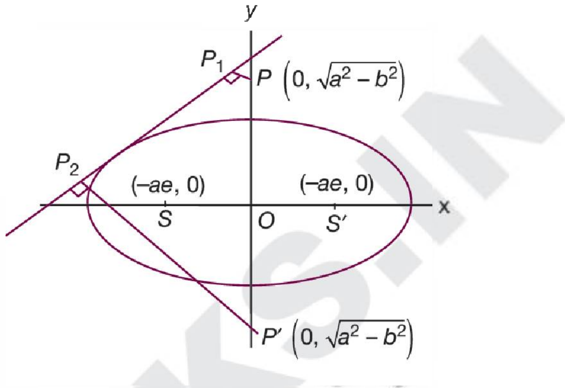
Or $\quad x^2+y^2=a^2$

∴ The required locus is $x^2+y^2=a^2$.

ILLUSTRATION 42

If two points are taken on the minor axis of an ellipse at the same distance from the centre as the foci, then prove that the sum of the squares of the perpendiculars from these points on any tangent to the ellipse is constant.

SOLUTION


y
P_1
$P\left(0, \sqrt{a^2-b^2}\right)$
P_2
(–ae, 0)
(–ae, 0)
S
O
S′
x
$P'\left(0, \sqrt{a^2-b^2}\right)$

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then, the distance of a focus from the centre,

$$ae=a\sqrt{1-\frac{b^2}{a^2}}=\sqrt{a^2-b^2}$$

So, the two points on minor axis are

$$P\left(0, \sqrt{a^2-b^2}\right) \text{ and } P'\left(0, -\sqrt{a^2-b^2}\right)$$

Now, any tangent to the ellipse is $y=mx+\sqrt{a^2m^2+b^2}$, where m is a parameter.

The sum of the squares of the perpendiculars on this tangent from the two points P and P' is

$$\left(\frac{\sqrt{a^2-b^2}-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right)^2+\left(\frac{-\sqrt{a^2-b^2}-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right)^2$$

$$=\frac{2\left(a^2-b^2+a^2m^2+b^2\right)}{\left(1+m^2\right)}=2a^2 \text{ (Constant).}$$

ILLUSTRATION 43

For all real p, the line $2px+y\sqrt{1-p^2}=1$ touches a fixed ellipse. Find the eccentricity of this ellipse.

SOLUTION

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The line that touches this ellipse for all m is $y=mx \pm \sqrt{a^2m^2+b^2} \qquad \ldots(1)$

Hence, it is identical with $y = -\frac{2px}{\sqrt{1-p^2}} + \frac{1}{\sqrt{1-p^2}}$ ....(2)

By equating the coefficients of (1) and (2), we get,

$$m = -\frac{2p}{\sqrt{1-p^2}} \text{ and } a^2m^2 + b^2 = \frac{1}{1-p^2}$$

$$\Rightarrow \quad a^2 \cdot \frac{4p^2}{1-p^2} + b^2 = \frac{1}{1-p^2}$$

$$\Rightarrow \quad p^2(4a^2 - b^2) + b^2 - 1 = 0$$

This equation is true for all real p, if $b^2 = 1$ and $4a^2 = b^2$. Which is true for all $p \Rightarrow b^2 = 1$ and $4a^2 = b^2 = 1$

$$\Rightarrow \quad b^2 = 1 \text{ and } a^2 = \frac{1}{4}$$

Hence, the ellipse is $\frac{x^2}{1/4} + \frac{y^2}{1} = 1$

If e is its eccentricity, then $\frac{1}{4} = 1 - e^2$

$$\Rightarrow \quad e^2 = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$$

ILLUSTRATION 44

There are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose distance from its centre is same and is equal to $\sqrt{\frac{a^2 + 2b^2}{2}}$. Find the eccentricity of the ellipse.

SOLUTION

Since there are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose distance from centre is same, the points would be either end points of the major axis or of the minor axis.

However, $\sqrt{\frac{a^2 + 2b^2}{2}} > b$ and hence the points are the vertices of major axis.

Hence, $a = \sqrt{\frac{a^2 + 2b^2}{2}} \Rightarrow a^2 = 2b^2$

$$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$$

ILLUSTRATION 45

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated through a right angle in its own plane about its centre, which is fixed. Prove that the locus of the point of intersection of the tangents to the ellipse in its original and in the new position is $(x^2 + y^2)(x^2 + y^2 - a^2 - b^2) = 2(a^2 - b^2)xy$.

SOLUTION

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Any point on it is $P(a \cos \theta, b \sin \theta)$.

The equation of the tangent is

$$\frac{x \cos\theta}{a} + \frac{y \sin\theta}{b} = 1 \qquad \text{...(1)}$$

When the ellipse is rotated through a right angle, its equation becomes (replace a and b and θ by $90° + \theta$)

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

The point P also change its position to $(-b \sin \theta, a \cos \theta)$

The equation of the tangent is $-\frac{x}{b} \sin \theta + \frac{y}{a} \cos \theta = 1$ ...(2)

Eliminating θ from (1) and (2) yields the locus of the point of intersection of the tangents (1) and (2).

From (1) and (2),

we find that, $\cos \theta = \frac{(x+y)a}{x^2 + y^2}$ and $\sin \theta = \frac{(y-x)b}{x^2 + y^2}$

$$\Rightarrow \quad (x^2 + y^2)^2 = (x + y)^2 a^2 + (y - x)^2 b^2$$
$$= (x^2 + y^2)(a^2 + b^2) + 2xy(a^2 - b^2)$$

Hence, the locus of the point of intersection of the two tangents is $(x^2 + y^2)(x^2 + y^2 - a^2 - b^2) = 2(a^2 - b^2)xy$.

Test Your Concepts-II

(Solutions on page H.196)

1. Find the area of the quadrilateral formed by the tangents at the ends of latus rectum of $5x^2 + 9y^2 = 45$.

2. Find the number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$.

3. Find the value of k, if the line $x - y + k = 0$ is a tangent to the ellipse $9x^2 + 16y^2 = 144$.

4. The straight line $3x + 4y - 12 = 0$ intersects the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at two points A and B. There is a point P on the ellipse such that area of ΔPAB is 3 sq. units. Find the number of such points P.

5. Let P and Q be two points on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, whose eccentric angles differ by a right angle. Tangents are drawn at P and Q to meet at R. If the chord PQ divides the joint of C and R in the ratio $m:n$ (C being centre of ellipse), then find $m + n$ ($m:n$ is in simplified form).

6. If PQ is focal chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, which passes through $S(3, 0)$ and $PS = 2$, then find the length of chord PQ.

7. $P(\theta)$, $Q\left(\theta + \frac{\pi}{2}\right)$ are two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then find the locus of point of intersection of the two tangents at P and Q to the ellipse.

8. Find the product of perpendiculars drawn from the points $(\pm 3, 0)$ to the line $y = mx - \sqrt{25m^2 + 16}$.

9. A tangent to the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$, having slope $-\frac{4}{3}$ cuts the x and y axes at the points A and B, respectively. If O is the origin, then find the area of triangle.

Answer Keys

1. 27 sq. units

2. 2

3. –5, 5

4. 2

5. 2

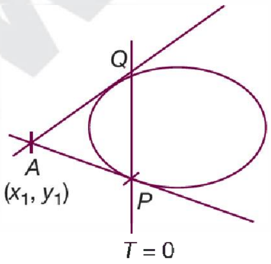
6. 10

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

8. 16

9. 24 sq. units

CHORD OF CONTACT

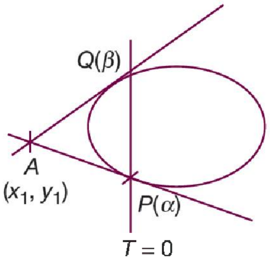


(i) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $T = 0$ or $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

(ii) Tangent drawn at point $P(\alpha)$ and $Q(\beta)$ intersect at

$$\text{point } A\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

PROOF:



Chord PQ: $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

Equation of chord of contact, $T=0$: $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

By comparing both equations, we get,

$$A(x_1, y_1)=\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

ILLUSTRATION 46

From a variable point on line $x-y+2=0$, a pair of tangents are drawn to ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$. Prove that, variable chord of contact passes through a fixed point.

SOLUTION

Let the variable point be (x_1, y_1) on the given line.

Then, $x_1-y_1+2=0$

Equation of chord of contact, $T=0$

$$\Rightarrow \quad \frac{xx_1}{4}+\frac{yy_1}{2}=1$$

$$\Rightarrow \quad \frac{xx_1}{4}+\frac{y}{2}(x_1+2)=1$$

$$\Rightarrow \quad x_1(x+2y)+4(y-1)=0$$

$$\Rightarrow \quad x_1L_1+L_2=0$$

Fixed point is point of intersection of lines L_1 and L_2.

L_1: $x+2y=0$; L_2: $y-1=0$

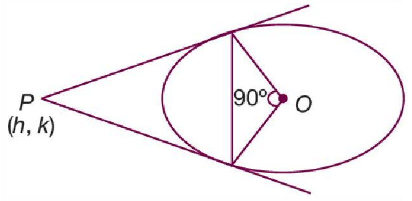
$\Rightarrow \quad y=1; x=-2$

$\therefore$ Fixed point is $(-2, 1)$.

ILLUSTRATION 47

Find the locus of a point, whose chord of contact with respect to the ellipse $x^2+2y^2=1$ subtends a right angle at the centre of the ellipse.

SOLUTION



Let $P(h, k)$ be the point.

Then equation of chord of contact of P with respect to ellipse is

$$x^2+2y^2=1 \quad \ldots(1)$$

$$xh+2yk=1 \quad \ldots(2)$$

Then the joint equation of lines joining origin to points of intersection of (1) and (2) is

$$x^2+2y^2=(xh+2yk)^2 \text{ (By homogenisation)}$$

$$\Rightarrow \quad (1-h^2)x^2+(2-4k^2)y^2-4hkxy=0$$

The lines will be perpendicular if,

coefficient of x^2 + coefficient of $y^2=0$

$$(1-h^2)+(2-4k^2)=0$$

$$\Rightarrow \quad h^2+4k^2=3$$

$\therefore$ Required locus is $x^2+4y^2=3$

ILLUSTRATION 48

P and Q are two points of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ such that sum of their ordinates is 3. Prove that the locus of the intersection of tangents at P and Q is $9x^2+25y^2=150y$.

SOLUTION

If (h, k) be point of intersection of tangents at θ and ϕ, then

$$\frac{h}{a}=\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}; \frac{k}{b}=\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\therefore \quad \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{1}{\cos^2\left(\frac{\theta-\phi}{2}\right)} \quad \ldots(1)$$

We are given that sum of ordinates is 3.

$$\therefore \quad b(\sin\theta+\sin\phi)=3$$

$$\text{Or} \quad 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=1 \quad \ldots(2)$$

We know that, $b=3$

$$\therefore \quad \frac{k}{b}=\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\cdot\frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$= \frac{1}{2\cos^2\left(\dfrac{\theta-\phi}{2}\right)} \qquad \text{(By (2))}$$

$$\therefore \quad \frac{2k}{b} = \frac{1}{\cos^2\left(\dfrac{\theta-\phi}{2}\right)} \qquad \ldots(3)$$

Hence, from (1) and (3) we get,

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{2k}{b}$$

Now, put $a^2=25$ and $b^2=9$

We get locus (h, k)

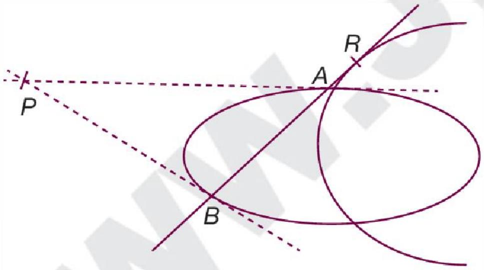
$$\frac{x^2}{25}+\frac{y^2}{9}=\frac{2y}{3}$$

$$\Rightarrow \quad 9x^2+25y^2=150y \qquad \text{Hence proved}$$

ILLUSTRATION 49

A tangent to parabola $y^2 = 4ax$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at points A and B. Find locus of point of intersection of tangent drawn at points A and B on the ellipse.

SOLUTION



Let the point be $P(x_1, y_1)$.

Equation of chord of contact on ellipse $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$$\Rightarrow \quad y=-\left(\frac{b^2x_1}{a^2y_1}\right)x+\frac{b^2}{y_1}$$

It is a tangent to the given parabola.

Then, $c=\frac{a}{m}$

$$\frac{b^2}{y_1}=\frac{a}{\left(\dfrac{-b^2}{a^2}\cdot\dfrac{x_1}{y_1}\right)}$$

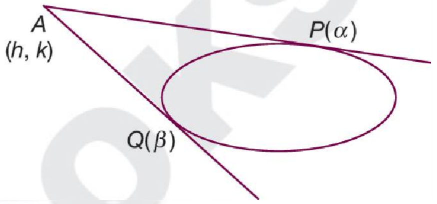
We get locus of point $P(x_1, y_1)$.

By replacing $x_1 = x$ and $y_1 = y$, we get the locus as parabola $b^4x+a^3y^2=0$.

ILLUSTRATION 50

Prove that locus of point of intersection of tangents drawn at two points on ellipse, whose difference of eccentric angle is constant.

SOLUTION



Let the tangent drawn at point $P(\alpha)$ and $Q(\beta)$ meet at point $A(h, k)$. Then, we have,

$$A(h,\ k)=\left(\frac{a\cos\left(\dfrac{\alpha+\beta}{2}\right)}{\cos\dfrac{(\alpha-\beta)}{2}},\ \frac{b\sin\left(\dfrac{\alpha+\beta}{2}\right)}{\cos\left(\dfrac{\alpha-\beta}{2}\right)}\right)$$

Given, $\alpha-\beta=\theta=\text{Constant}$

$$\text{We get, } h=\frac{a\cos\left(\dfrac{\alpha+\beta}{2}\right)}{\cos\dfrac{\theta}{2}}$$

$$\Rightarrow \quad \cos\left(\frac{\alpha+\beta}{2}\right)=\frac{h}{a\sec\dfrac{\theta}{2}}$$

$$\text{and} \quad k=\frac{b\sin\left(\dfrac{\alpha+\beta}{2}\right)}{\cos\dfrac{\theta}{2}}$$

$$\Rightarrow \quad \sin\left(\frac{\alpha+\beta}{2}\right)=\frac{k}{b\sec\dfrac{\theta}{2}}$$

By squaring and adding,

$$\text{we get, } \frac{h^2\cos^2(\theta/2)}{a^2}+\frac{k^2\cos^2(\theta/2)}{b^2}=1$$

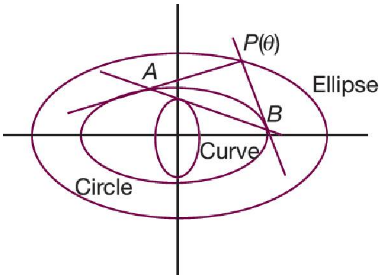
Locus $h=x$, $k=y$ we get locus as

$$\frac{x^2}{a^2\sec^2\theta/2}+\frac{y^2}{b^2\sec^2\theta/2}=1 \text{ (it represents ellipse)}$$

ILLUSTRATION 51

From any point on ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, a pair of tangents are drawn to circle $x^2+y^2=4$. Prove that chord of contact of circle will be a tangent to the curve $25x^2+16y^2=16$.

SOLUTION



Equation of chord of contact of circle from $P(\theta)=(5\cos\theta, 4\sin\theta)$ is $T=0$.

$\Rightarrow \quad 5x\cos\theta+4y\sin\theta=4$

Or $\quad y=\left(\frac{-5}{4}\cot\theta\right)x+\operatorname{cosec}\theta$

Now, it is a tangent to given curve $\frac{x^2}{(4/5)^2}+y^2=1$

It is an ellipse so

For tangency, $c^2=a^2m^2+b^2$

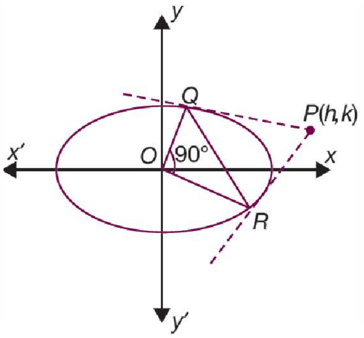
$$a^2m^2+b^2=\frac{16}{25}\left(\frac{-5}{4}\cot\theta\right)^2+1 \Rightarrow \operatorname{cosec}^2\theta=c^2$$

Hence proved.

ILLUSTRATION 52

If the chord of contact of tangents drawn from the point (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at the centre, then find the locus of (h, k).

SOLUTION



The equation of the chord of contact of tangents drawn from (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$T=0 \Rightarrow \frac{hx}{a^2}+\frac{ky}{b^2}=1 \qquad \ldots(1)$$

The equations of the straight lines, joining the centre of the ellipse, i.e., the origin to the points of intersection of the ellipse and (1) is obtained by making a homogeneous equation with the help of Equation (1) and the ellipse,
we get,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2=0$$

Or $\quad x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+y^2\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)-\frac{2hk}{a^2b^2}xy=0 \quad \ldots(2)$

If the chord of contact of tangents subtends a right angle at the centre, then the lines represented by (2) should be at right angles.

$\therefore \quad$ Coefficient of x^2 + Coefficient of $y^2=0$

$$\Rightarrow \quad \left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)=0$$

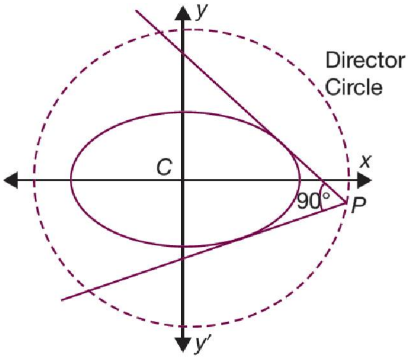
$$\Rightarrow \quad \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

So, the locus of (h, k) is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$

DIRECTOR CIRCLE

The locus of the point of intersection of perpendicular tangents to an ellipse is known as its director circle.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse and let (h, k) be a point on its director circle.



The equation of any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=mx+\sqrt{a^2m^2+b^2}$.

If it passes through (h, k), then $k = mh+\sqrt{a^2m^2+b^2}$

$\Rightarrow \quad (k-mh)^2 = a^2m^2+b^2$

$\Rightarrow \quad m^2(h^2-a^2)-2mhk+(k^2-b^2)=0$

This is quadratic equation in m. So, it gives two values of m, i.e., m_1 and m_2. If two tangents are perpendicular, then $m_1m_2=-1$.

$\Rightarrow \quad \frac{k^2-b^2}{h^2-a^2}=-1$

$\Rightarrow \quad h^2+k^2=a^2+b^2$

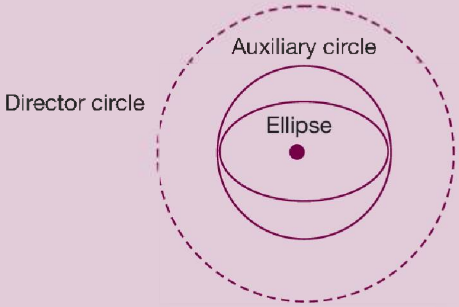
Thus, the equation of the director circle is $x^2+y^2=a^2+b^2$

Remark(s)

(i) Radius of director circle

$$=\sqrt{a^2+b^2}=\sqrt{l^2\text{semi-major axis}+l^2\text{semi-minor axis}}$$

(ii) For better understanding, let us draw ellipse, auxiliary circle and director circle in the same diagram, to understand their relative position.



Ellipse	Auxiliary circle	Director circle
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$	$x^2+y^2=a^2$	$x^2+y^2=a^2+b^2$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a<b)$	$x^2+y^2=b^2$	$x^2+y^2=a^2+b^2$

(iii) Ellipse and auxiliary circle are very much dependent on whether $a > b$ or $a < b$ but for director circle it does not matter.

(iv) Ellipse, Auxiliary circle and director circles are concentric.

ILLUSTRATION 53

From the point, $P\left(\sqrt{41}\cos\theta,\ \sqrt{41}\sin\theta\right)$ a pair of tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$. Find the angle between the tangents.

SOLUTION

Here, the point P lies on director circle $x^2+y^2=41$

$\therefore \quad \phi = 90°$

Alternate

General method let tangent to ellipse is

$$y = mx+\sqrt{a^2m^2+b^2}\ ;\ (y-mx)^2 = 16\,m^2+25$$

Point $P(h, k)$ lies on it.

$\Rightarrow \quad m^2(16-h^2)+2\,hkm+25-k^2=0$

Now, $h=\sqrt{41}\ \cos\theta$ and $k=\sqrt{41}\ \sin\theta$

$$m^2(16-41\cos^2\theta)+(2)41\cos\theta\sin\theta\, m$$
$$+25-41\sin^2\theta=0$$

$\Rightarrow \quad m^2(-25+41\sin^2\theta)+82\cos\theta\sin\theta\, m$
$\quad +(25-41\sin^2\theta)=0$

Angle between tangents, $\tan\phi=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

Here, $m_1m_2=-1$

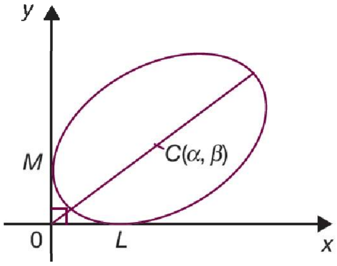
$\therefore \quad \phi = 90°$

ILLUSTRATION 54

An ellipse slides between the two lines at right angles to on another. Show that the locus of its centre is a circle.

SOLUTION

Consider the perpendicular lines as X and Y axes.
Let the co-ordinates of centre C be (α, β)
Consider the length of semi-major axis as a and length of semi-minor axis as b.
Hence, equation of its director circle will be



EBOOKS.IN

$$(x-\alpha)^2+(y-\beta)^2=a^2+b^2 \quad \ldots(1)$$

Clearly, (1) will pass through (0, 0)

$\Rightarrow \quad \alpha^2+\beta^2=a^2+b^2$

Hence, the required locus will be $x^2+y^2=a^2+b^2$.

POLE AND POLAR

Let $P(x_1, y_1)$ be any point inside, on or outside the ellipse. Let chords AB and $A'B'$ be passing through P. If tangents to the ellipse at A and B meet at $Q(h, k)$, then locus of Q is called polar of P with respect to ellipse and P is called the pole.

Let the pole be $P(x_1, y_1)$ and the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Then, the equation of polar is $T=0$ or $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

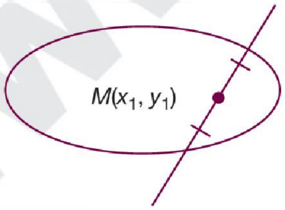
Properties of Pole and Polar

(i) The polar of any point on the directrix passes through the focus.

(ii) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

(iii) If the pole of a line $ax+by+c=0$ lies on the other line $a_1x+b_1y+c_1=0$, then the pole of second line will lie on the first line, then such lines are said to be conjugate lines.

Chord with given Middle Point



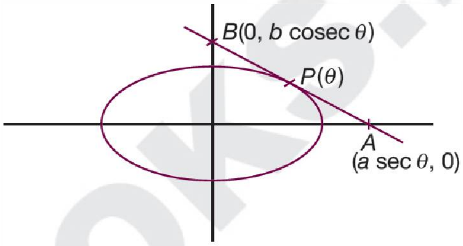
The equation of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ bisected at the point $M(x_1, y_1)$ is given by $T=S_1$

Or $\quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

ILLUSTRATION 55

Find the curve which is the locus of midpoints of the portions of the tangents to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ included between the axes.

SOLUTION



Equation of tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Tangent cuts the co-ordinate axes at points A and B

Midpoint of AB is $M(h, k)=\left(\frac{a\sec\theta}{2}, \frac{b\operatorname{cosec}\theta}{2}\right)$

$$h=\frac{a\sec\theta}{2} \Rightarrow \cos\theta=\frac{a}{2h}$$

$$k=\frac{b\operatorname{cosec}\theta}{2} \Rightarrow \sin\theta=\frac{b}{2k}$$

By squaring and adding, we get,

$$1=\frac{a^2}{4h^2}+\frac{b^2}{4k^2}$$

Locus $h=x$ and $k=y$

$\therefore$ The required locus is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$

ILLUSTRATION 56

Find the locus of the midpoint of chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which are tangents to the parabola $y^2=4ax$.

SOLUTION

Let $M(x_1, y_1)$ be the midpoint of the chord.

Equation of chord $T=S_1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

$$\Rightarrow \quad y = -\left(\frac{b^2x_1}{a^2y_1}\right)x + \left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)\frac{b^2}{y_1}$$

It is tangent to $y^2 = 4ax$, then it should be of form $y = mx + \frac{a}{m}$

On comparison, we get, $c = \frac{a}{m}$

$$\Rightarrow \quad \left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)\frac{b^2}{y_1} = \frac{a}{\left(\frac{-b^2x_1}{a^2y_1}\right)}$$

$$\Rightarrow \quad \left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right) = -\frac{a^3y_1^2}{b^4x_1}$$

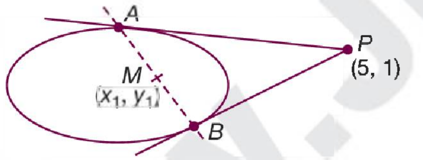
Locus of $M(x_1, y_1) = (x, y)$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = -\frac{a^3y^2}{b^4x}$$

ILLUSTRATION 57

Find the midpoint of the chord, the tangents at the extremities of which meets at (5, 1) for $\frac{x^2}{9}+\frac{y^2}{4}=1$.

SOLUTION



The given equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$

Equation of chord of contact AB is $T = 0$.

$$\frac{5x}{9}+\frac{y}{4}=1 \Rightarrow y = 4\left(1-\frac{5x}{9}\right) \quad \ldots(1)$$

Solving with $\frac{x^2}{9}+\frac{y^2}{4}=1$

$$\Rightarrow \quad \frac{x^2}{9}+\frac{1}{4}\left(16\left(1-\frac{5x}{9}\right)^2\right)=1$$

$$\Rightarrow \quad \frac{x^2}{9}+4\left(1+\frac{25}{81}x^2-\frac{10x}{9}\right)=1$$

$$\Rightarrow \quad \left(\frac{109}{81}\right)x^2-\frac{40}{9}x+3=0$$

Let the points of intersection of chord with ellipse are $A(x_1, y_1)$ and $B(x_2, y_2)$

Then, $\frac{x_1+x_2}{2} = \frac{180}{109}$

Now, from Equation (1), we get, $y_1+y_2 = 4\left(1-\frac{5x_1}{9}\right)+4\left(1-\frac{5x_2}{9}\right) = 8-\frac{20}{9}(x_1+x_2)$

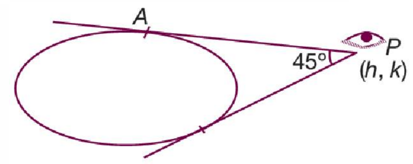
$$\frac{y_1+y_2}{2} = 4-\frac{20}{9}\cdot\frac{180}{109} = \frac{36}{109}$$

Now, the midpoint of chord is $\left(\frac{180}{109}, \frac{36}{109}\right)$

ILLUSTRATION 58

Find the locus of the point from which the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is seen at an angle of 45°.

SOLUTION



Equation of tangent is $y = mx + \sqrt{a^2m^2+b^2}$

It passes through point (h, k)

then $k = mh + \sqrt{a^2m^2+b^2}$

$$\Rightarrow \quad (k-mh)^2 = a^2m^2+b^2$$

$$\Rightarrow \quad k^2+m^2h^2-2mhk = a^2m^2+b^2$$

$$\Rightarrow \quad m^2(h^2-a^2)-2mhk+k^2-b^2=0$$

$$m_1+m_2 = \frac{2hk}{h^2-a^2}\,;\; m_1m_2 = \frac{k^2-b^2}{h^2-a^2}$$

Angle between tangents $\tan 45^\circ = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = 1$

$$(m_1-m_2)^2 = (1+m_1m_2)^2$$

⇒ $(m_1+m_2)^2 - 4m_1m_2 = (1+m_1m_2)^2$

⇒ $\left(\frac{2hk}{h^2-a^2}\right)^2 - \frac{4(k^2-b^2)}{h^2-a^2} = \left(1+\frac{k^2-b^2}{h^2-a^2}\right)^2$

⇒ $4h^2k^2 - 4(h^2-a^2)(k^2-b^2) = (h^2+k^2-a^2-b^2)^2$

⇒ $4(a^2k^2+h^2b^2-a^2b^2) = (h^2+k^2-a^2-b^2)^2$

Locus of $P(h, k)$ is given by

$$(x^2+y^2-a^2-b^2)^2 = 4(y^2a^2+x^2b^2-a^2b^2)$$

ILLUSTRATION 59

From a point O on the circle $x^2+y^2=25$, tangents OP and OQ are drawn to the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$. Show that the locus of the midpoint of the chord PQ describes the curve $x^2+y^2=25\left[\frac{x^2}{4}+\frac{y^2}{1}\right]^2$.

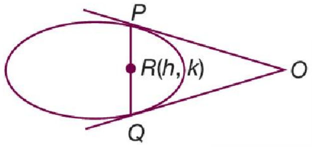
SOLUTION

Let $O(5\cos\theta, 5\sin\theta)$ and R be midpoint of PQ.

Let $R(h, k)$

Equation of the chord of contact PQ is $T=0$, i.e.,

$$\frac{5x\cos\theta}{4}+\frac{5y\sin\theta}{1}=1 \quad \ldots(1)$$



Equation of the chord PQ with midpoint $R(h, k)$ is

$T=S_1$, i.e.,

$$\frac{xh}{4}+\frac{yk}{1}-1=\frac{h^2}{4}+\frac{k^2}{1}-1 \text{ or } \frac{xh}{4}+\frac{yk}{1}=\frac{h^2}{4}+\frac{k^2}{1} \quad \ldots(2)$$

(1) and (2) represent the same straight line

$$\frac{5\cos\theta}{h}=\frac{5\sin\theta}{k}=\frac{1}{\frac{h^2}{4}+\frac{k^2}{1}}$$

⇒ $(5\cos\theta)^2+(5\sin\theta)^2=25$

⇒ $\frac{(h^2+k^2)}{\left(\frac{h^2}{4}+\frac{k^2}{1}\right)^2}=25$

Hence, locus of (h, k) is $(x^2+y^2)=25\left[\frac{x^2}{4}+\frac{y^2}{1}\right]^2$.

Test Your Concepts-III

(Solutions on page H.197)

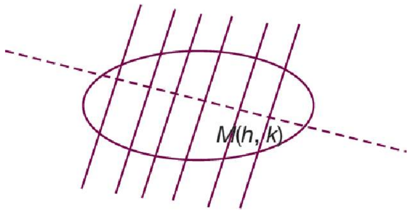
1. Tangents are drawn from the points on the line $x-y-5=0$ to $x^2+4y^2=4$, then all the chords of contact pass through a fixed point, find it.
2. Find the ratio of the area enclosed by the locus of midpoint of PS and area of the ellipse, where P is any point on the ellipse and S is the focus of the ellipse.
3. Find the locus of midpoint of chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which passes through the foot of the directrix from the focus.
4. If tangents are drawn to the ellipse $x^2+2y^2=2$, then find the locus of the midpoint of the intercept made by the tangents between the co-ordinate axes.

Answer Keys

1. $\left(\frac{4}{5}, -\frac{1}{5}\right)$
2. 1:4
3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\pm\frac{x}{ae}$
4. $\frac{1}{2x^2}+\frac{1}{4y^2}=1$

DIAMETER OF ELLIPSE

The locus of the midpoints of a system of parallel chords is called a diameter of a curve.



Let, $y = mx + c$ represents a system of parallel chords of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the midpoint of chord is $M(h, k)$.
Equation of chord $T = S_1$

Or $\quad \frac{xh}{a^2}+\frac{yk}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$

Or $\quad y=-\frac{b^2hx}{ka^2}+\frac{b^2}{k}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$, which also represents the system of parallel chords $y = mx + c$.

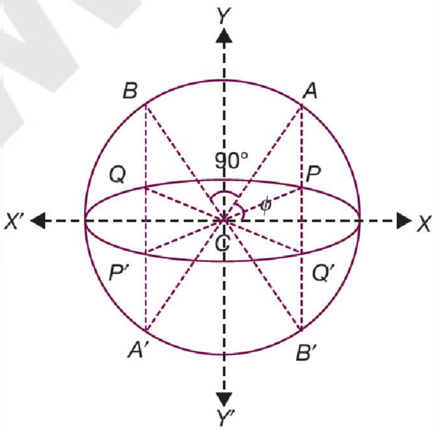
On comparison, we get, $m=\left(-\frac{b^2}{a^2}\right)\frac{h}{k}$

Now, the locus of midpoint of chord is M (h, k) $y=-\frac{b^2}{a^2m}x$

Line $y=-\frac{b^2}{a^2m}$ x is the equation of the diameter, as we can clearly see that it passes through the centre (0, 0).

Conjugate Diameter

Two diameters are said to be conjugate when each diameter bisects all chords parallel to the other diameter.



Conjugate diameters of auxiliary circle, i.e., AA' and BB' are perpendicular to each other. Hence, conjugate diameters of ellipse are PP' and QQ'.

If m_1 and m_2 be the slope of the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2=\frac{-b^2}{a^2}$

Hence, angle between conjugate diameters of ellipse is not 90°.

Now, the co-ordinates of the four extremities of two conjugate diameters are

$$P(a\cos\phi, b\sin\phi),\ P'(-a\cos\phi, -b\sin\phi)$$

$$Q(-a\sin\phi, b\cos\phi),\ Q'(a\sin\phi, -b\cos\phi)$$

ILLUSTRATION 60

Find eccentricity of ellipse having diameter and conjugate diameter as $y = x$ and $3x + 2y = 0$.

SOLUTION

Condition for diameter and conjugate diameter, we have $m_1m_2=-\frac{b^2}{a^2}$

$$\Rightarrow \quad (1)\left(-\frac{3}{2}\right)=\frac{-b^2}{a^2}$$

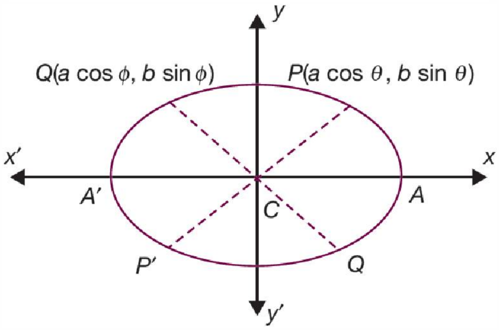
$$\Rightarrow \quad \frac{b^2}{a^2}=\frac{3}{2} \text{ (Clearly } b > a)$$

Now, eccentricity $e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$

ILLUSTRATION 61

Prove that the eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.

SOLUTION


y
Q(a cos ϕ, b sin ϕ)
P(a cos θ, b sin θ)
X′
x
A′
C
A
P′
Q
y′

Let PCP' and QCQ' be a pair of conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let the eccentric angles of the extremities P and Q be θ and ϕ, respectively. Then, the co-ordinates of P and Q are $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$, respectively.

$$\text{Now, } m_1 = \text{Slope of } CP = \frac{b}{a}\tan\theta$$

$$m_2 = \text{Slope of } CD = \frac{b}{a}\tan\phi$$

Since the diameters PCP' and QCQ' are conjugate diameters.

$$\therefore \quad m_1 m_2 = \frac{-b^2}{a^2}$$

$$\Rightarrow \quad \frac{b}{a}\tan\theta \times \frac{b}{a}\tan\phi = \frac{-b^2}{a^2}$$

$$\Rightarrow \quad \tan\theta\tan\phi = -1$$

$$\Rightarrow \quad \tan\theta = -\cot\phi$$

$$\Rightarrow \quad \tan\theta = \tan\left(\frac{\pi}{2}+\phi\right)$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}+\phi \Rightarrow \theta-\phi = \frac{\pi}{2}$$

ILLUSTRATION 62

Prove that the sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse, i.e., $CP^2 + CQ^2 = a^2 + b^2$.

SOLUTION

Let CP and CQ be two conjugate semi-diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let the eccentric angle of P be θ. Then the eccentric angle of Q is $\left(\frac{\pi}{2}+\theta\right)$. So, the co-ordinates of P and Q are $(a\cos\theta, b\sin\theta)$ and $\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$, i.e., $(-a\sin\theta, b\cos\theta)$, respectively.

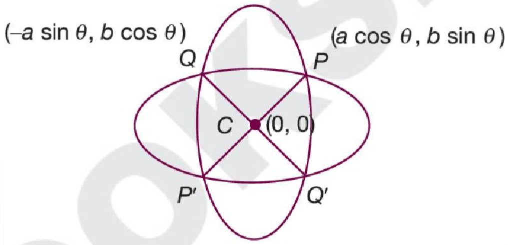
$$\therefore \quad CP^2 + CQ^2$$

$$= (a^2\cos^2\theta + b^2\sin^2\theta) + (a^2\sin^2\theta + b^2\cos^2\theta) = a^2 + b^2$$

ILLUSTRATION 63

Prove that the product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter, which is conjugate to the diameter through the point.

SOLUTION



Let S and S' be two foci of the ellipse. Then,

$$SP\cdot S'P = (a - ae\cos\theta)(a + ae\cos\theta)$$

$$= a^2 - a^2e^2\cos^2\theta$$

$$= a^2 - (a^2 - b^2)\cos^2\theta \quad (\because b^2 = a^2[1-e^2])$$

$$= a^2\sin^2\theta + b^2\cos^2\theta = CQ^2$$

ILLUSTRATION 64

If the points of intersection of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ be the extremities of the conjugate diameters of I$^{\text{st}}$ ellipse, then prove that $\frac{a^2}{p^2}+\frac{b^2}{q^2}=2$.

SOLUTION

Subtracting in order to find points of intersection, we get,

$$x^2\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{q^2}\right)=0$$

It passes through origin.

$$(y - m_1x)(y - m_2x) = 0$$

$$\Rightarrow \quad y^2 - (m_1 + m_2)x + m_1m_2x^2 = 0$$

$$\text{We get } m_1m_2 = \frac{\left(\frac{1}{a^2}-\frac{1}{p^2}\right)}{\left(\frac{1}{b^2}-\frac{1}{q^2}\right)}$$

Above equation will represent a pair of conjugate diameters of Ist ellipse if,

$$m_1 m_2 = -\frac{b^2}{a^2}$$

Now

$$\therefore \quad \frac{\left(\frac{1}{a^2}-\frac{1}{p^2}\right)}{\left(\frac{1}{b^2}-\frac{1}{q^2}\right)} = -\frac{b^2}{a^2}$$

Or $\quad a^2\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+b^2\left(\frac{1}{b^2}-\frac{1}{q^2}\right)=0$

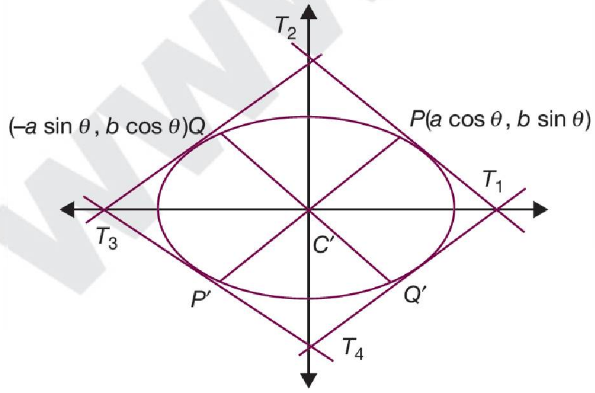
Or $\quad \frac{a^2}{p^2}+\frac{b^2}{q^2}=2$

ILLUSTRATION 65

Show that the tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.

SOLUTION

Let PCP' and QCQ' be a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let the eccentric angle of P be θ. Then the eccentric angle of Q is $\frac{\pi}{2}+\theta$. So the co-ordinates of P and Q are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ and co-ordinate of P' and Q' are $(-a\cos\theta, -b\sin\theta)$ and $(a\sin\theta, -b\cos\theta)$, respectively. Equation of tangents at P, Q, P' and Q' are respectively,



$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,\ \frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta=-1,$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=-1 \text{ and } \frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta=1$$

Clearly, the tangents at P and P' are parallel. Also, the tangents at Q and Q' are parallel. Hence, the tangents at P, Q, P', Q' form a parallelogram.

ILLUSTRATION 66

Prove that, the area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axes.

SOLUTION

We have written tangents in previous illustration.
Area of parallelogram $T_1\,T_2\,T_3\,T_4$ = 4 (Area of parallelogram CPT_2Q) = 4(2 Area of ΔCPQ)

$$= 8\ (\text{Area of } \Delta CPQ) = 8\times\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\cos\theta & b\sin\theta & 1 \\ -a\sin\theta & b\cos\theta & 1 \end{vmatrix}$$

$$= 4\,(ab\cos^2\theta + ab\sin^2\theta) = 4\,ab = 2a\times 2b$$

= Product of the axes of the ellipse

ILLUSTRATION 67

Find the angle between ellipse $\frac{x^2}{4}+y^2=1$ and the circle $x^2+y^2=2$.

SOLUTION

First, let us get the point of intersection of the given curves.

$$\frac{x^2}{4}+2-x^2=1$$

$$\Rightarrow \quad x^2+8-4x^2=4$$

$$\Rightarrow \quad 3x^2=4$$

$$\Rightarrow \quad x=\pm\frac{2}{\sqrt{3}};\ y^2=2-\frac{4}{3}=\frac{2}{3} \Rightarrow y=\pm\sqrt{\frac{2}{3}}$$

Let one point be $A\left(\frac{2}{\sqrt{3}},\sqrt{\frac{2}{3}}\right)$

Tangent at A on ellipse $\frac{x}{2\sqrt{3}}+y\sqrt{\frac{2}{3}}=1;\ m_1=-\frac{1}{2\sqrt{2}}$

Tangent at A on circle $\frac{2x}{\sqrt{3}}+y\sqrt{\frac{2}{3}}=2;\ m_2=-\sqrt{2}$

Angle between circle and ellipse = Angle between their tangents at their point of contact

Angle between the ellipse and the circle is $\tan\theta$

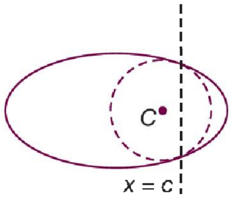
$$= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2\sqrt{2}} + \sqrt{2}}{1 + \frac{1}{2}}\right| = \frac{3}{2\sqrt{2}.\frac{3}{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

ILLUSTRATION 68

Find the largest circle with centre (1, 0) that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

SOLUTION



Given circle $(x - 1)^2 + y^2 = r^2$

Ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$\Rightarrow \quad \frac{x^2}{4} + y^2 = 4$

Taking difference, we get, $(x - 1)^2 - \frac{x^2}{4} = r^2 - 4$

$\Rightarrow \quad 3x^2 - 8x + 4 = 4r^2 - 16$

$\Rightarrow \quad 3x^2 - 8x + 20 - 4r^2 - 0$

Circle touches the ellipse and at points of contact and abscissae is equal.

$D = 0 \Rightarrow 64 - 4(3)(20 - 4r^2) = 0$

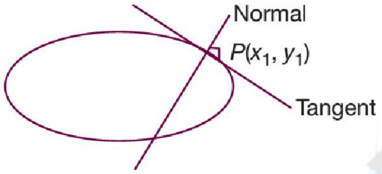
$\Rightarrow \quad 16 - 60 + 12r^2 = 0$

$\Rightarrow \quad r^2 = \frac{11}{3}$

$\therefore$ Required circle is $(x - 1)^2 + y^2 = \frac{11}{3}$

NORMAL OF AN ELLIPSE

A straight line passing through point of contact and perpendicular to tangent is known as normal of an ellipse.



Equation of tangent at point $P(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

Slope of tangent $= -\frac{b^2 x_1}{a^2 y_1}$

Slope of normal $= \frac{a^2 y_1}{b^2 x_1}$

Equation of normal, $y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

$\Rightarrow \quad \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

Now, let us write equation of normal in different forms.

(i) Point Form

The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

(ii) Parametric Form

The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is

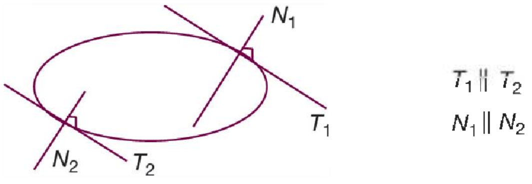
$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$.

(iii) Slope Form

The equation of the normal $y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2 m^2}}$

at the points is $\left(\mp\frac{a^2}{\sqrt{a^2 + b^2 m^2}}, \mp\frac{b^2 m}{\sqrt{a^2 + b^2 m^2}}\right)$.

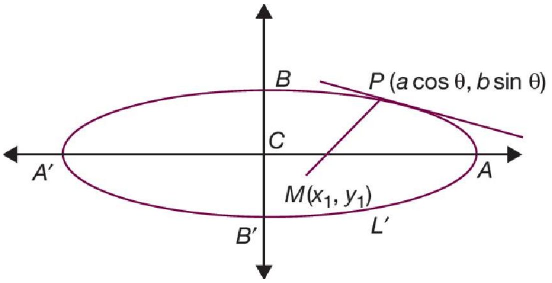
In ellipse, two normal can be drawn having same slope.



We know that, in circle, every normal passes through the center of circle. However, in ellipse, not every normal passes through the centre of the ellipse.

Co-normal Points

The points on the ellipse, the normals at which pass through a given point are called co-normal points.



The equation of the normal to the ellipse at the point θ is $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$.
If it passes through $M(x_1, y_1)$, then

$$ax_1 \sec\theta - by_1 \operatorname{cosec}\theta = a^2 - b^2 \qquad \ldots(1)$$

Putting $\tan\frac{\theta}{2} = t$, $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ in (1), we get,

$$ax_1\left(\frac{1+t^2}{1-t^2}\right) - by_1\left(\frac{1+t^2}{2t}\right) = a^2 - b^2$$

$$\Rightarrow \quad by_1 t^4 + 2t^3(ax_1 + a^2e^2) + 2t(ax_1 - a^2e^2) - by_1 = 0 \qquad \ldots(2)$$

This is a fourth degree equation in t. So, it gives four values of t namely, t_1, t_2, t_3, t_4, i.e., $\tan\frac{\theta_1}{2}$, $\tan\frac{\theta_2}{2}$, $\tan\frac{\theta_3}{2}$ and $\tan\frac{\theta_4}{2}$.

Thus, the four values of t correspond to four points on the ellipse.

Four normals can be drawn to an ellipse from a point in its plane. Let us say four points P, Q, R and S on the ellipse the normals at which will pass through a given point. These four points P, Q, R and S are called the co-normal points.

ILLUSTRATION 69

Find minimum and maximum distance between line $x + y = 10$ and $\frac{x^2}{16} + \frac{y^2}{9} = 1$

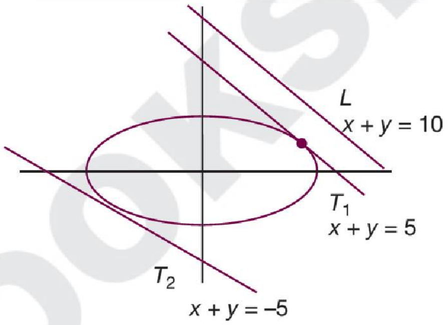
SOLUTION

First, let get tangents parallel to given line $x + y = 10$ $(m = -1)$

$$C^2 = 16 + (1)\,9 = a^2m^2 + b^2 \text{ (tangency)}$$

$$C = \pm 5$$

Equation of tangents T_1: $x + y = 5$ and T_2: $x + y = -5$ parallel to the given line.



Minimum distance = Distance between line L and T_1

$$= \frac{5}{\sqrt{2}}$$

Maximum distance = Distance between line L and T_2

$$= \frac{15}{\sqrt{2}}$$

Alternate

Any point on the ellipse is $P(4\cos\theta, 3\cos\theta)$.

Distance of point P from given line is

$$\left|\frac{4\cos\theta + 3\sin\theta - 10}{\sqrt{2}}\right| \qquad (\because -5 \le 4\cos\theta + 3\sin\theta \le 5)$$

$$\text{Minimum distance} = \frac{5}{\sqrt{2}} \text{ units}$$

$$\text{Maximum distance} = \frac{15}{\sqrt{2}} \text{ units}$$

ILLUSTRATION 70

If the normal at $P(\theta)$ to ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ meets ellipse again $Q(2\theta)$, then find $\cos\theta$.

SOLUTION

Equation of normal at P $(\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$

$$\sqrt{14}x\sec\theta - \sqrt{5}y\,\text{cosec}\,\theta = 9$$

It again meets ellipse at point $Q(\sqrt{14}\cos 2\theta, \sqrt{5}\sin 2\theta)$

$$14\frac{\cos 2\theta}{\cos\theta} - \frac{5\sin 2\theta}{\sin\theta} = 9$$

$$\Rightarrow \quad 14\cos 2\theta - 10\cos^2\theta = 9\cos\theta$$

$$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$$

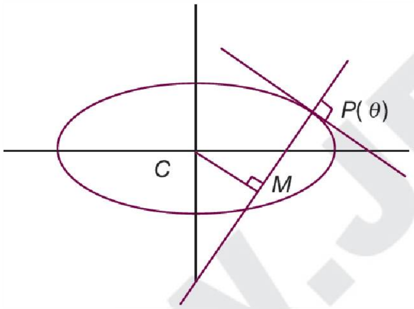
$$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta + 2) = 0$$

$$\Rightarrow \quad \cos\theta = \frac{7}{6}(\text{not possible}); \text{we get } \cos\theta = \frac{-2}{3}$$

ILLUSTRATION 71

Prove that length of perpendicular drawn from centre to any normal is always less than difference of semi principal axes.

SOLUTION



Normal at point $P(\theta)$ $xa\sec\theta - yb\sec\theta = a^2 - b^2$

Length of perpendicular from centre to normal is

$$CM = \left|\frac{0-(a^2-b^2)}{\sqrt{a^2\sec^2\theta + b^2\,\text{cosec}^2\theta}}\right|$$

$$= \left|\frac{a^2-b^2}{\sqrt{a^2+b^2+a^2\tan^2\theta + b^2\cot^2\theta}}\right|$$

For non-negative real numbers, $AM \geq GM$

Or $$\frac{a^2\tan^2\theta + b^2\cot^2\theta}{2} \geq \sqrt{a^2\tan^2\theta\, b^2\cot^2\theta}$$

Or $$a^2\tan^2\theta + b^2\cot^2\theta \geq 2ab$$

$$CM \leq \left|\frac{a^2-b^2}{\sqrt{a^2+b^2+2ab}}\right| = \left|\frac{a^2-b^2}{\sqrt{(a+b)^2}}\right|$$

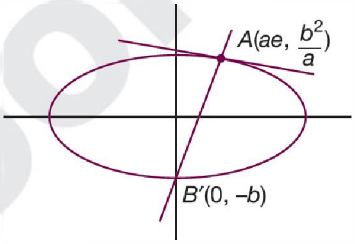
$$\Rightarrow \quad CM \leq |a-b|$$

$\therefore$ $CM \leq$ Difference of semi principal axes

ILLUSTRATION 72

If the normal at one end of latus rectum passes through the extremity of minor axis, then find the eccentricity of the ellipse.

SOLUTION



Normal at $(x_1, y_1) = (ae, \frac{b^2}{a})$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$$\Rightarrow \quad \frac{ax}{e} - ay = a^2 - b^2 = a^2e^2$$

It passes through $B'(0, -b)$

$$0 + ba = a^2 - b^2$$

$$\Rightarrow \quad b^2 + ab - a^2 = 0$$

$$\Rightarrow \quad \left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right) - 1 = 0, \text{ we get } \frac{b}{a} = \frac{-1\pm\sqrt{5}}{2}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{\sqrt{5}-1}{2}$$

$$e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\left(\frac{\sqrt{5}-1}{2}\right)^2}$$

$$= \sqrt{1-\frac{(6-2\sqrt{5})}{4}} = \sqrt{\frac{\sqrt{5}-1}{2}}$$

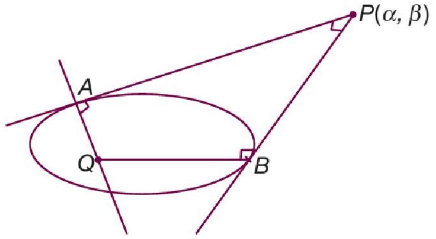
ILLUSTRATION 73

The tangents from (α, β) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect at right angle. Show that the locus of the point of intersection of normals at the point of contact of the two tangents is the line $y\alpha - x\beta = 0$.

SOLUTION

The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let θ and ϕ be the eccentric angles of the point of contact of the two tangents from (α, β).



Equation of the tangents at point θ and ϕ are

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

And $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$

Since lines (2) and (3) are at right angle, then

$$\left(-\frac{b}{a}\cot\theta\right)\left(-\frac{b}{a}\cot\phi\right)=-1$$

Or $\quad \tan\theta\tan\phi=-\frac{b^2}{a^2}$ ...(1)

Since (α, β) is the point of intersection of line (2) and (3), therefore,

$$\frac{\alpha\cos\theta}{a}+\frac{\beta\sin\theta}{b}=1 \text{ and } \frac{\alpha\cos\phi}{a}+\frac{\beta\sin\phi}{b}=1$$

By taking difference, we get,

$$\frac{\alpha}{a}(\cos\phi-\cos\theta)=-\frac{\beta}{b}(\sin\phi-\sin\theta) \qquad ...(2)$$

Also, equation of normals at the points θ and ϕ are $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$ and $ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2$.

In order to find the locus of point of intersection of normals, take difference of normal equations, we get,

$$ax(\sec\theta-\sec\phi)=by(\operatorname{cosec}\theta-\operatorname{cosec}\phi)$$

$$\Rightarrow \quad ax\left(\frac{\cos\phi-\cos\theta}{\cos\theta\cos\phi}\right)=by\left(\frac{\sin\phi-\sin\theta}{\sin\theta\sin\phi}\right) \qquad ...(3)$$

Taking ratio of Equations (2) and (3),

$$\frac{a^2x}{\alpha\cos\theta\cos\phi}=-\frac{b^2y}{\beta\sin\theta\sin\phi}$$

$$\Rightarrow \quad \tan\theta\tan\phi=\frac{b^2}{a^2}\,\frac{y}{x}\left(-\frac{\alpha}{\beta}\right)$$

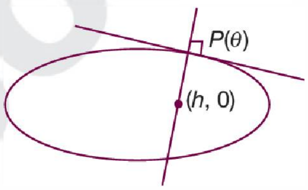
$$\Rightarrow \quad -\frac{b^2}{a^2}=-\frac{b^2}{a^2}\,\frac{y}{x}\,\frac{\alpha}{\beta} \qquad \text{(From (1))}$$

$\therefore \quad y\alpha-x\beta=0$ is the required locus. Hence proved

ILLUSTRATION 74

Find the range of parameter h, if for real normal can be drawn to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut each other at point $(h, 0)$.

SOLUTION



Equation of normal is $xa\sec\theta-yb\operatorname{cosec}\theta=a^2-b^2$, which passes through point $(h, 0)$.

Then, $ah\sec\theta-0=a^2-b^2$

$$\sec\theta=\left|\frac{a^2-b^2}{ah}\right|>1$$

$$a^2e^2>a|h|$$

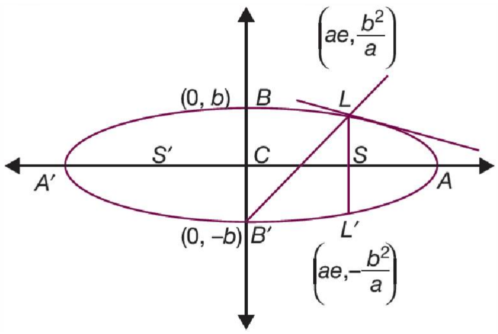
$$|h|<ae^2$$

$$\Rightarrow \quad -ae^2<h<ae^2$$

ILLUSTRATION 75

If the normal at an end of a latus rectum of an ellipse passes through one extremity of the minor axis, then show that the eccentricity of the ellipse is given by $e^4+e^2-1=0$.

SOLUTION



Let, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse. The co-ordinates of an end of the latus rectum is $\left(ae, \frac{b^2}{a}\right)$.

The equation of the normal at $\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2$$

Or $$\frac{ax}{e}-ay=a^2-b^2$$

It passes through one extremity of the minor axis, whose co-ordinates are (0, – b).

$\therefore \quad ab=a^2-b^2$

$\Rightarrow \quad a^2b^2=(a^2-b^2)^2 \qquad (\because\ b^2=a^2(1-e^2))$

$\Rightarrow \quad a^2\times a^2(1-e^2)=(a^2e^2)^2$

$\Rightarrow \quad 1-e^2=e^4 \Rightarrow e^4+e^2-1=0$ Hence proved

ILLUSTRATION 76

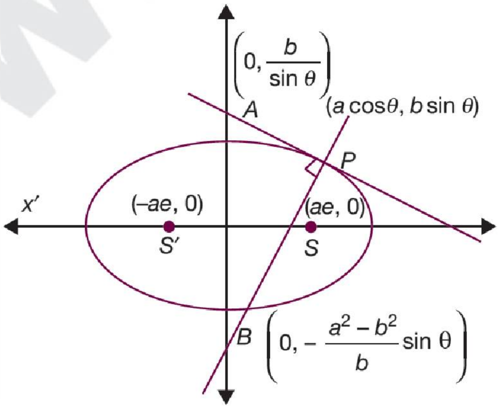
The tangent and normal at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the y-axis at A and B, respectively. Obtain the equation of circle on AB as diameter and show that it passes through P, as well as the two foci of the ellipse.

SOLUTION

Clearly, the equation of tangent and normal at any point $P(a\cos\theta, b\sin\theta)$ can be written as

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad \ldots(1)$$

And $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$ ...(2)


$\left(0, \frac{b}{\sin\theta}\right)$
A
$(a\cos\theta, b\sin\theta)$
P
x′
(–ae, 0)
(ae, 0)
S′
S
B
$\left(0, -\frac{a^2-b^2}{b}\sin\theta\right)$

Now, putting $x=0$ in equation of the tangent and normal the points A and B are

$$A\left(0, \frac{b}{\sin\theta}\right) \text{ and } B\left(0, -\frac{a^2-b^2}{b}\sin\theta\right)$$

Circle on AB as diameter is given by

$$(x-0)(x-0)+\left(y-\frac{b}{\sin\theta}\right)\left(y+\frac{a^2-b^2}{b}\sin\theta\right)=0$$

Or $$x^2+y^2-y\left(\frac{b^2-(a^2-b^2)\sin^2\theta}{b\sin\theta}\right)-(a^2-b^2)=0 \qquad \ldots(3)$$

$\therefore \quad \angle APB=90°$

$\therefore \quad P$ lies on circle.

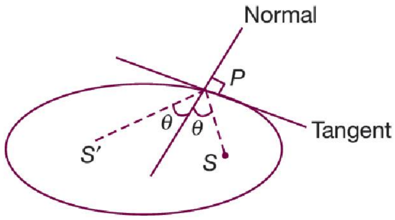
It passes through the foci (± ae, 0).

Since, $a^2e^2=a^2-b^2$

Highlights of Ellipse

1. The tangent and normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P.

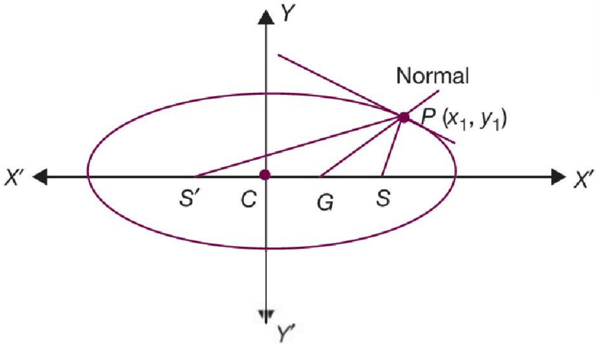
   Angle bisector of focal distance PS and PS' = Tangent and normal at point P



PROOF:

Let P be any point on the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore$ Equation of normal PG is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$



Putting $y = 0$ for the point G, we have

$$\therefore \quad x = CG = \left(\frac{a^2 - b^2}{a^2}\right)x_1 = \frac{a^2e^2}{a^2}x_1 = e^2x_1$$

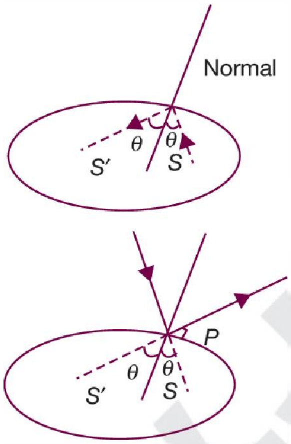
$$\therefore \quad SG = CS - CG = ae - e^2x_1 = e\,(a - ex_1) = eSP$$

Similarly, $S'G = eS'P$

$$\therefore \quad \frac{SG}{S'G} = \frac{eSP}{eS'P} = \frac{SP}{S'P}$$

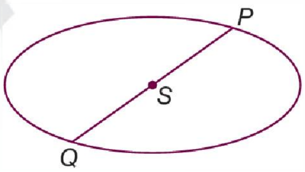
$\therefore$ The normal PG bisects the internal $\angle SPS'$ between the focal distances as tangent and normal are at right angles, the tangent PT bisects the external angle SPL between them.

2. A ray originated from one focus, after getting reflected by elliptical surface passes through other focus.



In other words a ray travelling towards a focus after getting reflected by elliptical mirror moves away from other focus.

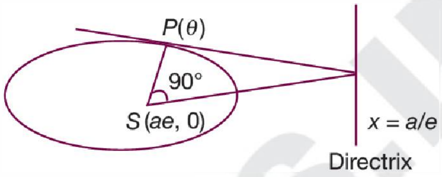
3. Semi-latus rectum is the harmonic mean of the segments of focal chord of an ellipse, where PQ is the focal chord through focus S. (Parabola also has same property)



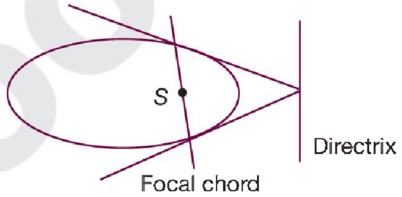
$$\frac{2SP.SQ}{SP + SQ} = \frac{1}{2}(\text{Length of latus rectum})$$

$$\text{Or} \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}, (a > b)$$

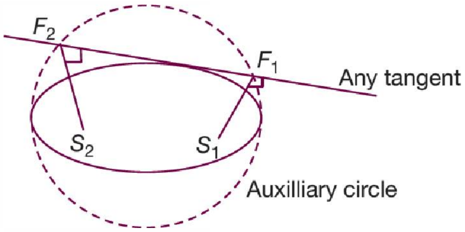
4. Segment of tangent intercepted between point of contact and directrix subtends right angle at corresponding focus.
(It is a similar property as in parabola)



5. From any point on directrix, chord of contact to an ellipse always passes through corresponding focus. In other words, from any point on directrix, every chord of contact is focal chord.
(It is a similar property as in parabola)



6. The locus of the feet of perpendiculars from the foci on any tangent to an ellipse is auxiliary circle.



PROOF:

The equation of any tangent in terms slope (m) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = mx + \sqrt{(a^2m^2 + b^2)}$.

$$\text{Or} \quad y - mx = \sqrt{(a^2m^2 + b^2)} \qquad \ldots(1)$$

Equation of perpendicular line on (1) and passing through ($\pm\, ae$, 0) is

$$my + x = \pm\, ae \qquad \ldots(2)$$

The locus of the point of intersection of the line given by equation (1) and (2) can be obtained by eliminating m between them.

Squaring and adding (1) and (2), we get,

$$y^2(1 + m^2) + x^2(1 + m^2) = a^2m^2 + b^2 + a^2e^2$$

$$\Rightarrow \quad (1 + m^2)(x^2 + y^2) = a^2m^2 + a^2 = a^2(1 + m^2)$$

$$\text{Or} \quad x^2 + y^2 = a^2$$

It is the equation of the auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Feet of perpendicular drawn from both focii on any tangent lie on auxiliary circle.

7. The product of the lengths of the perpendicular segments from the foci on any tangent to the ellipse is square of semi-minor axis.

$$(S_1F_1)(S_2F_2)=b^2 \text{ (When } a>b)$$

PROOF:

Let tangent $y-mx-a\sqrt{1+m^2}=0$

$S_1(ae, 0)$ $S_2(-ae, 0)$

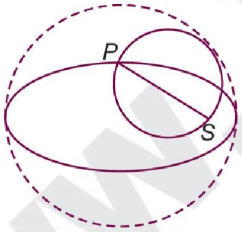
$S_1F_1\cdot S_2F_2$

$$=\left(\frac{aem+\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}\right)\left(\frac{-aem+\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}\right)$$

$$=\left|\frac{a^2m^2+b^2-a^2e^2m^2}{m^2+1}\right|=\left|\frac{a^2m^2+b^2-(a^2-b^2)m^2}{m^2+1}\right|$$

$=b^2$

8. The circle which takes focal distance as diameter touches the auxiliary circle of ellipse.



Auxiliary circle

ILLUSTRATION 77

Find the eccentricity of an ellipse, whose foci are (2, 4) and (14, 9) and touches the x-axis.

SOLUTION

Distance between focus $=2ae=\sqrt{144+25}=13$ units

$\Rightarrow\quad 4a^2e^2=169$

$\Rightarrow\quad 4(a^2-b^2)=169$

Tangent is x-axis ($y=0$)

Product of perpendiculars from focii to tangent $=4\times 9=b^2$

We get, $b=6$

$$a^2=36+\frac{169}{4}=\frac{313}{4}$$

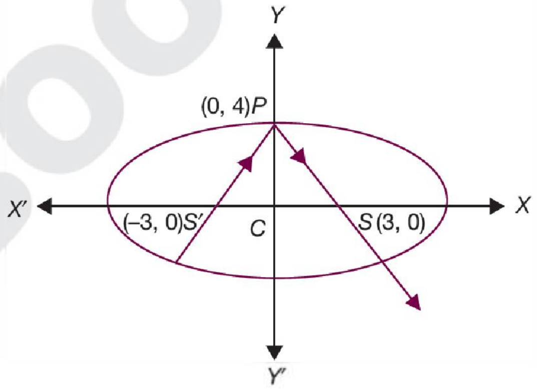
Now, eccentricity of ellipse $=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{36.4}{313}}$

$=\frac{13}{\sqrt{313}}$

ILLUSTRATION 78

A ray emanating from the point (–3, 0) is incident on the ellipse $16x^2+25y^2=400$ at the point P with ordinate 4. Find the equation of the reflected ray after first reflection.

SOLUTION



For point P, y-co-ordinate $=4$

∵ The given ellipse is

$16x^2+25y^2=400$, by substituting $y=4$, we get,

$16x^2+25(4)^2=400$

∴ Co-ordinate of P is (0, 4).

$$e^2=1-\frac{16}{25}=\frac{9}{25}\Rightarrow e=\frac{3}{5}$$

Foci ($\pm ae$, 0), i.e., ($\pm$ 3, 0)

A ray emanating from one focus (–3, 0) after striking ellipse passes through the other focus (3, 0).

Equation of reflected ray (i.e., PS) is $\frac{x}{3}+\frac{y}{4}=1$.

Or $\quad 4x+3y=12$

ILLUSTRATION 79

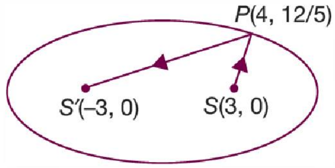
A ray of light is coming along the line $12x-5y-36=0$ and strikes an ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, in Ist quadrant. Find the equation of reflected ray.

SOLUTION

Incident ray $12x - 5y = 36$ intersect

ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ at $\left(4, \dfrac{12}{5}\right)$ in I$^{\text{st}}$ quadrant.

Focus of ellipse are $S(3, 0)$ $S'(-3, 0)$



As we can observe that incident ray passes through focus (3, 0), after reflection from elliptical surface, it passes through other focus (–3, 0).

Equation of reflected ray $S'P$ is

$$y - 0 = \frac{12/5}{7}(x + 3)$$

$\therefore$ The required equation is $12x - 35y + 36 = 0$.

Test Your Concepts-IV

(Solutions on page H.197)

1. Find the area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point, whose eccentric angle is $\dfrac{\pi}{4}$.
2. Find the number of tangents to the circle $x^2 + y^2 = 3$, which are normal to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.
3. A normal to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ meets the axes in L and M. If the perpendiculars to the axes through L and M intersect at P, then find the equation to the locus of P.
4. If a tangent of slope 2 of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then find the maximum value of ab.
5. If the points of intersection of two ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ be at the extremities of conjugate diameters of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then compute $\dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2}$.
6. The normal to the curve $x^2 + 3y^2 - 4 = 0$ at the point $P\left(\dfrac{\pi}{6}\right)$ intersects the curve again at the point $Q(\theta)$, θ being the eccentric angle of the point Q, then find θ.
7. The locus of the feet of the perpendiculars drawn from the foci of the ellipse $S = 0$ to any tangent to it is auxiliary circle (True/False).

Answer Keys

1. $\dfrac{ab(a^2 - b^2)}{a^2 + b^2}$

2. 0

3. $a^2x^2 + b^2y^2 = (a^2 - b^2)^2$

4. 4

5. 2

6. $\dfrac{3\pi}{2}$

7. True

SOLVED PROBLEMS

PROBLEM 1

If the circle $x^2 + y^2 = r^2$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at four points, then show that sum of eccentric angles of point of intersection is even multiple of π.

SOLUTION

Any point ($a\cos\theta, b\sin\theta$) on ellipse also lies on circle, as they are intersecting $a^2\cos^2\theta + b^2\sin^2\theta = r^2$

$$\Rightarrow \quad a^2\left[\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]^2 + b^2\left[\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]^2 = r^2$$

$$\Rightarrow \quad a^2\left(1+\tan^4\frac{\theta}{2} - 2\tan^2\frac{\theta}{2}\right) + 4b^2\tan^2\frac{\theta}{2}$$

$$= r^2\left(1+\tan^4\frac{\theta}{2} + 2\tan^2\frac{\theta}{2}\right)$$

$$\Rightarrow \quad (a^2-r^2)\tan^4\frac{\theta}{2} + 0.\tan^3\frac{\theta}{2} + \tan^2\frac{\theta}{2}(4b^2-2a^2-2r^2)$$

$$+0.\tan\frac{\theta}{2} + a^2 - r^2 = 0$$

Let them cut at four points namely, $P(\alpha)$, $Q(\beta)$, $R(\gamma)$ and $S(\delta)$ then

$$\Sigma\tan\frac{\alpha}{2} = 0 \text{ and } \Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2} = 0$$

$$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{r}{2}+\frac{\delta}{2}\right) = \frac{S_1 - S_3}{1-S_2+S_4}$$

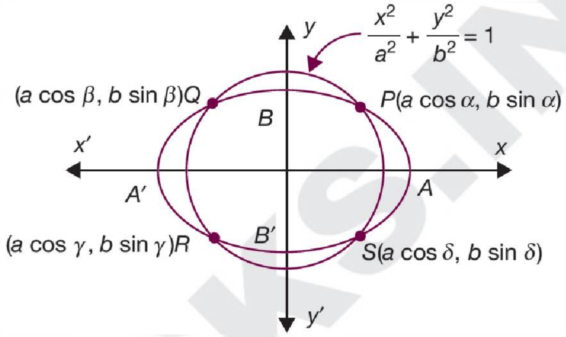
$$= \frac{\Sigma\tan\frac{\alpha}{2} - \Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}{1-\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}}$$

$$= \frac{0-0}{1-S_2+S_4} = 0$$

$$\Rightarrow \quad \frac{\alpha+\beta+r+\delta}{2} = \pi$$

$$\Rightarrow \quad \alpha+\beta+r+\delta = 2n\pi$$

Alternate solution



Let the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be cut by a concentric circle in points P, Q, R and S. Let the eccentric angles of P, Q, R and S be α, β, γ and δ respectively. Then, the equations of chords PQ and RS are respectively,

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2} \qquad \ldots(1)$$

$$\text{And} \quad \frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2} = \cos\frac{\gamma-\delta}{2} \qquad \ldots(2)$$

The equation of a curve passing through points P, Q, R and S, i.e., the points of intersection of the ellipse and lines (1) and (2) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left\{\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right\}$$

$$\left\{\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}\right\} = 0$$

Since this curve is given to be a circle. Therefore, coefficient of $xy = 0$.

$$\Rightarrow \quad \cos\frac{\alpha+\beta}{2}\sin\frac{\gamma+\delta}{2}+\cos\frac{\gamma+\delta}{2}\sin\frac{\alpha+\beta}{2} = 0$$

$$\Rightarrow \quad \sin\left\{\frac{\alpha+\beta+\gamma+\delta}{2}\right\} = 0$$

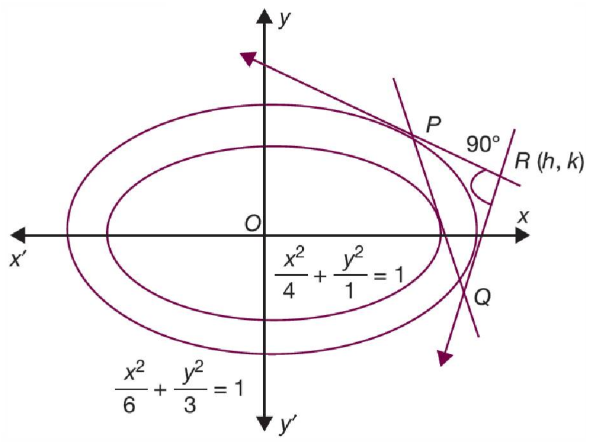
$$\Rightarrow \quad \frac{\alpha+\beta+\gamma+\delta}{2} = n\pi,\ n\in Z$$

$$\Rightarrow \quad \alpha+\beta+\gamma+\delta = 2n\pi,\ n\in Z.$$

PROBLEM 2

A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

SOLUTION


y
P
90°
R (h, k)
O
x
x′
$\frac{x^2}{4}+\frac{y^2}{1}=1$
Q
$\frac{x^2}{6}+\frac{y^2}{3}=1$
y′

The equations of two ellipses are respectively,

$$\frac{x^2}{4}+\frac{y^2}{1}=1 \qquad \ldots(1)$$

and $$\frac{x^2}{6}+\frac{y^2}{3}=1 \qquad \ldots(2)$$

Suppose the tangents at P and Q to the ellipse (2) intersect at $R\ (h, k)$. Then PQ is the chord of contact of tangents drawn from $R\ (h, k)$ to the ellipse (2). So, the equation of PQ is

$$\frac{hx}{6}+\frac{ky}{3}=1 \Rightarrow y=-\frac{hx}{2k}+\frac{3}{k}$$

This touches the ellipse given in (1). Therefore,

$$\frac{9}{k^2}=4\left(\frac{-h}{2k}\right)^2+1 \qquad (\text{Using, } c^2=a^2+b^2)$$

$$\Rightarrow \quad h^2+k^2=9$$

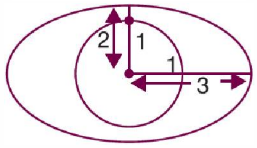
$\Rightarrow$ (h, k) lies on the circle $x^2+y^2=9$.

Clearly, $x^2+y^2=9$ is the director circle of the ellipse (2). Hence, the angle between the tangents at P and Q to the ellipse is a right angle.

PROBLEM 3

Find the length of the common chord of the ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$ and the circle $(x-1)^2+(y-2)^2=1$.

SOLUTION



Given ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$ and the circle $(x-1)^2+(y-2)^2=1$ both have centres at (1, 2).

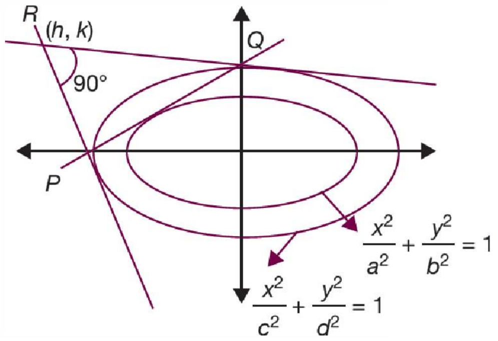
It is clear that they do not intersect in real points.

$\therefore$ There is no common chord.

PROBLEM 4

A tangent is drawn to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to cut $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ at points P and Q. If tangent at P and Q to the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ intersect at right angle, then prove that $\frac{a^2}{c^2}+\frac{b^2}{d^2}=1$.

SOLUTION


R
(h, k)
Q
90°
P
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
$\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$

Let tangents to $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ at P and Q intersect at $R(h, k)$ at right angle.

$$\therefore \quad h^2+k^2=c^2+d^2 \text{ (Director circle)} \qquad \ldots(1)$$

And chord of contact PQ is given by

$$\frac{hx}{c^2}+\frac{ky}{d^2}=1 \qquad \ldots(2)$$

It is a tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad \ldots(3)$$

Equation (2) and (3) represent the same line, thus

$$\frac{ha}{c^2\cos\theta}=\frac{kb}{d^2\sin\theta}=1$$

$$\Rightarrow \quad \frac{h^2a^2}{c^4}+\frac{k^2b^2}{d^4}=1$$

Comparing with equation (1), we get,

$$\frac{a^2}{c^4}=\frac{b^2}{d^4}=\frac{1}{c^2+d^2}$$

$$\Rightarrow \quad \frac{a^2}{c^2}=\frac{c^2}{c^2+d^2} \text{ and } \frac{b^2}{d^2}=\frac{d^2}{c^2+d^2}$$

Hence, $\frac{a^2}{c^2}+\frac{b^2}{d^2}=\frac{c^2+d^2}{c^2+d^2}=1$

PROBLEM 5

Find the shorter distance from the ellipse $x^2+2y^2=2$ and the circle $x^2+y^2-3x-2\sqrt{2}y+4=0$.

SOLUTION

The normal at $\left(\sqrt{2}.\cos\theta,\sin\theta\right)$ on the ellipse is

$$\sqrt{2}x\sec\theta-y\,\mathrm{cosec}\,\theta=1$$

It passes through the centre $\left(\frac{3}{2},\sqrt{2}\right)$ of the circle

$$\therefore \quad \frac{3}{\sqrt{2}}\sec\theta-\sqrt{2}\,\mathrm{cosec}\,\theta=1$$

Solving, we get, $\theta=\frac{\pi}{4}$

The nearest point of the ellipse $\left(1,\frac{1}{\sqrt{2}}\right)$

Its distance from the centre of the circle is $\frac{\sqrt{3}}{2}$.

The radius of the circle $=\sqrt{\left(\frac{3}{2}\right)^2+2-4}=\frac{1}{2}$

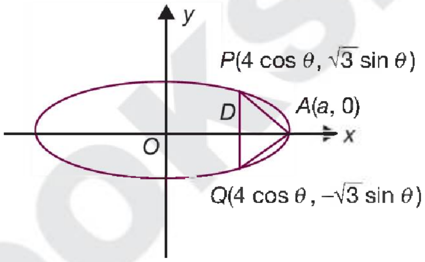
$\therefore$ The shortest distance is $\frac{\sqrt{3}-1}{2}$ units.

PROBLEM 6

Find the area of the greatest triangle that can be inscribed in an ellipse $\frac{x^2}{16}+\frac{y^2}{3}=1$ having its vertex coincident with an extremity of the major axis.

SOLUTION

Let P and Q be on ellipse



$$\Delta = \text{Area of } \Delta APQ$$

$$=\frac{1}{2}PQ\times AD$$

$$=\frac{1}{2}\left(2\sqrt{3}\sin\theta\right)\left(OA-OD\right)$$

$$\Delta=\left(\sqrt{3}\sin\theta\right)\left(4-4\cos\theta\right)$$

$$=2\sqrt{3}\left(2\sin\theta-\sin 2\theta\right)$$

So, $\frac{dA}{d\theta}=4\sqrt{3}\left(\cos\theta-\cos 2\theta\right)$

For max or min, $\frac{dA}{d\theta}=0$

$$\cos\theta-(2\cos^2\theta-1)=0$$

$$\Rightarrow \quad 2\cos^2\theta-\cos\theta-1=0$$

$$\Rightarrow \quad (2\cos\theta+1)(\cos\theta-1)=0$$

$$\cos\theta=-\frac{1}{2} \text{ or } 1$$

$\cos\theta=1$ not possible

$$\cos\theta=-\frac{1}{2}\Rightarrow\theta=\frac{2\pi}{3}$$

$$\frac{dA^2}{d\theta^2}=4\sqrt{3}\left(-\sin\theta+\sin 2\theta\right)<0$$

$\therefore$ Maximum area $=2\sqrt{3}\left[2\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}\right]=9$ sq. units

PROBLEM 7

If the locus of midpoints of portions of tangents intercepted between the axes of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=\lambda$, then find the numerical value of λ.

SOLUTION

Tangent at 'α' is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$

Let $A\left(a\sec\alpha,0\right)$ $B\left(0,b\operatorname{cosec}\alpha\right)$

Clearly, (h,k) is the midpoint.

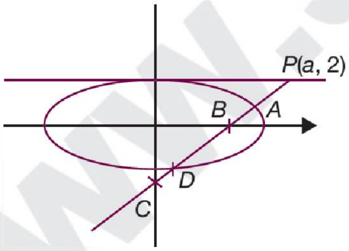
$$h=\frac{1}{2}a\sec\alpha \quad k=\frac{b}{2}\operatorname{cosec}\alpha$$

$$\Rightarrow \quad \frac{a^2}{4x^2}+\frac{b^2}{4y^2}=1$$

PROBLEM 8

If line through $P(a, 2)$ meets the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ at A and D and meets the axes at B and C, so that PA, PB, PC, PD are in G.P., then find the minimum value of $|a|$.

SOLUTION



Equation of line $\frac{x-a}{\cos\theta}=\frac{y-2}{\sin\theta}=r$

Point $\left(a+r\cos\theta,2+r\sin\theta\right)$ lies on ellipse for A and D.

$$\frac{\left(a+r\cos\theta\right)^2}{9}+\frac{\left(2+r\sin\theta\right)^2}{4}=1$$

$$r^2\left[\frac{\cos^2\theta}{9}+\frac{\sin^2\theta}{4}\right]+r(\alpha)+\frac{a^2}{9}=0$$

$$r_1r_2=\frac{a^2}{\left(\cos^2\theta+\frac{9\sin^2\theta}{4}\right)}=PA\cdot PD$$

$$r_3=-\frac{2}{\sin\theta}=PB;\ r_4=-\frac{a}{\cos\theta}=PC$$

PA, PB, PC, PD are in G.P. $\Rightarrow PA\cdot PD=PB\cdot PC$

$$\frac{a^2}{\left(\cos^2\theta+\frac{9}{4}\sin^2\theta\right)}=\frac{2a}{\sin\theta\cos\theta}$$

$$\text{or}\quad |a|=\left|\frac{2\left(\cos^2\theta+\frac{9}{4}\sin^2\theta\right)}{\sin\theta\cos\theta}\right|$$

$$|a|=2\left|\cot\theta+\frac{9}{4}\tan\theta\right|\geq 4\sqrt{\frac{9}{4}}=6$$

$\therefore$ Minimum value of $|a|=6$

PROBLEM 9

Let E_1 and E_2 be two ellipses. The area of the ellipse E_2 is one-third the area of the quadrilateral formed by the tangents at the ends of the latus rectum of the ellipse, E_3 (E_3: $5x^2+9y^2=45$). The eccentricities of E_1, E_2 and E_3 are equal. E_1 is inscribed in E_2 in such a way that both E_1 and E_2 touch each other at one end of their common major axis. If the length of the major axis of E_1 is equal to the length of the minor axis of E_2, then find the area of the ellipse E_2 outside the ellipse E_1.

SOLUTION

Given ellipse is E_3: $5x^2+9y^2=45$

$a=3$, $b=\sqrt{5}$, $e=\frac{2}{3}$, one end of latus rectum in I$^{\text{st}}$ quadrant $\left(2,\frac{5}{3}\right)$

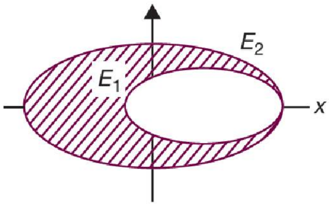
Equation of tangent is $2x+3y=9$.

It meets axes at $\left(\frac{9}{2},0\right)$ and $(0, 3)$

Area of the quadrilateral $=4\left(\frac{1}{2}\times\frac{9}{2}\times3\right)=27$ sq. units

Let the equation of the ellipses E_2 and E_1 be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{(x-\alpha)^2}{b^2}+\frac{y^2}{c^2}=1$$



Now, $b^2 = a^2(1-e^2)$ and $c^2 = b^2(1-e^2) = a^2(1-e^2)^2$

$\Rightarrow \quad c = a(1-e^2) \Rightarrow a - c = ae^2$

Area of $E_2 = \pi ab$ and Area of $E_1 = \pi bc$

Required area $= \pi b(a-c) = \pi bae^2$

$$= 9\times\frac{4}{9} = 4 \text{ sq. units}$$

(As $\pi ab = \frac{1}{3}\times$ Area of quadrilateral $= 9$)

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The equation $\frac{x^2}{10-a}+\frac{y^2}{4-a}=1$ represents an ellipse, if:

(A) $a<4$ (B) $a>4$

(C) $4<a<10$ (D) $a>10$

2. The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is:

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{\sqrt{3}}$

3. An ellipse having foci at (3, 3) and (– 4, 4) and passing through the origin has eccentricity is equal to:

(A) $\frac{3}{7}$ (B) $\frac{2}{7}$

(C) $\frac{5}{7}$ (D) $\frac{3}{5}$

4. The foci of an ellipse are (–2, 4) and (2, 1). The point $\left(1,\frac{23}{6}\right)$ is an extremity of the minor axis. What is the value of the eccentricity?

(A) $\frac{9}{13}$ (B) $\frac{3}{\sqrt{13}}$

(C) $\frac{2}{\sqrt{13}}$ (D) $\frac{4}{13}$

5. An ellipse has foci at (9, 20) and (49, 55) in the *xy*-plane and is tangent to the *x*-axis. The length of its major axis is:

(A) 85 (B) 75

(C) 65 (D) 55

6. The length of the latus rectum of the ellipse $(5x-10)^2+(5y+15)^2=\frac{(3x-4y+7)^2}{4}$ is:

(A) 10 (B) $\frac{20}{7}$

(C) $\frac{20}{3}$ (D) 4

7. If $\frac{x^2}{f(4a)}+\frac{y^2}{f(a^2-5)}$ represents an ellipse with major axis as *y*-axis and f is a decreasing function, then:

(A) $a\in(-\infty,1)$ (B) $a\in(5,\infty)$

(C) $a\in(1,4)$ (D) $a\in(-1,5)$

8. If $b^2=a^2(1-e^2)$, then for any α, $a,b\in R$,

$$\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha-2a^2e\cos\alpha+a^2e^2}$$

$+\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha+2a^2e\,\cos\alpha+a^2e^2}$ is equal to:

(A) $a+b$ (B) a

(C) $2a$ (D) $2b$

9. If an ellipse slides on the rectangular co-ordinate axes and foci of the ellipse are at $(2,y_1)$ and $(3,y_2)$, then y_1y_2 equals:

(A) 5 (B) 1

(C) 6 (D) $\frac{3}{2}$

10. The foci of the ellipse $3x^2-4xy+3y^2=5$ are:

(A) $(\sqrt{2},\sqrt{2}),(-\sqrt{2},-\sqrt{2})$

(B) $(\sqrt{2},-\sqrt{2}),(-\sqrt{2},\sqrt{2})$

(C) $(\pm 2,0)$

(D) $(2\sqrt{2},2\sqrt{2}),(-2\sqrt{2},-2\sqrt{2})$

11. The minimum value of $\left\{(r+5-4|\cos\theta|)^2+(r-3|\sin\theta|)^2\right\}$ $\forall r,\theta\in R$ is:

(A) 0 (B) 2

(C) 3 (D) none of these

12. An ellipse with major and minor axes of length $10\sqrt{3}$ units and 10 units, respectively, slides along the co-ordinate axes and always remains confined in the Ist quadrant. The locus of the center of the ellipse will be the arc of a circle. The length of this arc will be equal to:

(A) 10π units (B) 5π units

(C) $\frac{5}{4}\pi$ units (D) $\frac{5}{3}\pi$ units

13. The area of the triangle inscribed in an ellipse bears the ratio $\sqrt{5}:3$ to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the I^{st} triangle, then the eccentricity of the ellipse is:

(A) $\frac{2}{3}$ (B) $\frac{3}{4}$

(C) $\frac{\sqrt{5}}{3}$ (D) $\frac{3}{5}$

14. Let $S(3, 4)$ and $S'(9, 12)$ are the foci of an ellipse and the foot of perpendicular from S to a tangent to the ellipse is $(1, -4)$, then the eccentricity of the ellipse is:

(A) $\frac{3}{13}$ (B) $\frac{4}{13}$

(C) $\frac{5}{13}$ (D) $\frac{7}{13}$

15. The maximum distance of the centre of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ from the chord of contact of mutually perpendicular tangents of the ellipse is:

(A) $\frac{12}{5}$ (B) $\frac{4}{5}$

(C) $\frac{32}{5}$ (D) $\frac{16}{5}$

16. For an ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with vertices A and A', tangent drawn at the point P in the I^{st} quadrant meets the y-axis in Q and the chord $A'P$ meets the y-axis in M. If 'O' is the origin, then $OQ^2 - MQ^2$ equals to:

(A) 9 (B) 13

(C) 4 (D) 5

17. The line, $lx + my + n = 0$ will cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ $= 1$ in points, whose eccentric angles differ by $\frac{\pi}{2}$, if:

(A) $a^2l^2 + b^2n^2 = 2m^2$ (B) $a^2m^2 + b^2l^2 = 2n^2$

(C) $a^2l^2 + b^2m^2 = 2n^2$ (D) $a^2n^2 + b^2m^2 = 2l^2$

18. Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. If $OF = 6$ and the diameter of the inscribed circle of ΔOCF is 2, then the product $(AB)(CD)$ is equal to:

(A) 65 (B) 52

(C) 78 (D) none of these

19. Which one of the following is the common tangent to the ellipses, $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$?

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$

(B) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(D) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$

20. The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ $= 1$ of eccentricity e meets the axes of the ellipse in Q and R, then the locus of the midpoint of QR is a conic with an eccentricity e' such that:

(A) e' is independent of e (B) $e' = 1$

(C) $e' = e$ (D) $e' = \frac{1}{e}$

21. An ellipse is drawn with major and minor axes of lengths 10 units and 8 units, respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. The radius of the circle is:

(A) $\sqrt{3}$ (B) 2

(C) $2\sqrt{2}$ (D) $\sqrt{5}$

22. A circle touches the major axes of the two ellipse $\frac{x^2}{2086}+\frac{y^2}{2010}=1$ and $\frac{x^2}{2010}+\frac{y^2}{2086}=1$ and one of their common tangent has radius r then find $[r]$, where [.] is the greatest integer function.

(A) 16 (B) 17

(C) 18 (D) 19

23. If $\tan\theta_1.\tan\theta_2 = -\frac{a^2}{b^2}$, then the chord joining two points θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at:

(A) focus (B) centre

(C) end of the major axis (D) end of the minor axis

24. Equation of largest circle with centre $(1, 0)$ that can be inscribed in the ellipse $x^2 + 4y^2 = 16$ is:

(A) $2x^2 + 2y^2 - 4x + 7 = 0$

(B) $x^2 + y^2 - 2x + 5 = 0$

(C) $3x^2 + 3y^2 - 6x - 8 = 0$

(D) none of these

25. Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ in the Ist quadrant, whose foci are S_1 and S_2. Then the least possible value of the circumradius of ΔPS_1S_2 is:

(A) ae (B) be

(C) $\frac{ae}{b}$ (D) $\frac{ae^2}{b}$

26. In the standard ellipse, the lines joining the ends of the minor axis to one focus are at right angles. The distance between the focus and the nearer vertex is $\sqrt{10}-\sqrt{5}$. The equation of the ellipse is:

(A) $\frac{x^2}{36}+\frac{y^2}{18}=1$ (B) $\frac{x^2}{40}+\frac{y^2}{20}=1$

(C) $\frac{x^2}{20}+\frac{y^2}{10}=1$ (D) $\frac{x^2}{10}+\frac{y^2}{5}=1$

27. Two tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from the point $p(h, k)$. The points in which these tangents cut the axes are concyclic. The locus of P is:

(A) $xy=a^2b^2$ (B) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(C) $x^2-y^2=a^2-b^2$ (D) $x^2+y^2=a^2+b^2$

28. If the tangent at a point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in two points and the chord joining them subtends a right angle at the centre, then the eccentricity of the ellipse is given by:

(A) $\left(1+\cos^2\theta\right)^{-\frac{1}{2}}$ (B) $1+\sin^2\theta$

(C) $\left(1+\sin^2\theta\right)^{-\frac{1}{2}}$ (D) $\left(1+\cos^2\theta\right)$

29. If 'CF' is the perpendicular from the centre C of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the tangent at any point P and G is the point, where the normal at P meets the major axis, then $CF.PG$ is:

(A) b^2 (B) $2b^2$

(C) $\frac{b^2}{2}$ (D) $3b^2$

30. Let P be a point on the ellipse $9x^2+16y^2=144$, whose foci are S, S'. If A, B are feet of perpendiculars from S, S' respectively to the tangent at P and if normal at P passes through S meet the ellipse at Q and if O is origin, then $\frac{SP+SQ}{SP\cdot SQ}+\frac{OA\cdot OB}{SA\cdot S'B}$ equals:

(A) $\frac{9}{24}$ (B) $\frac{25}{9}$

(C) $\frac{24}{9}$ (D) $\frac{9}{16}$

31. If the ellipse $\frac{x^2}{4}+y^2=1$ meets the ellipse $x^2+\frac{y^2}{a^2}=1$ in four distinct points and $a=b^2-5b+7$, then b does not lie in:

(A) $[4,5]$ (B) $(-\infty,2)\cup(3,\infty)$

(C) $(-\infty,0)$ (D) $[2,3]$

32. If a variable tangent of the circle $x^2+y^2=1$ intersects the ellipse $x^2+2y^2=4$ at P and Q, then the locus of the points of intersection of the tangents at P and Q is:

(A) a circle of radius 2 units

(B) a parabola with focus as (2, 3)

(C) an ellipse with eccentricity $\frac{\sqrt{3}}{4}$

(D) an ellipse with length of latus rectrum as 2 units

33. A circle $S=0$ touches a circle $x^2+y^2-4x+6y-23=0$ internally and the circle $x^2+y^2-4x+8y+19=0$ externally. The locus of centre of the circle $S=0$ is conic, whose eccentricity is k then $\left[\frac{1}{k}\right]$ is where [.] denotes greatest integer function.

(A) 7 (B) 2

(C) 0 (D) 3

34. If circumcentre of an equilateral triangle inscribed in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with vertices having eccentric angles α, β, γ respectively is (x_1, y_1), then $\sum\cos\alpha\cos\beta+\sum\sin\alpha\sin\beta$ is

(A) $\frac{9x_1^2}{a^2}+\frac{9y_1^2}{b^2}+\frac{3}{2}$

(B) $9x_1^2-9y_1^2+a^2b^2$

(C) $\frac{9x_1^2}{2a^2}+\frac{9y_1^2}{2b^2}-\frac{3}{2}$

(D) $\frac{9x_1^2}{a^2}+\frac{9y_1^2}{b^2}+3$

35. Tangents are drawn to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ at the ends of latus rectum. The area of quadrilateral so formed is:

(A) 27 (B) $\frac{27}{2}$

(C) $\frac{27}{4}$ (D) $\frac{27}{55}$

36. For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, four normals cannot be drawn through a point $(h, 0)$ unless:

(A) $|h| \geq ae$ (B) $|h| \leq ae^2$

(C) $|h| \leq ae$ (D) none of these

37. If the ellipse $\frac{x^2}{a^2-3}+\frac{y^2}{a-4}=1$ is inscribed in a square of side length $a\sqrt{2}$ units, then a is:

(A) 4 (B) 2

(C) 1 (D) none of these

38. Let 'O' be the centre of ellipse for which A and B are end points of major axis and C and D are end points of minor axis, F is focus of the ellipse. If inradius of ΔOCF is '1', then $|AB| \times |CD|$ is:

(A) 65 (B) 52

(C) 78 (D) 47

39. Let there exist atleast one normal to an ellipse, which touches the concentric circle. Given that, circle intersects the ellipse at four distinct points. Then, the eccentricity of ellipse is:

(A) $e \in \left(0, \frac{1}{\sqrt{2}}\right)$ (B) $e \in \left(\frac{1}{\sqrt{2}}, 1\right)$

(C) $e \in \left(0, \frac{\sqrt{3}}{2}\right)$ (D) $e \in \left(\frac{\sqrt{3}}{2}, 1\right)$

40. Tangents are drawn from any point on the circle $x^2+y^2=41$ to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. Then, the angle between the two tangents is:

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

41. The area of the parallelogram formed by the tangents at the points whose eccentric angles are θ, $\theta+\frac{\pi}{2}$, $\theta+\pi$, $\theta+\frac{3\pi}{2}$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is:

(A) ab (B) $4ab$

(C) $3ab$ (D) $2ab$

42. A circle 'C' intersecting the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ such that common chord is of maximum length and intersecting the circle $x^2+y^2-6x-10y+9=0$ orthogonally, then the number of common tangents of circle 'C' and circle $x^2+y^2-10x-11=0$ is:

(A) 0 (B) 2

(C) 3 (D) 4

43. The points of intersection of the two ellipses $x^2+2y^2-6x-12y+23=0$, $4x^2+2y^2-20x-12y+35=0$:

(A) lie on a circle centred at $\left(\frac{8}{3}, 3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$

(B) lie on a circle centred at $\left(\frac{8}{3}, -3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{3}}$

(C) lie on a circle centred at $(8, 9)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$

(D) are not concyclic

44. In a model, it is shown that an arc of a bridge is semi elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal, then the height of the arch 2 m from the centre of the base is (in m):

(A) $\frac{8}{3}$ (B) $\frac{\sqrt{65}}{3}$

(C) $\sqrt{\frac{56}{3}}$ (D) $\frac{9}{3}$

45. The eccentric angle of a point p lying in the first quadrant on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is θ. If OP makes an angle ϕ with x-axis, then $\theta-\phi$ will be maximum, when θ is:

(A) $\tan^{-1}\sqrt{\frac{a}{b}}$ (B) $\tan^{-1}\sqrt{\frac{b}{a}}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$

46. If a tangent of slope $\frac{1}{3}$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ is normal to the circle $x^2+y^2+2x+2y+1=0$, then:

(A) minimum value of ab is $\frac{2}{3}$

(B) $a\in\left(\sqrt{\frac{2}{5}},2\right)$

(C) $a\in\left(\frac{2}{3},2\right)$

(D) maximum value of ab is 1

47. Area bounded by the circle, which is concentric with the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and which passes through $\left(4,-\frac{9}{5}\right)$, the vertical chord common to both circle and ellipse on the positive side of x-axis is:

(A) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{36}{5}$

(B) $2\tan^{-1}\left(\frac{9}{20}\right)$

(C) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)$

(D) none of these

48. Tangents to the ellipse $b^2x^2+a^2y^2=a^2b^2$ makes angles θ_1 and θ_2 with major axis such that $\cot\theta_1+\cot\theta_2=k$. Then the locus of the point of intersection is:

(A) $xy=2k(y^2+b^2)$

(B) $2xy=k(y^2-b^2)$

(C) $4xy=k(y^2-b^2)$

(D) $8xy=k(y^2-b^2)$

49. If PSQ is a focal chord of the ellipse $16x^2+25y^2=400$ such that $SP=8$, then the length of SQ is:

(A) 2

(B) $\frac{11}{3}$

(C) 16

(D) 25

50. A man running round a race course notes that the sum of the distances of two flag posts from him is 8 m. The area of the path he encloses in square metres, if the distance between flag posts is 4 m is:

(A) $15\sqrt{3}\pi$

(B) $12\sqrt{3}\pi$

(C) $18\sqrt{3}\pi$

(D) $8\sqrt{3}\pi$

51. The locus of point of intersection of the two tangents to the ellipse $b^2x^2+a^2y^2=a^2b^2$, which makes an angle 60° with one another is:

(A) $4(x^2+y^2-a^2-b^2)^2=3(b^2x^2+a^2y^2-a^2b^2)$

(B) $2(x^2+y^2-a^2-b^2)^2=4(b^2x^2+a^2y^2-a^2b^2)$

(C) $3(x^2+y^2-a^2-b^2)^2=2(b^2x^2+a^2y^2-a^2b^2)$

(D) $3(x^2+y^2-a^2-b^2)^2=(b^2x^2+a^2y^2-a^2b^2)$

52. If the equation of the chord joining the points $P(\theta)$ and $D\left(\theta+\frac{\pi}{2}\right)$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x\cos\alpha+y\sin\alpha=p$, then $a^2\cos^2\alpha+b^2\sin^2\alpha$ is:

(A) $4p^2$

(B) p^2

(C) $\frac{p^2}{2}$

(D) $2p^2$

53. From the point $A(4, 3)$, tangents are drawn to the ellipse $9x^2+16y^2=144$ to touch the ellipse at B and C. Let EF be a tangent to the ellipse parallel to the line BC and towards the point A. The distance of A from EF is:

(A) $\frac{4\sqrt{18}-12}{5}$

(B) $\frac{12-2\sqrt{18}}{5}$

(C) $\frac{24-4\sqrt{18}}{5}$

(D) $\frac{24-2\sqrt{18}}{5}$

54. Consider two points namely A and B on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. If the circles are drawn having segments of tangents at A and B in between tangents at the two ends of major axis of ellipse as diameter, then the length of common chord of the circles is:

(A) 8

(B) 6

(C) 10

(D) $4\sqrt{2}$

55. The line passing through the extremity A of the major axis and the extremity B of the minor axis of the ellipse $x^2+9y^2=9$, meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin 'O' is:

(A) $\frac{31}{10}$

(B) $\frac{29}{10}$

(C) $\frac{21}{10}$

(D) $\frac{27}{10}$

56. A circle of radius 'r' is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then inclination of common tangent with major axis is $(b<r<a)$.

(A) $\tan^{-1}\left(\frac{b}{a}\right)$

(B) $\tan^{-1}\left(\frac{rb}{a}\right)$

(C) $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$

(D) $\frac{\pi}{2}$

57. If $2x^2 + y^2 - 24y + 80 = 0$, then the maximum value of $x^2 + y^2$ is:

(A) 20 (B) 40

(C) 200 (D) 400

58. If a circle of radius 2 is concentric with the ellipse $\frac{x^2}{7} + \frac{y^2}{3} = 1$, then the inclination of common tangent with x-axis is:

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

59. A variable point P on the ellipse of eccentricity e is joined to foci S and S'. The eccentricity of the locus of the in centre of $\Delta PSS'$ is:

(A) $\sqrt{\frac{2e}{1+e}}$ (B) $\sqrt{\frac{e}{1+e}}$

(C) $\sqrt{\frac{1-e}{1+e}}$ (D) $\frac{e}{2(1+e)}$

60. From the focus (–5, 0) of the ellipse $\frac{x^2}{45} + \frac{y^2}{20} = 1$, a ray of light is sent, which makes an angle $\left(\cos^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$ with the positive direction of x-axis upon reacting the ellipse the ray is reflected from it. Slope of the reflected ray is:

(A) $\frac{-3}{2}$ (B) $\frac{-7}{3}$

(C) $\frac{-5}{4}$ (D) $\frac{-2}{11}$

61. Let $Q(3, \sqrt{5})$, $R(7, 3\sqrt{5})$. A point P in the xy-plane varies in such a way that the perimeter of ΔPQR is 16 units. Then the maximum area of ΔPQR is:

(A) 6 (B) 12

(C) 18 (D) 9

62. O is the centre of ellipse for which A and B are end points of major axis and C and D are the end points of minor axis. F is a focus of the ellipse. If $|OF| = 6$ and inradius of ΔOCF is 1, then $|AB| \times |CD|$ is:

(A) 65 (B) 52

(C) 78 (D) 47

63. A triangle is formed by a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the co-ordinate axes. The area of the triangle cannot be less than:

(A) $\frac{a^2+b^2}{2}$ sq. units

(B) $\frac{a^2+ab+b^2}{3}$ sq. units

(C) $\frac{a^2+2ab+b^2}{2}$ sq. units

(D) ab sq. units

64. Image of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ in the line $x + y = 10$ is:

(A) $\frac{(x-10)^2}{16} + \frac{(y-10)^2}{25} = 1$

(B) $\frac{(x-10)^2}{25} + \frac{(y-10)^2}{16} = 1$

(C) $\frac{(x-5)^2}{16} + \frac{(y-5)^2}{25} = 1$

(D) $\frac{(x-5)^2}{25} + \frac{(y-5)^2}{16} = 1$

65. Length of common tangent to $x^2 + y^2 = 16$ and $\frac{x^2}{25} + \frac{y^2}{7} = 1$ is:

(A) $\frac{9}{4\sqrt{2}}$ (B) $\frac{9}{4}$

(C) $\frac{9}{2\sqrt{2}}$ (D) $\frac{9}{2}$

66. An ellipse having foci S (3, 4) and S' (6, 8) passes through the point $P(0, 0)$. The equation of the tangent at P to the ellipse is:

(A) $4x + 3y = 0$ (B) $3x + 4y = 0$

(C) $x + y = 0$ (D) $x - y = 0$

67. An ellipse is drawn with major and minor axis of lengths 10 units and 8 units, respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of circle being outside the ellipse. The radius of circle is:

(A) 4 (B) 5

(C) 2 (D) 1

68. P_1, P_2 are the lengths of the perpendicular from the foci on the tangent to the ellipse and P_3, P_4 are perpendiculars from extremities of major axis and P from the centre of the ellipse on the same tangent, then $\frac{P_1P_2 - P^2}{P_3P_4 - P^2}$ equals (where, e is the eccentricity of the ellipse):

(A) e (B) $\sqrt{e}$

(C) e^2 (D) none of these

69. If the length of the major axis intercepted between the tangent and normal at a point $(a \cos \theta, b \sin \theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the length of semi-major axis, then the eccentricity of the ellipse is:

(A) $\frac{\cos\theta}{\sqrt{1-\cos\theta}}$ (B) $\frac{\sqrt{1-\cos\theta}}{\cos\theta}$

(C) $\frac{\sqrt{1-\cos\theta}}{\sin\theta}$ (D) $\frac{\sin\theta}{\sqrt{1-\sin\theta}}$

70. Tangents drawn to the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$ from the point P meet the co-ordinate axes at concyclic points. The locus of the point P is:

(A) $x^2 + y^2 = 7$ (B) $x^2 + y^2 = 25$

(C) $x^2 - y^2 = 7$ (D) $x^2 - y^2 = 25$

71. The line $2px + y\sqrt{1-p^2} = 1 (|p| < 1)$ for different value of p touches:

(A) an ellipse of eccentricity $\frac{2}{\sqrt{3}}$

(B) an ellipse of eccentricity $\frac{\sqrt{3}}{2}$

(C) hyperbola of eccentricity 2

(D) none of these

72. If the tangents PQ and PR are drawn from a point on the circle $x^2 + y^2 = 25$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ $(b < 4)$, so that the fourth vertex 'S' of parallelogram $PQSR$ lies on the circumcircle of ΔPQR, then the eccentricity of the ellipse is:

(A) $\frac{\sqrt{5}}{4}$ (B) $\frac{\sqrt{7}}{3}$

(C) $\frac{\sqrt{7}}{4}$ (D) $\frac{\sqrt{5}}{3}$

73. If A and B are the foci of ellipse $(x - 2y + 3)^2 + (8x + 4y + 4)^2 = 20$ and P is any point on it, then $PA + PB$ equals:

(A) 2 (B) 4

(C) $\sqrt{2}$ (D) $2\sqrt{2}$

74. If the curve $x^2 + 3y^2 = 9$ subtends an obtuse angle at the point $(2\alpha, \alpha)$, then the possible value of α^2 is:

(A) 1 (B) 2

(C) 3 (D) 4

75. If the tangent at point P to the ellipse $16x^2 + 11y^2 = 256$ is also the tangent to the circle $x^2 + y^2 - 2x = 15$, then the eccentric angle of point P is:

(A) $\pm \frac{\pi}{2}$ (B) $\pm \frac{\pi}{4}$

(C) $\pm \frac{\pi}{3}$ (D) $\pm \frac{\pi}{6}$

76. If an ellipse slides between two perpendicular straight lines $x = 0$ and $y = 0$ then, the locus of its foci is:

(A) a parabola (B) an ellipse

(C) a circle (D) none of these

77. From a point P, perpendicular tangents PQ and PR are drawn to ellipse $x^2 + 4y^2 = 4$. Locus of circumcentre of ΔPQR is:

(A) $x^2 + y^2 = \frac{16}{5}(x^2 + 4y^2)^2$

(B) $x^2 + y^2 = \frac{5}{16}(x^2 + 4y^2)^2$

(C) $x^2 + 4y^2 = \frac{16}{5}(x^2 + y^2)^2$

(D) $x^2 + 4y^2 = \frac{16}{5}(x^2 + y^2)^2$

78. Let PQ be a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which subtends an angle of $\frac{\pi}{2}$ radians at the centre. If L is the foot of perpendicular from $(0, 0)$ to PQ, then:

(A) locus of L is an ellipse

(B) locus of L is circle concentric with given ellipse

(C) locus of L is a hyperbola concentric with given ellipse

(D) a square concentric with given ellipse

79. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then $\left(\frac{x_1x_2}{y_1y_2}\right)$ is equal to:

(A) $\frac{a^2}{b^2}$ (B) $-\frac{b^2}{a^2}$

(C) $-\frac{a^4}{b^4}$ (D) $-\frac{b^4}{a^4}$

80. An ellipse has the point $(1, -1)$ and $(2, -1)$ as its foci and $x + y = 5$ as one of its tangent, then the value of $a^2 + b^2$, where a and b are the length of semi-major and semi-minor axes of the ellipse, respectively is:

(A) $\frac{41}{2}$
(B) 10
(C) 19
(D) $\frac{81}{4}$

81. On the ellipse $4x^2 + 9y^2 = 1$, the point at which the tangent is parallel to the line $8x = 9y$ is:

(A) $\left(\frac{2}{5}, \frac{1}{5}\right)$
(B) $\left(\frac{-2}{5}, \frac{1}{5}\right)$
(C) $\left(\frac{-2}{5}, \frac{-1}{5}\right)$
(D) none of these

82. The inclination to the major axis of the diameter of an ellipse the square of whose length is the harmonic mean between the squares of the major and minor axes is:

(A) $\frac{\pi}{4}$
(B) $\frac{\pi}{3}$
(C) $\frac{2\pi}{3}$
(D) $\frac{\pi}{2}$

83. From any point Plying in Ist quadrant on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, PN is drawn perpendicular to the major axis such that N lies on major axis. Now, PN is produced to the point Q such that NQ equals to PS, where S is a focus. The point Q lies on which of the following lines:

(A) $2y - 3x - 25 = 0$
(B) $3x + 5y + 25 = 0$
(C) $2x - 5y - 25 = 0$
(D) $2x - 5y + 25 = 0$

84. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:

(A) $\sqrt{ab}$
(B) $\frac{a}{b}$
(C) $2ab$
(D) ab

85. A tangent at any point to the ellipse $4x^2 + 9y^2 = 36$ is cut by the tangent at the extremities of the major axis at T and T'. The circle on $T\,T'$ as diameter passes through the point:

(A) $\left(0, \sqrt{5}\right)$
(B) $\left(\sqrt{5}, 0\right)$
(C) (2, 1)
(D) $\left(0, -\sqrt{5}\right)$

86. The circle $x^2 + y^2 = c^2$ contains the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ if:

(A) $c < a$
(B) $c < b$
(C) $c > a$
(D) $c > b$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If $x + y = 2$ is a tangent to the ellipse, whose foci are (2, 3) and (3, 5), then:

(A) length of minor axis is 6 units
(B) length of minor axis is 3 units
(C) length of major axis is $\frac{\sqrt{41}}{2}$ units
(D) eccentricity of ellipse is $\frac{\sqrt{5}}{\sqrt{41}}$

2. If the tangent at any point 'P' on a standard ellipse with foci as S and S' meets the tangents at the vertices A and A' in the points V and V', then:

(A) $l(AV).l(A'V') = b^2$
(B) $l(AV).l(A'V') = a^2$
(C) $\angle V'SV = 90°$
(D) $V'S'SV$ is a cyclic quadrilateral

3. Extremities of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, (a > b)$ having a major axis $2a$ lies on:

(A) $x^2 = a(a - y)$
(B) $x^2 = a(a + y)$
(C) $y^2 = a(a + x)$
(D) $y^2 = a(a - x)$

4. A point P moves on x-y plane such that $PS + PS' = 4$ where $S(K, 0)$ and $S'(-K, 0)$ then locus of P is:

(A) ellipse, if $K \in (-2, 2)$
(B) pair of coincidence lines, if $K = \pm 2$
(C) empty, if $K \in (-\infty, -2) \cup (2, \infty)$
(D) none of these

5. For the conic $3(x+y-5)^2+2(x-y+7)^2=6$:

(A) one focus is $\left(-1-\frac{\sqrt{3}}{2}, 6+\frac{\sqrt{3}}{2}\right)$

(B) point of intersection of directrix and major axis is $\left(-1-\frac{3}{2\sqrt{2}}, 6+\frac{3}{2\sqrt{2}}\right)$

(C) one focus is $\left(\frac{-3}{2}, \frac{13}{2}\right)$

(D) centre is $(-1, 6)$

6. If P is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose foci are S and S' and let $\angle PSS'=\alpha$ and $\angle PS'S=\beta$, then:

(A) $PS+PS'=2a$, if $a>b$

(B) $PS+PS'=2b$, if $a<b$

(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$

(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{\sqrt{a^2-b^2}}{b^2}\left[a-\sqrt{a^2-b^2}\right]$, when $a>b$

7. If the chord through the points, whose eccentric angles are θ and ϕ on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ passes through a focus, then $\tan\frac{\theta}{2}.\tan\frac{\phi}{2}$ is:

(A) $\frac{1}{9}$ (B) -9

(C) $-\frac{1}{9}$ (D) 9

8. The equation, $3x^2+4y^2-18x+16y+43=c$:

(A) cannot represent a real pair of straight lines for any value of c

(B) represents an ellipse, if $c>0$

(C) represent empty set, if $c<0$

(D) a point, if $c=0$

9. Consider an ellipse, whose distance between foci S and S' is 4 units and is inscribed in the ΔABC by touching the sides AB, AC and BC at P, Q and R. If centre of ellipse is at origin and major axis along x-axis $SP+S'P=6$, then:

(A) equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

(B) if $\angle BAC=90°$, then locus of vertex A is $x^2+y^2=14$

(C) if chord PQ subtends 90° angle at centre of ellipse, then locus of A is $25x^2+81y^2=630$

(D) eccentricity of the ellipse is $\frac{2}{3}$

10. A point on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at a distance equal to the mean of the lengths of the semi-major axes and semi-minor axis from the centre is:

(A) $\left(\frac{2\sqrt{91}}{7}, \frac{3\sqrt{105}}{14}\right)$

(B) $\left(\frac{2\sqrt{91}}{7}, \frac{-3\sqrt{105}}{14}\right)$

(C) $\left(\frac{-2\sqrt{91}}{7}, \frac{-3\sqrt{105}}{14}\right)$

(D) $\left(\frac{-2\sqrt{91}}{7}, \frac{3\sqrt{105}}{14}\right)$

11. Let F_1, F_2 be two foci of the ellipse and PT and PN be the tangent and the normal, respectively to the ellipse at point P. Then:

(A) PN bisects $\angle F_1PF_2$

(B) PT bisects $\angle F_1PF_2$

(C) PT bisects angle $(180°-\angle F_1PF_2)$

(D) PN bisects angle $(180°-\angle F_1PF_2)$

12. If a variable tangent to the circle $x^2+y^2=1$ intersects the ellipse $x^2+2y^2=4$ at points P and Q, then the locus of the point of intersection of tangents to the ellipse at P and Q is a conic, whose

(A) eccentricity is $\frac{\sqrt{3}}{2}$

(B) eccentricity is $\frac{\sqrt{5}}{2}$

(C) latus rectum is of length 2 units

(D) foci are $(\pm2\sqrt{5}, 0)$

13. If $(a\cos\alpha, b\sin\alpha), (a\cos\beta, b\sin\beta)$ are the end points of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then which of the following options are correct.

(A) $e=\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$

(B) $e=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$

(C) $\frac{e-1}{e+1}=\tan\left(\frac{\alpha}{2}\right).\tan\left(\frac{\beta}{2}\right)$

(D) $e=\frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}$

14. For the curve $\frac{x^2}{\sin\sqrt{2}-\sin\sqrt{3}}-\frac{y^2}{\cos\sqrt{3}-\cos\sqrt{2}}=1$, it is

(A) an ellipse
(B) a hyperbola
(C) having foci on x-axis
(D) having foci on y-axis

15. If A_1 is the area of the quadrilateral formed by foci and the ends of minor axis of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b)$ and A_2, A_3 are the maximum area of a rectangle and a triangle that can be inscribed in the same ellipse, respectively then:

(A) $e=\frac{1}{2}$, if $2A_1=A_2$

(B) $e=\frac{3\sqrt{3}}{8}$, if $A_1=A_3$

(C) $e=\frac{\sqrt{3}}{8}$, if $2A_1=A_2$

(D) $e=\frac{1}{\sqrt{2}}$, if there exist exactly two points on the ellipse at which line joining foci subtends a right angle

16. Consider an ellipse, whose major axis is parallel to x-axis is such that the segments of a focal chord are 1 and 3 units. The lines $ax+by+c=0$ are the chords of the ellipse such that a, b, c are in A.P and bisected by the point at which they are concurrent. If the equation of auxiliary circle is $x^2+y^2+2\alpha x+2\beta y-2\alpha-1=0$, then:

(A) the locus of perpendicular tangents to the ellipse is $x^2+y^2=7$

(B) length of the double ordinate, which is conjugate to directrix is 3 units

(C) area of an auxiliary circle is 2π sq. units

(D) eccentricity of the ellipse is $\frac{1}{2}$

17. If foci of an ellipse be $(-1, 2)$ and $(-2, 3)$ and its tangent at a point A is $2x+3y+9=0$, then:

(A) length of the minor axis of the ellipse will be $2\sqrt{14}$ units

(B) co-ordinate of the point 'A' will be $\left(\frac{-32}{9}, \frac{-17}{27}\right)$

(C) distance between the foci is $2\sqrt{2}$ units

(D) product of the perpendiculars from foci to any tangent is 56

18. Let $P(x_1, y_1)$, $Q(x_2, y_2)$, such that $y_1>0, y_2>0$ are the end points of the latus rectum of the ellipse $3x^2+4y^2=12$. The equations of the parabolas with latus rectum PQ are:

(A) $x^2-2y-2=0$ (B) $x^2-2y+2=0$

(C) $x^2+2y-4=0$ (D) $x^2-2y+4=0$

19. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, such that $y_1<0, y_2<0$ are the end points of the latus rectum of the ellipse $x^2+4y^2=4$. The equations of parabolas with latus rectum PQ are:

(A) $x^2+2\sqrt{3}\ y=3+\sqrt{3}$
(B) $x^2-2\sqrt{3}\ y=3+\sqrt{3}$
(C) $x^2+2\sqrt{3}\ y=3-\sqrt{3}$
(D) $x^2-2\sqrt{3}\ y=3-\sqrt{3}$

20. If a tangent of slope $\frac{1}{3}$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ is normal to the circle $x^2+y^2+2x+2y+1=0$, then:

(A) maximum value of ab is $\frac{2}{3}$

(B) $a \in \left(\sqrt{\frac{2}{5}}, 2\right)$

(C) $a \in \left(\frac{2}{3}, 2\right)$

(D) maximum value of ab is 1

21. If a quadrilateral formed by four tangents to the ellipse $3x^2 + 4y^2 = 12$ is a square, then:

(A) the vertices of the square lie on $y = \pm x$

(B) the vertices of the square lie on $x^2 + y^2 = 7$

(C) the area of all such squares is constant

(D) only two such squares are possible

22. If the distance of a point P on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ from centre C is 2 units, then the eccentric angle of P maybe:

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$

(C) $\frac{3\pi}{4}$ (D) $\frac{7\pi}{4}$

23. If the focal chord to $y^2 = 16x$ is a tangent to $(x-6)^2 + y^2 = 2$, then the slope of focal chord is:

(A) 1 (B) $\frac{1}{2}$

(C) $-\frac{1}{2}$ (D) -1

24. A point on the ellipse $x^2 + 3y^2 = 37$, where the normal is parallel to the line $6x - 5y = 2$ is:

(A) (5, –2) (B) (5, 2)

(C) (–5, 2) (D) (–5, –2)

25. P and Q are two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose eccentric angles are differing by $90°$, then:

(A) locus of point of intersection of tangents at P and Q is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

(B) locus of midpoint (P, Q) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$

(C) product of slopes of OP and OQ, where O is the centre is $\frac{-b^2}{a^2}$

(D) maximum area of ΔOPQ is $\frac{1}{2}ab$ sq. units

26. The equations of the common tangents of the curves $x^2 + 4y^2 = 8$ and $y^2 = 4x$ are:

(A) $x + 2y + 4 = 0$ (B) $x - 2y + 4 = 0$

(C) $2x + y = 4$ (D) $2x - y + 4 = 0$

27. If latus rectum of the ellipse $x^2 \tan^2\alpha + y^2 \sec^2\alpha = 1$ is $\frac{1}{2}$, then $\alpha(0 < \alpha < \pi)$ is:

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$

(C) $\frac{5\pi}{12}$ (D) none of these

28. An ellipse passes through the point (4, –1) and its axes are along the axes of co-ordinates. If the line $x + 4y - 10 = 0$ is a tangent to it, then its equation can be:

(A) $\frac{x^2}{100} + \frac{y^2}{5} = 1$ (B) $\frac{x^2}{80} + \frac{y^2}{5/4} = 1$

(C) $\frac{x^2}{20} + \frac{y^2}{5} = 1$ (D) none of these

29. The normal at an end of latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ pass through an end of minor axis, if:

(A) $e^4 + e^2 = 1$ (B) $e = \sqrt{\frac{\sqrt{5}-1}{2}}$

(C) $e = 2\sin 18°$ (D) $e = 2\cos 36°$

30. AB and CD are two equal and parallel chords of their ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If the tangents to the ellipse at A and B intersect at P and at C and D at Q, then line PQ:

(A) passes through the origin

(B) is bisected at the origin

(C) cannot pass through the origin

(D) is not bisected at the origin

31. Consider the ellipse $\frac{x^2}{\tan^2\alpha} + \frac{y^2}{\sec^2\alpha} = 1$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$. Which of the following quantities would vary, as α varies?

(A) Eccentricity
(B) Ordinate of the vertex
(C) Ordinates of the foci
(D) Length of the latus rectum

32. If the tangent at the point $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x = 15$, then find θ.

(A) $\frac{\pi}{3}$
(B) $\frac{2\pi}{3}$
(C) $\frac{-\pi}{3}$
(D) $\frac{5\pi}{3}$

33. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which of the following statements are correct?

(A) The point of intersection of tangents drawn at the points $\left(ae, \frac{b^2}{a}\right)$ and $\left(ae, \frac{-b^2}{a}\right)$ is $\left(\frac{a}{e}, 0\right)$
(B) The foot of the perpendicular drawn from focus onto a tangent lies on $x^2 + y^2 = a^2$
(C) Equation of the directrix corresponds to the focus $(ae, 0)$ is $x - \frac{a}{e} = 0$
(D) If F_1 and F_2 are the two foci and B is one end of the minor axis such that ΔBF_1F_2 is equilateral, then the eccentricity of the ellipse is $\frac{1}{2}$

34. Equation of the locus at point of intersection of perpendicular tangents to $\frac{(x+y-2)^2}{9} + \frac{(x-y)^2}{16} = 1$ is:

(A) $(x-1)^2 + (y-1)^2 = \frac{25}{2}$
(B) $(x-1)^2 + (y-1)^2 = 50$
(C) $(x-1)^2 + (y-1)^2 = 25$
(D) $(x+y-2)^2 + (x-y)^2 = 25$

35. If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the major and minor axes at G and g such that $PG{:}Pg$ = 1:5 and area of $\Delta CSB = 10$ sq.units, where C is centre of ellipse S is focus and B is one end of minor axis, then:

(A) equation of ellipse is $\frac{x^2}{50} + \frac{y^2}{10} = 1$
(B) equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{5} = 1$
(C) equation of director circle of ellipse is $x^2 + y^2 = 60$
(D) equation of director circle of ellipse is $x^2 + y^2 = 30$

36. For the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$:

(A) the foci of each ellipse always lies within the other ellipse
(B) their auxiliary circles are the same
(C) their director circles are the same
(D) the ellipses encloses the same area

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Consider an ellipse $(x-1)^2 + (y-2)^2 = \left(\frac{5x+4y-20}{7}\right)^2$.

1. The eccentricity is:

(A) $\frac{5}{7}$
(B) $\frac{4}{7}$
(C) $\frac{7}{41}$
(D) $\frac{\sqrt{41}}{7}$

2. Equation of major axis is:

(A) $4x - 5y + 8 = 0$
(B) $3x + 4y - 6 = 0$
(C) $4x - 5y + 6 = 0$
(D) $5x + 4y - 10 = 0$

3. Length of latus rectum is:

(A) 1
(B) 2
(C) 3
(D) 4

Comprehension 2

Consider an ellipse $25(3x-4y+7)^2+16(4x+3y-6)^2=10000$.

4. The length of major axis is:

(A) 50
(B) 8
(C) 16
(D) 10

5. The length of minor axis is:

(A) 5
(B) 8
(C) 40
(D) 50

6. Equation of minor axis of ellipse:

(A) $x+7y-13=0$
(B) $7x-y+1=0$
(C) $3x-4y+7=0$
(D) $4x+3y-6=0$

Comprehension 3

d is a diameter of an ellipse, whose principal axes have lengths $2a$ units and $2b$ units. The inclination of the diameter to the major axis is θ.

7. If d^2 is the A.M of $(2a)^2$ and $(2b)^2$, then $\tan\theta$ equals:

(A) $\pm\sqrt{\frac{a}{b}}$
(B) $\pm\frac{b}{a}$
(C) ± 1
(D) $\pm\sqrt{3}$

8. If d^2 is the G.M of $(2a)^2$ and $(2b)^2$, then $\tan\theta$ equals:

(A) $\pm\sqrt{\frac{b}{a}}$
(B) $\pm\frac{b}{a}$
(C) ± 1
(D) $\pm\sqrt{3}$

9. If d^2 is the H.M of $(2a)^2$ and $(2b)^2$, then θ equals:

(A) $\pm\tan^{-1}\sqrt{\frac{a}{b}}$
(B) $\pm\tan^{-1}\left(\frac{b}{a}\right)$
(C) $\pm\frac{\pi}{4}$
(D) $\pm\frac{\pi}{3}$

Comprehension 4

Consider the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, where $b>a>0$. Let, the points are $A(-a,0)$ and $B(a,0)$. A parabola passes through the points A, B and its directrix is a tangent to $x^2+y^2=b^2$. The locus of focus of the parabola is a conic.

10. The eccentricity of the conic is:

(A) $\frac{2a}{b}$
(B) $\frac{b}{a}$
(C) $\frac{a}{b}$
(D) 1

11. The foci of the conic are:

(A) $(\pm 2a, 0)$
(B) $(0, \pm a)$
(C) $(\pm a, 2a)$
(D) $(\pm a, 0)$

12. Area of triangle formed by a latus rectum and the lines joining the end points of the latus rectum and the centre of the conic is:

(A) $\frac{a}{b}\left(b^2-a^2\right)$
(B) $2ab$
(C) $\frac{ab}{2}$
(D) $\frac{4ab}{3}$

Comprehension 5

Let an ellipse having major axis and minor axis parallel to x-axis and y-axis, respectively. Its two foci S and S' are (2, 1), (4, 1) and a line $x+y=9$ is a tangent to this ellipse at point P.

13. Eccentricity of the ellipse is:

(A) $\frac{1}{\sqrt{12}}$
(B) $\frac{1}{\sqrt{13}}$
(C) $\frac{1}{2}$
(D) none of these

14. Length of major axis:

(A) $\sqrt{13}$
(B) $2\sqrt{11}$
(C) $\sqrt{52}$
(D) $2\sqrt{12}$

15. The latus rectum of ellipse:

(A) $\frac{12}{13}$
(B) $\frac{12}{\sqrt{13}}$
(C) $\frac{24}{\sqrt{13}}$
(D) $\frac{25}{\sqrt{13}}$

Comprehension 6

Let P, Q, R be three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let P', Q', R' be their corresponding points on its auxiliary circle.

16. The maximum area of the ΔPQR is:

(A) $\frac{3\sqrt{3}}{4}ab$
(B) $\frac{3\sqrt{3}}{2}ab$
(C) $\frac{\sqrt{3}}{4}ab$
(D) πab

17. $\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P'Q'R'}$ is:

(A) $\frac{a}{b}$
(B) $\frac{b}{a}$
(C) $\frac{1}{2}$
(D) depends on points taken

18. When the area of ΔPQR is maximum, the centroid of $\Delta P'Q'R'$ lies at:

(A) focus
(B) vertex
(C) centre
(D) directrix

Comprehension 7

'P' is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and S_1 and S_2 are foci of the ellipse.

19. Locus of incentre of ΔPS_1S_2 will be:

(A) a straight line
(B) a circle
(C) a parabola
(D) an ellipse

20. If $e=\frac{1}{2}$ and $\angle PS_1S_2=\alpha, \angle PS_2S_1=\beta, \angle S_1PS_2=\gamma$, then $\cot\frac{\alpha}{2}$, $\cot\frac{\gamma}{2}$ and $\cot\frac{\beta}{2}$ are in:

(A) A.P
(B) G.P
(C) H.P
(D) none of these

21. Maximum area of the ΔPS_1S_2 is:

(A) b^2e sq. units
(B) a^2e sq. units
(C) ab sq. units
(D) abe sq. units

Comprehension 8

An ellipse E has its centre C (1, 3), focus at S (6, 3) and passes through the point P(4, 7).

22. The product of the perpendicular distances of foci from tangent at P to the ellipse is:

(A) 20
(B) 45
(C) 40
(D) 60

23. If the normal at a variable point on the ellipse (E) meets its axes in Q and R, then the locus of the midpoint of QR is a conic with eccentricity is:

(A) $\frac{3}{\sqrt{10}}$
(B) $\frac{\sqrt{5}}{3}$
(C) $\frac{3}{\sqrt{5}}$
(D) $\frac{\sqrt{10}}{3}$

Comprehension 9

A sequence of ellipse $E_1, E_2, ...E_n$ is constructed as follows. Ellipse E_n is drawn, so as to touch ellipse E_{n-1} as the extremities of the major axis of E_{n-1} and to have its foci at the extremities of the minor axis of E_{n-1}.

24. If E_n is independent of n, then the eccentricity of the ellipse E_{n-2} is:

(A) $\frac{3-\sqrt{5}}{2}$
(B) $\frac{\sqrt{5}-1}{2}$
(C) $\frac{2-\sqrt{3}}{2}$
(D) $\frac{\sqrt{3}-1}{2}$

25. If eccentricity of ellipse E_n is e_n, then the locus of $\left(e_n^2, e_{n-1}^2\right)$ is:

(A) a parabola
(B) an ellipse
(C) a circle
(D) a rectangular hyperbola

26. If eccentricity of E_n is independent of n, then the locus of the midpoint of chords of slope -1 of E_n (If axis of E_n is along y-axis) is:

(A) $\left(\sqrt{5}-1\right)x=2y$
(B) $\left(\sqrt{5}+1\right)x=2y$
(C) $\left(3-\sqrt{5}\right)x=2y$
(D) $\left(3+\sqrt{5}\right)x=2y$

Comprehension 10

Let $S:\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ be the ellipse, whose major axis along x-axis and minor axis along y-axis. Let e be its eccentricity.

27. If the normal at one end of a latus rectum of $s=0$ passes through one end of its minor axis, then:

(A) $e=\frac{1}{2}$
(B) $e^4+e^2+1=0$
(C) $e^2=\frac{\sqrt{5}-1}{2}$
(D) $e^2=\frac{\sqrt{5}-1}{4}$

28. If ϕ is one of the angle between the normals to $S=0$ at the points whose eccentric angles are θ and $\frac{\pi}{2}+\theta$, then $\cot\phi \operatorname{cosec} 2\theta$ is:

(A) $\sqrt{1-e^2}$
(B) $\frac{\sqrt{1-e^2}}{2e^2}$
(C) $\frac{2e^2}{\sqrt{1-e^2}}$
(D) $\frac{e^2}{2\sqrt{1-e^2}}$

29. Let 'C' be the centre of the ellipse $S=0$ and P, Q are the corresponding points on the ellipse and its auxiliary circle, respectively. If the normal at P to the ellipse meets CQ in R, then CR is:

(A) $a-b$
(B) $a+b$
(C) $2b$
(D) $\sqrt{1-e^2}$

Comprehension 11

P is any point of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. S and S' are foci and e is the eccentricity of ellipse. $\angle PSS' = \alpha$ and $\angle PS'S = \beta$

30. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is:

(A) $\frac{2e}{1-e}$ (B) $\frac{1+e}{1-e}$

(C) $\frac{1-e}{1+e}$ (D) $\frac{2e}{1+e}$

31. Eccentricity of conic, which is locus of incentre of $\Delta PSS'$ is:

(A) $\sqrt{\frac{e}{1+e}}$ (B) $\sqrt{\frac{2e}{1+e}}$

(C) $\sqrt{\frac{2e}{1-e}}$ (D) $\sqrt{\frac{e}{1-e}}$

Comprehension 12

To find out the lengths and positions of the axes of the conic, whose equation is $ax^2+2hxy+by^2=1$ …. (1), where the axes of co-ordinates being rectangular, consider a circle of radius 'r' with its centre at the centre of the conic, whose equation is $\frac{x^2+y^2}{r^2}=1$ … (2). Subtracting (2) from (1), we obtain $\left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0$ …. (3). Equation (3) represents a pair of straight lines through the origin and the intersection of Equations (1) and (2). These straight lines will be coincident when and only when they lie along the axes of the conic, the condition for which is $\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2$ …. (4). If r_1^2 and r_2^2 be the root and both be positive, then the conic is an ellipse with $2r_1$ and $2r_2$ as the length of its axes.

Given, a conic $5x^2-6xy+5y^2+22x-26y+29=0$ and the axes being rectangular.
Now, answer the following questions.

32. Length of major axis is:

(A) 4 (B) 6

(C) 3 (D) 2

33. Length of minor axis is:

(A) 3 (B) 4

(C) 2 (D) 1

34. Equation of major and minor axes, respectively is:

(A) $2x+y-1=0, x-2y+3=0$

(B) $2x-y+3=0, x+2y+4=0$

(C) $x-y+5=0, x+y+3=0$

(D) $x-y+3=0, x+y-1=0$

Comprehension 13

An ellipse, whose distance between foci S and S' is 4 units is inscribed in the ΔABC, touching the sides AB, AC and BC at P, Q and R. If centre of ellipse is at origin 'O' and major axis along x-axis and $SP+S'P=6$.

35. If $\angle BAC = 90°$, then the locus of vertex A is:

(A) $x^2+y^2=12$ (B) $x^2+y^2=16$

(C) $x^2+y^2=14$ (D) $x^2+y^2=25$

36. If $\angle POQ = 90°$, then locus of vertex A is:

(A) $25x^2+81y^2=330$ (B) $81x^2+25y^2=630$

(C) $25x^2+81y^2=630$ (D) $25x^2+81y^2=230$

37. If $AB = BC$ and $\angle B = 90°$, then the locus of vertex A is:

(A) $(x^2+y^2-14)^2=20x^2+36y^2-180$

(B) $(x^2+y^2+14)^2=20x^2+36y^2-180$

(C) $(x^2+y^2-14)=20x^2-36y^2-180$

(D) $(x^2+y^2-14)^2=20x^2-36y^2-180$

Comprehension 14

Let the equation $ax^2+2hxy+by^2=1$ represent an ellipse, then $h^2-ab<0$. The equation of ellipse can be changed to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, by putting $x = X\cos\theta - Y\sin\theta$ and $y = X\sin\theta + Y\cos\theta$.

38. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ must be:

(A) $\frac{1}{a^2}+\frac{1}{b^2}$ (B) $\frac{1}{a}+\frac{1}{b}$

(C) $a+b$ (D) $a-b$

39. $\frac{1}{\alpha^2\beta^2}$ equals:

(A) h^2-ab (B) $\sqrt{h^2-ab}$

(C) h^2+ab (D) $ab-h^2$

40. If e is eccentricity, then e^2 :

(A) $\dfrac{\sqrt{(a-b)^2+4h^2}}{a+b}$

(B) $\dfrac{2\sqrt{(a-b)^2+4h^2}}{a+b+\sqrt{(a-b)^2+4h^2}}$

(C) $\dfrac{\sqrt{(a-b)^2+4h^2}}{a+b+\sqrt{(a-b)^2+4h^2}}$

(D) $\dfrac{2\sqrt{(a-b)^2+4h^2}}{a+b}$

Comprehension 15

The normal to an ellipse at $P(\theta)$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{ax}{\cos\theta}-\frac{by}{\cos\theta}=a^2-b^2$. If S and S' are the foci, it bisects the angle $S\hat{P}S'$. On this bases, answer the following questions.

41. P is a variable point on $\frac{x^2}{96}+\frac{y^2}{72}=1$, whose foci are S and S'. If the locus of incentre of $\Delta SPS'$ is a conic, then its eccentricity is:

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{\frac{2}{3}}$

(C) $\sqrt{\frac{3}{4}}$ (D) $\frac{1}{2}$

42. If α is the angle between the normals drawn at the points θ and $\frac{\pi}{2}+\theta$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\cot\alpha\cdot\operatorname{cosec}2\theta$ is:

(A) $\frac{e^2}{\sqrt{1-e^2}}$ (B) $\frac{e^2}{2\sqrt{1-e^2}}$

(C) $\frac{2e^2}{\sqrt{1-e^2}}$ (D) $\frac{e^2}{\sqrt{2-e^2}}$

43. C is the centre of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at P meets the major axis at G. CF is the perpendicular from C to the normal at P, then $PF.\,PG$ is:

(A) a^2 (B) b^2

(C) a^2-b^2 (D) a^2b^2

Comprehension 16

Consider an ellipse $\frac{x^2}{4}+y^2=\alpha$, (α is parameter > 0) and a parabola $y^2=8x$. A common tangent to the ellipse and the parabola meets the co-ordinate axes at A and B, respectively.

44. Locus of midpoint of AB:

(A) $y^2=-2x$ (B) $y^2=-x$

(C) $y^2=-\frac{x}{2}$ (D) $\frac{x^2}{4}+\frac{y^2}{2}=1$

45. If the eccentric angle of a point on the ellipse, where the common tangent meets it is $\left(\frac{2\pi}{3}\right)$, then α is:

(A) 4 (B) 5

(C) 26 (D) 36

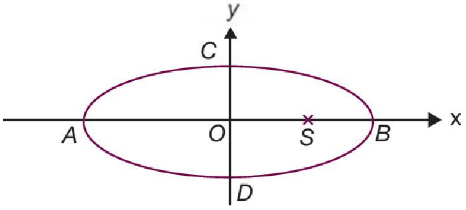
46. If two of the three normals drawn from the point $(h, 0)$ on the ellipse to the parabola $y^2=8x$ are perpendicular, then:

(A) $h=2$ (B) $h=3$

(C) $h=4$ (D) $h=6$

Comprehension 17

Consider an ellipse, as shown in the figure.
Such that OS = 6 units and the inradius of the ΔOCS is 1 unit.



47. The equation of the director circle of the director circle of the ellipse is:

(A) $x^2+y^2=\frac{97}{2}$ (B) $x^2+y^2=\sqrt{97}$

(C) $x^2+y^2=97$ (D) $x^2+y^2=\sqrt{\frac{97}{2}}$

48. The semi-perimeter of the ΔOCS is:

(A) 10 (B) 5

(C) 7.5 (D) 12.5

49. The area of the ellipse is:

(A) $\frac{65\pi}{4}$ (B) $\frac{64\pi}{5}$

(C) 64π (D) 65π

Comprehension 18

If $S = 0$ is a parabola or ellipse, then the equation of the chord of conic with midpoint $P(x_1, y_1)$ is $T = S_1$ and $S_1 < 0$.

For the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ and the parabola, $y^2 = 4x$.

50. If $Q(a, 0)$ is a point, where $a > 4$, is such that three distinct chords through Q of the ellipse are drawn and each of which is bisected by the parabola, then:

(A) $a \in (4, 10+2\sqrt{5})$ (B) $a \in (16, 8+4\sqrt{5})$

(C) $a \in (4, 20)$ (D) $a \in (4, \infty)$

51. The length of the common chord of ellipse and parabola is:

(A) $2\sqrt{5-\sqrt{3}}$ (B) $4\sqrt{2-\sqrt{3}}$

(C) $8\sqrt{\sqrt{5}-2}$ (D) $2\sqrt{5}$

52. The slopes of common tangents to the ellipse and parabola are:

(A) $\pm\sqrt{\sin 9^\circ \cos 9^\circ}$ (B) $\pm\sqrt{\sin 9^\circ}$

(C) $\pm\sqrt{\cos 9^\circ}$ (D) $\pm\sqrt{\tan 9^\circ}$

Comprehension 19

$C: x^2 + y^2 = 9$, $E: \frac{x^2}{9}+\frac{y^2}{4}=1$, $L: y = 2x$

53. If P is a point on the circle C, the perpendicular PL to the major axis of the ellipse E meets the ellipse at M, then $\frac{ML}{PL}$ is:

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $\frac{1}{2}$ (D) $\frac{3}{2}$

54. Equation of the diameter of the ellipse E conjugate to the diameter represented by L is:

(A) $9x + 2y = 0$ (B) $2x + 9y = 0$

(C) $4x + 9y = 0$ (D) $4x - 9y = 0$

55. If R is the point of intersection of the line L with the line $x = 1$, then:

(A) R lies inside both C and E

(B) R lies outside both C and E

(C) R lies on E

(D) R lies inside C but outside E

Comprehension 20

Consider the ellipse $E: \frac{x^2}{4}+\frac{y^2}{1}=1$, S, S' are foci and P is any point on E such that $SP \cdot S'P$ be maximum, among the two values of P with lesser ordinate be name it as A and other as B.

56. Possible co-ordinates of P are:

(A) $(0, \pm 1)$ (B) $(\pm 2, 0)$

(C) $(\pm\sqrt{2}, 0)$ (D) none of these

57. Ray of light is passing through point A. It gets reflected from curve $X\left(\frac{-x^2}{a^2}+\frac{y^2}{b^2}=1\right)$ such that reflected ray appear to pass through B, curve X is passing through the point $D(2, 1)$, then the curve X is:

(A) $\frac{-x^2}{2(\sqrt{2}-1)}+\frac{y^2}{(\sqrt{2}-1)^2}=1$

(B) $\frac{x^2}{2(\sqrt{2}-1)^2}-\frac{y^2}{2(\sqrt{2}-1)}=1$

(C) $\frac{x^2}{2}-\frac{y^2}{3}=1$

(D) $\frac{x^2}{(\sqrt{2}+1)^2}+\frac{y^2}{2(\sqrt{2}+1)}=1$

58. From point D normal be drawn to curve X cut axis at point E, then area of ΔBDE is:

(A) $2\sqrt{2}-2$ (B) $\sqrt{2}+1$

(C) $2(\sqrt{2}+1)$ (D) $2(\sqrt{2}+3)$

Comprehension 21

The two common tangents to the parabola $y^2 = 4x$ and the ellipse $x^2 + 4y^2 = 8$ meet at M. Of the two tangents, one meets the parabola and the ellipse at P_1 and E_1, respectively and the other meets them at P_2 and E_2, respectively.

59. The length of the segment P_1E_2 equals:

(A) $\sqrt{51}$ (B) $\sqrt{61}$

(C) 8 (D) 7

60. The ratio $\frac{ME_1}{MP_1}$ equals:

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$

(C) $\frac{3}{4}$ (D) $\frac{1}{3}$

61. The area of the quadrilateral $P_1E_1E_2P_2$ is:

(A) 30 (B) 45

(C) 15 (D) 20

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	ⓠ	ⓡ	●	●
B	ⓟ	●	●	ⓢ	ⓣ
C	●	●	ⓡ	ⓢ	ⓣ
D	ⓟ	ⓠ	ⓡ	●	●

1. **COLUMN-I** contains various equations of ellipse and **COLUMN-II** contains different parameter. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) $2x^2+3y^2=6$	(p) $e=\frac{1}{\sqrt{3}}$
(B) $3x^2+2y^2=6$	(q) Centre $C(0,0)$
(C) $2x^2+3y^2-4x-12y+8=0$	(r) Latus rectum $=\frac{4}{\sqrt{3}}$
(D) $3x^2+2y^2+6x+8y+5=0$	(s) Focus $S(0,1)$

2. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) If P is a point on the ellipse, $\frac{x^2}{16}+\frac{y^2}{20}=1$ whose foci are S, S', then $SP+S'P$	(p) $\frac{4}{5}$
(B) The eccentricity of the ellipse $2x^2+3y^2-4x-12y+13=0$ is	(q) $4\sqrt{5}$
(C) Tangent are drawn from the points on the line $x-y-5=0$ to $x^2+4y^2=4$, then all the chords of contact pass through a fixed point, whose abscissa is	(r) 8
(D) The sum of distance of any point on the ellipse $3x^2+4y^2=12$ from its directix is	(s) $\frac{1}{\sqrt{3}}$

(Continued)

3. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Eccentricity of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{1}{2}\cdot S_1,S_2$ are focii, 'P' is any point on ellipse then eccentricity of locus of incentre of ΔPS_1S_2 is	(p) $\sqrt{\frac{2}{3}}$
(B) The maximum distance of the centre of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ from the chord of contact of mutually perpendicular tangents of the ellipse is	(q) 4
(C) If the line $x+2y=4$ cutting the ellipse at points, whose eccentric angles 30° and 60° subtends right angle at the origin. Then, the length of minor axis is	(r) $\frac{9}{5}$
(D) If locus of points of intersection of tangents drawn at the ends of all normal chords to the parabola $y^2=8(x-1)$ is $y^2(x+5)=k$, then k equals	(s) 32

4. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The equation of an ellipse is given by $5x^2+5y^2-6xy-8=0$. If r_1, r_2 are distances of points on the ellipse, which are at the maximum and minimum distance from the origin, then r_1+r_2 is	(p) 15
(B) If 1 and 8 are the length of perpendicular chords of the parabola $y^2=4ax$ drawn through the vertex, then a^2 equals	(q) $\frac{1}{5}$
(C) From a point A, common tangents are drawn to the circle $x^2+y^2=8$ and the parabola $y^2=8x$. Area of quadrilateral formed by the common tangents and the chords of contacts of the circle and parabola is	(r) 3
(D) PQ is a variable focal chord of the parabola $y^2=36x$, whose vertex is A. If the locus of centroid of ΔAPQ is parabola, then its length of latus rectum is	(s) 12

5. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Length of common tangent to $x^2+y^2=16$ and $9x^2+25y^2=225$ is	(p) $\frac{3\sqrt{7}}{4}$
(B) If p_1, p_2 be the length of perpendicular drawn from the two foci of ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ to any tangent, then $p_1.p_2$ equals	(q) 16
(C) If normals drawn to parabola $y^2=8x$ at 'P' and 'Q' meet the parabola again at 'R', then the product of abscissae of P and Q is	(r) $\frac{13}{18}$
(D) If the line $2x+3y=1$ touches the parabola $y^2=4ax$ at 'P', then focal distance of 'P' is	(s) 9

(Continued)

6. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The co-ordinates of the point on the parabola $y=x^2+7x+2$, which is nearest to the straight line $y=3x-3$ are	(p) (2, 1)
(B) $y=x+2$ is a tangent to the parabola $y^2=8x$. The point on this line, the other tangent from which it is perpendicular to this tangent is	(q) (–2, 0)
(C) The point on the ellipse $x^2+2y^2=6$, whose distance from the line $x+y=7$ is the least is	(r) $\left(2, \frac{2}{\sqrt{3}}\right)$
(D) The foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ are S and S'. P is a point on the ellipse, whose eccentric angle is $\frac{\pi}{3}$. The incentre of the $\Delta SPS'$ is	(s) (–2,–8)
	(t) (2, 2)

7. Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) A tangent to the ellipse $\frac{x^2}{27}+\frac{y^2}{48}=1$ having slope as $-\frac{4}{3}$ and cuts the x and y axes at the points A and B, respectively. If O is the origin, then the area of ΔOAB is	(p) 36
(B) Product of the perpendiculars drawn from the points $(\pm 3, 0)$ to the line $y=mx-\sqrt{25m^2+16}$ is	(q) $10\sqrt{2}$

(Continued)

COLUMN-I	COLUMN-II
(C) An ellipse passing through the origin has its foci (3, 4) and (6, 8), then the length of its minor axis is	(r) 10
(D) If PQ is the focal chord of ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, which passes through $S(3,0)$ and $PS=2$, then the length of chord PQ is	(s) 16

8. Consider the ellipse

$$(3x-6)^2+(3y-9)^2=\frac{4}{169}(5x+12y+6)^2.$$

COLUMN-I contains the distances associated with this ellipse and **COLUMN-II** gives their value.

Match the expressions or statements in **COLUMN-I** with those in **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) The length of major axis	(p) $\frac{72}{5}$
(B) The length of minor axis	(q) $\frac{16}{\sqrt{5}}$
(C) The length of latus rectum	(r) $\frac{16}{3}$
(D) The distance between the directrices	(s) $\frac{48}{5}$

9. Match the following loci for the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b).$$

COLUMN-I	COLUMN-II
(A) Locus of point of intersection of two perpendicular tangents is	(p) $\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$
(B) Locus of foot of perpendicular from foci upon any tangent is	(q) $4(x^2+y^2)^2=(a^2x^2+b^2y^2)$

(Continued)

COLUMN-I	COLUMN-II
(C) Locus of foot of perpendicular from centre on any tangent is	(r) $x^2+y^2=a^2$
(D) Locus of mid-point of segment OM, where M is the foot of the perpendicular from O to any tangent (O is centre) is	(s) $x^2+y^2=a^2+b^2$

10. Consider an ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ with centre C and point P on it with eccentric angle $\frac{\pi}{4}$. Normal drawn at P intersects the major and minor axes at A and B, respectively N_1 and N_2 are the feet of the perpendiculars from the foci S_1 and S_2, respectively on the tangent at P and N is the foot of the perpendicular from the centre of the ellipse on the normal at P. Tangent at 'P' intersect the axis of x at T.

Match the column using the above information.

COLUMN-I	COLUMN-II
(A) $(CA)\,(CT)$ is equal to	(p) 9
(B) $(PN)\,(PB)$ is equal to	(q) 16
(C) $(S_1N_1)(S_2N_2)$ is equal to	(r) 17
(D) $(S_1P)(S_2P)$ is equal to	(s) 25
	(t) 30

11. Match **COLUMN-I** with **COLUMN-II**

COLUMN-I	COLUMN-II
(A) A stick of length 10 m slides on co-ordinate axes, then locus of a point dividing this stick reckoning from x-axis in the ratio 6:4 is a curve, whose eccentricity is e, then $9e$ is	(p) $\sqrt{6}$
(B) AA' is major axis of on ellipse $3x^2+2y^2+6x-4y-1=0$ and P is a variable point on it, then the greatest area of $\Delta APA'$ is	(q) $2\sqrt{7}$

(Continued)

COLUMN-I	COLUMN-II
(C) Distance between foci of the curve represented by the equations $x = 1 + 4\cos\theta$, $y = 2 + 3\sin\theta$ is	(r) $\dfrac{128}{3}$
(D) Tangents are drawn to the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{7}=1$ at end points of latus rectum. The area of the equilateral so formed is	(s) $3\sqrt{5}$

12. Match the following.

COLUMN-I	COLUMN-II
(A) The number of rational points on the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ is	(p) ∞
(B) Number of integral points on the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ is	(q) 4
(C) Number of rational points on the ellipse $\dfrac{x^2}{3}+y^2=1$ is	(r) 0

COLUMN-I	COLUMN-II
(D) Number of integral points on the ellipse $\dfrac{x^2}{3}+y^2=1$ is	(s) 2
	(t) 3

13. Let $C_1 = 9x^2 + 8y^2 - 36x - 16y - 28 = 0$ and $C_2 = 4x^2 + 9y^2 + 8x - 36y + 4 = 0$.

Match **COLUMN-I** with **COLUMN-II**.

COLUMN-I	COLUMN-II
(A) Minimum distance of (2, 6) from C_1	(p) 1
(B) Maximum distance of (2, 6) from C_1	(q) 2
(C) Minimum distance of (1, 2) from C_2	(r) 5
(D) Maximum distance of (1, 2) from C_2	(s) 8

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Find the number of integral points on the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$.

2. If m is the slope of the common tangents of the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$ and the circle $x^2+y^2=3$, then find the value of m^2.

3. If m is the length of the side of square, which can be made by four perpendicular tangents to the ellipse $\dfrac{x^2}{7}+\dfrac{2y^2}{11}=1$, then find the value of m.

4. Find the number of points on the ellipse $\dfrac{x^2}{50}+\dfrac{y^2}{20}=1$, from which a pair of perpendicular tangents are drawn to the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$.

5. If $a = 1$ and $b = 2$, then find the minimum value of the segment of a tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ intercepted by the co-ordinate axes.

6. The tangent of any point P on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ meets the tangents at the ends of major axis namely, A and A_1 at Q and R, respectively. If $a=2$ and $b=3$, then find $AQ\cdot A_1R$.

7. An ellipse is drawn with major and minor axes of lengths 10 and 8 units, respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Find the radius of the circle.

8. Let P_i and P_i' be the feet of the perpendiculars drawn from foci S, S' on a tangent T_i to an ellipse, whose length of semi-major axis is 20 units. If $\sum_{i=1}^{40}(SP_i)(S'P_i') = 2560$, then find the value of $5e$ (where e denotes eccentricity).

9. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a normal to the circle $x^2+y^2+4x+1=0$, then find the maximum value of ab.

10. A common tangent to $9x^2+16y^2=144, y^2=x-4$ and $(x-6)^2+y^2=4$ is $x=k$, then find the value of k.

11. A circle concentric to the ellipse $\frac{4x^2}{289}+\frac{y^2}{\lambda^2}=1, \lambda<\frac{17}{2}$ passes through foci S_1, S_2 and cuts ellipse at point P. If area of ΔPS_1S_2 is 30 sq. units and if $S_1S_2=d^2$, then find $[d]$, where [.] is the greatest integer function.

12. The area of the quadrilateral formed by the tangents at the end point of latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is 3K. Then find K.

13. If the curve $x^2+3y^2=9$ subtends an obtuse angle at the point $(2\alpha, \alpha)$, then find the possible integral value of α^2.

14. The straight line $\frac{x}{4}+\frac{y}{3}=1$ intersects the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at two points namely, A and B. There is a point P on this ellipse such that the area of ΔPAB is equal to $6(\sqrt{2}-1)$. Then find the number of such points P.

15. If the chord through the points, whose eccentric angles are α and β on the ellipse $9x^2+25y^2=225$ passes through a focus. Then find the positive value of $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)$.

16. If PSP' and QSQ' are two perpendicular focal chords of $\frac{x^2}{6}+\frac{y^2}{3}=1$, then find $\frac{1}{SP.SP'}+\frac{1}{SQ.SQ'}$.

17. If the equation of the curve on reflection of the ellipse $\frac{(x-4)^2}{16}+\frac{(y-3)^2}{9}=1$ about the line $x-y-2=0$ is $16x^2+9y^2+k_1x-36y+k_2=0$, then what will be the unit digit of k_1+k_2.

18. If the distance between the directrices of the ellipse $(4x-8)^2+16y^2=\left(x+\sqrt{3}y+10\right)^2$ is K, then find $\frac{K}{2}$.

19. If F_1 and F_2 are the feet of the perpendiculars from the foci S_1 and S_2 of the ellipse $16x^2+25y^2=400$ on the tangent at any point P on the ellipse, then find the minimum value of $S_1F_1+S_2F_2$.

20. The equation of an ellipse is given by $5x^2+5y^2-6xy-8=0$. If r_1 and r_2 are distances of points on the ellipse, which are at the maximum and minimum distance from origin, then find r_1+r_2.

21. If L be the length of common tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=16$ intercepted by the co-ordinate axis, then find $\frac{\sqrt{3}L}{2}$.

22. An ellipse is sliding along the co-ordinate axes. If the foci of the ellipse are (1, 1) and (3, 3) and then the area of the director circle of the ellipse is $K\pi$, find K.

23. If the tangents are drawn from points on the line $x-y+2=0$ to the ellipse $x^2+2y^2=2$, and all the chords of contact pass through a point, find its distance from point $\left(2, \frac{1}{2}\right)$.

24. If P and Q are the ends of a pair of conjugate diameters and C is the centre of the ellipse $4x^2+9y^2=36$, then find the area of ΔCPQ in sq. units.

25. Find the maximum distance from the origin to any normal chord drawn to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$.

26. The number of values of c such that the straight line $y=4x+c$ touches the curve $\frac{x^2}{4}+y^2=1$ is K, then find K.

27. Tangent is drawn to the ellipse $\frac{x^2}{27}+y^2=1$ at $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$ (where $\theta\in\left(0, \frac{\pi}{2}\right)$). Then the value of θ such that sum of intercepts on co-ordinate axes made by this tangent is least is $\frac{\pi}{K}$, then find K.

28. P_1, P_2....., P_i..... P_n are the points on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and Q_1, Q_2 ,....Q_i ,.... Q_n are the corresponding points on the auxiliary circle of the ellipse. If the line joining C to Q_i meets the normal at P_i with respect to the given ellipse at K_i and $\sum_{i=1}^{n}(CK_i)=175$, then find the value of $\frac{n}{5}$.

29. Co-ordinates of the vertices B and C are $(2, 0)$ and $(8, 0)$, respectively. The vertex 'A' is varying in such a way that $4\tan\frac{B}{2}\tan\frac{C}{2}=1$. If the locus of '$A$' is an ellipse, then find the length of its semi-major axis.

30. Let P and Q be two points on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, whose eccentric angles differ by a right angle. Tangents are drawn at P and Q to meet at R. If the chord PQ divides the joint of C and R in the ratio $m:n$ (C being centre of ellipse), then find $m + n$, where $m:n$ is in its simplified form.

31. A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+\lambda y^2=6$ at P and Q. If tangents at P and Q of ellipse $x^2+\lambda y^2=6$ are at right angles, then find λ.

32. If PSP' is a focal chord of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and $SP=3$, then find $61S'P'-56$, where S and S' are foci of ellipse.

33. Tangents are drawn to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ at the ends of latus rectum. If the area of an quadrilateral be λ sq. units, then find the value of $\frac{\lambda}{9}$.

34. Let P be a point on ellipse $\frac{x^2}{9}+\frac{y^2}{7}=1$, whose parametric angle is $\frac{\pi}{4}$, then find the abscissa of the incentre of $\Delta SPS'$.

35. If the ellipse $\frac{x^2}{4}+y^2=1$ meets the ellipse $x^2+\frac{y^2}{a^2}=1$ in four distinct points and if $a=b^2-3b-9$, then find the total number of integers which are not a solution of b.

36. Suppose F_1, F_2 are the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. P is a point on ellipse such that $PF_1:PF_2=2:1$. Find the area of the ΔPF_1F_2.

37. Let P, Q and R be any three points on the ellipse $\frac{x^2}{16}+\frac{y^2}{3}=1$. Find the maximum area of ΔPQR.

38. From a point, perpendicular tangents are drawn to the ellipse $x^2+2y^2=2$. The chord of contact touches a circle, which is concentric with the given ellipse. Find the ratio of maximum and minimum area of circle.

39. P be a point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ with foci S and S'. If A be the area of $\Delta SPS'$, then find the maximum value of $\frac{A}{2}$.

40. A circle is inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$, whose centre is one of the foci. If the circle touches the ellipse, then find the radius of the circle.

41. y-axis is a tangent to one ellipse with foci (2, 0) and (6, 4). If the length of the major axis is x, then find the greatest integer of x.

ARCHIVE: JEE MAIN

1. **[JEE MAIN 2020: Jan Attempt]**

The length of the minor axis (along, y-axis) of an ellipse in the standard form is $\frac{4}{\sqrt{3}}$. If this ellipse touches the line, $x + 6y = 8$; then its eccentricity is :

(A) $\frac{1}{2}\sqrt{\frac{5}{3}}$ (B) $\frac{1}{2}\sqrt{\frac{11}{3}}$

(C) $\frac{1}{3}\sqrt{\frac{11}{3}}$ (D) $\sqrt{\frac{5}{6}}$

2. **[JEE MAIN 2020: Jan Attempt]**

If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{9}=1$ for some $a \in R$, then the distance between the foci of the ellipse is :

(A) 4 (B) $2\sqrt{7}$

(C) $2\sqrt{5}$ (D) $2\sqrt{2}$

3. **[JEE MAIN 2020: Jan Attempt]**

If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is :

(A) $\sqrt{3}$ (B) $2\sqrt{3}$

(C) $3\sqrt{2}$ (D) $\frac{3}{\sqrt{2}}$

4. [JEE MAIN 2020: Jan Attempt]
Let the line $y = mx$ and the ellipse $2x^2 + y^2 = 1$ intersect at a point P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(-\frac{1}{3\sqrt{2}}, 0\right)$ and $(0, \beta)$, then β is equal to :

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{2\sqrt{2}}{3}$

(C) $\frac{2}{3}$ (D) $\frac{\sqrt{2}}{3}$

5. [JEE MAIN 2019]
If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points $(1, 2)$ and (a, b) are perpendicular to each other, then a^2 is:

(A) $\frac{2}{17}$ (B) $\frac{64}{17}$

(C) $\frac{128}{17}$ (D) $\frac{4}{17}$

6. [JEE MAIN 2019]
In an ellipse with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 units and one of the foci is at $(0, 5\sqrt{3})$, then the length of its latus rectum is:

(A) 5 (B) 6

(C) 8 (D) 10

7. [JEE MAIN 2019]
If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to:

(A) $\sqrt{2} + 1$ (B) $2\sqrt{2} + 1$

(C) $\sqrt{2} - 1$ (D) $2\sqrt{2} - 1$

8. [JEE MAIN 2019]
If the line $x - 2y = 12$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\left(3, \frac{-9}{2}\right)$, then the length of the latus rectum of the ellipse is:

(A) $8\sqrt{3}$ (B) $12\sqrt{2}$

(C) 5 (D) 9

9. [JEE MAIN 2019]
If the tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point $P(2, 2)$ meet the x-axis at Q and R, respectively, then the area (in sq. units) of the ΔPQR is:

(A) $\frac{14}{3}$ (B) $\frac{16}{3}$

(C) $\frac{68}{15}$ (D) $\frac{34}{15}$

10. [JEE MAIN 2019]
If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices, then the midpoints of the tangents intercepted between the co-ordinate axes lie on the curve is:

(A) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (B) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

(C) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$ (D) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

11. [JEE MAIN 2019]
Let the length of the latus rectum of an ellipse with its major axis along x-axis and centre at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it?

(A) $(4\sqrt{2}, 2\sqrt{3})$ (B) $(4\sqrt{3}, 2\sqrt{3})$

(C) $(4\sqrt{3}, 2\sqrt{2})$ (D) $(4\sqrt{2}, 2\sqrt{2})$

12. [JEE MAIN 2019]
If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent to the ellipse at P passes through $Q(4, 4)$ then PQ is equal to:

(A) $\frac{\sqrt{61}}{2}$ (B) $\frac{\sqrt{221}}{2}$

(C) $\frac{\sqrt{157}}{2}$ (D) $\frac{5\sqrt{5}}{2}$

13. [JEE MAIN 2019]
Let S and S' be the foci of an ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS$ is a right angled triangle with right angle at B and area of $\Delta S'BS$ = 8 sq. units, then the length of a latus rectum of the ellipse is:

(A) 2 (B) $4\sqrt{2}$

(C) 4 (D) $2\sqrt{2}$

14. [JEE MAIN 2019]

An ellipse, with foci at (0, 2) and (0, –2) and minor axis of length 4 units, passes through which of the following points?

(A) $(2, \sqrt{2})$ (B) $(2, 2\sqrt{2})$

(C) $(\sqrt{2}, 2)$ (D) $(1, 2\sqrt{2})$

15. [JEE MAIN 2018]

Two sets A and B are as follows.

$A = \{(a, b) \in R \times R: |a - 5| < 1 \text{ and } |b - 5| < 1\};$

$B = \{(a, b) \in R \times R: 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$. Then:

(A) $A \subset B$

(B) $A \cap B = \phi$ (an empty set)

(C) neither $A \subset B$ nor $B \subset A$

(D) $B \subset A$

16. [JEE MAIN 2017]

The eccentricity of an ellipse, whose centre is at the origin, is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is:

(A) $x + 2y = 4$ (B) $2y - x = 2$

(C) $4x - 2y = 1$ (D) $4x + 2y = 7$

17. [JEE MAIN 2015]

The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is:

(A) $\frac{27}{2}$ (B) 27

(C) $\frac{27}{4}$ (D) 18

18. [JEE MAIN 2014]

The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it, is:

(A) $(x^2 - y^2)^2 = 6x^2 + 2y^2$

(B) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

(C) $(x^2 + y^2)^2 = 6x^2 + 2y^2$

(D) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

19. [JEE MAIN 2013]

The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at (0, 3) is:

(A) $x^2 + y^2 - 6y - 7 = 0$ (B) $x^2 + y^2 - 6y + 7 = 0$

(C) $x^2 + y^2 - 6y - 5 = 0$ (D) $x^2 + y^2 - 6y + 5 = 0$

20. [AIEEE 2012]

An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the co-ordinate axes, then the equation of the ellipse is:

(A) $x^2 + 4y^2 = 16$ (B) $4x^2 + y^2 = 4$

(C) $x^2 + 4y^2 = 8$ (D) $4x^2 + y^2 = 8$

21. [AIEEE- 2012]

Statement-1: An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement-2: If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

(A) Statement-1 is True, Statement-2 is True and Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True and Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is False and Statement-2 is True.

(D) Statement-1 is True and Statement-2 is False.

22. [AIEEE- 2011]

Equation of the ellipse, whose axes are the axes of co-ordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is:

(A) $3x^2 + 5y^2 - 15 = 0$ (B) $5x^2 + 3y^2 - 32 = 0$

(C) $3x^2 + 5y^2 - 32 = 0$ (D) $5x^2 + 3y^2 - 48 = 0$

23. [AIEEE- 2009]

The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the co-ordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is:

(A) $4x^2 + 48y^2 = 48$ (B) $4x^2 + 64y^2 = 48$

(C) $x^2 + 16y^2 = 16$ (D) $x^2 + 12y^2 = 16$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE 2012]

The ellipse E_1: $\frac{x^2}{9}+\frac{x^2}{4}=1$ is inscribed in a rectangle R, whose sides are parallel to the co-ordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is:

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$ (D) $\frac{3}{4}$

2. [JEE 2009]

The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is:

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$

(C) $\frac{21}{10}$ (D) $\frac{27}{10}$

3. [JEE 2009]

The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the midpoint of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the point is:

(A) $\left(\pm\frac{3\sqrt{5}}{2},\pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2},\pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3},\pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3},\pm\frac{4\sqrt{3}}{7}\right)$

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. [JEE Advanced-2019]

Define the collections $\{E_1, E_2, E_3 \ldots\}$ of ellipse and $\{R_1, R_2, R_3 \ldots\}$ of rectangles as follows.

E_1: $\frac{x^2}{9}+\frac{y^2}{4}=1$

R_1: Rectangle of largest area, with sides parallel to the axes, inscribed in E_1

E_n: Ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of larger area inscribed in R_{n-1}, $n > 1$

R_n: Rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$

Then which of the following options is/are correct?

(A) $\sum_{n=1}^{N}$ (Area of R_n) < 24, for each positive integer N

(B) The eccentricities of E_{18} and E_{19} are not equal

(C) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(D) The length of latus rectum of E_9 is $\frac{1}{6}$ units

2. [JEE Advanced-2018]

Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose centre is at the origin $O(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) true?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1 unit

(B) For the all ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$ unit

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{\pi-2}{4\sqrt{2}}$ sq. units

(D) The area of the region bounded by the ellipse between the lies $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{\sqrt{2}}$ $(\pi - 2)$ sq. units

3. [JEE Advanced 2015]

Let E_1 and E_2 be two ellipses, whose centres are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle

$x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$ and if e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is/are:

(A) $e_1^2 + e_2^2 = \frac{43}{40}$
(B) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
(C) $|e_1^2 - e_2^2| = \frac{5}{8}$
(D) $e_1 e_2 = \frac{\sqrt{3}}{4}$

4. **[JEE 2009]**
In a ΔABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2 \frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then:
(A) $b + c = 4a$
(B) $b + c = 2a$
(C) locus of point A is an ellipse
(D) locus of point A is a pair of straight lines

5. **[JEE 2008]**
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are:
(A) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
(B) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
(C) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
(D) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension I

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

1. **[JEE Advanced- 2016]**
The orthocenter of ΔF_1MN is:
(A) $\left(-\frac{9}{10}, 0\right)$
(B) $\left(\frac{2}{3}, 0\right)$
(C) $\left(\frac{9}{10}, 0\right)$
(D) $\left(\frac{2}{3}, \sqrt{6}\right)$

2. **[JEE Advanced-2016]**
If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the ΔMQR to area of the quadrilateral MF_1NF_2 is:
(A) 3:4
(B) 4:5
(C) 5:8
(D) 2:3

Comprehension 2

Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B.

3. **[JEE 2010]**
The co-ordinates of A and B are:
(A) $(3, 0)$ and $(0, 2)$
(B) $\left(-\frac{8}{5}, \frac{2\sqrt{261}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$
(C) $\left(-\frac{8}{5}, \frac{2\sqrt{261}}{15}\right)$ and $(0, 2)$
(D) $(3, 0)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

4. **[JEE 2010]**
The orthocenter of the ΔPAB is:
(A) $\left(5, \frac{8}{7}\right)$
(B) $\left(\frac{7}{5}, \frac{25}{8}\right)$
(C) $\left(\frac{11}{5}, \frac{8}{5}\right)$
(D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

5. **[JEE 2010]**
The equation of the locus of the point, whose distances from the point P and the line AB are equal, is:
(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
(B) $9x^2 + y^2 + 6xy - 54x + 62y - 241 = 0$
(C) $9x^2 + y^2 - 6xy - 54x - 62y - 241 = 0$
(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [JEE Advanced 2015]

Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$, where $f_1 > 0$ and $f_2 < 0$. Let, P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let, T_1 be a tangent to P_1, which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2, which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is:

2. [JEE-Advanced 2013]

A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the point P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the ΔPQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$ equals

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** B	**3.** C	**4.** B	**5.** A	**6.** C	**7.** D	**8.** C	**9.** C	**10.** A
11. A	**12.** D	**13.** A	**14.** C	**15.** D	**16.** C	**17.** C	**18.** A	**19.** B	**20.** C
21. B	**22.** C	**23.** B	**24.** C	**25.** A	**26.** D	**27.** C	**28.** C	**29.** A	**30.** C
31. D	**32.** D	**33.** A	**34.** C	**35.** A	**36.** B	**37.** D	**38.** A	**39.** D	**40.** D
41. B	**42.** B	**43.** A	**44.** B	**45.** A	**46.** B	**47.** A	**48.** B	**49.** A	**50.** D
51. B	**52.** D	**53.** C	**54.** A	**55.** D	**56.** C	**57.** D	**58.** D	**59.** A	**60.** D
61. B	**62.** A	**63.** D	**64.** A	**65.** B	**66.** B	**67.** C	**68.** C	**69.** B	**70.** C
71. B	**72.** C	**73.** B	**74.** B	**75.** C	**76.** D	**77.** B	**78.** B	**79.** C	**80.** D
81. B	**82.** A	**83.** B	**84.** C	**85.** B	**86.** C				

Multiple Correct Choice Type Questions

1. A, D	**2.** A, C, D	**3.** A, B	**4.** A, B, C	**5.** B, C, D
6. A, B, C	**7.** C, D	**8.** A, B, C, D	**9.** A, B, C, D	**10.** A, B, C, D
11. A, C	**12.** A, C	**13.** A, B, C, D	**14.** A, D	**15.** A, B, D
16. A, B, C, D	**17.** A, B	**18.** B, C	**19.** B, C	**20.** A, B
21. B, C	**22.** A, C, D	**23.** A, D	**24.** B, D	**25.** A, B, C, D
26. A, B	**27.** A, C	**28.** B, C	**29.** A, B	**30.** A, B
31. A, B, D	**32.** A, C, D	**33.** A, B, C, D	**34.** A, D	**35.** A, C
36. B, C, D				

Linked Comprehension Type Questions

1. D	**2.** C	**3.** B	**4.** D	**5.** B	**6.** D	**7.** B	**8.** A	**9.** C	**10.** C
11. D	**12.** A	**13.** B	**14.** C	**15.** C	**16.** A	**17.** B	**18.** C	**19.** D	**20.** A
21. D	**22.** A	**23.** B	**24.** B	**25.** D	**26.** B	**27.** C	**28.** D	**29.** B	**30.** C
31. B	**32.** A	**33.** C	**34.** D	**35.** C	**36.** C	**37.** A	**38.** C	**39.** D	**40.** B
41. B	**42.** B	**43.** B	**44.** B	**45.** D	**46.** D	**47.** C	**48.** C	**49.** A	**50.** B
51. C	**52.** A	**53.** B	**54.** B	**55.** D	**56.** A	**57.** A	**58.** C	**59.** B	**60.** A
61. A									

Matrix Match/Column Match Type Questions

1. A $\rightarrow$ (p, q, r)	B $\rightarrow$ (p, q, r, s)	C $\rightarrow$ (p, r)	D $\rightarrow$ (p, r)
2. A $\rightarrow$ (q)	B $\rightarrow$ (s)	C $\rightarrow$ (p)	D $\rightarrow$ (r)
3. A $\rightarrow$ (p)	B $\rightarrow$ (r)	C $\rightarrow$ (q)	D $\rightarrow$ (s)
4. A $\rightarrow$ (r)	B $\rightarrow$ (q)	C $\rightarrow$ (p)	D $\rightarrow$ (s)
5. A $\rightarrow$ (p)	B $\rightarrow$ (s)	C $\rightarrow$ (q)	D $\rightarrow$ (r)

6. A → (s)	B → (q)	C → (p)	D → (r)
7. A → (p)	B → (s)	C → (q)	D → (r)
8. A → (s)	B → (q)	C → (r)	D → (p)
9. A → (s)	B → (r)	C → (p)	D → (q)
10. A → (q)	B → (s)	C → (p)	D → (r)
11. A → (s)	B → (p)	C → (q)	D → (r)
12. A → (p)	B → (q)	C → (p)	D → (s)
13. A → (q)	B → (s)	C → (p)	D → (r)

Integer/Numerical Answer Type Questions

1. 4	**2.** 2	**3.** 5	**4.** 4	**5.** 3
6. 4	**7.** 2	**8.** 3	**9.** 4	**10.** 4
11. 3	**12.** 9	**13.** 2	**14.** 3	**15.** 9
16. 1	**17.** 2	**18.** 8	**19.** 8	**20.** 3
21. 7	**22.** 8	**23.** 3	**24.** 3	**25.** 3
26. 2	**27.** 6	**28.** 5	**29.** 5	**30.** 2
31. 2	**32.** 7	**33.** 3	**34.** 1	**35.** 8
36. 4	**37.** 9	**38.** 4	**39.** 6	**40.** 2
41. 8				

ARCHIVE: JEE MAIN

1. B	**2.** B	**3.** C	**4.** D	**5.** A	**6.** A	**7.** A	**8.** D	**9.** C	**10.** B
11. C	**12.** D	**13.** C	**14.** C	**15.** A	**16.** C	**17.** B	**18.** C	**19.** A	**20.** A
21. A	**22.** C	**23.** D							

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. C	**2.** D	**3.** C

Multiple Correct Choice Type Problems

1. A, D	**2.** A, C	**3.** A, B	**4.** B, C	**5.** B, C

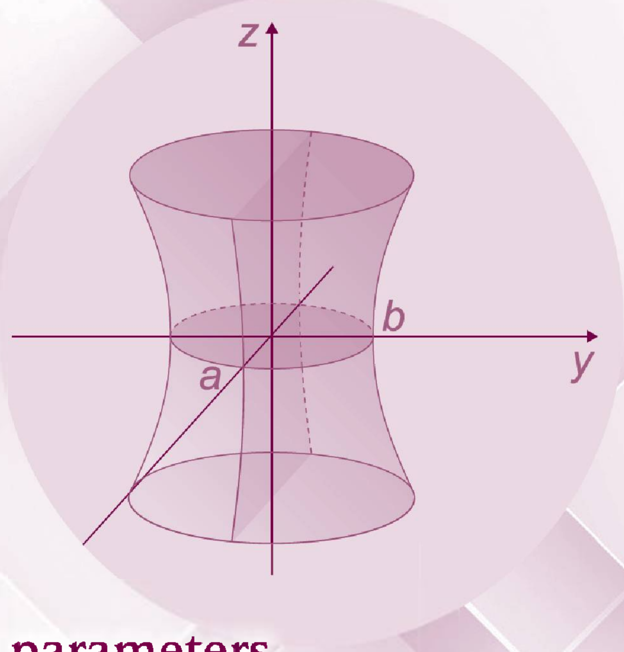
Linked Comprehension Type Questions

1. A	**2.** C	**3.** D	**4.** C	**5.** A

Integer/Numerical Answer Type Questions

1. 4	**2.** 9

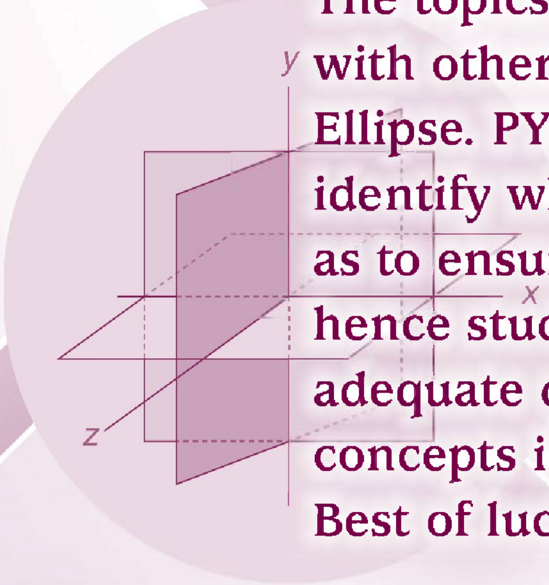
From the Author's Desk

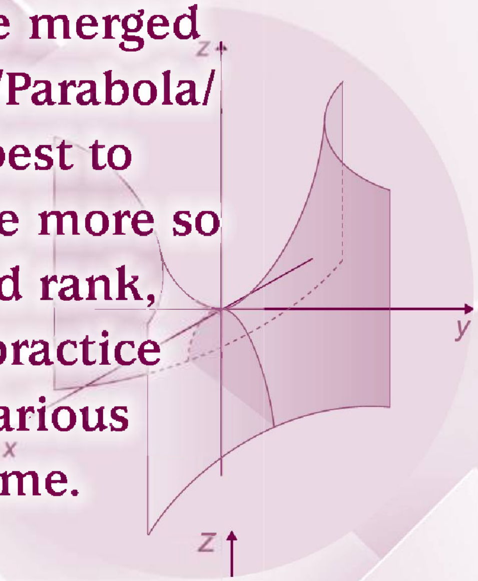


"Dear Learners,

In hyperbola chapter, its parameters and their relative position should be understood with clarity before attempting the related questions. Problems based on topics, such as calculation of parameters of hyperbola and conjugate hyperbola, auxiliary circle, tangent, chord of contact, focal chord, normal and its properties and rectangular hyperbola are important concepts as far as the examination is concerned. The topics in Hyperbola may be merged with other chapters, like Circle/Parabola/Ellipse. PYQs of any exam are best to identify which topics to practice more so as to ensure selection with good rank, hence students are advised to practice adequate questions based on various concepts in prescribed time frame.

Best of luck!!"





CHAPTER

6 Hyperbola

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Hyperbola and parameters

(b) Tangent, normal, chord of contact

(c) Conjugate hyperbola

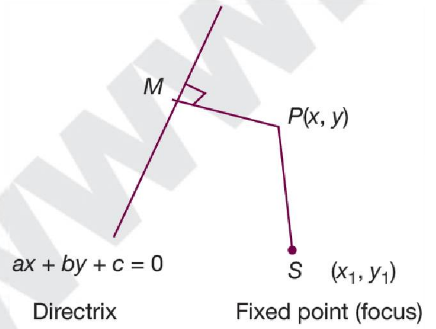
(d) Asymptotes and its properties

(e) Rectangular hyperbola and parameters

(f) Properties of hyperbola

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

Hyperbolic curves are of special importance in the field of Science and Technology, especially in Astronomy and Space Studies. In this chapter, we are going to study the characteristics of hyperbola.



A hyperbola is the locus of a point which moves in the plane in such a way that the ratio of its distance from a fixed point (**focus**) to its distance from a fixed straight line (**directrix**) is always constant (**eccentricity**), which is always greater than unity.

This constant ratio is generally denoted by e, which is known as the eccentricity of the hyperbola.

$$\frac{PS}{PM} = e \Rightarrow PS = ePM$$

As $e > 1$, then $PS > PM$, which tells that directrix is closer to the hyperbola than focus.

If $S(x_1, y_1)$ is the focus, $ax + by + c = 0$ is the directrix and $P(x, y)$ is any point on the hyperbola, then by definition, $PS = ePM$

$$\Rightarrow \quad (x - x_1)^2 + (y - y_1)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}$$

$$\Rightarrow \quad (a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = e^2(ax + by + c)^2,$$

which is the equation of second degree form.

Remark(s)

Any equation of second degree in x and y as $F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents equation of a hyperbola.

(i) $h^2 - ab > 0; e > 1$.

(ii) $\Delta \neq 0 \Rightarrow abc + 2hgf - bg^2 - ch^2 - af^2 \neq 0$

(iii) Focus does not lie on directrix

(iv) Directrix is closer to hyperbola than focus.

(v) For centre $\frac{\partial f(x,y)}{\partial x}=0$ and $\frac{\partial f(x,y)}{\partial y}=0$, solve them and get centre.

ILLUSTRATION 1

Find the hyperbola, whose directrix is $2x + y - 1 = 0$, focus is (1, 2) and $e = \sqrt{3}$.

SOLUTION

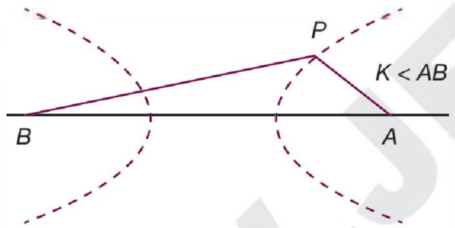
Equation of hyperbola, $PS = ePM$

$$\sqrt{(x-1)^2+(y-2)^2} = \sqrt{3}\left|\frac{2x+y-1}{\sqrt{4+1}}\right|$$

On squaring, $(x-1)^2 + (y-2)^2 = \frac{3}{5}(2x+y-1)^2$

DEFINITION OF HYPERBOLA

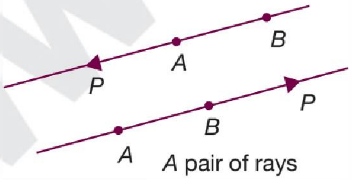
Hyperbola is locus of a point moving in a plane, whose difference of distances from two fixed point remains constant.



$|PA - PB| = K$ (constant)

(i) If $K < AB$, then locus of P is a hyperbola.

(ii) If $K = AB$, then locus of P is a pair of rays joining the points A and B.



A pair of rays

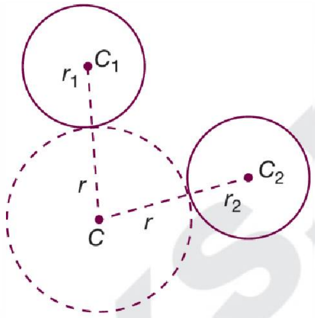
(iii) If $K > AB$, then no locus of P is possible.

ILLUSTRATION 2

Show that the locus of the centre of circle touches two non-intersecting given circles externally is hyperbola.

SOLUTION

Let there are two fixed circles, whose centres are C_1 and C_2 and radius r_1 and r_2 and variable circle with centre C and radius r, touches these two circles externally.



Then, $CC_1 = r_1 + r$ and $CC_2 = r_2 + r$

Now, taking difference, we get,

$$|CC_1 - CC_2| = |r_1 - r_2| = \text{Constant}$$

Locus of the centre of variable circle C is hyperbola, whose foci are C_1 and C_2.

ILLUSTRATION 3

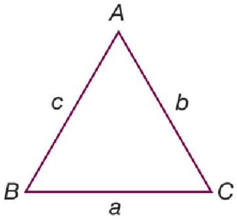
A ΔABC, whose base BC is fixed, then find the locus of vertex A of the triangle moving in such a way that,

$$\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = \lambda \text{ (Constant).}$$

SOLUTION

Using formula, $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$;

$BC = a$ = Constant (Given)



$$\frac{\tan B/2}{\tan C/2} = \frac{\Delta}{s(s-b)} \Big/ \frac{\Delta}{s(s-c)} = \lambda$$

$$\Rightarrow \quad \frac{s-c}{s-b} = \lambda$$

Where, $s = \frac{a+b+c}{2}$; we get $\frac{a+b-c}{a+c-b} = \lambda$

$$\Rightarrow \quad a + b - c = \lambda a + \lambda c - \lambda b$$

$$\Rightarrow \quad -(\lambda + 1)c + (1 + \lambda)b = (\lambda - 1)a$$

$$\Rightarrow \quad b-c=\left(\frac{\lambda-1}{\lambda+1}\right)a=\text{Constant}$$

$$AC-AB=\text{Constant}$$

Locus of A is hyperbola.

STANDARD HYPERBOLA

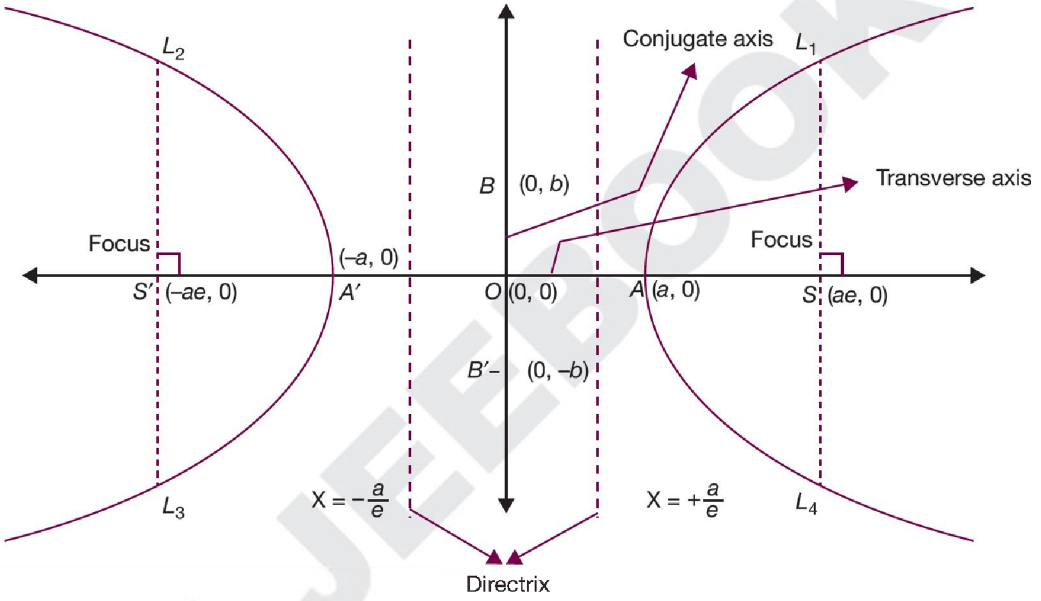
Let's consider the equation of standard hyperbola as

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

(i) It does not matter in hyperbola, whether $a>b$ or $b>a$

(ii) Replace b^2 by $-b^2$ in parameters of standard ellipse

(iii) Diagram of hyperbola explains about relative position of parameters, hence it is advised to students to understand it very well.



Principal Axes

Transverse axis

- It cuts hyperbola
- Both foci lie on it
- AA' transverse axis
- $\ell_{\text{transverse axis}}=AA'=2a$
- Equation $Y=0$

Conjugate axis

- It does not cut hyperbola
- Parallel to directrix and passing through centre
- BB' conjugate axis
- $\ell_{\text{conjugate axis}}=BB'=2b$
- Equation $X=0$

Centre: A point which bisects all chords passing through $X=0$, $Y=0$

Vertex: Point of intersection of conic and axis (Transverse axis)

$$A(\pm a, 0)(X=\pm a, Y=0)$$

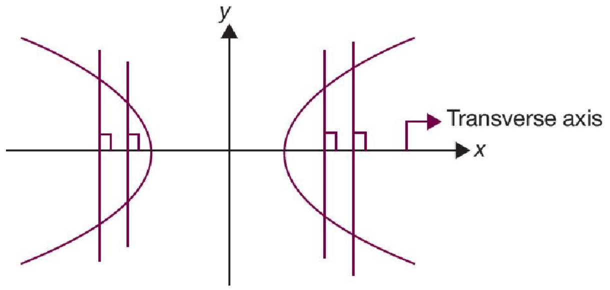
Focus: $(\pm ae, 0)$

$$X=\pm ae,\ Y=0$$

Directrix: $X=\pm\dfrac{a}{e}$

Eccentricity: $e=\sqrt{1+\dfrac{b^2}{a^2}}=\sqrt{1+\left(\dfrac{\ell_{\text{conjugate axis}}}{\ell_{\text{transverse axis}}}\right)^2}$

Double ordinate: Chord perpendicular to transverse axis.



Latus rectum: Double ordinate passing through foci. Extremities of latus rectum are L_1, L_2, L_3, L_4

$$\left(X = \pm ae,\ Y = \pm \frac{b^2}{a}\right)$$

$$\text{Length of latus rectum } = \frac{2b^2}{a} = \frac{(\ell_{\text{conjugate axis}})^2}{(\ell_{\text{transverse axis}})}$$

$= 2e$ (Distance between focus and corresponding directrix)

ILLUSTRATION 4

From the equation $\frac{(x+2)^2}{16} - \frac{(y-2)^2}{9} = 1$, find centre, eccentricity, focus, directrix, vertex, latus rectum and axes.

SOLUTION

Given hyperbola is $\frac{(x+2)^2}{16} - \frac{(y-2)^2}{9} = 1$

or $\frac{X^2}{4^2} - \frac{Y^2}{3^2} = 1$

Where $X = x + 2$, $Y = y - 2$, $a = 4$, $b = 3$

Centre $X = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$

$Y = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

Centre (–2, 2)

Transverse axis $Y = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

Conjugate axis $X = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$

Eccentricity, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

Foci $(\pm ae, 0)$ $X = \pm ae \Rightarrow x + 2 = \pm 4\left(\frac{5}{4}\right)$

$\Rightarrow x = -2 \pm 5 \Rightarrow x = 3, -7$

$Y = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

Foci (3, 2) and (–7, 2)

Directrix $X = \pm \frac{a}{e} \Rightarrow x + 2 = \pm \frac{16}{5}$

$\Rightarrow x = \frac{6}{5}$ or $x = -\frac{26}{5}$

Vertex $(\pm a, 0)$ $X = \pm a \Rightarrow x + 2 = \pm 4 \Rightarrow x = 2, -6$

$Y = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

Vertices (2, 2) and (–6, 2)

Length of latus rectum $= \frac{2b^2}{a} = \frac{2.9}{4} = \frac{9}{2}$

Extremities of latus rectum $= \left(\pm ae, \pm \frac{b^2}{a}\right)$

$X = \pm ae \Rightarrow x + 2 = \pm 4\left(\frac{5}{4}\right) \Rightarrow x = -2 \pm 5 \Rightarrow x = 3, -7$

$Y = \pm \frac{b^2}{a} \Rightarrow y - 2 = \pm \frac{9}{4} \Rightarrow y = 2 \pm \frac{9}{4} \Rightarrow y = -\frac{1}{4}, \frac{17}{4}$

$\left(3, -\frac{1}{4}\right)\left(3, \frac{17}{4}\right)\left(-7, \frac{-1}{4}\right)\left(-7, \frac{17}{4}\right)$ are extremities of latus rectum.

ILLUSTRATION 5

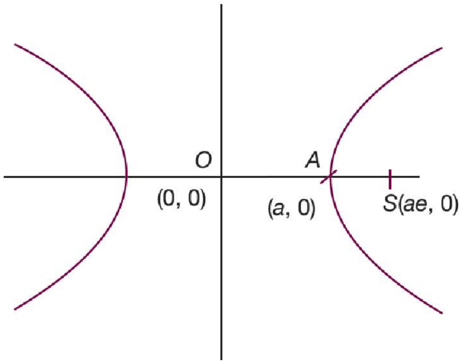
Find the eccentricity of hyperbola, as per the following:

(i) Vertex is midpoint of centre and focus.

(ii) Transverse axis subtends 90° at the extremities of conjugate axis.

SOLUTION

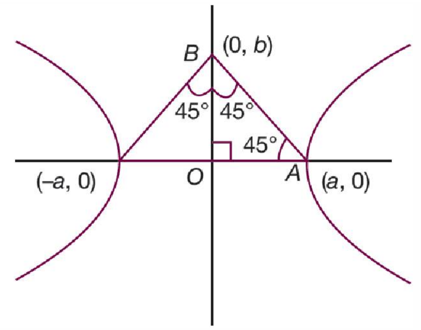
(i)



We get vertex A as the midpoint of centre (0, 0) and focus $(ae, 0)$.

Hence, $\frac{0 + ae}{2} = a \Rightarrow e = 2$

(ii)



ΔOAB is an isosceles right angled triangle.

$$OA = OB \Rightarrow a = b$$

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$

ILLUSTRATION 6

Find the equation of a standard hyperbola of given transverse axis, whose vertex bisects the distance between the centre and the focus.

SOLUTION

Let the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and length of transverse axis $= 2a$ units (Given)

Vertex $(a, 0)$ = Mid point of centre $(0, 0)$ and focus $(ae, 0)$

$$a = \frac{0+ae}{2} \Rightarrow e = 2$$

We get, $e = 2 = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow 3 = \frac{b^2}{a^2} \Rightarrow b^2 = 3a^2$

Required equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or $\frac{x^2}{a^2} - \frac{y^2}{3a^2} = 1$

ILLUSTRATION 7

If the distance between the foci of a hyperbola is 16 units and its eccentricity is $\sqrt{2}$, then find the equation of standard hyperbola.

SOLUTION

As $e = \sqrt{2}$ and $2ae = 16$, thus $\sqrt{2}a = 8$

$\Rightarrow \quad a = 4\sqrt{2}$

$$e = \sqrt{2} = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow a = b$$

$$a = b = 4\sqrt{2}$$

Thus, $x^2 - y^2 = 32$

ILLUSTRATION 8

Find the equation of hyperbola in standard, form whose conjugate axis is 5 and distance between foci are 13 units.

SOLUTION

Let the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Length of conjugate axis $= 2b = 5 \Rightarrow 4b^2 = 25$

$$2ae = 13 \Rightarrow 4a^2e^2 = 169$$

$\Rightarrow \quad 4a^2 + 4b^2 = 169 \Rightarrow 4a^2 + 25 = 169$

$\Rightarrow \quad 4a^2 = 144 \Rightarrow a^2 = 36$

$\therefore$ The required hyperbola is $\frac{x^2}{36} - \frac{4y^2}{25} = 1$

ILLUSTRATION 9

Prove that the locus of point of intersection of the lines $\frac{x}{a} - \frac{y}{b} = m$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$, is a hyperbola, where m is a parameter.

SOLUTION

Let (h, k) is the point of intersection of the given lives, it means

$$\frac{h}{a} - \frac{k}{b} = m; \; \frac{h}{a} + \frac{k}{b} = \frac{1}{m}$$

Multiply them, we get,

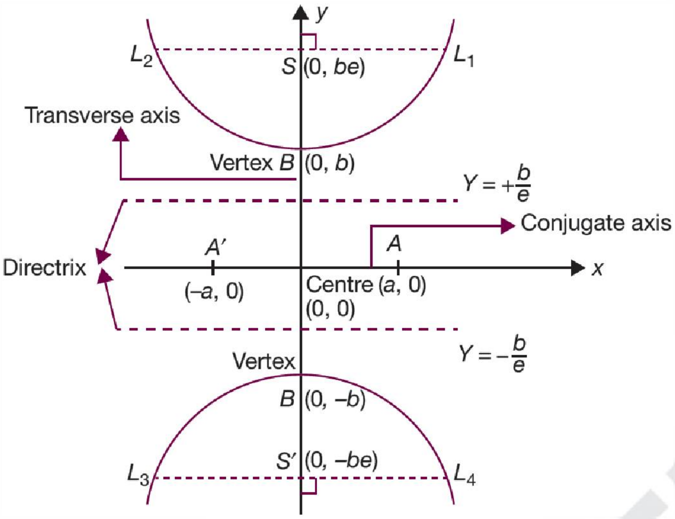
$$\left(\frac{h}{a} - \frac{k}{b}\right)\left(\frac{h}{a} + \frac{k}{b}\right) = m\left(\frac{1}{m}\right)$$

$\Rightarrow \quad \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$

Locus of (h, k) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which is a hyperbola.

CONJUGATE HYPERBOLA

Let us consider an equation $\frac{-X^2}{a^2}+\frac{Y^2}{b^2}=1$, we can rewrite it as $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=-1$



Replace $X \leftrightarrow Y$ and $a \leftrightarrow b$ in standard hyperbola parameters.

Transverse axis $(BB')X=0$

Length of transverse axis $=2b$ units

Conjugate axis $(AA')Y=0$

Length of conjugate axis $=2a$ units

$$e=\sqrt{1+\frac{a^2}{b^2}},$$

Focus $(0, \pm be)$; $X=0, Y=\pm be$

Directrix $(Y=\pm b/e)$

Vertex $(0, \pm b)$ $\quad X=0, Y=\pm b$

Centre $(0, 0)$ $\quad X=0, Y=0$

$$\ell_{\text{L.R}}=\frac{2a^2}{b}=\frac{(\ell_{\text{conjugate axis}})^2}{(\ell_{\text{transverse axis}})}$$

End points of latus rectum are $\left(\pm\frac{a^2}{b}, \pm be\right)$

$$X=\pm\frac{a^2}{b}; Y=\pm be$$

L_1, L_2, L_3, L_4 are extremities of latus rectum

Hyperbola	Conjugate hyperbola
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
$e_1=\sqrt{1+\frac{b^2}{a^2}}$	$e_2=\sqrt{1+\frac{a^2}{b^2}}$

Now

$b^2=a^2\left(e_1^2-1\right)$ and $a^2=b^2\left(e_2^2-1\right)$

$$1=\left(e_1^2-1\right)\left(e_2^2-1\right)$$

$$\Rightarrow \quad 1=e_1^2e_2^2-e_1^2-e_2^2+1$$

$$\Rightarrow \quad e_1^2+e_2^2=e_1^2e_2^2 \Rightarrow \frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

Remark(s)

(i) If e_1 and e_2 are eccentricities of hyperbola and conjugate hyperbola then

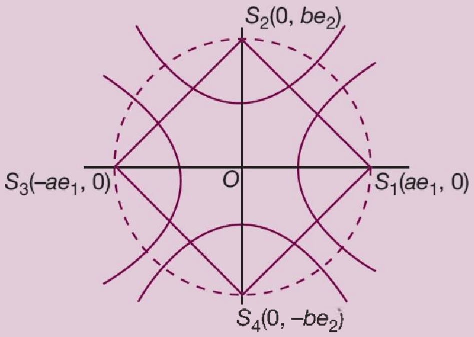
$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

(ii) If the centre of hyperbola is (h, k) and the axes are parallel to the co-ordinates axes,

Transverse axis	Conjugate axis	Hyperbola
Parallel to x-axis	Parallel to y-axis	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$
Parallel to y-axis	Parallel to x-axis	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1$

(iii) Foci of hyperbola and conjugate hyperbola construct a square and they are concyclic.

PROOF:



$$OS_1 = ae_1 = \sqrt{a^2+b^2} \text{ and } OS_2 = be_2 = \sqrt{a^2+b^2}$$

Diagonals are equal and perpendicular.
Hence, it is a square and every square is concyclic.

ILLUSTRATION 10

If eccentricity of hyperbola is sec θ, then find the eccentricity of conjugate hyperbola.

SOLUTION

$e_1 = \sec\theta$; we have $\dfrac{1}{e_1^2} + \dfrac{1}{e_2^2} = 1$

$$\Rightarrow \quad \frac{1}{\sec^2\theta} + \frac{1}{e_2^2} = 1$$

$$\Rightarrow \quad \frac{1}{e_2^2} = 1 - \cos^2\theta = \sin^2\theta$$

$$\Rightarrow \quad e_2 = |\operatorname{cosec}\theta|$$

ILLUSTRATION 11

Find the centre, vertices, eccentricity, directrix, focus and length latus rectum of hyperbola,

$$9x^2 - 16y^2 - 18x + 32y + 137 = 0.$$

SOLUTION

Rewriting the equation

$$9(x-1)^2 - 16(y-1)^2 = -144$$

$$\Rightarrow \quad \frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = -1 \text{ or } \frac{X^2}{4^2} - \frac{Y^2}{3^2} = -1$$

Where $X = x-1$, $Y = y-1$, $a = 4$, $b = 3$

Eccentricity, $e = \sqrt{1+\dfrac{16}{9}} = \dfrac{5}{3}$

Centre, $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$

Hence, Centre (1, 1)

Foci $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$$Y = \pm be \Rightarrow y - 1 = \pm 3\left(\frac{5}{3}\right) \Rightarrow y = 1 \pm 5 \text{ or } y = 6, -4$$

Now, foci are (1, 6) and (1, –4)

Directrix, $Y = \pm\dfrac{b}{e} \Rightarrow y - 1 = \pm\dfrac{9}{5} \Rightarrow y = 1 \pm \dfrac{9}{5}$

$$\Rightarrow \quad y = \frac{14}{5} \text{ or } y = -\frac{4}{5}$$

Conjugate axis, $Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$

Transverse axis, $X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$$\ell_{\text{latus rectum}} = \frac{2a^2}{b} = \frac{2(16)}{3} = \frac{32}{3}$$

ILLUSTRATION 12

Find the centre, eccentricity, foci, directrices and the lengths of the transverse and conjugate axes of the hyperbola, whose equation is $(x-1)^2 - 2(y-2)^2 + 6 = 0$.

SOLUTION

The equation of the hyperbola can be rewritten as $(x-1)^2 - 2(y-2)^2 + 6 = 0$.

$$\Rightarrow -\frac{(x-1)^2}{(\sqrt{6})^2} + \frac{(y-2)^2}{(\sqrt{3})^2} = 1 \Rightarrow \frac{X^2}{(\sqrt{6})^2} - \frac{Y^2}{(\sqrt{3})^2} = -1.$$

Where $Y = (y-2)$ and $X = (x-1)$ ...(1)

$\therefore$ Centre, $X = 0$, $Y = 0$, i.e., (1, 2).

So, $a = \sqrt{6}$ and $b = \sqrt{3}$

Hence, transverse axis $= 2\sqrt{3}$

and conjugate axis $= 2\sqrt{6}$.

$$\Rightarrow \quad e = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{6}{3}} = \sqrt{3}$$

In (X, Y) co-ordinates, foci are $(0, \pm be)$, i.e., $(0, \pm 3)$.

$\therefore$ Foci are $(1+0, 2\pm 3)$, i.e., (1, 5) and (1, –1).

Equations of directrices, $Y = \pm\dfrac{b}{e}$

$$\Rightarrow \quad y - 2 = \pm\frac{\sqrt{3}}{\sqrt{3}} = \pm 1$$

$$\Rightarrow \quad y = 3, y = 1$$

ILLUSTRATION 13

Show that the equation $7y^2 - 9x^2 + 54x - 28y - 116 = 0$ represents a hyperbola. Find the co-ordinates of the centre, lengths of transverse and conjugate axes, eccentricity, latus rectum, co-ordinate of foci, vertices and equations of the directrices of the hyperbola.

SOLUTION

We have, $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

$$\Rightarrow \quad 7(y^2 - 4y) - 9(x^2 - 6x) - 116 = 0$$

$$\Rightarrow \quad 7(y^2 - 4y + 4) - 9(x^2 - 6x + 9) = 116 + 28 - 81$$

$$\Rightarrow \quad 7(y-2)^2 - 9(x-3)^2 = 63$$

$$\Rightarrow \quad \frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$$

Hence, we get, $\frac{Y^2}{9} - \frac{X^2}{7} = 1$ or $\frac{X^2}{7} - \frac{Y^2}{9} = -1$

It is an equation of hyperbola

$\therefore \quad X = x - 3$ and $Y = y - 2$

$\therefore \quad a = \sqrt{7}$ and $b = 3$

Centre: $X = 0, Y = 0$

$\Rightarrow \quad x - 3 = 0, y - 2 = 0 \qquad \therefore C(3, 2)$

Length of transverse axis $= 2b = 6$ units

Length of conjugate axis $= 2a = 2\sqrt{7}$

The eccentricity e is given by

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{7}{9}} = \frac{4}{3}$$

Length of latus rectum $= \frac{2a^2}{b} = \frac{2(7)}{3} = \frac{14}{3}$ units

Foci: The co-ordinates of foci are $(0, \pm be)$

$$\Rightarrow \quad x - 3 = 0, y - 2 = \pm 3 \times \frac{4}{3} = \pm 4$$

$\Rightarrow \quad (3, -2)$ and $(3, 6)$

Vertices: The co-ordinates of vertices are $(0, \pm b)$

$\Rightarrow \quad x - 3 = 0, y - 2 = \pm 3$

$\therefore$ The vertices are $(3, -1)$ and $(3, 5)$

Equation of directrices: $Y = \pm \frac{b}{e}$

$$y - 2 = \pm \frac{3}{4/3}$$

$$\Rightarrow \quad y = \left(2 \pm \frac{9}{4}\right)$$

i.e., $y = \frac{17}{4}$ and $y = -\frac{1}{4}$

ILLUSTRATION 14

Find the equation of hyperbola, whose foci are (8, 3) and (0, 3) and $e = \frac{4}{3}$

SOLUTION

Given foci are (8, 3) and (0, 3) and $e = \frac{4}{3}$

Line joining foci is parallel to x-axis. Hence hyperbola is of type $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$\therefore SS' = 2ae = 8$

$$\Rightarrow \quad a\left(\frac{4}{3}\right) = 4 \Rightarrow a = 3$$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{4}{3} \Rightarrow 1 + \frac{b^2}{9} = \frac{16}{9} \Rightarrow b^2 = 7$$

Midpoint of foci = Centre (4, 3)

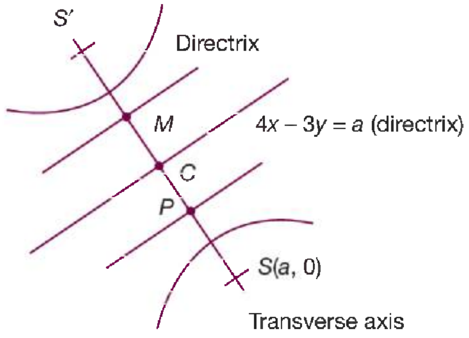
$\therefore$ Equation of required hyperbola is

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1.$$

ILLUSTRATION 15

Find the equation to hyperbola, whose eccentricity is $\frac{5}{4}$, whose focus is $(a, 0)$ and whose directrix is $4x - 3y = a$. Find also the co-ordinates of the centre and the equation to other directrix.

SOLUTION



C ≡ Centre of Hyperbola

Equation of hyperbola,

$$(x-a)^2 + (y-0)^2 = e^2 \frac{(4x-3y-a)^2}{25}$$

$$x^2 - 2ax + a^2 + y^2 = \frac{25}{16}(16x^2 + 9y^2 + a^2 - 24xy - 8ax + 6ay) \times \frac{1}{25}$$

$$\Rightarrow \quad f(x, y) = 7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0 \qquad \ldots(1)$$

To find the centre of the hyperbola differentiate (1)

Differentiating with respect to 'x'

$$\frac{\partial(f(x,y))}{\partial x} = 24y - 24a = 0 \Rightarrow y = a \qquad \ldots(2)$$

Differentiating with respect to 'y'

$$\frac{\partial(f(x,y))}{\partial y} = 14y + 24x - 6a = 0 \qquad \ldots(3)$$

Solving (2) and (3), we get centre C $\left(-\frac{a}{3}, a\right)$

Transverse axis perpendicular to directrix and passing through centre

$$3x + 4y = 3a \qquad \ldots(4)$$

'P' is point of intersection of transverse axis and directrix

$$\therefore \quad P\left(\frac{13a}{25}, \frac{9a}{25}\right)$$

'C' is midpoint of MP

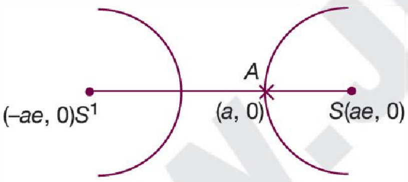
$$\therefore \quad M\left(\frac{-89a}{75}, \frac{41a}{25}\right)$$

$\therefore$ Equation of the other directrix $4x - 3y = \lambda$, passes through the 'M' is $12x - 9y + 29a = 0$.

ILLUSTRATION 16

Obtain the equation of a hyperbola with co-ordinates axes as principal axes given that the distances of one of its vertices from foci are 9 and 1 units.

SOLUTION



CASE-1:

Let the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$AS = ae - a = 1$$
$$AS' = a + ae = 9$$

We get, $2ae = 10 \Rightarrow ae = 5$

$$2a = 8 \Rightarrow a = 4;\ ae = 5$$

$\Rightarrow \quad a^2e^2 = 25$ or $a^2 + b^2 = 25$

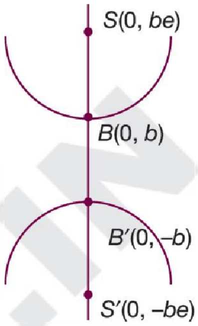
$\Rightarrow \quad 16 + b^2 = 25$

$\Rightarrow \quad b^2 = 9 \Rightarrow b = 3$

$$\text{Hyperbola } \frac{x^2}{16} - \frac{y^2}{9} = 1$$

CASE-2:

Let the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$



$$BS = be - b = 1$$
$$BS' = b + be = 9$$
$$2b = 8 \Rightarrow b = 4$$
$$be = 5 \Rightarrow b^2 + a^2 = 25$$

$\Rightarrow \quad 16 + a^2 = 25 \Rightarrow a^2 = 9$

$\therefore$ The required hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{16} = -1.$$

Remark(s)

- Two hyperbolas are said to be similar, if their eccentricities are equal.
- Two hyperbolas are said to be equal, if they have equal latus rectums.
- If a hyperbola is equilateral (a = b), then the conjugate hyperbola is also equilateral.
- If an ellipse and a hyperbola are orthogonal, then they are confocal (focus of both conics are coincident) and converse is also true.

ILLUSTRATION 17

Find the equation of hyperbola with eccentricity 2 which is confocal with ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

SOLUTION

Foci of ellipse ($\pm$ 4, 0) ($\because$ $a = 5; b = 3; e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$; foci $(\pm ae, 0) = (\pm 4, 0)$)

As both foci lie on x-axis then required

$$\text{Hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ae = 4 \Rightarrow a = 2$$

$$e = 2 = \sqrt{1 + \frac{b^2}{4}} \Rightarrow b^2 = 12$$

$$\therefore \quad \text{Required hyperbola } \frac{x^2}{4} - \frac{y^2}{12} = 1$$

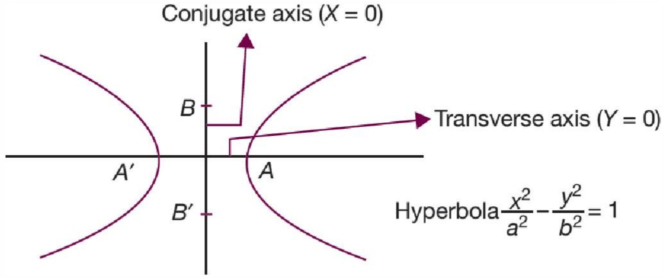
EQUATION OF HYPERBOLA USING PRINCIPAL AXES

Let's closely observe equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can also rewrite it as,

Equation of hyperbola is

$$\left(\frac{\text{Equation of conjugate axis}}{\ell_{\text{semi}TA}}\right)^2-\left(\frac{\text{Equation of transverse axis}}{\ell_{\text{semi}CA}}\right)^2=1$$


Conjugate axis ($X = 0$)
Transverse axis ($Y = 0$)
B
A'
A
B'
Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

NOTE:

Equation of principal axes are written in normalised form

General form $ax + by + c = 0$

Normalised form $\frac{ax+by+c}{\sqrt{a^2+b^2}}=0$

ILLUSTRATION 18

The equation of transverse axis of hyperbola is $x - y + 2 = 0$ and conjugate axis is $x + y - 1 = 0$. If transverse axis and conjugate axis are equal and the hyperbola passes through (1, 2), then find the equation of the hyperbola.

SOLUTION

Required equation of hyperbola is

$$\left(\frac{\frac{x+y-1}{\sqrt{2}}}{a}\right)^2-\left(\frac{\frac{x-y+2}{\sqrt{2}}}{a}\right)^2=1$$

$$\Rightarrow \quad \frac{(x+y-1)^2}{2a^2}-\frac{(x-y+2)^2}{2a^2}=1$$

It passes through (1, 2), $\frac{4}{2a^2}-\frac{1}{2a^2}=1 \Rightarrow 2a^2=3$

$\therefore$ Equation of hyperbola is $(x+y-1)^2-(x-y+2)^2=3$.

Test Your Concepts-I

(Solutions on page H.245)

1. Find the curve, which is represented by $x^2 - y^2 + 5x + 8y - 4 = 0$.
2. If the eccentricity of the hyperbola $x^2 - y^2\sec^2\alpha = 5$. is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then find α.
3. Find eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$.
4. Find the length of the transverse axis of the hyperbola $9x^2-16y^2-18x-32y-151=0$.
5. Find the eccentricity of hyperbola $\frac{x^2}{k}+\frac{y^2}{k^2}=1$ $(k < 0)$.
6. Find hyperbola, having the transverse axis of length $2\sin\theta$ units and is confocal with the ellipse $3x^2+4y^2=12$.
7. The equations of the transverse and conjugate axes of a hyperbola respectively are $x + 2y - 3 = 0$ and $2x - y + 4 = 0$ and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$ units, then find the equation of the hyperbola.
8. If the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then find the value of b^2.
9. If S_1 and S_2 are the foci of the hyperbola, whose transverse axis length is 4 units and conjugate axis length is 6 units and S_3 and S_4 are the foci of the conjugate hyperbola, then find the area of the quadrilateral $S_1S_3S_2S_4$.

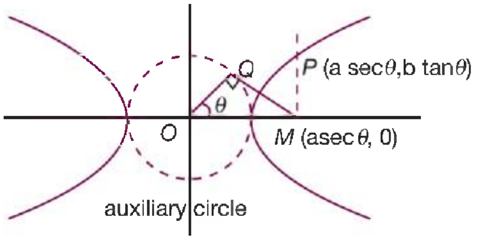
Answer Keys

1. Rectangular hyperbola **2.** $\pm\frac{\pi}{4}$ **3.** $\frac{2}{\sqrt{3}}$ **4.** 8 **5.** $\sqrt{1-\frac{1}{k}}$

6. $\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$ **7.** $\frac{2}{5}(2x-y+4)^2-\frac{3}{5}(x+2y-3)^2=1$ **8.** 16 **9.** 26

AUXILIARY CIRCLE

- A circle whose diameter is transverse axis is auxiliary circle.
- Auxiliary circle and hyperbola are concentric.



Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

Draw PM perpendicular to x-axis.

Let MQ be a tangent to the auxiliary circle $x^2+y^2=a^2$.

Join OQ, so that $\angle QOM=\theta.$

Here P and Q are the corresponding points of the hyperbola and the auxiliary circle. θ is the eccentric angle of P. $(0 \le \theta < 2\pi)$

Eccentric angle (θ) = Angle made by OQ with positive x-axis

Thus, $Q(a\cos\theta, a\sin\theta)$

Now, $x = OM = \frac{OM}{OQ}\cdot OQ = \sec\theta\cdot a$

$\therefore$ $P(x, y)(a\sec\theta, y)$ which lies on the hyperbola

Thus, $\frac{a^2\sec^2\theta}{a^2}-\frac{y^2}{b^2}=1$

$\Rightarrow$ $\frac{y^2}{b^2}=\sec^2\theta-1=\tan^2\theta$

$\therefore$ $y=\pm b\tan\theta$

$\therefore$ $y=b\tan\theta$ (as P lies in Ist quadrant)

The equation $x=a\sec\theta$ and $y=b\tan\theta$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

Conceptual Note(s)

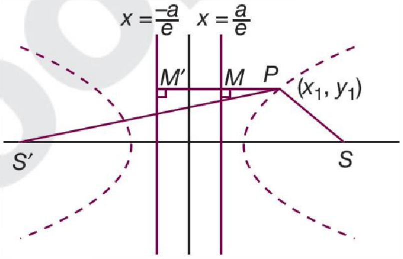
Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	**Hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Auxiliary circle $x^2+y^2=a^2$	Auxiliary circle $x^2+y^2=b^2$
$P(a\sec\theta, b\tan\theta)=P(\theta)$ on hyperbola	$P(a\tan\theta, b\sec\theta)=P(\theta)$ on hyperbola

Hyperbola whose centre is (α, β) and principal axes are parallel to co-ordinate axes

Hyperbola	Parametric Equation
$\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$	$x=\alpha+a\sec\theta$ $y=\beta+b\tan\theta$
$\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=-1$	$x=\alpha+a\tan\theta$ $y=\beta+b\sec\theta$

FOCAL DISTANCE

Distance of a point on hyperbola from focus is known as focal distances.



Let point $P(x_1, y_1)$ lies on hyperbola, then focal distances PS and PS' using definition of conic.

$$PS = ePM = e\left(x_1-\frac{a}{e}\right)=ex_1-a$$

$$PS' = ePM' = e\left(x_1+\frac{a}{e}\right)=ex_1+a$$

Now, from triangle rule, we know that difference of any two sides is less than the third side.

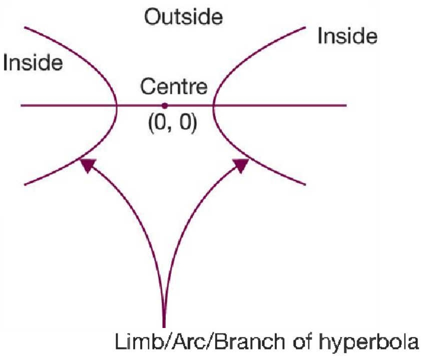
$\Rightarrow$ $|PS-PS'|<SS'$

$\Rightarrow$ $2a<2ae$

$\therefore$ $e>1$, which is true for hyperbola.

LOCATION OF POINT

Let the equation of hyperbola is $S: \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$



For point $P(x_1, y_1)$, we get expression S_1: $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

As we know that centre (0, 0) lies outside the hyperbola. Now we get $S_1 = 0 - 0 - 1 \Rightarrow S_1 < 0$

Condition	Location of point
$S_1 < 0$	Outside
$S_1 = 0$	On the hyperbola
$S_1 > 0$	Inside

NOTE:
In case of hyperbola, condition on S_1 is not same we received in circle/parabola/ellipse.

ILLUSTRATION 19

Find the condition for which (x_1, y_1) lies inside $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

SOLUTION

Centre (0, 0) lies outside the hyperbola.

$$S_1 = 0 - 0 - 1 < 0$$

For inside point, $S_1 > 0$

Hence, we get, $\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2} - 1 > 0$

ILLUSTRATION 20

Find the position of the point (1, 2) and (3, –4) relative to hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$.

SOLUTION

Let $f(x, y) = 2x^2 + 5xy + 2y^2 + 4x + 5y = 0$

For centre,

$$\frac{\partial f(x,y)}{\partial x} = 4x + 5y + 4 = 0$$

$$\frac{\partial f(x,y)}{\partial y} = 5x + 4y + 5 = 0$$

Solving above relations, we get, centre (–1, 0).

As we know centre lies outside of hyperbola,

$$S_1 = 2 + 0 + 0 - 4 + 0 = -2 < 0$$

For point (1, 2),

$S_1 = 2 + 10 + 8 + 4 + 10 = 34 > 0$, which lies inside the hyperbola.

For point (3, – 4),

$S_1 = 18 - 60 + 32 + 12 - 20 = -18 < 0$, which lies outside the hyperbola.

POSITION OF LINE

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Straight line $L: y = mx + c$

Solving line with hyperbola, we get, $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$

$$\Rightarrow \quad b^2x^2 - a^2(m^2x^2 + c^2 + 2mcx) = a^2b^2$$

$$\Rightarrow \quad (b^2 - a^2m^2)x^2 - 2mca^2x - a^2(c^2 + b^2) = 0$$

It is a quadratic equation in x.

Now discriminant $D = 4m^2c^2a^4 + 4a^2(b^2 - a^2m^2)(c^2 + b^2)$

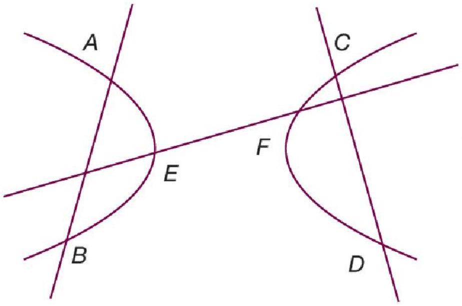
$= 4a^2[m^2c^2a^2 + b^2c^2 + b^4 - a^2m^2c^2 - a^2b^2m^2]$

$= 4a^2b^2[c^2 + b^2 - a^2m^2]$

Working Rule

For chord,	$D > 0$	(when straight line cuts at two distinct points)
For tangent,	$D = 0$	(when straight touches the hyperbola)
For no intersection,	$D < 0$	(when straight line never cut hyperbola)

For tangency, $D = 0$.
We get condition for tangency, $c^2 = a^2m^2 - b^2$



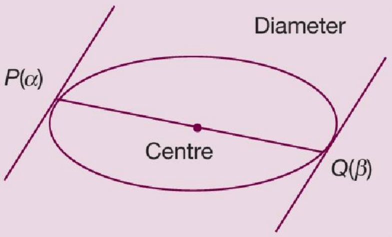
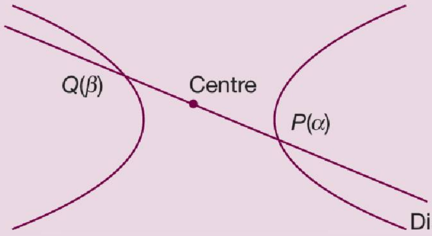
AB, CD, EF are chords

As we know equation of standard ellipse and hyperbola only differ by b^2 and $-b^2$ hence most of the concepts and parameters are similar let's study a comparison chart between ellipse and hyperbola on different concepts.

Working Rule

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	**Hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
$P(a\cos\theta, b\sin\theta)=P(\theta)$	$P(a\sec\theta, b\tan\theta)=P(\theta)$
Tangent $y=mx\pm\sqrt{a^2m^2+b^2}$ at $\left(\mp\frac{a^2m}{c},\pm\frac{b^2}{c}\right)$ where $c=\sqrt{a^2m^2+b^2}$	Tangent $y=mx\pm\sqrt{a^2m^2-b^2}$ at $\left(\mp\frac{a^2m}{c},\mp\frac{b^2}{c}\right)$ where $c=\sqrt{a^2m^2-b^2}$
Point $(x_1, y_1)S_1=0$ lies on ellipse Tangent $T:\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$ Point $(a\cos\theta, b\sin\theta)$ Tangent $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$	Point $(x_1, y_1)S_1=0$ lies on hyperbola Tangent $T:\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$ Point $(a\sec\theta, b\tan\theta)$ Tangent $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$
$(x_1, y_1)S_1>0$ point lies outside ellipse **I^st^ Method** Tangent $y-y_1=m(x-x_1)$ $D=0$ with equation of ellipse We get two values of m_1 and m_2 **II^nd^ Method** Equation of pair of tangents $SS_1=T^2$ where $S:\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ $\quad S_1:\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$	$(x_1, y_1)S_1<0$ point lies outside hyperbola **I^st^ Method** Tangent $y-y_1=m(x-x_1)$ $D=0$ with equation of hyperbola We get two values of m_1 and m_2 **II^nd^ Method** Equation of pair of tangents $SS_1=T^2$ where $S:\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$ $\quad S_1:\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$
Chord of contact T: $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$	Chord of contact T: $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$
Equation of chord with midpoint $M(x_1, y_1)$: $\boldsymbol{T=S_1}$	Equation of chord with mid point $M(x_1, y_1)$: $\boldsymbol{T=S_1}$
Normal : $P(x_1, y_1)$ $\quad \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ $P(a\cos\theta, b\sin\theta)$ $\quad xa\sec\theta-yb\,\mathrm{cosec}\,\theta=a^2-b^2=a^2e^2$	Normal : $P(x_1, y_1)$ $\quad \frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ $P(a\sec\theta, b\tan\theta)$ $\quad xa\cos\theta+yb\cot\theta=a^2+b^2=a^2e^2$

(Continued)

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	**Hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Equation of chord joining the points $P(\alpha)$ and $Q(\beta)$ is given by, $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$	Equation of chord joining the points $P(\alpha)$ and $Q(\beta)$ is given by, $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\frac{\alpha+\beta}{2}$
Point of intersection of the tangents at $P(\alpha)$ and $Q(\beta)$ is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$	Point of intersection of the tangents at $P(\alpha)$ and $Q(\beta)$ is $\left(a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}, b\tan\left(\frac{\alpha+\beta}{2}\right)\right)$
 $\lvert\alpha-\beta\rvert=\pi$	 $\lvert\alpha+\beta\rvert=\pi$
Director circle: $x^2+y^2=a^2+b^2$	Director circle: $x^2+y^2=a^2-b^2$ (If $a>b$, then only it exists)

Common chord: A line joining points of intersection of given two curves.

ILLUSTRATION 21

Find the condition for the line joining the points $P(\phi_1)$ and $Q(\phi_2)$ to be a focal chord.

SOLUTION

Equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

If the chord is focal chord, then it passes through $(ae, 0)$ or $(-ae, 0)$.

Suppose it passes through $(ae, 0)$ then,

$$e\cos\left(\frac{\phi_1-\phi_2}{2}\right)-0=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}=\frac{1}{e}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right)+\cos\left(\frac{\phi_1+\phi_2}{2}\right)}=\frac{1-e}{1+e}$$

$$\Rightarrow \quad \tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right)=\frac{1-e}{1+e}$$

Similarly, it passes through the other foci $(-ae, 0)$.

We get, $\tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right)=\frac{1+e}{1-e}$

ILLUSTRATION 22

If the tangent at the point (2 secθ, 3 tanθ) of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x-y+4=0$, then find the value of θ.

SOLUTION

Equation of tangent at (2 secθ, 3 tanθ)) is

$$\frac{x}{2}\sec\theta-\frac{y}{3}\tan\theta=1 \qquad \ldots(1)$$

Given line is $3x–y+4=0$, both lines are parallel so equating their slopes $\frac{3}{2}\left(\frac{\sec\theta}{\tan\theta}\right)=3$

$$\therefore \quad \sin\theta=\frac{1}{2}\,;\ \theta=30^\circ \text{ or } 150^\circ$$

ILLUSTRATION 23

Show that the chord joining two points, whose sum of parametric angle is $\frac{\pi}{2}$ passes through a fixed point for hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$.

SOLUTION

Equation of chord joining the points $P(\alpha)$ and $Q(\beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$$

Given, $\alpha+\beta=\frac{\pi}{2}$, $a=3$, $b=2$

$$\text{Chord } \frac{x}{3}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{2}\sin 45^\circ=\cos 45^\circ$$

$$\Rightarrow \quad \frac{x}{3}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{2\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}(y+2)-\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{3}x=0$$

We can write it as, $L_1+\lambda L_2=0$

For fixed point, $L_1=0 \Rightarrow y+2=0 \Rightarrow y=-2$

$$L_2=0 \Rightarrow x=0 \Rightarrow x=0$$

$\therefore$ The fixed point is $(0, -2)$.

ILLUSTRATION 24

Find the value of λ for which the line $2x+\lambda=y$ touches the hyperbola $9x^2-5y^2=45$.

SOLUTION

Equation of tangent $y=2x+\lambda$;

Hyperbola $\frac{x^2}{5}-\frac{y^2}{9}=1$

Where $m=2$, $a^2=5$, $b^2=9$

For tangency, $c^2=a^2m^2-b^2$

$$\Rightarrow \quad \lambda^2=5(2)^2-9=20-9=11$$

$$\lambda=\pm\sqrt{11}$$

ILLUSTRATION 25

Find the equation of the tangent to the hyperbola $x^2-4y^2=36$, which is perpendicular to the line $x-y+4=0$.

SOLUTION

Slope of tangent, which is perpendicular to $x-y+4=0$, then we get, $m=-1$

Tangent $y=mx\pm\sqrt{a^2m^2-b^2}$, where $m=-1$, $a=6$, $b=3$

$$y=-x\pm\sqrt{36(-1)^2-9}$$

$$\Rightarrow \quad x+y=\pm 3\sqrt{3}$$

ILLUSTRATION 26

Find the locus of the midpoint of the chord of the circle $x^2+y^2=16$, which are tangents to the hyperbola $9x^2-16y^2=144$.

SOLUTION

Let the midpoint of chord of circle $x^2+y^2=16$ is $M(x_1, y_1)$.

Equation of chord, $T=S_1$

$$\Rightarrow \quad xx_1+yy_1-16=x_1^2+y_1^2-16$$

$$\Rightarrow \quad yy_1=-xx_1+x_1^2+y_1^2$$

$$\Rightarrow \quad y=\left(-\frac{x_1}{y_1}\right)x+\frac{x_1^2+y_1^2}{y_1}=mx+c$$

It is a tangent to hyperbola, $\frac{x^2}{16}-\frac{y^2}{9}=1$

$$\Rightarrow \quad c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad \left(\frac{x_1^2 + y_1^2}{y_1}\right)^2 = 16\left(\frac{-x_1}{y_1}\right)^2 - 9$$

$$(x_1^2 + y_1^2)^2 = 16x_1^2 - 9y_1^2$$

Required locus, $x_1 = x, y_1 = y$

$$(x^2 + y^2)^2 = 16x^2 - 9y^2$$

ILLUSTRATION 27

Find the equations of the tangents to the hyperbola $2x^2 - 3y^2 = 6$, which are parallel to the line $y = 3x + 4$.

SOLUTION

Slope of given line, $m = 3$

Hyperbola $\frac{x^2}{3} - \frac{y^2}{2} = 1$

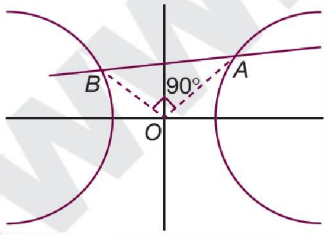
Equation of tangent $y = 3x \pm \sqrt{(3)(9) - 2}$ or $y = 3x \pm 5$

ILLUSTRATION 28

If a variable straight line $x\cos\alpha + y\sin\alpha = p$, which is a chord of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ (b > a)$ subtend a right angle at the centre of the hyperbola, then it always touches a fixed circle. Find its radius.

SOLUTION

Let given line cuts hyperbola at A and B



Homogenising the given straight line with hyperbola, it is combined equation of OA and OB.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$$

$$\Rightarrow \quad x^2\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) - \left(\frac{1}{b^2} - \frac{\sin^2\alpha}{p^2}\right)y^2 - \frac{2\sin\alpha\cos\alpha\ xy}{p^2} = 0$$

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2}$$

$$\Rightarrow \quad p = \frac{ab}{\sqrt{b^2 - a^2}}$$

ILLUSTRATION 29

If tangent at a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the axis at A and B and rectangle $OAPB$ is completed, then find the locus of point P. (When O is the centre of the hyperbola).

SOLUTION

Equation of tangent is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

$x = 0, y = -b\cot\theta \qquad B(0, -b\cot\theta)$

$y = 0, x = a\cos\theta \qquad A(a\cos\theta, 0)$

Now $P(h, k)$ is forming rectangle $OAPB$.

Then $h = a\cos\theta, k = -b\cot\theta$

$$\because \quad \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad \left(\frac{a}{h}\right)^2 - \left(\frac{-b}{k}\right)^2 = 1$$

$$\Rightarrow \quad \frac{a^2}{h^2} - \frac{b^2}{k^2} = 1$$

Locus of $P(h, k)$ is $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

ILLUSTRATION 30

Find the locus of the midpoint of chord of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, which is parallel to $y = 4x + 6$.

SOLUTION

Equation of a chord with midpoint (h, k) to $\frac{x^2}{25} - \frac{y^2}{16} = 1$ is $T = S_1$

$$\text{Or} \quad \frac{xh}{25} - \frac{yk}{16} = \frac{h^2}{25} - \frac{k^2}{16} \qquad \ldots(1)$$

Now slope of line (1) is 4

$$\frac{h}{25}\frac{16}{k} = 4 \Rightarrow h = \frac{25k}{4}$$

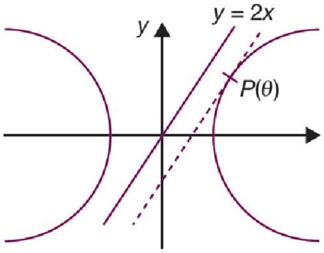
Hence, the locus is $x = \frac{25y}{4}$.

ILLUSTRATION 31

Let P be a point on the hyperbola $x^2 - y^2 = a^2$, where a is a parameter, such that P is nearest to the line $y = 2x$. Find the locus of P.

SOLUTION

Any point on $x^2 - y^2 = a^2$ is P (a secθ, a tanθ). This point will be nearest to $y = 2x$, if tangent at this point is parallel to $y = 2x$.



Equation of tangent at $P(\theta)$

$$x \sec\theta - y \tan\theta = a$$

$$\text{Slope of tangent} = \frac{\sec\theta}{\tan\theta} = \operatorname{cosec}\theta = 2$$

$$\Rightarrow \quad \theta = 30°$$

Thus, $P(a \sec\dfrac{\theta}{6}, a \tan\dfrac{\theta}{6})$

$$\left(\frac{2a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = (h, k)$$

$$h = \frac{2a}{\sqrt{3}} \text{ and } k = \frac{a}{\sqrt{3}} \Rightarrow k = \frac{h}{2}$$

So, the required locus is $2y - x = 0$.

ILLUSTRATION 32

Find the condition for which the line $\ell x + my = n$ is a normal to hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

SOLUTION

Normal at $p(\theta)$ $xa \cos\theta + by \cot\theta = a^2 + b^2$

Given line $\ell x + my = n$ also represent same normal then on comparison

$$\frac{a\cos\theta}{\ell} = \frac{b\cot\theta}{m} = \frac{a^2 + b^2}{n};$$

$$\sec\theta = \frac{an}{\ell(a^2 + b^2)}; \tan\theta = \frac{bn}{m(a^2 + b^2)}$$

$$\Rightarrow \sec^2\theta - \tan^2\theta = 1 \Rightarrow \left(\frac{a^2n^2}{\ell^2} - \frac{b^2n^2}{m^2}\right) = (a^2 + b^2)^2$$

$$\Rightarrow \left(\frac{a^2}{\ell^2} - \frac{b^2}{m^2}\right)n^2 = (a^2 + b^2)^2$$

ILLUSTRATION 33

From a point, four normals are drawn to hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. If eccentric angles of co-normal points are $\alpha, \beta, \gamma, \delta$ then prove that $\alpha + \beta + \gamma + \delta =$ odd multiple of π.

SOLUTION

Normal at $P(\theta)$ $xa\cos\theta + by \cot\theta = a^2 + b^2$

$$xa\left(\frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}\right) + by\left(\frac{1 - \tan^2\theta/2}{2\tan\theta/2}\right) = a^2 + b^2$$

$\Rightarrow$ It passes through $A(h, k)$

$$ha\left(1 - \tan^2\frac{\theta}{2}\right)2\tan\frac{\theta}{2} + bk\left(1 - \tan^4\frac{\theta}{2}\right)$$

$$= (a^2 + b^2)\left(2\tan\frac{\theta}{2} + 2\tan^3\frac{\theta}{2}\right)$$

$$\Rightarrow \quad bk\tan^4\frac{\theta}{2} + (2a^2 + 2b^2 + 2ah)\tan^3\frac{\theta}{2} + 0.\tan^2\frac{\theta}{2}$$

$$+ (2a^2 + 2b^2 - 2ah)\tan\frac{\theta}{2} - bk = 0$$

It has roots $\tan\dfrac{\alpha}{2}, \tan\dfrac{\beta}{2}, \tan\dfrac{\gamma}{2}, \tan\dfrac{\delta}{2}$

Now

$$S_2 = \Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 0$$

$$S_4 = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = -1$$

$$\text{Now } \tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4} \Rightarrow \infty$$

$$\Rightarrow \quad \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} = (2n + 1)\frac{\pi}{2}$$

$$\Rightarrow \quad \alpha + \beta + \gamma + \delta = (2n + 1)\pi$$

ILLUSTRATION 34

Find the locus of the midpoint of the focal chord of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

SOLUTION

Let the midpoint of chord is (x_1, y_1).

Then equation of chord, $T = S_1$

$$\Rightarrow \quad \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

It passes through foci ($\pm ae$, 0)

$$\pm\frac{ex_1}{a}-0=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

Locus of (x_1, y_1) is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm\frac{ex}{a}$

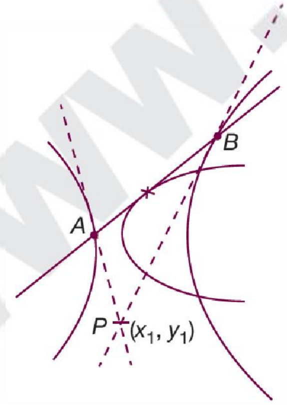
ILLUSTRATION 35

If tangents to the parabola $y^2 = 4ax$ intersects the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at A and B, then find the locus of point of intersection of tangents at A and B.

SOLUTION

From point $P(x_1, y_1)$, tangents are drawn to hyperbola and chord of contact is tangent to parabola.

Now, equation of chord of contact of hyperbola $T = 0$.



$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \Rightarrow \frac{yy_1}{b^2}=\frac{xx_1}{a^2}-1$$

$$\Rightarrow \quad y=\left(\frac{b^2x_1}{a^2y_1}\right)x-\frac{b^2}{y_1}$$

$$y = mx + c$$

For tangency with parabola,

$$c=\frac{a}{m} \Rightarrow -\frac{b^2}{y_1}=\frac{a}{\left(\frac{b^2x_1}{a^2y_1}\right)}$$

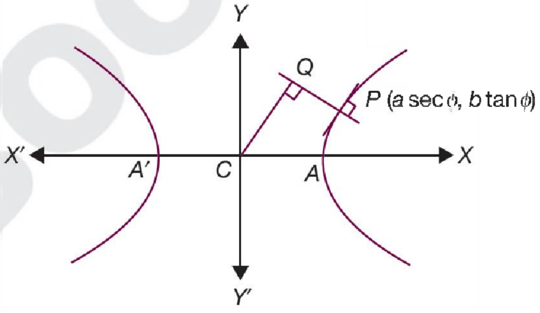
$$\Rightarrow \quad -\frac{b^4x_1}{a^2y_1^2}=a \Rightarrow a^3y_1^2=-b^4x_1$$

$\therefore$ Required locus of $P(x_1, y_1)$ is $a^3y^2 = -b^4x$.

ILLUSTRATION 36

Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

SOLUTION



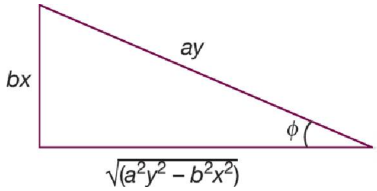
Normal at $P(a \sec\phi, b \tan\phi)$ is

$$ax \cos\phi + by \cot\phi = a^2 + b^2 \qquad \text{...(1)}$$

and equation of line perpendicular to (1) and passing through origin is

$$bx – ay \sin\phi = 0 \Rightarrow \sin\phi=\frac{bx}{dy} \qquad \text{...(2)}$$

Eliminating ϕ from (1) and (2), we will get the equation of locus of Q, as from (2),



Putting value of $\cos\phi$ and $\cot\phi$ from (2) in (1), we get,

$$ax\left(\frac{\sqrt{a^2y^2-b^2x^2}}{ay}\right)+by\left(\frac{\sqrt{(a^2y^2-b^2x^2)}}{bx}\right)=a^2+b^2$$

$$\Rightarrow \quad (x^2+y^2)\sqrt{(a^2y^2-b^2x^2)}=(a^2+b^2)xy$$

$\therefore \quad (x^2+y^2)^2(a^2y^2-b^2x^2)=(a^2+b^2)^2x^2y^2$ is the required locus.

ILLUSTRATION 37

Find the locus of the midpoints of the chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which subtend a right angle at the origin.

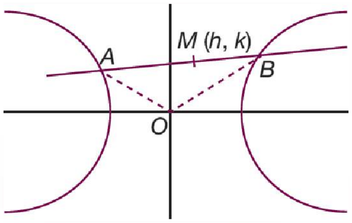
SOLUTION

Let $M(h, k)$ be the midpoint of a chord AB of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Then, its equation is $T = S_1$

$$\Rightarrow \quad \frac{hx}{a^2}-\frac{ky}{b^2}-1=\frac{h^2}{a^2}-\frac{k^2}{b^2}-1$$

$$\Rightarrow \quad \frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2} \qquad \ldots(1)$$



The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (1) are given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{\frac{hx}{a^2}-\frac{ky}{b^2}}{\frac{h^2}{a^2}-\frac{k^2}{b^2}}\right)^2$.

$$\Rightarrow \quad \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2=\left(\frac{hx}{a^2}-\frac{ky}{b^2}\right)^2$$

The lines represented by the above equation will be at right angle, if

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{h^2}{a^4}-\frac{k^2}{b^4}=0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{h^2}{a^4}+\frac{k^2}{b^4}$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}.$$

Test Your Concepts-II

(Solutions on page H.246)

1. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the x-axis at $(9, 0)$, then find the eccentricity of the hyperbola.
2. Find the equation to the common tangents to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.
3. Find the product of focal distances of the point $(4, 3)$ on the hyperbola $x^2 - y^2 = 7$.
4. If P is a point on the rectangular hyperbola $x^2 - y^2 = a^2$, C being the centre and S, S' are two foci, then prove that $SP \cdot S'P = (CP)^2$.
5. If a tangent drawn to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ sq. units, with the co-ordinate axes, then find the square of its eccentricity.
6. If a chord joining $P(a \sec\theta, a\tan\theta)$, $Q(a\sec\alpha, a\tan\alpha)$ on the hyperbola $x^2 - y^2 = a^2$ is the normal at P, then show that $\tan\alpha = \tan\theta(4\sec^2\theta - 1)$.
7. If the line $5x + 12y - 9 = 0$ is a tangent to the hyperbola $x^2 - 9y^2 = 9$, then find its point of contact.
8. Find the foot of the normal $3x + 4y = 7$ to the hyperbola $4x^2 - 3y^2 = 1$.
9. Find the locus of the midpoints of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.

Answer Keys

1. $\sqrt{\frac{3}{2}}$ **2.** $y = \pm x \pm \sqrt{a^2 - b^2}$ **3.** 25 **5.** 17 **7.** 5, –4/3

8. (1, 1) **9.** $3x - 4y = 4$

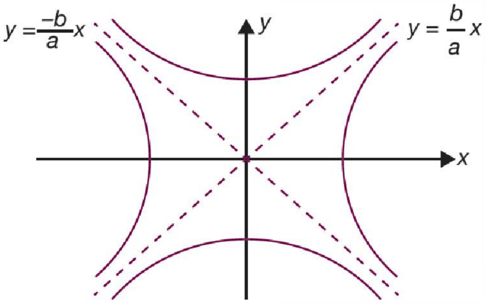
ASYMPTOTES

A line whose distance from a curve approaches to zero as we move towards infinity.

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Tangent, $y=mx\pm\sqrt{a^2m^2-b^2}$ at $\left(\mp\frac{a^2m}{c},\mp\frac{b^2}{c}\right)$

Let's assume that it passes through centre (0, 0) so, $c=0$ that means point of contact is not finite.



Equation of asymptotes $\frac{x}{a}+\frac{y}{b}=0$ and $\frac{x}{a}-\frac{y}{b}=0$

Pair of asymptotes: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

Conceptual Note(s)

(i) Pair of asymptotes always passes through centre.

(ii) Hyperbola and conjugate hyperbola have same pair of asymptotes.

(iii) Principal axes are angle bisectors of pair of asymptotes.

(iv) Equation of hyperbola, conjugate hyperbola, pair of asymptotes are same but only differ in constant term.

Hyperbola: $ax^2+2hxy+by^2+2gx+2fy+c_H=0$

Conjugate hyperbola:

$ax^2+2hxy+by^2+2gx+2fy+c_{CH}=0$

Pair of asymptotes:

$f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c_A=0$

Equation of hyperbola + conjugate hyperbola

= 2 [Equation of Asymptotes]

* $C_A=\frac{C_H+C_{C.H}}{2}$

* To find C_A(constant term in asymptotes)

I

Ist Method

Pair of asymptotes is pair of straight line

$$\Delta=\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c_A\end{vmatrix}=0$$

IInd Method

Pair of asymptotes always passes through centre

$$\left.\begin{aligned}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{aligned}\right\}$$ On solving we get centre

ILLUSTRATION 38

For hyperbola $xy-3y-2x=0$, find the pair of asymptotes.

SOLUTION

Let pair of asymptotes is $xy-3y-2x+\lambda=0$

$$\Delta=\begin{vmatrix} 0 & \frac{1}{2} & -1\\ \frac{1}{2} & 0 & \frac{-3}{2}\\ -1 & \frac{-3}{2} & \lambda\end{vmatrix}=\frac{3}{4}+\frac{3}{4}-\frac{\lambda}{4}=0\Rightarrow\lambda=6$$

$\therefore\quad xy-3y-2x+6=0$

$\Rightarrow\quad (x-3)(y-2)=0$

$\therefore$ Equation of asymptotes are $x=3$ and $y=2$.

ILLUSTRATION 39

A hyperbola whose centre is (1, 2) has asymptotes parallel to $2x+3y=0$, $3x+2y=0$. If hyperbola passes through (5, 3), find the equation of hyperbola and conjugate hyperbola.

SOLUTION

Required asymptotes are $2x+3y+\lambda=0$ and $3x+2y+\mu=0$

It passes through centre (1, 2).

We get $\lambda=-8$ and $\mu=-7$

Equations of asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$

Equation of hyperbola is

$$(2x + 3y - 8)(3x + 2y - 7) + C = 0$$

(5, 3) lies on hyperbola, then

$$11(14) + C = 0 \Rightarrow C = -154$$

Equation of required hyperbola is

$$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$$

Equation of conjugate hyperbola is

$$(2x + 3y - 8)(3x + 2y - 7) + C' = 0$$

$$\because \quad \frac{C' + C}{2} = C_{\text{Asymptotes}} \Rightarrow \frac{C' - 154}{2} = 0 \Rightarrow C' = 154$$

Equation of conjugate hyperbola is

$$(2x + 3y - 8)(3x + 2y - 7) + 154 = 0$$

ILLUSTRATION 40

Find the equation to the hyperbola, conjugate to the hyperbola $2x^2 + 3xy - 2y^2 - 5x + 5y + 2 = 0$.

SOLUTION

Let asymptotes $2x^2 + 3xy - 2y^2 - 5x + 5y + \lambda = 0$

$$\therefore \quad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \quad -4\lambda - \frac{75}{4} + \frac{25}{2} - \frac{25}{2} - \frac{9}{4}\lambda = 0 \Rightarrow \lambda = -3$$

Equation hyperbola + Conjugate hyperbola = 2 (Asymptotes)

$\therefore$ Conjugate hyperbola = 2 (Asymptotes) – Hyperbola

$\therefore$ Required hyperbola $2x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$

ILLUSTRATION 41

Find the equation and angle between the asymptotes of the hyperbola $x^2 + 2xy - 3y^2 + x + 7y + 9 = 0$.

SOLUTION

Let the combined equation of asymptotes is

$$x^2 + 2xy - 3y^2 + x + 7y + \lambda = 0.$$

If it represents pair of straight lines, then,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

$$-3\lambda + \frac{7}{2} + \frac{3}{4} - \frac{49}{4} - \lambda = 0$$

$$\Rightarrow \quad \lambda = -2$$

$$\therefore \quad \text{Asymptotes } x^2 + 2xy - 3y^2 + x + 7y - 2 = 0$$

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{1 + 3}}{1 - 3}\right| = 2$$

Required angle = $\pm \tan^{-1} 2$

ILLUSTRATION 42

The product of the perpendicular from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is $\frac{a^2b^2}{a^2 + b^2}$

SOLUTION

Let $(a\sec\theta, b\tan\theta)$ be the any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The equations of the asymptotes of the given hyperbola are

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0$$

Now, p_1 = Length of the perpendicular from

$(a\sec\theta, b\tan\theta)$ on $\frac{x}{a} - \frac{y}{b} = 0$

$$= \frac{\sec\theta - \tan\theta}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

p_2 = Length of the perpendicular from

$(a\sec\theta, b\tan\theta)$ on $\frac{x}{a} + \frac{y}{b} = 0$

$$= \frac{\sec\theta + \tan\theta}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}},$$

$$\therefore \quad p_1p_2 = \frac{\sec^2\theta - \tan^2\theta}{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{a^2b^2}{a^2 + b^2}$$

ILLUSTRATION 43

Consider the hyperbola, $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find

(i) Centre
(ii) Pair of asymptotes
(iii) Conjugate hyperbola
(iv) Transverse axis
(v) Conjugate axis
(vi) Vertices
(vii) Eccentricity

SOLUTION

Let $f(x,y) = 2x^2 + 5xy + 2y^2 + 4x + 5y = 0$

For centre,

$$\frac{\partial f(x,y)}{\partial x} = 0 \Rightarrow 4x + 5y + 4 = 0$$

$$\frac{\partial f(x,y)}{\partial y} = 0 \Rightarrow 5x + 4y + 5 = 0$$

On solving, we get, centre $(-1, 0)$

Pair of asymptotes, $2x^2 + 5xy + 2y^2 + 4x + 5y + k = 0$

It passes through centre $(-1, 0)$

$$2 + 0 + 0 - 4 + 0 + k = 0$$

$$\Rightarrow \quad k = 2$$

Pair of asymptotes $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

$$\Rightarrow \quad (2x + y + 2)(x + 2y + 1) = 0$$

Conjugate hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y + 4 = 0$

Principal axes = angle bisectors of asymptotes

$$\frac{2x+y+2}{\sqrt{4+1}} = \pm\frac{(x+2y+1)}{\sqrt{1+4}}$$

By taking + sign, we get $x - y + 1 = 0$

By taking − sign, we get $x + y + 1 = 0$

As we know transverse axis cuts hyperbola at vertices but conjugate axis does not, let $x - y + 1 = 0$ is transverse axis solving it with hyperbola

$$2x^2 + 5x(x+1) + 2(x+1)^2 + 4x + 5(x+1) = 0$$

$$\Rightarrow \quad 9x^2 + 18x + 7 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{2}}{3}\,;\ y = \pm\frac{\sqrt{2}}{3}$$

Vertex of hyperbola

$$A\left(-1+\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right) \quad A'\left(-1-\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}\right);\ 2a = AA' = \frac{4}{3}$$

$$= \ell_{\text{Transverse axis of hyperbola}}$$

Solving $x + y + 1 = 0$ with conjugate hyperbola, we get

Vertex of conjugate hyperbola

$$B\left(-1+\sqrt{2}, -\sqrt{2}\right) B'\left(-1-\sqrt{2}, \sqrt{2}\right) \ 2b = 4$$

$$= \ell_{\text{Conjugate axis of hyperbola}}$$

Principal axes $x - y + 1 = 0$

Transverse axis of hyperbola and conjugate axis of conjugate hyperbola

$x + y + 1 = 0$ conjugate axis of hyperbola and transverse axis of conjugate hyperbola

$$e_H = \sqrt{10}\ ;\ e_{CH} = \frac{\sqrt{10}}{3}$$

ILLUSTRATION 44

The tangent to a hyperbola makes with the asymptotes a triangle of area 6 sq. units and the length of the latus rectum is $\frac{1}{3}$ units. Find the equation of the hyperbola.

SOLUTION

Let the equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Equation of asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

The equation of the tangent at $(a \sec\theta, b \tan\theta)$ is $bx\sec\theta - ay\tan\theta = ab$.

The co-ordinate of the point of intersection of tangent and the asymptotes are

$$P\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right) \text{and } Q\left(\frac{a\cos\theta}{1+\sin\theta}, \frac{-b\cos\theta}{1+\sin\theta}\right)$$

The asymptotes meet in origin (0, 0).

Area of the ΔOPQ =

$$\frac{1}{2}\left[\frac{a\cos\theta}{1+\sin\theta}\cdot\frac{b\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1-\sin\theta}\cdot\frac{b\cos\theta}{1+\sin\theta}\right]$$

$$= \frac{1}{2}\cdot 2\,ab = ab \text{ sq. units}$$

We have, $ab = 6$ and $\frac{2b^2}{a} = \frac{1}{3}$

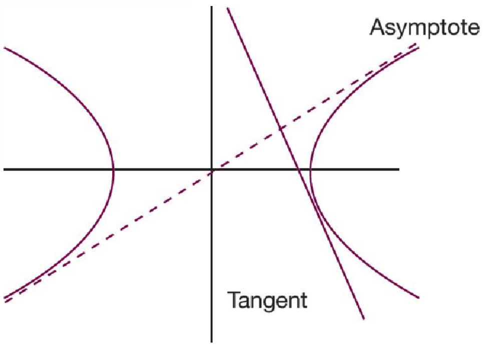
$$\Rightarrow \quad a = 6, b = 1$$

Equation is $\frac{x^2}{36} - y^2 = 1$.

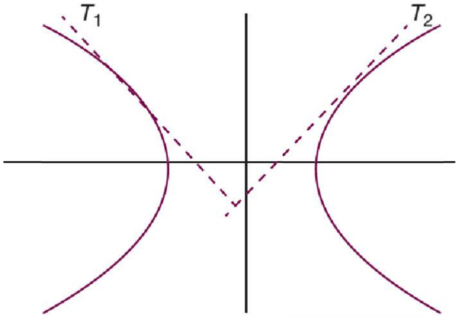
PROPERTIES OF HYPERBOLA AND ASYMPTOTES

(a) No tangents can be drawn to hyperbola from centre.

(b) From any point on asymptotes except centre, only one tangent can be drawn to hyperbola.

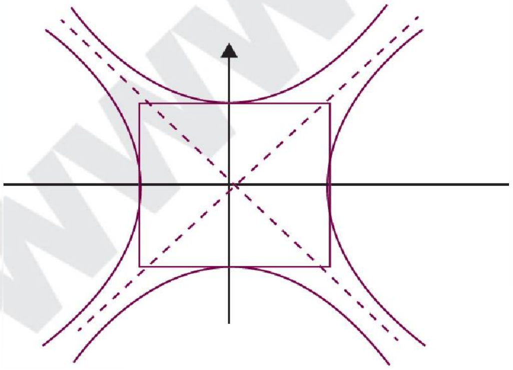


(c) From a point which lies outside the hyperbola and not on any of asymptotes, then two tangents are drawn to hyperbola.

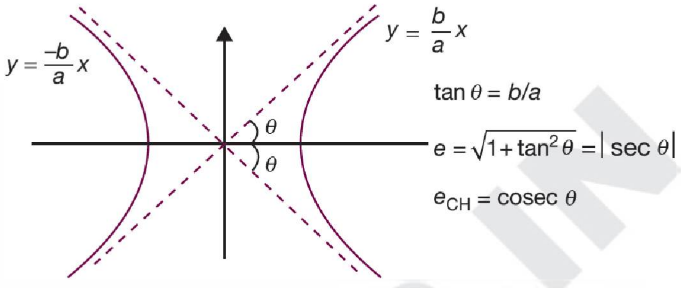


(d) A line drawn parallel to asymptotes intersect hyperbola only at one point. In other words, if a line intersects hyperbola only at one point, then it is parallel to an asymptote.

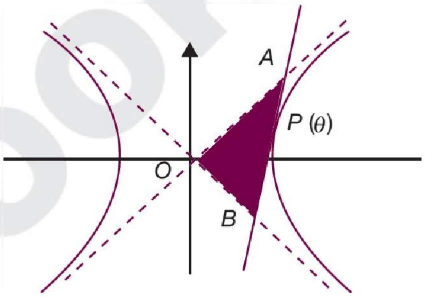
(e) Asymptotes are diagonals of rectangle formed by lines drawn parallel to principal axes and passing through extremities of principal axes.



(f) If angle between asymptotes is 2θ, then eccentricity of hyperbola can be $\sec\theta$ or $\operatorname{cosec}\theta$


$y = \frac{-b}{a}x$
$y = \frac{b}{a}x$
$\tan\theta = b/a$
$e = \sqrt{1+\tan^2\theta} = |\sec\theta|$
$e_{CH} = \operatorname{cosec}\theta$
θ
θ

(g) Area of triangle formed by any tangent on hyperbola and pair of asymptotes is constant and segment of tangent intercept between asymptotes is bisected at point of contact.



PROOF:

Tangent at $P(\theta)$ is $\frac{x}{a}\sec\theta + \frac{y}{b}\tan\theta = 1$

Now get points of intersection of tangent with asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ as A and B respectively.

$$A(a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta))$$

and $B(a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta))$

Midpoint of AB = Point of contact (P)

Area of ΔOAB

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a(\sec\theta + \tan\theta) & b(\sec\theta + \tan\theta) & 1 \\ a(\sec\theta - \tan\theta) & -b(\sec\theta - \tan\theta) & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left| -ab(\sec^2\theta - \tan^2\theta) - ab(\sec^2\theta - \tan^2\theta) \right|$$

$$= ab$$

$$\Delta OAB = \ell_{\text{semiTA}} \times \ell_{\text{semiCA}} = ab = \text{Constant}$$

Diameter: The locus of the midpoints of a system of parallel chords of the hyperbola with slope m is $y=\frac{b^2}{a^2m}x$.

PROOF:

Let mid point of chord (h, k)

Equation of chord $T=S_1$

$$\frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}$$

Slope of chord $= m = \frac{b^2h}{a^2k}$

Locus of $P(h, k)$ $m=\frac{b^2x}{a^2y} \Rightarrow y=\frac{b^2}{a^2m}x$

If m_1 and m_2 be the slope of the conjugate diameters of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $m_1m_2=\frac{b^2}{a^2}$.

ILLUSTRATION 45

Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that CP and CD are conjugate diameters of the hyperbola, where D lies on conjugate hyperbola and C is centre then show that the co-ordinates of D are $(a\tan\theta, b\sec\theta)$.

SOLUTION

The equation of CP is

$$y=\frac{b\tan\theta}{a\sec\theta}x \Rightarrow y=\left(\frac{b}{a}\sin\theta\right)x \qquad \ldots(1)$$

Let $y = mx$ be the diameter conjugate to (1).

Then $m_1m_2=\frac{b^2}{a^2}$

$$\Rightarrow \quad m\times\left(\frac{b}{a}\sin\theta\right)=\frac{b^2}{a^2}$$

$$\Rightarrow \quad m=\frac{b}{a}\operatorname{cosec}\theta$$

So, the equation of CD is $y=\left(\frac{b}{a}\operatorname{cosec}\theta\right)x$

This cuts the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ at the points, whose abscissae are given by

$$\frac{x^2}{a^2}-\frac{\operatorname{cosec}^2\theta}{a^2}x^2=-1$$

$$\Rightarrow \quad x^2(\operatorname{cosec}^2\theta-1)=a^2$$

$$\Rightarrow \quad x=a\tan\theta$$

Putting $x=a\tan\theta$ in $y=\left(\frac{b}{a\sin\theta}\right)x$, we get, $y=b\sec\theta$.

Hence, the co-ordinates of D are $(a\tan\theta, b\sec\theta)$.

ILLUSTRATION 46

If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D, respectively, then show that $CP^2-CD^2=a^2-b^2$ where C is centre.

SOLUTION

Since CP and CD are conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Therefore, the co-ordinates of P and D are $(a\sec\theta, b\tan\theta)$ and $(a\tan\theta, b\sec\theta)$, respectively.

Now, $CP^2-CD^2=(a^2\sec^2\theta+b^2\tan^2\theta)-(a^2\tan^2\theta+b^2\sec^2\theta)$

$$=a^2(\sec^2\theta-\tan^2\theta)-b^2(\sec^2\theta-\tan^2\theta)$$

$$=a^2-b^2$$

ILLUSTRATION 47

If CP and CD represent a pair of conjugate diameters of the hyperbola $x^2-y^2=a^2$, then prove that ΔCPD is isosceles and has constant area.

SOLUTION

Since, CP and CD are conjugate diameters of the hyperbola $x^2-y^2=a^2$. So, the co-ordinates of P and D are $(a\sec\theta, a\tan\theta)$ and $(a\tan\theta, a\sec\theta)$, respectively.

$$\therefore \quad CP^2=a^2\sec^2\theta+a^2\tan^2\theta \text{ and}$$

$$CD^2=a^2\tan^2\theta+a^2\sec^2\theta$$

$$\Rightarrow \quad CP^2=CD^2 \Rightarrow CP=CD$$

$\therefore$ ΔCPD is an isosceles triangle.

Now,

Δ = Area of ΔCPD

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\sec\theta & a\tan\theta & 1 \\ a\tan\theta & a\sec\theta & 1 \end{vmatrix} = \frac{1}{2}(ab) \text{ sq. units}$$

ILLUSTRATION 48

Show that the parallelogram formed by the tangents at the extremities of conjugate diameters of a hyperbola has its vertices on the asymptotes and is of constant area.

SOLUTION

The co-ordinates are $P(a\sec\phi, b\tan\phi)$, $D(a\tan\phi, b\sec\phi)$, $P'(-a\sec\phi, -b\tan\phi)$ and $D'(-a\tan\phi, -b\sec\phi)$.

The asymptotes Q and CR are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

The equation of tangents at P, P', D and D' are

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1 \quad \ldots(1)$$

$$-\frac{x}{a}\sec\phi + \frac{y}{b}\tan\phi = 1 \quad \ldots(2)$$

$$\frac{x}{a}\tan\phi + \frac{y}{b}\sec\phi = 1 \quad \ldots(3)$$

$$-\frac{x}{a}\tan\phi + \frac{y}{b}\sec\phi = 1 \quad \ldots(4)$$

Area of parallelogram $QRQ'R'$ = 4(Area of parallelogram $QDCP$)

$= 4 \times 2$ Area of $\Delta\,(CDP)$

$$= 8\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\sec\phi & b\sec\phi & 1 \\ a\sec\phi & b\sec\phi & 1 \end{vmatrix} = 2a \times 2b = 4ab \text{ sq. units}$$

Test Your Concepts-III

(Solutions on page H.246)

1. If e is the eccentricity of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is the angle between the asymptotes then find $\cos\frac{\theta}{2}$.

2. From a point $P(1, 2)$, a pair of tangents are drawn to a hyperbola 'H', in which one tangent to each arm of hyperbola. Equations of asymptotes of hyperbola H are $\sqrt{3}x - y + 5 = 0$ and $\sqrt{3}x + y - 1 = 0$, then find the eccentricity of 'H'.

3. The pair of asymptotes of $\frac{x^2}{10} - \frac{y^2}{4} = 1$ and the pair of asymptotes of $\frac{x^2}{10} - \frac{y^2}{4} = -1$ coincide. State whether true or false.

4. If the equation to the hyperbola is $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$, then find equation to the conjugate hyperbola.

5. The asymptotes of a hyperbola are $3x - 4y + 2 = 0$ and $5x + 12y - 4 = 0$. If the hyperbola passes through the point (1, 2), then find the slope of transverse axis of the hyperbola.

6. The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle, whose area is $a^2\tan\lambda$ in magnitude, then find its eccentricity.

7. If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ from its asymptotes is equal to 6, then find the length of the transverse axis of the hyperbola.

8. Find the area of the triangle formed by the asymptotes and any tangent to the hyperbola $x^2 - y^2 = a^2$.

9. Find the combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$.

10. If angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 45°, then find the value of eccentricity e.

11. Find the area of triangle formed by the lines $x-y=0, x+y=0$ and any tangent to the hyperbola $x^2-y^2=a^2$.

12. If two distinct tangents can be drawn from the point $(\alpha, 2)$ to different branches of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$, then find α.

Answer Keys

1. $1/e$
2. $\frac{2}{\sqrt{3}}$
3. True
4. $3x^2-5xy-2y^2+5x+11y-16=0$
5. -8
6. $\sec\lambda$
7. 6
8. a^2
9. $2x^2+5xy+2y^2+4x+5y+2=0$
10. $\sqrt{4-2\sqrt{2}}$
11. a^2
12. $\left(\frac{-3}{2},\frac{3}{2}\right)$

RECTANGULAR HYPERBOLA: (EQUILATERAL HYPERBOLA)

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

If $a=b$, then it is known as rectangular hyperbola $x^2-y^2=a^2; e=\sqrt{2}$

Asymptotes $y=\pm\frac{b}{a}x$ or $y=\pm x$

Referred to the transverse and conjugate axes along the axes of co-ordinates, the equation of the rectangular hyperbola is

$$x^2-y^2=a^2 \qquad \ldots(1)$$

The asymptotes of hyperbola are $y=x$ and $y=-x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis. so, if we rotate the co-ordinate axes through an angle of $-\frac{\pi}{4}$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola.

Therefore, we have, $x=X\cos\left(-\frac{\pi}{4}\right)-Y\sin\left(-\frac{\pi}{4}\right)=\frac{X+Y}{\sqrt{2}}$

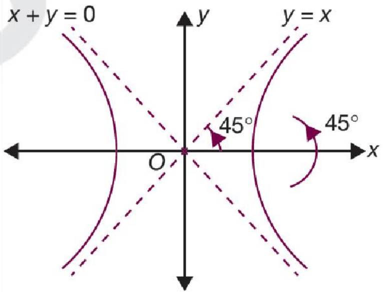
$$y=X\sin\left(-\frac{\pi}{4}\right)+Y\cos\left(-\frac{\pi}{4}\right)=\frac{Y-X}{\sqrt{2}}$$

Substituting the values of x and y in (1), we obtain,

$$\left(\frac{X+Y}{\sqrt{2}}\right)^2-\left(\frac{Y-X}{\sqrt{2}}\right)^2=a^2$$

$$\Rightarrow \quad XY=\frac{a^2}{2}$$

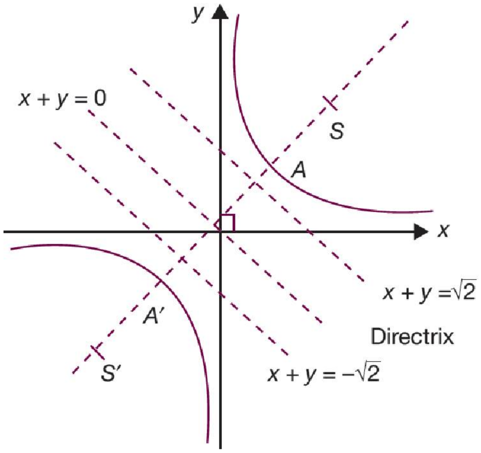
$$\Rightarrow \quad XY=c^2, \text{ where } c^2=\frac{a^2}{2}$$



Thus, the equation of the hyperbola referred to its asymptotes as the co-ordinate axes is

$$xy=c^2, \text{ where } c^2=\frac{a^2}{2}.$$

RECTANGULAR HYPERBOLA $xy=c^2$



Transverse axis: $y = x$
Conjugate axis: $x + y = 0$
Asymptotes $x = 0, y = 0$
Vertex A, A' $(\pm c, \pm c)$

Foci $S, S'\left(\pm\sqrt{2}c, \pm\sqrt{2}c\right)$

Length of Transverse axis = length of conjugate axis

$$2a = 2b = 2\sqrt{2}\,c$$

$$\Rightarrow \quad a = b = \sqrt{2}\,c$$

Directrix: $x + y = \pm\sqrt{2}c$

Any point on hyperbola is $\left(ct, \frac{c}{t}\right)$.

In this section, we shall discuss various forms of tangent and normal to a rectangular hyperbola which will be used to solve the related problems on tangents to a rectangular hyperbola.

Tangent

(i) Point Form

The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

(ii) Parametric Form

The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$.

Normal

(i) Point Form

The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(ii) Parametric Form

The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

PROOF:

Tangent: $T = 0$, $\frac{1}{2}\left(x\frac{c}{t} + cty\right) = c^2$, $\frac{x}{t} + yt = 2c$

$$m_T = \frac{-1}{t^2}, \qquad m_N = t^2$$

Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

$$\Rightarrow \quad t^3x - yt + c - ct^4 = 0$$

NOTE:

If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$, then equation of hyperbola is

$$(x - \alpha)(y - \beta) = c^2 \text{ or } xy - \alpha y - \beta x + \lambda = 0$$

ILLUSTRATION 49

Find eccentricity of the hyperbola, whose asymptotes are given by $3x + 2y - 5 = 0$ and $2x - 3y - 5 = 0$.

SOLUTION

The asymptotes $3x + 2y = 5$ and $2x - 3y = 5$ are perpendicular to each other.

So, eccentricity = $\sqrt{2}$.

ILLUSTRATION 50

Consider the system of hyperbola $xy = k$, $k \in R$. Let e_1 be the eccentricity, when $k = 4$ and e_1 be the eccentricity, when $k = 9$. Then, find $e_1 - e_2$.

SOLUTION

For hyperbola $xy = k$, it has constant $e = \sqrt{2}$

(i) $k = 4$, $e_1 = \sqrt{2}$

(ii) $k = 9$, $e_2 = \sqrt{2}$

We get $e_1 - e_2 = 0$

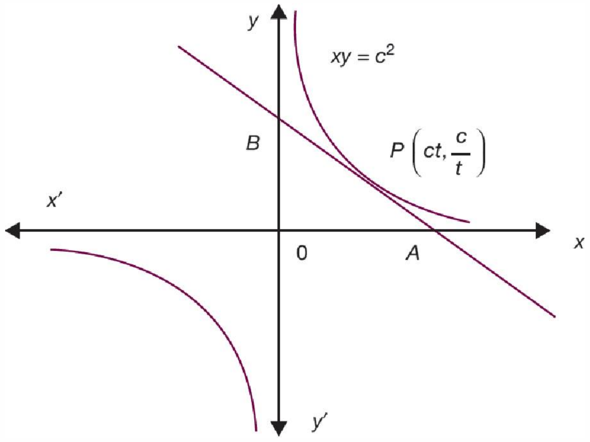
ILLUSTRATION 51

Prove that the portion of the tangent to the rectangular hyperbola intercepted between the asymptotes is bisected at the point of contact and the area of the triangle formed by the tangent and the two asymptotes is constant.

SOLUTION

The equation of the rectangular hyperbola referred to its asymptotes as the co-ordinates axes is

$$xy = c^2$$



The equation of the tangent at any point $P\left(ct, \frac{c}{t}\right)$ is

$$\frac{x}{t} + yt = 2c$$

It cuts the co-ordinate axes at $A(2ct, 0)$ and $B\left(0, \frac{2c}{t}\right)$.

Clearly, AB is bisected at P.

Now,

$$\text{Area of } \Delta OAB = \frac{1}{2}(OA \times OB)$$

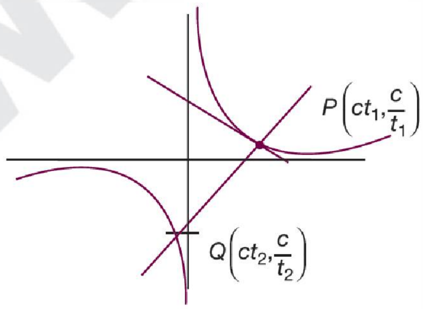
$$= \frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2 = \text{Constant}$$

ILLUSTRATION 52

If normal to hyperbola $xy = c^2$ at point t_1 cuts curve again at t_2 then prove that $t_2 = -\frac{1}{t_1^3}$.

SOLUTION

Since the equation of normal at $\left(ct_1, \frac{c}{t_1}\right)$ to the hyperbola $xt_1^3 - yt_1 - ct_1^4 + c = 0$


$P\left(ct_1, \frac{c}{t_1}\right)$
$Q\left(ct_2, \frac{c}{t_2}\right)$

However, this passes through $\left(ct_2, \frac{c}{t_2}\right)$.

Then, $ct_2t_1^3 - \frac{c}{t_2}t_1 - ct_1^4 + c = 0$

$$\Rightarrow \quad t_2^2\, t_1^3 - t_1 - t_1^4 t_2 + t_2 = 0$$

$$\Rightarrow \quad t_2 t_1^3 (t_2 - t_1) + (t_2 - t_1) = 0$$

$$\Rightarrow \quad (t_1^3 t_2 + 1)\,(t_2 - t_1) = 0$$

$$\Rightarrow \quad t_1^3 t_2 + 1 = 0 \qquad [\because t_2 \neq t_1]$$

$$\therefore \quad t_1^3 t_2 = -1 \Rightarrow t_2 = -\frac{1}{t_1^3}$$

ILLUSTRATION 53

If the normal at t_1, t_2, t_3 meet on the hyperbola itself show that the centroid of the triangle joining t_1, t_2, t_3 is centre of hyperbola itself.

SOLUTION

Let all normal meet at $\left(ct_0, \frac{c}{t_0}\right)$ on hyperbola. Then,

$$t_0 = -\frac{1}{t^3} \Rightarrow t^3 t_0 + 1 = 0$$

Roots are t_1, t_2, t_3

Now, $t_1 + t_2 + t_3 = 0$ and $\sum t_1 t_2 = 0$

$$\text{Centroid} \left(\frac{\sum ct_1}{3}, \frac{\sum \frac{c}{t_1}}{3}\right) = (0, 0)$$

ILLUSTRATION 54

If a circle cuts hyperbola $xy = c^2$ at four points t_1, t_2, t_3, t_4 then prove that $t_1t_2t_3t_4 = 1$.

SOLUTION

Let Circle $x^2 + y^2 + 2gx + 2fy + c' = 0$

Point $\left(ct, \frac{c}{t}\right)$ on hyperbola also lies on circle.

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + c' = 0$$

$$\Rightarrow \quad c^2t^4 + 2gct^3 + c^1t^2 + 2fct + c^2 = 0\,;$$

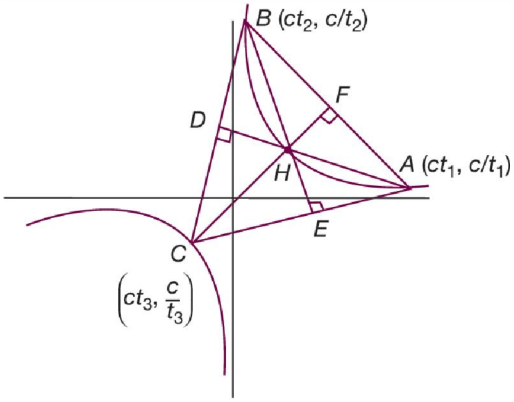
Circle cuts hyperbola at four points t_1, t_2, t_3 and t_4.

$t_1t_2t_3t_4 = 1$ Hence proved

ILLUSTRATION 55

If three points $A, B, C\left(ct_i, \dfrac{c}{t_i}\right) i = 1, 2, 3$ are on hyperbola $xy = c^2$, then prove that orthocenter of ΔABC also lies on hyperbola.

SOLUTION


B (ct2, c/t2)
F
D
H
A (ct1, c/t1)
E
C
(ct3, c/t3)

Three points A, B, C lie on hyperbola,

$$m_{AB} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} = \frac{-1}{t_1t_2}; \quad m_{CF} = t_1t_2$$

Altitude $CF \quad y - \dfrac{c}{t_3} = t_1t_2(x - ct_3)$

$\Rightarrow \quad t_3y - c = t_1t_2t_3x - ct_1t_2t_3^{\,2}$

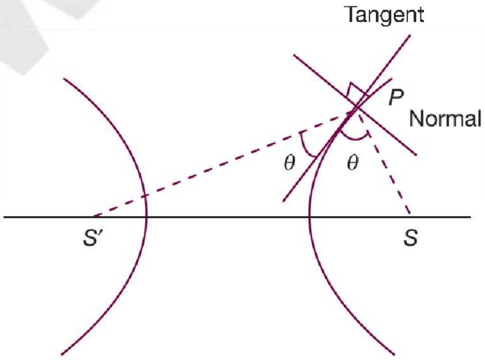
Altitude $AD \quad t_1y - c = t_1t_2t_3x - ct_1^2t_2t_3$

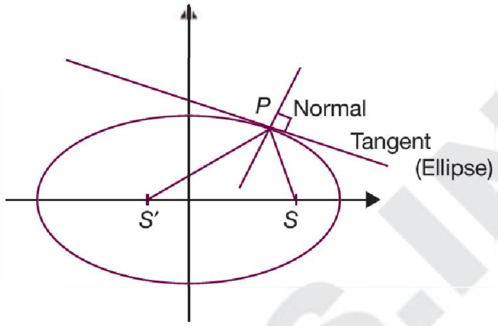
Orthocenter $H\left(\dfrac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right) H$ lies on hyperbola

Highlights of Hyperbola

(i) $|PS - PS'| = 2a = \ell_{T.A.} = \text{constant}$

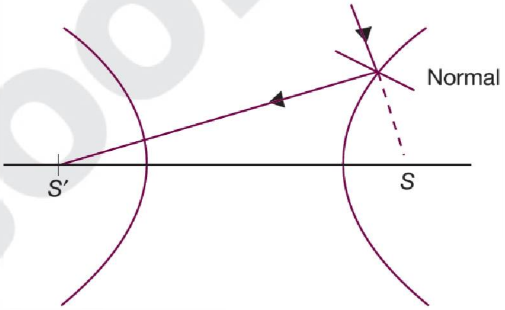
(ii) Angle bisector of focal distances PS and PS' = Tangent and normal at P.


Tangent
P
Normal
θ
θ
S′
S

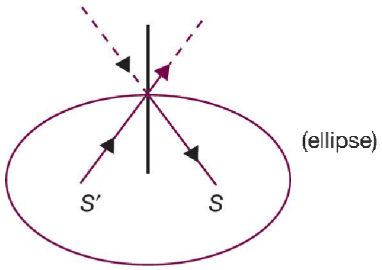

P
Normal
Tangent
(Ellipse)
S′
S

Ellipse also follows same property.

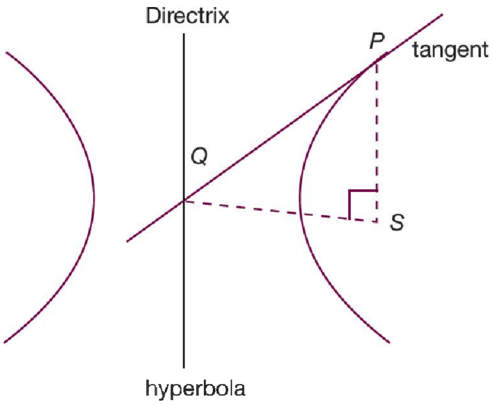
(iii) A ray travelling towards a focus, after reflection from hyperbolic mirror move towards other focus.



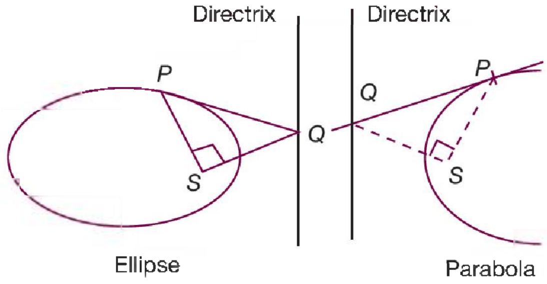
Ellipse also follows same property.



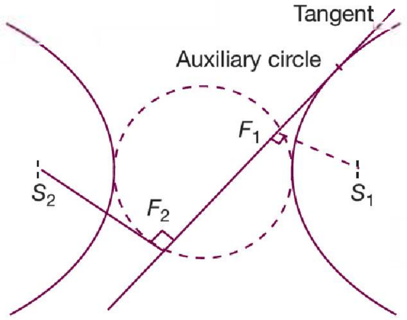
(iv) Segment of any tangent between point of contact and directrix subtends right angle at corresponding focus.


Directrix
P
tangent
Q
S
hyperbola

Ellipse and parabola also follow this property.

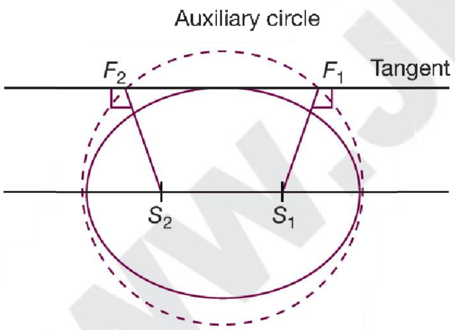

Directrix
Directrix
P
Q
S
Q
P
S
Ellipse
Parabola

(v) Feet of perpendicular drawn from foci to any tangent lie on auxiliary circle and product of length of perpendiculars from both foci to any tangent is square of semi-conjugate axis.


Tangent
Auxiliary circle
F_1
S_2
F_2
S_1

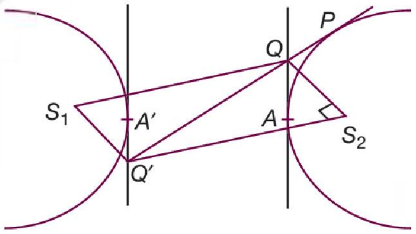
$$S_1F_1 \cdot S_2F_2 = \ell^2_{\text{semi-conjugate axis}} = b^2$$

Ellipse also follows the same property.

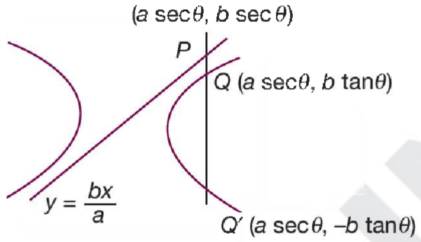

Auxiliary circle
F_2
F_1
Tangent
S_2
S_1

$$S_1F_1 \cdot S_2F_2 = \ell^2_{\text{semi-minor axis}} = b^2 (a > b)$$

(vi) The portion of tangent intercepted between the tangents at vertices subtends right angle on foci. So, the points Q, S_1, Q', S_2 are concyclic.



(vii) If any double ordinate QQ' intersect the any asymptote at P, then $PQ \cdot PQ' = b^2$.


(a secθ, b secθ)
P
Q (a secθ, b tanθ)
$y = \frac{bx}{a}$
Q′ (a secθ, –b tanθ)

$$PQ = b(\sec\theta - \tan\theta)$$
$$PQ' = b(\sec\theta + \tan\theta)$$
$$PQ \cdot PQ' = b^2$$

We can rewrite as, from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi-conjugate axis.

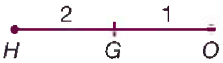
ILLUSTRATION 56

If a given circle of variable radius intersects the hyperbola $x^2 - y^2 = a^2$ at four points $PQRS$, then find the locus of centroid of ΔPQR.

SOLUTION

Let centroid (h, k), circumcentre $(-g, -f)$ is given, let orthocenter be (α, β).

$\Rightarrow \quad \alpha = 3h + 2g, \beta = 3k + 2f$



$\therefore$ Orthocenter of ΔPQR lies on hyperbola.

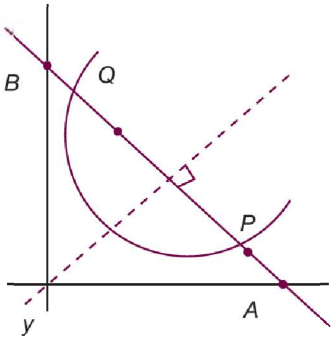
So, we get, $(3h + 2g)^2 - (3k + 2f)^2 = a^2$

$\therefore$ Required locus is $(3x + 2g)^2 - (3y + 2f)^2 = a^2$

ILLUSTRATION 57

If the line $x + y = c$ meets the hyperbola $xy = 1$ at two points P and Q and the axes at A and B, respectively such that P is close to A (on the x-axis). If $PA = BQ = 1$, then find AB.

SOLUTION



$\therefore \quad AP \cdot AQ = b^2 \; (AP = BQ = 1)$

$\Rightarrow \quad AQ = b^2$

$\quad\;\; BP \cdot BQ = b^2$

$\Rightarrow \quad BP = b^2 \quad (\because b^2 = 2)$

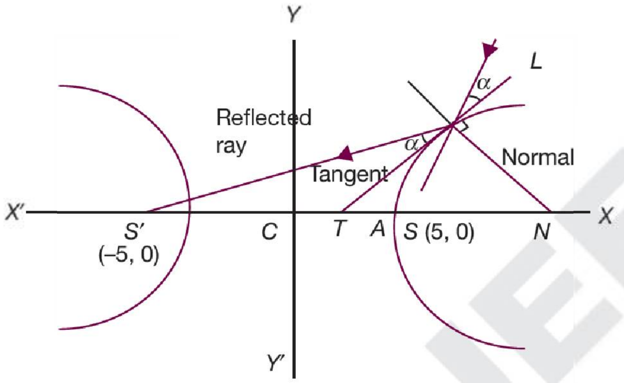
$\Rightarrow \quad BP = 2$

$\quad\;\; BQ = 1, AQ = 2, AP = 1, AB = 3$

ILLUSTRATION 58

A ray travelling towards the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lies in Ist quadrant.

SOLUTION



Given hyperbola is $9x^2 - 16y^2 = 144$, has foci $S(5, 0)$ and $S'(-5, 0)$

This equation can be rewritten as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \ldots(1)$$

Since, x-co-ordinate of P is 8.

Let y-co-ordinate of P is α.

$\therefore \quad (8, \alpha)$ lies on (1)

$\therefore \quad \dfrac{64}{16} - \dfrac{\alpha^2}{9} = 1 \Rightarrow \alpha^2 = 27$

$\Rightarrow \quad \alpha = 3\sqrt{3} \qquad (\because P$ lies in Ist quadrant)

Hence co-ordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passing through $P\left(8, 3\sqrt{3}\right)$ and $S'(-5, 0)$, its equation is $3\sqrt{3}x - 13y + 15\sqrt{3} = 0$

Test Your Concepts-IV

(Solutions on page H.247)

1. Find co-ordinates of foci of the hyperbola $xy = 4$.
2. Find the equation of latus rectum of the rectangular hyperbola $xy = 2$.
3. If a right angled triangle is inscribed in the rectangular hyperbola $xy = c^2$, then find the angle between the tangent at the right angled vertex and the hypotenuse of the triangle.
4. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then show that a and b are of opposite sign.
5. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Find the locus of the point, which divides the line segment between these two points in the ratio 1:2.
6. Find the locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent.
7. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, and $S(x_4, y_4)$ are 4 concyclic points on the rectangular hyperbola $xy = c^2$, then find co-ordinates of the orthocentre of the ΔPQR.
8. Find the number of normal(s) of a rectangular hyperbola, which can touch its conjugate.
9. Find the length of the transverse axis of the rectangular hyperbola $xy = 18$.
10. Find area of the triangle formed by any arbitrary tangent of the hyperbola $xy = c^2$, with co-ordinate axes.

11. In the hyperbola $xy = 4$, then find locus of mid-points of chords of constant length 2 units.

12. A rectangular hyperbola whose centre is C is cut by any circle of radius r in four points P, Q, R and S, then find $CP^2 + CQ^2 + CR^2 + CS^2$.

13. If the point $(3\tan(\theta + 60°), 2\tan(\theta + 30°))$ lies on the hyperbola, then find its centre.

Answer Keys

1. $(2\sqrt{2}, 2\sqrt{2}), (-2\sqrt{2}, -2\sqrt{2})$

2. $x + y = 4$

3. 90°

5. $16x^2 + 10xy + y^2 = 2$

6. $(x^2 + y^2)^2 = 4c^2xy$

7. $(-x_4, -y_4)$

8. 4

9. 12

10. $2c^2$

11. $(x^2 + y^2)(xy - 4) = xy$

12. $4r^2$

13. $(-3\sqrt{3}, 2\sqrt{3})$

SOLVED PROBLEMS

PROBLEM 1

Find the equation and the length of the common tangents to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

SOLUTION

Tangent at (a secϕ, b tanϕ) on the Ist hyperbola is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \qquad \ldots(1)$$

Similarly, tangent at any point (b tanθ, a secθ) on IInd hyperbola is

$$\frac{y}{a}\sec\theta-\frac{x}{b}\tan\theta=1 \qquad \ldots(2)$$

If (1) and (2) are common tangents, then they should be identical. Comparing the co-efficient of x and y, we get,

$$\Rightarrow \quad \frac{\sec\theta}{a}=-\frac{\tan\phi}{b} \text{ or } \sec\theta=-\frac{a}{b}\tan\phi \qquad \ldots(3)$$

$$-\frac{\tan\theta}{b}=\frac{\sec\phi}{a} \text{ or } \tan\theta=-\frac{b}{a}\sec\phi \qquad \ldots(4)$$

As $\sec^2\theta-\tan^2\theta=1$

$$\Rightarrow \quad \frac{a^2}{b^2}\tan^2\phi-\frac{b^2}{a^2}\sec^2\phi=1 \qquad \text{(From (3) and (4))}$$

$$\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)\tan^2\phi=1+\frac{b^2}{a^2}$$

$$\Rightarrow \quad \tan^2\phi=\frac{b^2}{a^2-b^2} \text{ and } \sec^2\phi=\frac{a^2}{a^2-b^2}$$

Hence, the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right) \text{ and}$$

$$\left(\mp\frac{b^2}{\sqrt{(a^2-b^2)}}, \mp\frac{a^2}{\sqrt{(a^2-b^2)}}\right) \text{(From (3) and (4))}$$

Length of common tangent, i.e., the distance between the above points is $\sqrt{2}\frac{(a^2+b^2)}{\sqrt{(a^2-b^2)}}$ units and equation of common tangent on putting the values of secϕ and tanϕ in (1) is

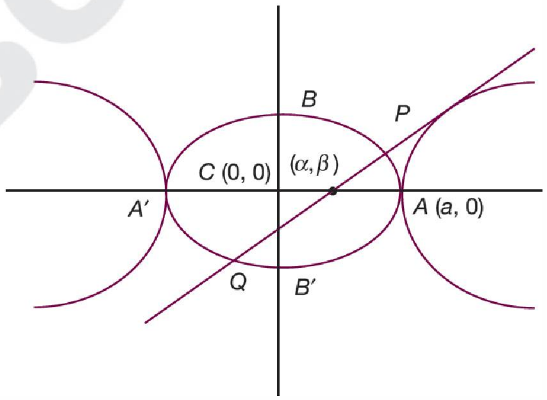
$$\pm\frac{x}{\sqrt{(a^2-b^2)}}\mp\frac{y}{\sqrt{(a^2-b^2)}}=1$$

$$\Rightarrow \quad x\mp y=\pm\sqrt{(a^2-b^2)}$$

PROBLEM 2

A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points P and Q. Find the locus of the midpoint of PQ.

SOLUTION



Let (α, β) be the midpoint of PQ. The equation of the chord of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with (α, β) as its midpoint is

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}$$

Or $$y=-\frac{b^2\alpha}{a^2\beta}x+\frac{b^2}{\beta}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}\right) \qquad \ldots(1)$$

The equation of the general tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

then $a^2m^2-b^2=c^2 \qquad \ldots(2)$

$$\Rightarrow \quad a^2\cdot\frac{b^4\alpha^2}{a^4\beta^2}-b^2=\frac{b^4}{\beta^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}\right)^2$$

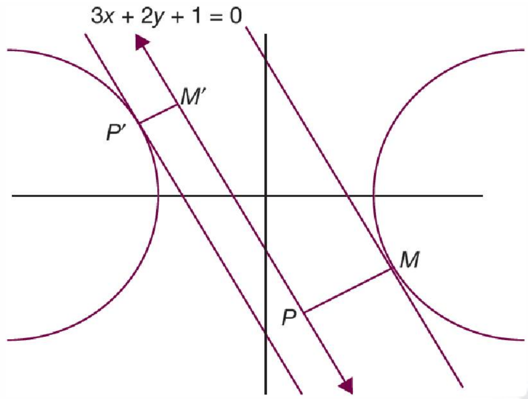
Hence, locus of (α, β) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$

PROBLEM 3

Find the point of the hyperbola $3x^2 - 4y^2 = 72$, which is nearest to the line $3x + 2y + 1 = 0$ and compare the distance between the point and line.

SOLUTION



The hyperbola is $\frac{x^2}{24}-\frac{y^2}{18}=1$

Any point on the hyperbola is $P(a \sec\theta, b \tan\theta)$

Slope of normal at this point is $-\frac{a}{b}\sin\theta$

$$\Rightarrow \quad -\frac{\sqrt{24}}{\sqrt{18}}\sin\theta$$

Now, the shortest distance occurs along the common normal.

$$\therefore \quad -\sqrt{\frac{4}{3}}\sin\theta=\frac{2}{3} \text{ or } \sin\theta=-\frac{1}{\sqrt{3}}$$

$$\therefore \quad \sec\theta=\pm\frac{\sqrt{3}}{\sqrt{2}} \text{ and } \tan\theta=\pm\frac{1}{\sqrt{2}}$$

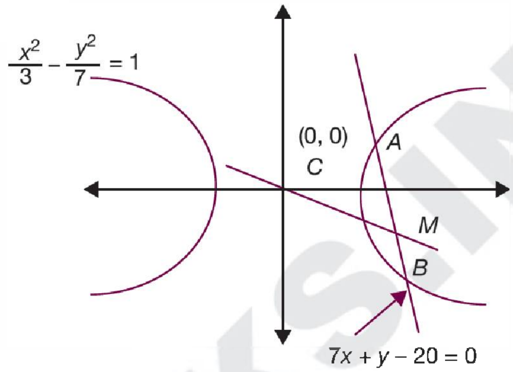
$\therefore$ Points on hyperbola are $(-6, 3)$ or $(6, -3)$

From this, $(-6, 3)$ is the nearest point to the line.

PROBLEM 4

Find the equation of the line passing through origin and bisects the chord $7x + y - 20 = 0$ of the hyperbola $\frac{x^2}{3}-\frac{y^2}{7}=1$.

SOLUTION



The ends $A(x_1, y_1)$ and $B(x_2, y_2)$ of the chords are found by solving

$$7x + y - 20 = 0 \text{ and } \frac{x^2}{3}-\frac{y^2}{7}=1.$$

We get, $\frac{x^2}{3}-\frac{1}{7}(20-7x)^2=1$

$\Rightarrow \quad 140x^2 - 840x + 1221 = 0$

Its roots are x_1 and x_2

So, $\quad x_1 + x_2 = \frac{840}{140} = 6$

Also, $y = 20 - 7x$

$\Rightarrow \quad y_1 + y_2 = 40 - 7(x_1 + x_2) = -2$

The midpoint of the chord is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) =$ $M(3, -1)$

Now the equation of the line OM is $y = -\frac{1}{3}x$.

PROBLEM 5

Chords of contact of a point P w.r.t the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and its auxiliary circle $x^2 + y^2 = a^2$ are at right angle. Prove that P lies on one of the asymptotes.

SOLUTION

Let P be (x_1, y_1), then the equation of chords of contact are

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \text{ and } xx_1 + yy_1 = a^2$$

They are perpendicular. So, $\left(\frac{x_1}{a^2}\cdot\frac{b^2}{y_1}\right)\left(-\frac{x_1}{y_1}\right)=-1$

That is, $b^2 x_1^2 = a^2 y_1^2$ or, $\left(\frac{x_1^2}{a^2}\right) - \left(\frac{y_1^2}{b^2}\right) = 0$

∴ Locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ Hence proved.

PROBLEM 6

Tangents are drawn from point P on the curve $x^2 - 4y^2 = 4$ to the curve $x^2 + 4y^2 = 4$ touching it in the points Q and R. Prove that the midpoint of QR lies on $\frac{x^2}{4} - y^2 \left(\frac{x^2}{4} + y^2\right)^2$.

SOLUTION

Any point on the curve $x^2 - 4y^2 = 4$ can be expressed as $(2\sec\theta, \tan\theta)$.

Let (h, k) be the midpoint of QR, which is a chord to the curve $x^2 + 4y^2 = 4$

Equation of QR is $T = S_1$

$$\Rightarrow \quad \frac{hx}{4} + ky = \frac{h^2}{4} + k^2$$

QR is also the chord of contact to the curve $x^2 + 4y^2 = 4$ w.r.t the point $(2\sec\theta, \tan\theta)$.

∴ Equation of chord of contact of $(2\cos\theta, \tan\theta)$ w.r.t $x^2 + 4y^2 = 4$ is $\sec\theta \cdot \frac{x}{2} + \tan\theta\, y = 1$.

Comparing the two equation

$$\sec\theta = \frac{h}{2\left(\frac{h^2}{4} + k^2\right)}; \tan\theta = \frac{k}{\left(\frac{h^2}{4} + k^2\right)}$$

Eliminating θ, $\sec^2\theta - \tan^2\theta = 1$

$$\frac{h^2}{4\left(\frac{h^2}{4} + k^2\right)^2} - \frac{k^2}{\left(\frac{h^2}{4} + k^2\right)^2} = 1$$

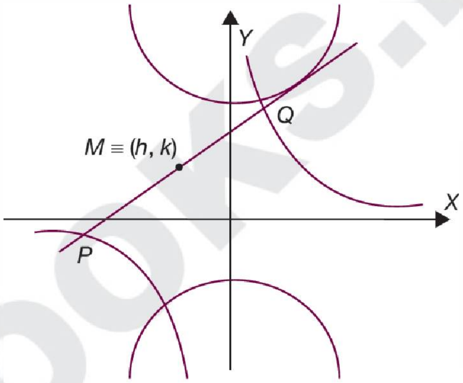
$$\Rightarrow \quad \frac{h^2}{4} - k^2 = \left(\frac{h^2}{4} + k^2\right)^2$$

∴ Required locus of (h, k) is $\frac{x^2}{4} - y^2 = \left(\frac{x^2}{4} + y^2\right)^2$

PROBLEM 7

A chord PQ of the hyperbola $xy = c^2$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. Find the locus of the mid-point of PQ.

SOLUTION



Equation of chord PQ of the hyperbola is $T = S_1$

$$\Rightarrow \quad \frac{1}{2}(hy + xk) - c^2 = hk - c^2$$

$$\Rightarrow \quad y = \left(-\frac{k}{h}\right)x + 2k$$

⇒ Condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $c^2 = b^2 - a^2m^2$.

$$\Rightarrow \quad (2k)^2 = b^2 - a^2\left(-\frac{k}{h}\right)^2$$

$$\Rightarrow \quad 4h^2k^2 = b^2h^2 - a^2k^2$$

⇒ Locus of $M(h, k)$ is $\frac{b^2}{y^2} - \frac{a^2}{x^2} = 4$

PROBLEM 8

Find the locus of a point, which moves such that the chord of contact of this point with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION

Let (h, k) be the variable point, so its chord of contact with respect to the hyperbola is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \qquad \ldots(1)$$

This is a normal to the ellipse.
Equation of the normal to the ellipse is

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2 \qquad \ldots(2)$$

From (1) and (2),

$$\frac{h\cos\theta}{a^3} = \frac{k\sin\theta}{b^3} = \frac{1}{(a^2 - b^2)}$$

$\because \quad \cos^2\theta + \sin^2\theta = 1$

$\therefore \quad (a^2 - b^2)^2 = \dfrac{a^6}{h^2} + \dfrac{b^6}{k^2}$

Locus is $\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2} = (a^2 - b^2)^2$

PROBLEM 9

Find the locus of a point, whose chord of contact touches the circle described on the straight line joining the foci of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ as diameter.

SOLUTION

Circle on the join of foci $(ae, 0)$ and $(-ae, 0)$ as diameter is $(x - ax)(x + ae) + (y - 0)(y - 0) = 0$.

$\Rightarrow \quad x^2 + y^2 = a^2e^2 = a^2 + b^2 \qquad \ldots(1)$

$(\because a^2e^2 = a^2 + b^2)$

Let chord of contact of $P(x_1, y_1)$ touch the circle (1).
Equation of chord of contact of P is $(T = 0)$.

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$\Rightarrow \quad b^2x_1x - a^2y_1y - a^2b^2 = 0 \qquad \ldots(2)$

$\therefore \quad \dfrac{a^2 b^2}{\sqrt{(b^4 x_1^2 + a^4 y_1^2)}} = \pm\sqrt{(a^2 + b^2)}$

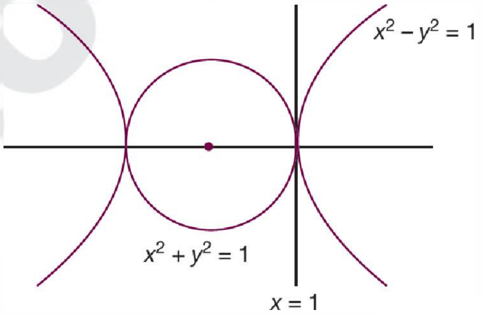
Hence, locus of $P(x_1, y_1)$ is $(b^4x^2 + a^4y^2)(a^2 + b^2) = a^4b^4$.

PROBLEM 10

An ellipse has eccentricity $\dfrac{1}{2}$ and one focus at the point $P\left(\dfrac{1}{2}, 1\right)$. It's one directrix is the common tangent, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$, nearer to P. Then, find the equation of the ellipse in the standard form.

SOLUTION

The circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$ touch each other at the points $(\pm 1, 0)$ and the common tangent at these points are $x = \pm 1$. Since $x = 1$ is near to the focus $P\left(\dfrac{1}{2}, 1\right)$, this is the directrix of the required ellipse.



Now required ellipse

$$PS = ePM$$

$$\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \frac{1}{4}(x-1)^2$$

$\Rightarrow \quad 4x^2 - 4x + 1 + 4(y - 1)^2 = x^2 - 2x + 1$

$\Rightarrow \quad 3x^2 - 2x + 4(y - 1)^2 = 0$

$\Rightarrow \quad 3\left(x - \dfrac{1}{3}\right)^2 + 4(y-1)^2 = \dfrac{1}{3}$

$\Rightarrow \quad 9\left(x - \dfrac{1}{3}\right)^2 + 12(y-1)^2 = 1$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. Which of the following is incorrect for the hyperbola $x^2 - 2y^2 - 2x + 8y - 1 = 0$?

(A) Its eccentricity is $\sqrt{2}$

(B) Length of the transverse axis is $2\sqrt{3}$ units

(C) Length of the conjugate axis is $2\sqrt{6}$ units

(D) Latus rectum $4\sqrt{3}$

2. The eccentricity of the conic defined by $\left|\sqrt{(x-1)^2+(y-2)^2} - \sqrt{(x-5)^2+(y-5)^2}\right| = 3$ is:

(A) $\frac{5}{2}$ (B) $\frac{5}{3}$

(C) $\sqrt{2}$ (D) $\frac{\sqrt{11}}{3}$

3. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is:

(A) 1 (B) 5

(C) 7 (D) 9

4. Chords of the parabola $y^2 = 4x$ touch the hyperbola $x^2 - y^2 = 1$. The locus of the point of intersection of the tangents drawn to the parabola at the extremities of such chords is:

(A) a circle

(B) a parabola

(C) an ellipse

(D) a rectangular hyperbola

5. The normal at 'P' be a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M, respectively. If locus of the midpoint of LM is hyperbola, then eccentricity of the hyperbola is:

(A) $\frac{e+1}{e-1}$ (B) $\frac{e}{\sqrt{e^2-1}}$

(C) e (D) $\frac{2e}{\sqrt{e^2-1}}$

6. If a ray of light incident along the line $3x + \left(5 - 4\sqrt{2}\right)y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then its reflected ray goes along the line:

(A) $\sqrt{2}x - y + 5 = 0$

(B) $\sqrt{2}y - x + 5 = 0$

(C) $\sqrt{2}y - x - 5 = 0$

(D) $3x - y\left(4\sqrt{2} + 5\right) + 15 = 0$

7. If the eccentricity of the hyperbola $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 5$ is 5 times the eccentricity of the ellipse $x^2 \operatorname{cosec}^2\theta + y^2 \sec^2\theta = 25$, then θ equals:

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) $\cot^{-1}\left(\pm\sqrt{\frac{2}{3}}\right)$ (D) $\tan^{-1}\left(\frac{4}{5}\right)$

8. If m is the slope of a tangent to the hyperbola $\frac{x^2}{a^2-b^2} - \frac{y^2}{a^3-b^3} = 1$, where $a > b > 1$, then:

(A) $(a+b)m^2 + ab \ge (a+b)^2$

(B) $(a+b)^2 m + ab \ge (a+b)$

(C) $ab\,m^2 + (a+b) \ge (a+b)^2$

(D) $(a+b)m + a^2b^2 \ge (a+b)^2$

9. $S(y^2 - 4x + 4 = 0)$ and $S' = 0$ is a hyperbola, whose asymptotes are the tangents drawn from origin to $S = 0$ and length of transverse axis is equal to length of latus rectum of $S = 0$. If S, S' intersect at two points A and B, then the distance of AB is:

(A) $2\sqrt{3}$ (B) $4\sqrt{3}$

(C) $6\sqrt{3}$ (D) 8

10. Let $P(x_1,y_1), Q(x_2,y_2)$ be any two points on a rectangular hyperbola, whose asymptotes are co-ordinate axes. If $R(x_3,y_3)$ is the point of intersection of tangents at P and Q to the hyperbola, then:

(A) x_3 is the A.M. between x_1 and x_2

(B) x_3 is the H.M. between x_1 and x_2

(C) x_3 is the G.M. between x_1 and x_2

(D) y_3 is the G.M. between y_1 and y_2

11. The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$ such that the portion of the tangent intercepted by the hyperbola $\alpha^2 x^2 - y^2 = 1$ subtends a right angle at the centre of the curve is:

(A) $\left[\frac{\sqrt{5}+1}{2}, 2\right]$

(B) $(1, 2)$

(C) $\left[\frac{\sqrt{5}-1}{2}, 2\right)$

(D) $\left[\frac{\sqrt{5}-1}{2}, 1\right) \cup \left(1, \frac{\sqrt{5}+1}{2}\right]$

12. A normal to the hyperbola $x^2 - 4y^2 = 4$ has equal intercepts on positive x and y axes. If this normal touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $a^2 + b^2$ is equal to:

(A) 5 (B) 16

(C) $\frac{25}{9}$ (D) $\frac{16}{9}$

13. Shortest distance between the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $4x^2 + 4y^2 = a^2 (b > a)$ is:

(A) $\frac{b}{2}$ (B) $\frac{b}{\sqrt{2}}$

(C) $\frac{a}{2}$ (D) $\frac{a}{\sqrt{2}}$

14. The equation of a hyperbola, which is conjugate to the hyperbola $x^2 + 3xy + 2y^2 + 2x + 3y = 0$ is:

(A) $x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$

(B) $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

(C) $x^2 + 3xy + 2y^2 + 2x + 3y + 3 = 0$

(D) $x^2 + 3xy + 2y^2 + 2x + 3y + 4 = 0$

15. Find the eccentricity of the conic formed by the locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$.

(A) 4 (B) $\frac{1}{4}$

(C) 2 (D) $\frac{1}{2}$

16. Let $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ be the points of a rectangular hyperbola $xy = c^2$, which subtends 90° at another point 'R' of the given hyperbola, if 'S' be the midpoint of PQ and 'O' centre of the hyperbola then area of the ΔROS is equal to:

(A) $\frac{c^2\left(1+(t_1t_2)^2\right)|t_1+t_2|}{4|t_1t_2|\sqrt{|t_1t_2|}}$

(B) $\frac{c^2|t_1+t_2|}{4|t_1t_2|\sqrt{|t_1t_2|}}$

(C) $\frac{c^2|1+t_1t_2||t_1+t_2|}{2|t_1t_2|\sqrt{|t_1t_2|}}$

(D) none of these

17. Locus of the midpoints of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is:

(A) $y + mx = 0$ (B) $y - mx = 0$

(C) $my - x = 0$ (D) $my + x = 0$

18. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT \cdot ON$ is equal to:

(A) e^2 (B) a^2

(C) b^2 (D) b^2/a^2

19. If PN is the perpendicular from a point on a rectangular hyperbola $x^2 - y^2 = a^2$ on any of its asymptotes, then the locus of the midpoint of PN is:

(A) a circle (B) a parabola
(C) an ellipse (D) a hyperbola

20. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A, C is the midpoint of PQ and 'O' is the origin. Then the ΔACO is:

(A) equilateral (B) isosceles
(C) right angled (D) right isosceles

21. A conic passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the co-ordinate axes is bisected at the point of tangency. Then the foci of the conic are:

(A) $\left(2\sqrt{2}, 0\right)$ and $\left(-2\sqrt{2}, 0\right)$

(B) $\left(2\sqrt{2}, 2\sqrt{2}\right)$ and $\left(-2\sqrt{2}, -2\sqrt{2}\right)$

(C) (4, 4) and (–4, –4)

(D) $\left(4\sqrt{2}, 4\sqrt{2}\right)$ and $\left(-4\sqrt{2}, -4\sqrt{2}\right)$

22. A straight line which is normal to the parabola $y^2 = 4ax$ is tangent to the hyperbola $x^2 - y^2 = a^2$. The number of such lines possible is:

(A) 1 (B) 2
(C) 6 (D) none of these

23. With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is:

(A) less than 2 (B) 2
(C) $\frac{11}{3}$ (D) none of these

24. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies:

(A) $e > \sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$

(C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$

25. $P(x, y)$ satisfies $x^2 + y^2 = 1$. If the maximum value of $(x+y)^2$ is λ, then number of tangents drawn from the point $(\lambda, 0)$ to hyperbola $(x-2)^2 - y^2 = 1$ is:

(A) 0 (B) 1
(C) 2 (D) 4

26. The tangent to the hyperbola $xy = c^2$ at the point P intersects the x-axis at T and the y-axis at T'. The normal to the hyperbola at P intersects the x-axis at N and the y-axis at N'. The areas of the triangles PNT and $PN'T'$ are Δ and Δ', respectively, then $\frac{1}{\Delta} + \frac{1}{\Delta'}$ is:

(A) equal to 1 (B) depends on t
(C) depends on c (D) equal to 2

27. If the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ have the same foci and they intersect at right angles, then the equation of the circle through the points of intersection of two conics is:

(A) $x^2 + y^2 = 5$

(B) $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$

(C) $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$

(D) $x^2 + y^2 = 25$

28. At the point of intersection of the rectangular hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$ tangents to the rectangular hyperbola and the parabola make an angle θ and ϕ, respectively with the axis of X, then:

(A) $\theta = \tan^{-1}(-2\tan\phi)$ (B) $\phi = \tan^{-1}(-2\tan\theta)$

(C) $\theta = \frac{1}{2}\tan^{-1}(-\tan\phi)$ (D) $\phi = \frac{1}{2}\tan^{-1}(-\tan\theta)$

29. Area of the quadrilateral formed with the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is:

(A) $4(a^2 + b^2)$ (B) $2(a^2 + b^2)$

(C) $(a^2 + b^2)$ (D) $\frac{1}{2}(a^2 + b^2)$

30. A tangent to the circle $x^2 + y^2 = 4$ intersects the hyperbola $x^2 - 2y^2 = 2$ at P and Q. If the locus of midpoint of PQ is $(x^2 - 2y^2)^2 = \lambda\,(x^2 + 4y^2)$, then λ equals:

(A) 2 (B) 4
(C) 6 (D) 8

31. r is the radius of a circle, whose centre is the origin. If the circle does not intersect the hyperbola $xy = 1$, then the range of r lies in the interval:

(A) (0, 1) (B) $\left(0, \sqrt{2}\right)$

(C) $\left(0, \frac{1}{\sqrt{2}}\right)$ (D) (0, 2)

32. A line is such that its intercepts on the co-ordinate axes are e and e', where e and e' are the eccentricities of a hyperbola and its conjugate hyperbola. If this line is touching a fixed circle, then the area of the circle is:

(A) $\frac{\pi}{2}$ sq. units (B) π sq. units

(C) $\frac{3\pi}{2}$ sq. units (D) 2π sq. units

33. If the asymptote of the hyperbola $(x + y + 1)^2 - (x - y - 3)^2 = 5$ cuts each other at A and the co-ordinate axis at B and C, then the radius of circle passing through the points A, B, C is:

(A) 3 (B) $\frac{\sqrt{5}}{2}$

(C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{3}$

34. If a variable straight line with slope $m(m \neq 0)$ intersects the hyperbola $xy = 1$ at two distinct points, then the locus of the point which divides the line segment between these two points in the ratio 1:2 is:

(A) an ellipse (B) a hyperbola

(C) a circle (D) a parabola

35. If two tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{9}=1$, having slopes 2 and m, where $(m \neq 2)$ cuts the axes at four concyclic points, then the slope m is/are:

(A) $-\frac{1}{2}$ (B) -2

(C) $\frac{1}{2}$ (D) 2

36. A hyperbola passing through origin has $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its asymptotes. Then the equations of its transverse and conjugate axis are:

(A) $x - y - 5 = 0$ and $x + y + 1 = 0$

(B) $x - y = 0$ and $x + y + 5 = 0$

(C) $x + y - 5 = 0$ and $x - y - 1 = 0$

(D) $x + y - 1 = 0$ and $x - y - 5 = 0$

37. An ellipse whose length of semi-major axis is l_1 and a confocal hyperbola with length of semi- transverse axis l_2 meet at P. If S and S' are the foci, then $(SP)(S'P)$ is equal to:

(A) $l_1^2+l_2^2$ (B) $\frac{l_1^2+l_2^2}{l_1^2-l_2^2}$

(C) $\frac{l_1^2-l_2^2}{l_1^2+l_2^2}$ (D) $l_1^2-l_2^2$

38. The tangent at any point $P(x_1, y_1)$ on the hyperbola $xy = c^2$ meets the co-ordinate axes at points Q and R. The circumcentre of ΔOQR has co-ordinates:

(A) $(0, 0)$ (B) (x_1, y_1)

(C) $\left(\frac{x_1}{2},\frac{y_1}{2}\right)$ (D) $\left(\frac{2x_1}{3},\frac{2y_1}{3}\right)$

39. The locus of the point of intersection of the lines $ax \sec\theta + by \tan\theta = a$ and $ax \tan\theta + by \sec\theta = b$, where θ is the parameter, is:

(A) a straight line (B) a circle

(C) an ellipse (D) a hyperbola

40. PM and PN are the perpendiculars from any point P on the rectangular hyperbola $xy = 8$ to the asymptotes. If the locus of the midpoint of MN is a conic, then the least distance of $(1, 1)$ to director circle of the conic is:

(A) $\sqrt{3}$ (B) $\sqrt{2}$

(C) $2\sqrt{3}$ (D) $2\sqrt{5}$

41. The locus of the midpoints of the chords passing through a fixed point (α, β) of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is:

(A) a circle with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

(B) an ellipse with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

(C) a hyperbola with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

(D) a straight line passing through $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

42. If two conics $a_1x^2 + 2h_1xy + b_1y^2 = c_1$ and $a_2x^2 + 2h_2xy + b_2y^2 = c_2$ intersect in four concyclic points, then:

(A) $(a_1 - b_1)h_2 = (a_2 - b_2)h_1$

(B) $(a_1 - b_1)h_1 = (a_2 - b_2)h_2$

(C) $(a_1 + b_1)h_2 = (a_2 + b_2)h_1$

(D) $(a_1 + b_1)h_1 = (a_2 + b_2)h_2$

43. The transverse axis of a hyperbola is of length $2a$ and a vertex divides the segment of the axis between the centre and the corresponding focus in the ratio 2:1, the equation of the hyperbola is:

(A) $4x^2 - 5y^2 = 4a^2$
(B) $4x^2 - 5y^2 = 5a^2$
(C) $5x^2 - 4y^2 = 4a^2$
(D) $5x^2 - 4y^2 = 5a^2$

44. Number of points from where perpendicular tangents to the curve $\frac{x^2}{16} - \frac{y^2}{25} = 1$ can be drawn is:

(A) 1
(B) 2
(C) 0
(D) 3

45. If the curve $xy = R^2 - 16$ represents a rectangular hyperbola, whose branches lies only in the quadrant in which abscissa and ordinate are opposite in sign but not equal in magnitude, then:

(A) $|R| < 4$
(B) $|R| \geq 4$
(C) $|R| = 4$
(D) $|R| = 5$

46. If $f(x) = ax^3 + bx^2 + cx + d$ (a, b, c, d are rational numbers) and roots of $f(x) = 0$ are eccentricities of a parabola and rectangular hyperbola then the value of $a + b + c + d$ equals:

(A) 0
(B) 1
(C) 2
(D) 3

47. The product of perpendiculars from any point $P(\theta)$ on the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ to its asymptotes is:

(A) $\frac{6}{5}$
(B) $\frac{36}{13}$
(C) depending on θ
(D) $\frac{5}{6}$

48. The foot of the perpendicular from the focus to an asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is:

(A) (ae, be)
(B) $\left(\frac{a}{e}, \frac{b}{e}\right)$
(C) $\left(\frac{e}{a}, \frac{e}{b}\right)$
(D) (be, ae)

49. If from the point (α, α^2) two tangents drawn to any one branch of hyperbola $x^2 - 4y^2 = 1$, then:

(A) $-\frac{1}{2} < \alpha < \frac{1}{2}$
(B) $0 < \alpha < \frac{1}{2}$
(C) $-1 < \alpha < 1$
(D) none of these

50. The equation of the transverse axis of the hyperbola $(x-3)^2 + (y+1)^2 = (4x+3y)^2$ is:

(A) $x + 3y = 0$
(B) $4x + 3y = 9$
(C) $3x - 4y = 13$
(D) $4x + 3y = 0$

51. Tangent at the point $\left(2\sqrt{2}, 3\right)$ to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ meets its asymptotes at A and B, then area of the ΔOAB, O being the origin is:

(A) 6 sq. units
(B) 3 sq. units
(C) 12 sq. units
(D) 2 sq. units

52. If the circle $x^2 + y^2 = a^2$ intersect the hyperbola $xy = c^2$ in four points $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3), S(x_4, y_4)$ then which of the following does not hold:

(A) $x_1 + x_2 + x_3 + x_4 = 0$
(B) $x_1x_2x_3x_4 = y_1y_2y_3y_4 = c^4$
(C) $y_1 + y_2 + y_3 + y_4 = 0$
(D) $x_1 + y_2 + x_3 + y_4 = 0$

53. For which of the hyperbola we can have more than one pair of perpendicular tangents?

(A) $\frac{x^2}{4} - \frac{y^2}{9} = 1$
(B) $\frac{x^2}{4} - \frac{y^2}{9} = -1$
(C) $x^2 - y^2 = 4$
(D) $xy = 4$

54. Let $y = 4x^2$ and $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$ intersect iff:

(A) $|a| \leq \frac{1}{\sqrt{2}}$
(B) $a > \frac{1}{\sqrt{2}}$
(C) $a > -\frac{1}{\sqrt{2}}$
(D) $a > \sqrt{2}$

55. From point (2, 2) tangents are drawn to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then the point of contact lie in:

(A) I and II quadrants
(B) I and IV quadrants
(C) I and III quadrants
(D) III and IV quadrants

56. The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is:

(A) $\dfrac{x}{x_1+x_2}+\dfrac{y}{y_1+y_2}=1$

(B) $\dfrac{x}{x_1-x_2}+\dfrac{y}{y_1-y_2}=1$

(C) $\dfrac{x}{y_1+y_2}+\dfrac{y}{x_1+x_2}=1$

(D) $\dfrac{x}{y_1-y_2}+\dfrac{y}{x_1-x_2}=1$

57. An ellipse intersects the hyperbola $2x^2-2y^2=1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the co-ordinates axes, then:

(A) equation of ellipse is $x^2+2y^2=1$

(B) the foci of ellipse are $(\pm1, 0)$

(C) equation of ellipse is $x^2+2y^2=4$

(D) the foci of ellipse are $(\pm\sqrt{2}, 0)$

58. If the curves $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $(a>b)$ and $x^2-y^2=c^2$ cut at right angles then:

(A) $a^2-b^2=c^2$ (B) $a^2+b^2=c^2$

(C) $a^2-b^2=2c^2$ (D) $a^2+b^2=2c^2$

59. For hyperbola $\dfrac{x^2}{\cos^2\alpha}-\dfrac{y^2}{\sin^2\alpha}=1$, which of the following remains constant with change in 'α'?

(A) Abscissae of vertices (B) Abscissae of foci

(C) Eccentricity (D) Directrix

60. Equation of a common tangent to the curves $y^2=8x$ and $xy=-1$ is:

(A) $3y=9x+2$ (B) $y=2x+1$

(C) $2y=x+8$ (D) $y=x+2$

61. The locus of a point, from where tangents to the rectangular hyperbola $x^2-y^2=a^2$ contain an angle of 45°, is:

(A) $(x^2+y^2)+a^2(x^2-y^2)=4a^2$

(B) $2(x^2+y^2)+4a^2(x^2-y^2)=4a^2$

(C) $(x^2+y^2)^2+4a^2(x^2-y^2)=4a^4$

(D) $(x^2+y^2)^2+a^2(x^2-y^2)=a^4$

62. From a point P on the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{4}=1$ straight lines are drawn parallel to the asymptotes of the hyperbola. Then the area of parallelogram formed by the asymptotes and the two lines through P is:

(A) dependent on co-ordinates of P

(B) 4

(C) 6

(D) $8\sqrt{2}$

63. An equation of common tangent to the parabola $y^2=8x$ and the hyperbola $3x^2-y^2=3$ is:

(A) $2x-y+1=0$ (B) $x-y+2=0$

(C) $x+y+2=0$ (D) $2x+y-1=0$

64. Let $P(\theta)$ and $Q(\phi)$ be two points on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ satisfying $\theta+\phi=\dfrac{\pi}{2}$. If (h, k) be the point of intersection of normals at P and Q, then k is equal to:

(A) $\dfrac{a^2+b^2}{a}$ (B) $-\dfrac{a^2+b^2}{a}$

(C) $-\dfrac{a^2+b^2}{b}$ (D) $\dfrac{a^2+b^2}{b}$

65. A chord of the hyperbola $x^2-2y^2=1$ is bisected at the point $(-1, 1)$. Then the area of the triangle formed by the chord and the co-ordinate axes is:

(A) 1 (B) 2

(C) $\dfrac{1}{2}$ (D) $\dfrac{1}{4}$

66. The angle of intersection between the curves $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and $\dfrac{x^2}{a^2-k^2}-\dfrac{y^2}{k^2-b^2}=1, (a>k>b>0)$ is:

(A) $\tan^{-1}\left(\frac{b}{a}\right)$ (B) $\tan^{-1}\left(\frac{b}{ka}\right)$

(C) $\tan^{-1}\left(\frac{a}{kb}\right)$ (D) none of these

67. Let A is the number of tangents drawn from a point on the asymptote of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (except origin) to the hyperbola itself. B is the number of normals, which can be drawn from centre of $xy=c^2$ to the $xy=c^2$. C is the maximum number of normals, which can be drawn from a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. D is the number of tangent common to both branches of a hyperbola. Then number of normals which can be drawn from the point (ABD, BC) to $y^2-48y-4x+616=0$ is (If $A=3$, $B=5$, $C=4$, then $ABC=354$):

(A) 1 (B) 0

(C) 2 (D) 3

68. Locus of the midpoints of the chords of the hyperbola $x^2-y^2=a^2$ that touch the parabola $y^2=4ax$ is:

(A) $x^2(x-a)=y^3$ (B) $y^2(x-a)=x^3$

(C) $x^3(x-a)=y^2$ (D) $y^3(x-a)=x^2$

69. If radii of director circles of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $2r$ and r, respectively and e_e and e_h be the eccentricities of the ellipse and the hyperbola, respectively then:

(A) $2e_h^2-e_h^2=6$ (B) $e_e^2-4e_h^2=6$

(C) $4e_h^2-e_e^2=6$ (D) none of these

70. Consider a branch of the hyperbola $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the ΔABC is:

(A) $1-\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}}-1$

(C) $1+\sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}}+1$

71. If a variable straight line $x\cos\alpha+y\sin\alpha=P$, which is a chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1(b>a)$, subtend a right angle at the centre of the hyperbola, then it always touches a fixed circle, whose radius is:

(A) $\frac{ab}{\sqrt{b-2a}}$ (B) $\frac{a}{\sqrt{a-b}}$

(C) $\frac{ab}{\sqrt{b^2-a^2}}$ (D) $\frac{ab}{b\sqrt{b+a}}$

72. If $P(\theta_1)$ and $Q(\theta_2)$ are the extremities of any focal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\cos^2\left(\frac{\theta_1+\theta_2}{2}\right)$ $=\lambda\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)$, where λ is equal to:

(A) $\frac{a^2+b^2}{a^2}$ (B) $\frac{a^2+b^2}{b^2}$

(C) $\frac{a^2+b^2}{ab}$ (D) $\frac{a^2+b^2}{2ab}$

73. If the normal at the points $P_i\,(x_i, y_i)$, $i=1$ to 4 on the hyperbola $xy=c^2$ are concurrent at the point $Q(h, k)$, then $\frac{(x_1+x_2+x_3+x_4)(y_1+y_2+y_3+y_4)}{x_1x_2x_3x_4}$ is equal to:

(A) $\frac{hk}{c^4}$ (B) $\frac{h^2k^2}{c^6}$

(C) $\frac{\sqrt{|hk|}}{c^3}$ (D) $-\frac{hk}{c^4}$

74. A tangent to the hyperbola $y=\frac{x+9}{x+5}$ passing through the origin is:

(A) $x+25y=0$ (B) $5x+y=0$

(C) $5x-y=0$ (D) $x-25y=0$

75. The circle $x^2+y^2-8x=0$ and hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ intersect at the points A and B. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is:

(A) $2x-\sqrt{5}\,y-20=0$ (B) $2x-\sqrt{5}\,y+4=0$

(C) $3x-4y+8=0$ (D) $4x-3y+4=0$

76. There is a point P on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ such that its distance to the right directrix is the average of its distance to the two foci. Let the x-co-ordinate of P

be $\frac{m}{n}$ with m and n being integers, ($n > 0$) having no common factor except 1. Then $n - m$ equals:

(A) 59
(B) 69
(C) –59
(D) –69

77. The reflection of the hyperbola $xy = 1$ in the line $y = 2x$ is the curve $12x^2 + rxy + sy^2 + t = 0$, then the value of '$r$' is:

(A) –7
(B) 25
(C) –175
(D) 90

78. A pair of tangents with inclinations α, β are drawn from an external point P to the parabola $y^2 = 16x$. If the point P varies in such a way that $\tan^2\alpha + \tan^2\beta = 4$ then the locus of P is a conic, whose eccentricity is:

(A) $\frac{\sqrt{5}}{2}$
(B) $\sqrt{5}$
(C) 1
(D) $\frac{\sqrt{3}}{2}$

79. If a variable line has its intercepts on the co-ordinates axes e, e', where $\frac{e}{2}, \frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where r equals:

(A) 1
(B) 2
(C) 3
(D) cannot be decided

80. A hyperbola has centre 'C' and one focus at $P(6,8)$. If its two directrices are $3x + 4y + 10 = 0$ and $3x + 4y - 10 = 0$, then CP equals:

(A) 14
(B) 8
(C) 10
(D) 6

81. A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $x^2 - y^2 = a^2$ at two points P and Q, then midpoint of P and Q lies on the curve:

(A) $y^3 = x(y - a)$
(B) $y^3 = x^2(y - a)$
(C) $y^2 = x^2(y - a)$
(D) $y^2 = x^3(a - y)$

82. The equation of a tangent to the hyperbola $3x^2 - y^2 = 3$, parallel to the line $y = 2x + 4$ is:

(A) $y = 2x + 3$
(B) $y = 2x + 1$
(C) $y = 2x + 4$
(D) $y = 2x + 2$

83. A circle cuts the x-axis and y-axis such that intercept on x-axis is a constant a and intercept on i-axis is a constant b. Then eccentricity of locus of centre of circle is:

(A) 1
(B) $\frac{1}{2}$
(C) $\sqrt{2}$
(D) $\frac{1}{\sqrt{2}}$

84. Consider a hyperbola $xy = 4$ and a line $2x + y = 4$. Let the given line intersect the x-axis at R. If a line through 'R' intersects the hyperbola at S and T. The minimum value of $RS \times RT$ is:

(A) 24
(B) 16
(C) 8
(D) 4

85. The normal at 'P' on a hyperbola of eccentricity 'e' intersects its transverse and conjugate axes at L and M, respectively. If the locus of the midpoint of LM is a hyperbola, then its eccentricity is:

(A) $\frac{e+1}{e-1}$
(B) $\frac{e}{\sqrt{e^2-1}}$
(C) e
(D) $\frac{2e}{\sqrt{e^2-1}}$

86. If angle between asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 120° and product of perpendiculars drawn from foci upon its any tangent is 9, then locus of point of intersection of perpendicular tangents of the hyperbola can be:

(A) $x^2 + y^2 = 6$
(B) $x^2 + y^2 = 9$
(C) $x^2 + y^2 = 3$
(D) $x^2 + y^2 = 18$

87. 'C' be a curve which is locus of point of intersection of lines $x = 2 + m$ and $my = 4 - m$. A circle $(x-2)^2 + (y+1)^2 = 25$ intersects the curve C at four points P, Q, R and S. If O is centre of the curve 'C', then $OP^2 + OQ^2 + OR^2 + OS^2$ is:

(A) 50
(B) 100
(C) 25
(D) 25/2

88. If $\alpha + \beta = 3\pi$ then the chord joining the points α and β for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through:

(A) focus
(B) centre
(C) one of the end points of the transverse axis
(D) one of the end points of the conjugates axis

89. For a given non-zero value of m each of the lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=m$ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point. Sum of the ordinates of these points is:

(A) $\frac{a\left(1+m^2\right)}{m}$ (B) $\frac{b\left(1-m^2\right)}{m}$

(C) 0 (D) $\frac{a+b}{2m}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. A rectangular hyperbola of latus rectum 4 units passes through (0, 0) and has (2, 0) as its one focus. The equation of locus of the other focus is/are not

(A) $x^2+y^2=36$ (B) $x^2+y^2=4$

(C) $x^2-y^2=4$ (D) $x^2+y^2=9$

2. If pair of asymptotes of a hyperbola H is $xy-2x-2y=0$. |If its conjugate hyperbola H' passes through (3, 3), then:

(A) eccentricity of H' is $\sqrt{2}$

(B) focus of H' is $\left(2+\sqrt{2},2+\sqrt{2}\right)$

(C) focus of H is $\left(\sqrt{2},-\sqrt{2}\right)$

(D) transverse axis of H is $x+y=4$

3. Circles are drawn on chords of the rectangular hyperbola $xy=c^2$ parallel to the line $y=x$ as diameters. All such circles pass through two fixed points, whose co-ordinates are:

(A) (c, c) (B) $(c, -c)$

(C) $(-c, c)$ (D) $(-c, -c)$

4. Equations of a common tangent to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is:

(A) $y=x+\sqrt{a^2-b^2}$ (B) $y=x-\sqrt{a^2-b^2}$

(C) $y=-x+\sqrt{a^2-b^2}$ (D) $y=-x-\sqrt{a^2-b^2}$

5. The tangent to the hyperbola, $x^2-3y^2=3$ at the point $\left(\sqrt{3},0\right)$ when associated with two asymptotes constitutes:

(A) isosceles triangle

(B) an equilateral triangle

(C) a triangles whose area is $\sqrt{3}$ sq. units

(D) a right isosceles triangle

6. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then:

(A) $x_1+x_2+x_3+x_4=0$ (B) $y_1+y_2+y_3+y_4=0$

(C) $x_1x_2x_3x_4=c^4$ (D) $y_1y_2y_3y_4=c^4$

7. The director circle of a hyperbola is $x^2+y^2-4y=0$. If one end of the transverse axis is (2, 0), then the focus is:

(A) $\left(\sqrt{3},2-\sqrt{3}\right)$ (B) $\left(-\sqrt{3},2+\sqrt{3}\right)$

(C) $\left(\sqrt{6},2-\sqrt{6}\right)$ (D) $\left(-\sqrt{6},2+\sqrt{6}\right)$

8. If in a rectangular hyperbola, normal at any point p meet the axes in G and g and C be the centre of hyperbola, then:

(A) $PG=Pg$ (B) $PG=PC$

(C) $Pg=PC$ (D) $PG=2.PC$

9. For the hyperbola $\frac{x^2}{9}-\frac{y^2}{3}=1$, the incorrect statement is:

(A) the acute angle between its asymptotes is 60°

(B) its eccentricity is $\frac{4}{3}$

(C) length of the latus rectum is 2 units

(D) product of the perpendicular distances from any point on the hyperbola on its asymptotes is less than the length of its latus rectum.

10. If foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ coincide with the foci of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and eccentricity of the hyperbola is 2, then:

(A) $a^2+b^2=16$

(B) there is no director circle to the hyperbola

(C) centre of the director circle is (0, 0)

(D) length of latus rectum of the hyperbola = 12 units

11. Equations of the line of latus rectum of $xy = 1$ is:

(A) $x - y = \sqrt{2}$ (B) $x + y = 2\sqrt{2}$

(C) $x + y = -2\sqrt{2}$ (D) $x + y = 0$

12. From point (2, 2) if the tangents are drawn to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then the point of contact lie in:

(A) I^{st} quadrant

(B) II^{nd} quadrant

(C) III^{rd} quadrant

(D) IV^{th} quadrant

13. If the equation of pair of asymptotes of a hyperbola H is $xy - 2x - y + 2 = 0$ and its conjugate hyperbola H' passes through (3, 0), then:

(A) eccentricity of H' is $\sqrt{2}$

(B) focus of H' is $\left(2 + \frac{1}{\sqrt{2}}, 2 + \frac{1}{\sqrt{2}}\right)$

(C) focus of H is $\left(\sqrt{2}, -\sqrt{2}\right)$

(D) equation of transverse axis of H is $x - y + 1 = 0$

14. Point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}m = 0$ and $\sqrt{3}mx + my - 4\sqrt{3} = 0$ describes:

(A) a hyperbola

(B) a parabola

(C) a conic of eccentricity 2

(D) a conic of eccentricity $\frac{1}{2}$

15. For the hyperbola $9x^2 - 16y^2 - 18x + 32y - 151 = 0$:

(A) one of the directrix is $x = \frac{21}{5}$

(B) length of latus rectum $= \frac{9}{2}$ units

(C) foci are (6, 1) and (– 4, 1)

(D) eccentricity is $\frac{5}{4}$

16. The directrices of the hyperbola $xy = c^2$ are:

(A) $x - y + c\sqrt{2} = 0$ (B) $x + y + c\sqrt{2} = 0$

(C) $x + y - c\sqrt{2} = 0$ (D) $x - y - c\sqrt{2} = 0$

17. Three points A, B and C are taken on rectangular hyperbola $xy = 4$ where $B(-2, -2)$ and $C\left(6, \frac{2}{3}\right)$. If the normal at A is parallel to BC, then:

(A) circumcentre of ΔABC is $\left(2, -\frac{2}{3}\right)$

(B) equation of circumcentre of ΔABC is $3x^2 + 3y^2 - 12x + 4y - 40 = 0$

(C) orthocenter of ΔABC can be $\left(\frac{2}{\sqrt{3}}, 2\sqrt{3}\right)$

(D) none of these

18. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn.

(A) If $n = 1$ then $e = \sqrt{2}$

(B) If $n = 0$ then $e < \sqrt{2}$

(C) If $n > 1$ then $1 < e < \sqrt{2}$

(D) If $n = 0$ then $e > \sqrt{2}$

19. A straight line touches the rectangular hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$. An equation of the line is:

(A) $9x + 3y - 8 = 0$ (B) $9x - 3y + 8 = 0$

(C) $9x + 3y + 8 = 0$ (D) $9x - 3y - 8 = 0$

20. If the circle $x^2 + y^2 = a^2$ cuts a rectangular hyperbola $xy = c^2$ in $A\left(ct_1, \frac{c}{t_1}\right)$, $B\left(ct_2, \frac{c}{t_2}\right)$, $C\left(ct_3, \frac{c}{t_3}\right)$ and $D\left(ct_4, \frac{c}{t_4}\right)$, then:

(A) $t_1t_2t_3t_4 = 1$ (B) $t_1 + t_2 + t_3 + t_4 = 0$

(C) $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0$ (D) $\Sigma \frac{1}{t_1t_2} = 0$

21. If two tangents can be drawn to the different branches of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ from the point (α, α^2), then:

(A) $\alpha \in (-2, 0)$ (B) $\alpha \in (0, 2)$

(C) $\alpha \in (-\infty, -2)$ (D) $\alpha \in (2, \infty)$

22. If $P(\alpha, B)$ is the point of intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2(E^2-1)} = \frac{1}{4}$, is equidistant from the foci of the two curves (all lying in the right of y-axis) then:

(A) $2\alpha = a(2e + E)$ (B) $a - e\alpha = E\alpha - \frac{a}{2}$

(C) $E = \frac{\sqrt{e^2 + 24} - 3e}{2}$ (D) $E = \frac{\sqrt{e^2 + 12} - 3e}{2}$

23. If a hyperbola having the transverse axis of length $\frac{1}{2}$ unit is confocal with the ellipse $3x^2 + 4y^2 = 12$, then:

(A) equation of the hyperbola is $\frac{x^2}{15} - \frac{y^2}{1} = \frac{1}{16}$

(B) eccentricity of the hyperbola is 4

(C) distance between the directrices of the hyperbola is $\frac{1}{8}$ units

(D) length of latus rectum of the hyperbola is $\frac{15}{2}$ units

24. The equation of tangent to the hyperbola $5x^2 - y^2 = 5$ passing through the point (2, 8) is/are:

(A) $3x - y + 2 = 0$ (B) $3x + y - 14 = 0$

(C) $23x - 3y - 22 = 0$ (D) $3x - 23y + 178 = 0$

25. The co-ordinates of a point common to a directrix and an asymptote of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ are:

(A) $\left(\frac{25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right)$ (B) $\left(-\frac{25}{\sqrt{41}}, -\frac{20}{\sqrt{41}}\right)$

(C) $\left(\frac{25}{3}, \frac{20}{3}\right)$ (D) $\left(-\frac{25}{3}, -\frac{20}{3}\right)$

26. If the foci of hyperbola lies on the line $y = x$, one asymptote is $y = 2x$ and it is passing through the point (3, 4), then:

(A) equation of hyperbola is $2x^2 - xy + 2y^2 = 38$

(B) equation of hyperbola is $2x^2 - 5xy + 2y^2 + 10 = 0$

(C) eccentricity of hyperbola is $\frac{\sqrt{17}}{4}$

(D) eccentricity of hyperbola is $\frac{\sqrt{10}}{3}$

27. The equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola with:

(A) length of transverse axis $= 2\sqrt{3}$ units

(B) length of conjugate axis = 8 units

(C) centre at $(1, -2)$

(D) eccentricity $= \sqrt{19}$

28. The equation of tangents to the hyperbola $3x^2 - y^2 = 3$ parallel to $y = 2x + 4$ is:

(A) $y = 2x + 3$ (B) $y = 2x + 1$

(C) $y = 2x - 1$ (D) $y = 2x + 2$

29. The locus of the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is:

(A) director circle (B) $x^2 + y^2 = a^2$

(C) $x^2 + y^2 = a^2 + b^2$ (D) $x^2 + y^2 = a^2 - b^2$

30. If (5, 12) and (24, 7) are the foci of a conic passing through the origin, then the eccentricity of conic is:

(A) $\frac{\sqrt{386}}{12}$ (B) $\frac{\sqrt{386}}{13}$

(C) $\frac{\sqrt{386}}{25}$ (D) $\frac{\sqrt{386}}{38}$

31. The co-ordinates of the foci of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ are:

(A) (4, 1) (B) (–4, 1)

(C) (6, –1) (D) (–6, 1)

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension I

$H: x^2 - y^2 = 9$; $P: y^2 = 4(x - 5)$, $L: x = 9$

1. If L is the chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents is:

(A) $9x^2 - 8y^2 + 18x - 9 = 0$ (B) $9x^2 - 8y^2 + 18x + 9 = 0$

(C) $9x^2 - 8y^2 - 18x + 9 = 0$ (D) $9x^2 - 8y^2 - 18x + 9 = 0$

2. If R is the point of intersection of the tangents to H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is:

(A) $x = 7$ (B) $x = 9$
(C) $y = 7$ (D) $y = 9$

3. If the chord of contact of R with respect to the parabola P meets the parabola at T and T', S is the focus of the parabola, then area of the $\Delta STT'$ is equal to:
(A) 8 sq. units (B) 9 sq. units
(C) 12 sq. units (D) 16 sq. units

Comprehension 2

If the normals at (x_i, y_i), such that $i = 1, 2, 3, 4$ on the rectangular hyperbola $xy = C^2$ meet at the point (α, β), then answer the following questions.

4. $\sum y_i^2 =$
(A) α^2 (B) c^2
(C) β^2 (D) $-c^2$

5. If $\Pi x_i = k_1$, $\Pi y_i = k_2$, then $k_1^2 - k_2^2 =$
(A) $2c^2$ (B) $2c^4$
(C) $-2c^4$ (D) 0

6. If $\sum x_i^2 - \sum x_i = 0$, then the number of possible 2×3 matrices, whose entries are the values of 'α' is:
(A) 6 (B) 0
(C) 32 (D) 64

Comprehension 3

The normal at any point (x_1, y_1) of curve is a line perpendicular to tangent at (x_1, y_1). In case of parabola $y^2 = 4ax$, the equation of normal is $y = mx - 2am - am^3$ (m is slope of normal). In case of rectangular hyperbola $xy = c^2$, the equation of normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - yt - ct^4 + c = 0$ and the shortest distance between any two curve always exist along the common normal.

7. If normal at (5, 3) of hyperbola $xy - y - 2x - 2 = 0$ intersects the curve again at, $(\alpha, \beta - 29)$, then $\frac{\beta}{\alpha}$ is:
(A) 10 (B) 20
(C) 30 (D) 40

8. If the shortest distance between $2y^2 - 2x + 1 = 0$ and $2x^2 - 2y + 1 = 0$ is d, then the number of solution of $|\sin\alpha| = 2\sqrt{2}d\,(\alpha \in [-\pi, 2\pi])$ is:
(A) 3 (B) 4
(C) 5 (D) none of these

9. Number of normals drawn from $\left(\frac{7}{6}, 4\right)$ to $y^2 = 2x - 1$ is:
(A) 0 (B) 1
(C) 2 (D) 3

Comprehension 4

Considering the rectangular hyperbola $xy = 15!$. The number of points (α, β) lying on it, where

10. $\alpha, \beta \in I$ is:
(A) 2032 (B) 4032
(C) 8066 (D) 8064

11. $\alpha, \beta \in I^+$ and HCF $(\alpha, \beta) = 1$ is:
(A) 64 (B) 625
(C) 8064 (D) 49

12. $\alpha, \beta \in I^+$ and α divides β is:
(A) 96 (B) 64
(C) 49 (D) 2032

Comprehension 5

P and Q are two points on the rectangular hyperbola $(x - 1)(y - 2) = c^2$ and O is the centre of hyperbola. Also, tangent at P is perpendicular to OQ and meets OQ at N such that $(OQ)\,(ON) = 4$.

13. One of the equations of directrix of the hyperbola is:
(A) $x + y - 2 = 0$ (B) $x + y - 2\sqrt{2} = 0$
(C) $x + y - 4 = 0$ (D) $x + y - 5 = 0$

14. An ellipse confocal with the hyperbola and with eccentricity equal to $\frac{1}{\sqrt{2}}$ is intersected by the line $x + y - 7 = 0$ at A and B, then intersection point of tangents at A and B will lie on:
(A) $x + y - 8 = 0$ (B) $x + y - 9 = 0$
(C) $x + y - 11 = 0$ (D) $x + y - 13 = 0$

15. If S is a focus of ellipse (defined in previous question) and P is a moving point on ellipse, then circle on SP as diameter always touches a circle, whose area is:
(A) 32π sq. units (B) 30π sq. units
(C) 25π sq. units (D) 16π sq. units

Comprehension 6

A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy = 16$ is equal to the sum of ordinates of feet of normals. Then the locus of $P(h, k)$ is a curve 'C'.
Based on these information, answer the following questions.

16. The equation of the curve C is:

(A) $x^2 = 12y$ (B) $x^2 = 16y$

(C) $y^2 = 16x$ (D) $y^2 = 12x$

17. If the tangent to the curve 'C' cut the co-ordinate axes at A and B. Then the locus of the midpoint of AB is:

(A) $x^2 + 2y = 0$ (B) $x^2 - 2y = 0$

(C) $y^2 + 2x = 0$ (D) $y^2 - 2x = 0$

18. Area of the equilateral triangle inscribed in the curve 'C' having one vertex as the vertex of curve 'C' is:

(A) $76\sqrt{3}$ (B) $786\sqrt{3}$

(C) $768\sqrt{3}$ (D) $687\sqrt{3}$

Comprehension 7

If from a given point P on a hyperbola $\frac{x^2}{2} - \frac{y^2}{1} = 1$ chords are drawn at right angles to meet the hyperbola at R and T such that all the chords RT are concurrent at the point Q.

19. If the eccentric angle of points P is $\frac{\pi}{4}$, then is the co-ordinates of point Q is given by:

(A) $(-6, 3)$ (B) $(6, -3)$

(C) $(4, -2)$ (D) $(8, 6)$

20. If C is the centre of hyperbola, then one of the bisector of $\angle PCQ$ is given by:

(A) $x = 4$ (B) $y = 4$

(C) $y = 0$ (D) $x + 2y = 2$

21. If the point P is variable, then the locus of Q is given by:

(A) $\frac{x^2}{18} - \frac{y^2}{9} = 1$ (B) $\frac{x^2}{32} - \frac{y^2}{16} = 1$

(C) $\frac{x^2}{16} - \frac{y^2}{8} = 1$ (D) $\frac{x^2}{16} - \frac{y^2}{4} = 2$

Comprehension 8

In hyperbola portion of tangent intercept between asymptotes is bisected at the point of contact. Consider a hyperbola whose centre is at origin. A line $x + y = 2$ touches this hyperbola at $P(1, 1)$ and interests the asymptotes at A and B such that $AB = 6\sqrt{2}$ units.

22. Equation of asymptotes are:

(A) $5xy + 2x^2 + 2y^2 = 0$ (B) $3x^2 + 2y^2 + 6xy = 0$

(C) $2x^2 + 2y^2 - 2xy = 0$ (D) $2x^2 + 3y^2 + 5x = 0$

23. Equation of the tangent to the hyperbola at $\left(-1, \frac{7}{2}\right)$ is:

(A) $5x + 2y = 2$ (B) $3x + 2y = 4$

(C) $3x + 4y = 11$ (D) $3x + 2y = 6$

Comprehension 9

If the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $A(a\sec\theta, b\tan\theta)$ meets the transverse and conjugate axes in D and E, respectively and F is the foot of the perpendicular to the normal at A from the centre 'C', then:

24. The value of AD^2 is:

(A) $\frac{b^2}{a^2}\left(b^2 \operatorname{cosec}^2\theta + a^2 \cot^2\theta\right)$

(B) $\frac{a^2}{b^2}\left(b^2 \operatorname{cosec}^2\theta - a^2 \cot^2\theta\right)$

(C) $\frac{b^2}{a^2}\left(b^2 \sec^2\theta + a^2 \tan^2\theta\right)$

(D) $\frac{b^2}{a^2}\left(b^2 \tan^2\theta + a^2 \sec^2\theta\right)$

25. Minimum value of AD is:

(A) $\frac{b^2}{a}$ (B) $\left|\frac{a}{b}(a+b)\right|$

(C) $\left|\frac{b}{a}(a-b)\right|$ (D) $\left|\frac{a}{b}(a-b)\right|$

26. The value of AE^2 is:

(A) $\frac{a^2}{b^2}\left(a^2 \tan^2\theta + b^2 \sec^2\theta\right)$

(B) $\frac{a^2}{b^2}\left(a^2 \tan^2\theta + b^2 \sec^2\theta\right)$

(C) $\frac{b^2}{a^2}\left(a^2 \sec^2\theta + b^2 \tan^2\theta\right)$

(D) $\frac{b^2}{a^2}\left(a^2 \tan^2\theta + b^2 \sec^2\theta\right)$

Comprehension 10

A line drawn through $P(-1, 2)$ meets the hyperbola $xy = c^2$ at the points A and B (points A and B lie on the same side of P).

27. A point 'Q' is chosen on this line such that PA, PQ and PB are in A.P, then the locus of point Q is:

(A) $x+3+y(1+2x)=0$ (B) $y=(1-x)$

(C) $2x=y(1+2x)$ (D) $x=2y(1+2x)$

28. If PA, PQ and PB are in G.P., then locus of point Q is:

(A) $xy-y+2x-C^2=0$

(B) $xy+y-2x+C^2=0$

(C) $xy+y+2x+C^2=0$

(D) $xy-y-2x-C^2=0$

29. If PA, PQ and PB are in H.P., then locus of point Q is:

(A) $2x-y=2C^2$ (B) $x-2y=2C^2$

(C) $2x+y+2C^2=0$ (D) $x+2y=2C^2$

Comprehension 11

If the second degree curve representing by a hyperbola $S=0$. The difference between the equations of hyperbola and pair of asymptotes is constant. That is $A=S+\lambda=0$, where λ is constant. By using the condition of pair of straight lines, we get λ. If the equation of conjugate hyperbola of S is represented by S_1. Also, asymptotes is the arithmetic mean of S and S_1.

30. Pair of asymptotes of the hyperbola $xy-3y-2x=0$ is:

(A) $xy-3y-2x+2=0$ (B) $xy-3y-2x+4=0$

(C) $xy-3y-2x+6=0$ (D) $xy-3y-2x+12=0$

31. A variable chord $x\cos\theta+y\sin\theta=p$ of $\dfrac{x^2}{a^2}-\dfrac{y^2}{2a^2}=1$ subtends a right angle at the origin. This chord always touches a curve whose radius is:

(A) a (B) $\dfrac{a}{\sqrt{2}}$

(C) $a\sqrt{2}$ (D) $2a\sqrt{2}$

Comprehension 12

Consider a hyperbola $xy=4$ and a line $y+2x=4$. O is the centre of hyperbola. Tangent at any point P of hyperbola intersect the co-ordinate axes at A and B.

32. Locus of circumcentre of ΔOAB is:

(A) an ellipse with eccentricity $\dfrac{1}{\sqrt{2}}$

(B) an ellipse with eccentricity $\dfrac{1}{\sqrt{3}}$

(C) a hyperbola with eccentricity $\sqrt{2}$

(D) a circle

33. Shortest distance between the line and hyperbola is:

(A) $\dfrac{8\sqrt{2}}{\sqrt{5}}$ (B) $\dfrac{4\left(\sqrt{2}-1\right)}{\sqrt{5}}$

(C) $\dfrac{2\sqrt{2}}{\sqrt{5}}$ (D) $\dfrac{4\left(\sqrt{2}+1\right)}{\sqrt{5}}$

34. Let the given line intersects the x-axis at R. If a line through R, intersect the hyperbolas at S and T then, then minimum value of $RS \times RT$ is:

(A) 2 (B) 4

(C) 6 (D) 8

Comprehension 13

Consider the conic defined by $x^2+y^2=(3x+4y+10)^2$.

35. If (α, β) is the centre of the conic, then $4\alpha+3\beta$ is:

(A) –8 (B) –10

(C) –6 (D) –9

36. If (p, q) is a vertex of the conic, then $2p-q$ is:

(A) –1 (B) 1

(C) –3 (D) 2

37. The eccentricity of the conic is:

(A) 5 (B) 4

(C) 3 (D) 2

38. The number of points through which a pair of real perpendicular tangents can be drawn to the conic is:

(A) infinite (B) 1

(C) 0 (D) 4

Comprehension 14

If a circle with centre $C(\alpha,\beta)$ intersects a rectangular hyperbola with centre $L(h, k)$ at four points $P(x_1,y_1), Q(x_2,y_2), R(x_3,y_3)$ and $S(x_4,y_4)$, then the mean of the four points P, Q, R, S is the mean of the points C and L. In other words, the midpoints of CL coincides with the mean point of P, Q, R, S. Analytically, $\dfrac{x_1+x_2+x_3+x_4}{4}=\dfrac{\alpha+h}{2}$ and $\dfrac{y_1+y_2+y_3+y_4}{4}=\dfrac{\beta+k}{2}$.

39. Five points are selected on a circle of radius a. The centres of the rectangular hyperbola, each passing through four of these points, all lie on a circle of radius is/are:

(A) a (B) $2a$

(C) $\dfrac{a}{\sqrt{2}}$ (D) $\dfrac{a}{2}$

40. A, B, C, D are the points of intersection of a circle and a rectangular hyperbola, which have different centres. If AB passes through the centre of the hyperbola, then CD passes through:

(A) centre of the hyperbola
(B) centre of the circle
(C) midpoint of the centres of circle and hyperbola
(D) none of the points mentioned in the three options.

41. If the normals drawn at four concylic points on a rectangular hyperbola $xy = c^2$ meet at point (α, β) then the centre of the circle has the co-ordinates:

(A) (α, β)
(B) $(2\alpha, 2\beta)$
(C) $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$
(D) $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right)$

Comprehension 15

Consider the conic defined by the equation,

$$\left|\sqrt{(x-1)^2+(y-2)^2}-\sqrt{(x-5)^2+(y-5)^2}\right| = 3$$

42. The equation of an axis of the conic is:

(A) $6x+8y=45$
(B) $3x-4y-5=0$
(C) $8x+6y=45$
(D) $3x+4y+5=0$

43. The distance between the directrices of the conic is:

(A) $\frac{9}{5}$
(B) $\frac{3}{5}$
(C) $\frac{5}{3}$
(D) $\frac{5}{9}$

44. The eccentricity of the conic conjugate to the given one is:

(A) $\frac{5}{3}$
(B) $\frac{5}{4}$
(C) $\frac{5}{2}$
(D) 5

Comprehension 16

The difference between the second degree curve and pair of asymptotes is constant. If second degree curve represented by a hyperbola is $S=0$, then the equation of its asymptotes is $S+\lambda=0$, where λ is constant, which will be a pair of straight lines, then we get λ. Then equation of asymptotes is $A \equiv S+\lambda=0$ and if equation of conjugate hyperbola of S represented by S_1, then A is the A.M. of S and S_1.

45. The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x+3y=0$ and $3x+2y=0$. If the hyperbola passes through the point (5, 3), then its equation is:

(A) $(2x+3y-3)(3x+2y-5)=256$
(B) $(2x+3y-7)(3x+2y-8)=156$
(C) $(2x+3y-5)(3x+2y-3)=252$
(D) $(2x+3y-8)(3x+2y-7)=154$

46. If the angle between the asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{\pi}{3}$, then the eccentricity of conjugate hyperbola is:

(A) $\sqrt{2}$
(B) 2
(C) $\frac{2}{\sqrt{3}}$
(D) $\frac{4}{\sqrt{3}}$

47. A hyperbola passing through origin has $3x-4y-1=0$ and $4x-3y-6=0$ as its asymptotes. Then the equations of its transverse and conjugate axes are:

(A) $x-y-5=0$ and $x+y+1=0$
(B) $x-y=0$ and $x+y+5=0$
(C) $x+y-5=0$ and $x-y-1=0$
(D) $x+y-1=0$ and $x-y-5=0$

Comprehension 17

If P is a variable point and F_1 and F_2 are two fixed points such that $|PF_1-PF_2|=2a$. Then the locus of the point P is a hyperbola, with points F_1 and F_2 as the two foci $(F_1F_2>2a)$. If $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a hyperbola, then its conjugate hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$. Let $P(x, y)$ is a variable point such that

$$\left|\sqrt{(x-1)^2+(y-2)^2}-\sqrt{(x-5)^2+(y-5)^2}\right| = 3$$

48. If the locus of the point P represents a hyperbola of eccentricity e, then the eccentricity e' of the corresponding conjugate hyperbola is:

(A) $\frac{5}{3}$
(B) $\frac{4}{3}$
(C) $\frac{5}{4}$
(D) $\frac{3}{\sqrt{7}}$

49. Locus of intersection of two perpendicular tangents to the given hyperbola is:

(A) $(x-3)^2+\left(y+\frac{7}{2}\right)^2=\frac{55}{4}$
(B) $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{25}{4}$

(C) $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{7}{4}$

(D) none of these

50. If origin is shifted to point $\left(3,\frac{7}{2}\right)$ and the axes are rotated through an angle θ is clockwise sense so that equation of given hyperbola changes to the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then θ is:

(A) $\tan^{-1}\left(\frac{4}{3}\right)$ (B) $\tan^{-1}\left(\frac{3}{4}\right)$

(C) $\tan^{-1}\left(\frac{5}{3}\right)$ (D) $\tan^{-1}\left(\frac{3}{5}\right)$

Comprehension 18

For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then normal at P meets the transverse axis AA' in G and the conjugate and BB' in g and CF is perpendicular to the normal from the centre.

51. $PF\cdot PG=KCB^2$, then K equals:

(A) 2 (B) 1

(C) $\frac{1}{2}$ (D) 4

52. $PF\cdot Pg$ equals to:

(A) CA^2 (B) CF^2

(C) CB^2 (D) $CA\cdot CB$

53. Locus of midpoint of G and g is a hyperbola of eccentricity is:

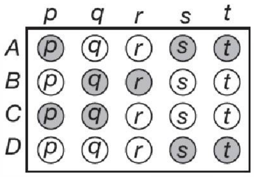
(A) $\frac{1}{\sqrt{e^2-1}}$ (B) $\frac{e}{\sqrt{e^2-1}}$

(C) $2\sqrt{e^2-1}$ (D) $\frac{e}{2}$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following

COLUMN-I	COLUMN-II
(A) $P(\theta)$ is a point on Ist quadrant on ellipse $\frac{x^2}{9}+y^2=1, \angle POX=\phi$, then for maximum value of $\theta-\phi$, θ is equal to (O is origin OX is major axis	(p) $\frac{\pi}{2}$
(B) If the eccentricity of the hyperbola $x^2-y^2\sec^2\theta=5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\theta+y^2=25$, then the smallest positive value of θ	(q) $\frac{\pi}{4}$

(Continued)

(Continued)

COLUMN-I	COLUMN-II
(C) The angle between asymptotes of hyperbola $5x^2 - 2\sqrt{7}xy - y^2 - 2x + 1 = 0$ is	(r) $\dfrac{\pi}{3}$
(D) If the portion of asymptotes of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ between centre and the tangent at vertex in I$^{\text{st}}$ quadrant is cut by the line $y + (\tan\theta)(x - a) = 0$, then θ can be	(s) $\dfrac{\pi}{6}$

2. Let the circle $(x - 1)^2 + (y - 2)^2 = 25$ cuts a rectangular hyperbola with transverse axis along $y = x$ at four points A, B, C and D having co-ordinates (x_i, y_i), such that $i = 1, 2, 3, 4$, respectively and O being the centre of the hyperbola. Now, match the entries from the following two columns.

COLUMN-I	COLUMN-II
(A) $x_1 + x_2 + x_3 + x_4$	(p) 2
(B) $x_1^2 + x_2^2 + x_3^2 + x_4^2$	(q) 4
(C) $OA^2 + OB^2 + OC^2 + OD^2$	(r) 44
(D) $y_1^2 + y_2^2 + y_3^2 + y_4^2$	(s) 56
	(t) 100

3. Let the foci of the hyperbola $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$ be the vertices of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the foci of the ellipse be the vertices of the hyperbola. Let the eccentricities of the ellipse and hyperbola be e_E and e_H, respectively, then match the following.

COLUMN-I	COLUMN-II
(A) $\dfrac{b}{B}$ is equal to	(p) 1
(B) $e_H + e_E$ is always greater than	(q) 2
(C) If the angle between the asymptotes of the hyperbola is $\dfrac{2\pi}{3}$, then $4e_E$ is equal to	(r) 3

(Continued)

COLUMN-I	COLUMN-II
(D) If $e_E^2 = \dfrac{1}{2}$ and (x, y) is a point of intersection of the ellipse and the hyperbola, then $\dfrac{x^2}{y^2}$ is	(s) 4
	(t) 5

4. Match **COLUMN-I** with **COLUMN-II.**

COLUMN-I	COLUMN-II
(A) The length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 144$ is	(p) $\dfrac{2}{3}$
(B) The product of the perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 = 2$ to its asymptotes is	(q) 3
(C) The length of the latus rectum of the hyperbola $xy = 18$ is	(r) $\dfrac{32}{3}$
(D) The product of the lengths of the perpendiculars drawn from the foci of $3x^2 - 4y^2 = 12$ on any of its tangents is	(s) 12

5. Match **COLUMN-I** with **COLUMN-II.**

COLUMN-I	COLUMN-II
(A) The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is	(p) 7
(B) If the foci of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{b^2} = 1$ and the hyperbola $\dfrac{x^2}{144} - \dfrac{y^2}{81} = \dfrac{1}{25}$ coincide, then b^2 is	(q) 0
(C) The product of lengths of perpendiculars from any point t of the hyperbola $x^2 - y^2 = 8$ to its asymptotes is	(r) $\sqrt{2}$

(Continued)

COLUMN-I	COLUMN-II
(D) The number of points outside the hyperbola $\frac{x^2}{25} - \frac{y^2}{36} = 1$ from where two perpendicular tangents can be drawn to the hyperbola is / are	(s) 4

6. Match the following

COLUMN-I	COLUMN-II
(A) The area of the triangle that a tangent at a point of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ makes with its asymptotes is	(p) 12
(B) If the line $y = 3x + \lambda$ touches the curve $9x^2 - 5y^2 = 45$, then $\lvert\lambda\rvert$ is	(q) 6
(C) If the chord $x\cos\alpha + y\sin\alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, then the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent is	(r) 24
(D) If λ be the length of the latus rectum of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, then 3λ is equal to	(s) 32
	(t) 3

7. Match the following

COLUMN-I	COLUMN-II
(A) A tangent drawn to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ sq. units, with co-ordinate axes, then the square of its eccentricity is equal to	(p) 17
(B) If the eccentricity of the hyperbola $x^2 - y^2\sec^2\theta = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$ then $25 - 2\cos^2\theta$ is	(q) 32
(C) For the hyperbola $\frac{x^2}{3} - y^2 = 3$, if the acute angle between its asymptotes is $\frac{l\pi}{24}$, then the value of 'l' is	(r) 16
(D) For the hyperbola $xy = 8$ any tangent of it at P meets co-ordinate axes at Q and R, then find the area of ΔCQR, where 'C' is the centre of the hyperbola is	(s) 24
	(t) 8

8. Match the following

COLUMN-I	COLUMN-II
(A) Value of c for which $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ are the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$ is	(p) 3
(B) If locus of a point, whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is $xy = c^2$, then value of c^2 is	(q) –4
(C) If equation of a hyperbola, whose conjugate axis is 5 and distance between its foci is 13 units, is $ax^2 - by^2 = c$ where a and b are co-prime natural numbers, then the value of $\frac{ab}{c}$ is	(r) –12

(*Continued*)

COLUMN-I	COLUMN-II
(D) If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then the ratio of square of its conjugate axis to the square of its transverse axis is	(s) 4
	(t) –6

9. Match the following

COLUMN-I	COLUMN-II
(A) Given that $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $\frac{x^2}{a}+\frac{y^2}{b}=1$ intersect orthogonally, then the equation of conjugate of $x^2-y^2=a-b$ is	(p) $\sqrt{3}$
(B) A variable straight line $x\cos\alpha+y\sin\alpha=p$ intercepts of chord on $\frac{x^2}{2}-\frac{y^2}{6}=1$ that subtends a right angle at centre of hyperbola. Then the chord always touches a fixed circle with radius	(q) $x^2-y^2=7$
(C) Area of the triangle formed by a tangent at any point on $x^2-y^2-2x-2y-6=0$ and its asymptotes is	(r) 6
(D) If the equation $2x^2+5xy+3y^2-11x-14y+24=0$ represents a hyperbola, then constant in the equation of conjugate hyperbola is	(s) $x^2-y^2=-7$

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. A line is such that its intercepts on the co-ordinate axes are e and e', where e and e' are the eccentricities of a hyperbola and its conjugate hyperbola. If this line is touching a fixed circle, then the area of the circle is k sq. units, then find $[k]$ (where [.] is the greatest integer function).

2. If the equation of asymptotes of hyperbola $xy+2x+4y+6=0$ is $xy+2x+4y+c=0$, then find c.

3. If PSP^1 and QSQ^1 are two perpendicular focal chords of $\frac{x^2}{6}+\frac{y^2}{3}=1$ then find $\frac{1}{SP.SP'}+\frac{1}{SQ.SQ'}$.

4. Vertices of ellipse $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ are the foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and vertices of hyperbola are the foci of ellipse. If eccentricity of ellipse is $\frac{1}{\sqrt{2}}$ and (t_1,t_2) is the points of intersection of ellipse and hyperbola, then find the value of $\frac{9t_1^2}{t_2^2}$.

5. Find the number of possible tangents, which can be drawn to the curve $4x^2-9y^2=36$ and which are perpendicular to the straight line $5x+2y-10=0$.

6. Find the number of points having integral co-ordinates lying outside the hyperbola $\frac{x^2}{13}-\frac{y^2}{8}=1$ from where two perpendicular tangents could be drawn to the hyperbola.

7. The graphs of $x^2+y^2+6x-24y+72=0$ and $x^2-y^2+6x+16y-46=0$ intersect at 4 points. If the sum of the distances of these four points from the point $(-3, 12)$ is d, then find the number of positive divisors of d is.

8. A lattice point (x, y) is a point in xy plane both of whose co-ordinates are integers. If the number of such lattice points on hyperbola $2xy-5x+y=55$ is λ, then find $\left[\sqrt{\lambda}\right]$ where $[\lambda]$ is the greatest integer function.

9. A variable chord of the hyperbola $\frac{x^2}{4}-\frac{y^2}{8}=1$ subtends a right angle at the centre of the hyperbola. If this chord touches a fixed circle concentric with hyperbola, then the equation of the circle is $x^2 + y^2 = p^2$, then find p^2.

10. PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then find the product of the slopes of CP, CQ, CR and CS.

11. Let PN be the ordinate of a point P on the hyperbola $\frac{x^2}{(97)^2}-\frac{y^2}{(79)^2}=1$ and the tangent at P meets the transverse axis in T, O is the origin. Then find $\left[\frac{ON.OT}{2011}\right]$, where [.] denotes greatest integer function.

12. If the length of the subtangent to the hyperbola $x^2 - 4y^2 = 4$ corresponding to the normal having slope unity is $\frac{1}{\sqrt{k}}$, then find k.

13. A chord AB which bisected at (1, 1) is drawn to the hyperbola $7x^2+8xy-y^2-4=0$ with centre C, which intersects its asymptotes in E and F. If equation of circumcircle of ΔCEF is $x^2+y^2-ax-by+c=0$, then find the value of $\frac{23(a-b+c)}{12}$.

14. If e is eccentricity of a hyperbola, whose asymptotes are $3x + 4y = 2$ and $4x - 3y + 5 = 0$, then find e^2

15. If the equation of the hyperbola whose conjugate axis is 5 and distance between its foci is 13 units, is $ax^2 - by^2 = c$ where a and b are coprime natural numbers, then find the value of $\frac{ab}{c}$.

16. Find the number of normals of a rectangular hyperbola which can touch its conjugate.

17. The latus rectum subtends a right angle at the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the eccentricity is e, then find the value of $[e]$, where [.] denotes greatest integer functions.

18. If the equation of common tangent to the curves $y^2 = 8x$ and $xy = -1$ is $x - y + k = 0$ then find k.

19. The ordinate of any point 'P' on the hyperbola $25x^2 - 16y^2 = 400$ is produced to meet the asymptotes in the points Q and R, then find $\sqrt{|PQ.PR|}$.

20. If the acute angle between the asymptotes of the hyperbola $x^2 + 2xy - 3y^2 + x + 7y + 9 = 0$ is θ. Then find $\tan\theta$.

21. If e is the eccentricity of the hyperbola $(5x - 10)^2 + (5y + 15)^2 = (12x - 5y + 1)^2$, then find $\frac{25e}{13}$.

22. If a variable tangent of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ cuts the circle $x^2 + y^2 = 4$ at point A, B and locus of midpoint of AB is $9x^2 - 4y^2 - \lambda(x^2 + y^2)^2 = 0$ then find λ.

23. If the angle between the asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{\pi}{3}$. Then find the eccentricity of conjugate hyperbola.

24. If the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then find b.

25. The equation $\frac{x^2}{9-\lambda}+\frac{y^2}{4-\lambda}=1$ represents a hyperbola when $a<\lambda<b$, then find $\left[\frac{b+a}{b-a}\right]$, where [.] denotes the greatest integer function.

26. If CP, CD are semi conjugate diameters of $5(x-2)^2+4(y-3)^2=20$, then find CP^2+CD^2.

27. If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its transverse and conjugate axes coincides with the major and minor axes of the ellipse and the product of eccentricities is 1, which is represented by the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then find the value of b^2-a^2.

28. If the product of perpendiculars from any point on the hyperbola $\frac{x^2}{4}-\frac{3y^2}{4}=1$ to its asymptotes is $\frac{1}{K}$, then find K.

29. If e_1 and e_2 are the eccentricities of the hyperbola $xy = c^2$ and $x^2-y^2=a^2$ then find $(e_1+e_2)^2$.

30. The product of perpendicular distances from any point on the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ on the asymptotes is k, then find $3k$.

31. If the values of m for which the line $y=mx+2\sqrt{5}$ touches the hyperbola $16x^2-9y^2=144$ are the roots of the equation $x^2-kx-4=0$, then find the value of k.

32. If a hyperbola is confocal and coaxial with ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ and intersects it at $\left(\sqrt{3},\frac{1}{2}\right)$, then find the length of transverse axis of hyperbola.

ARCHIVE: JEE MAIN

1. **[JEE MAIN 2020: Jan Attempt]**
Let the normal at a point P on the curve $y^2-3x^2+y+10=0$ intersect then y-axis at $\left(0,\frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then $|m|$ is equal to ________.

2. **[JEE MAIN 2020: Jan Attempt]**
If a hyperbola passes through the point $P(10, 16)$ and it has vertices at $(\pm 6, 0)$, then the equation of the normal to it at P is :
(A) $2x+5y=100$ (B) $x+3y=58$
(C) $x+2y=42$ (D) $3x+4y=94$

3. **[JEE MAIN 2020: Jan Attempt]**
If e_1 and e_2 are the eccentricities of the ellipse $\frac{x^2}{18}+\frac{y^2}{4}=1$ and the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ respectively and (e_1, e_2) is a point on the ellipse $15x^2+3y^2=k$ then k is equal to:
(A) 14 (B) 15
(C) 16 (D) 17

4. **[JEE MAIN 2019]**
If the eccentricity of the standard hyperbola passing through the point (4, 6) is 2, then the equation of the tangent to the hyperbola at (4, 6) is:
(A) $2x-y-2=0$ (B) $3x-2y=0$
(C) $2x-3y+10=0$ (D) $x-2y+8=0$

5. **[JEE MAIN 2019]**
If the line $y=mx+7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24}-\frac{y^2}{18}=1$, then the value of m is:
(A) $\frac{3}{\sqrt{5}}$ (B) $\frac{2}{\sqrt{5}}$
(C) $\frac{\sqrt{5}}{2}$ (D) $\frac{\sqrt{15}}{2}$

6. **[JEE MAIN 2019]**
If a directrix of a hyperbola centred at the origin and passing through the point $(4, -2\sqrt{3})$ is $5x=4\sqrt{5}$ and its eccentricity is e, then:
(A) $4e^4-24e^2+27=0$ (B) $4e^4-24e^2+35=0$
(C) $4e^4-12e^2-27=0$ (D) $4e^4+8e^2-35=0$

7. **[JEE MAIN 2019]**
If $5x+9=0$ is the directrix of the hyperbola $16x^2-9y^2=144$, then its corresponding focus is:
(A) $\left(\frac{5}{3},0\right)$ (B) $(5, 0)$
(C) $(-5, 0)$ (D) $\left(-\frac{5}{3},0\right)$

8. **[JEE MAIN 2019]**
Let $S=\left\{(x,y)\in R^2:\frac{y^2}{1+r}-\frac{x^2}{1-r}\right\}$; $r\neq\pm 1$. Then S represents:
(A) an ellipse, whose eccentricity is $\frac{1}{\sqrt{r+1}}$, where $r>1$
(B) a hyperbola, whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0<r<1$
(C) a hyperbola, whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0<r<1$
(D) a ellipse, whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r>1$

9. **[JEE MAIN 2019]**
Equation of a common tangent to the parabola $y^2=4x$ and the hyperbola $xy=2$ is:
(A) $x+y+1=0$ (B) $4x+2y+1=0$
(C) $x-2y+4=0$ (D) $x+2y+4=0$

10. [JEE MAIN 2019]
If a hyperbola has length of its conjugate axis equal to 5 units and the distance between its foci is 13, then the eccentricity of the hyperbola is:

(A) $\frac{13}{6}$ (B) 2

(C) $\frac{13}{12}$ (D) $\frac{13}{8}$

11. [JEE MAIN 2019]
Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and S' denote the foci of the hyperbola, where S lies on the positive x-axis, then P divides SS' in a ratio:

(A) 14:13 (B) 13:11

(C) 5:4 (D) 2:1

12. [JEE MAIN 2019]
If the vertices of a hyperbola be at (–2, 0) and (2, 0) and one of its foci be at (–3, 0), then which one of the following points does not lie on this hyperbola?

(A) $\left(6, 5\sqrt{2}\right)$ (B) $\left(2\sqrt{6}, 5\right)$

(C) $\left(-6, 2\sqrt{10}\right)$ (D) $\left(4, \sqrt{15}\right)$

13. [JEE MAIN 2019]
The equation of a tangent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is:

(A) $x - y + 9 = 0$ (B) $x - y - 3 = 0$

(C) $x - y + 1 = 0$ (D) $x - y + 7 = 0$

14. [JEE MAIN 2019]
A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse axis of length 4 along the x-axis. Then the eccentricity of the hyperbola is:

(A) $\frac{3}{2}$ (B) $\frac{2}{\sqrt{3}}$

(C) 2 (D) $\sqrt{3}$

15. [JEE MAIN 2019]
The equation of common tangent to the curves $y^2 = 16x$ and $xy = -4$, is:

(A) $x - y + 4 = 0$ (B) $x + y + 4 = 0$

(C) $x - 2y + 16 = 0$ (D) $2x - y + 2 = 0$

16. [JEE MAIN 2019]
Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola, $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval:

(A) $(3, \infty)$ (B) $\left(1, \frac{3}{2}\right]$

(C) $\left(\frac{3}{2}, 2\right]$ (D) $(2, 3]$

17. [JEE MAIN 2019]
Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point $T(0, 3)$, then the area (in sq. units) of ΔPTQ is:

(A) $54\sqrt{3}$ (B) $60\sqrt{3}$

(C) $60\sqrt{3}$ (D) $45\sqrt{5}$

18. [JEE MAIN 2017]
A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at P also passes through the point:

(A) $\left(-\sqrt{2}, -\sqrt{3}\right)$ (B) $\left(3\sqrt{2}, 2\sqrt{3}\right)$

(C) $\left(2\sqrt{2}, 3\sqrt{3}\right)$ (D) $\left(\sqrt{3}, \sqrt{2}\right)$

19. [JEE-Main 2016]
The eccentricity of the hyperbola, whose length of the latus rectum is equal to 8 units and the length of its conjugate axis is equal to half of the distance between its foci, is:

(A) $\sqrt{3}$ (B) $\frac{4}{3}$

(C) $\frac{4}{\sqrt{3}}$ (D) $\frac{2}{\sqrt{3}}$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE 2011]

Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is:

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$

(C) $\sqrt{2}$ (D) $\sqrt{3}$

2. **[JEE 2008]**
Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents:

(A) four straight lines, when $c = 0$ and a, b are of the same sign

(B) two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

3. **[JEE 2008]**
Consider a branch of the hyperbola, $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is:

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$

(C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

4. **[JEE 2007]**
A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is:

(A) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

(B) $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$

(C) $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$

(D) $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. **[JEE Advanced 2017]**

If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$.

Then which of the following cannot be the sides of a right angled triangle?

(A) a, 4, 1 (B) a, 4, 2

(C) $2a$, 8, 1 (D) $2a$, 4, 1

2. **[JEE advanced 2015]**
Consider the hyperbola H: $x^2 - y^2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the ΔPMN, then the correct expression(s) is/are:

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

3. **[JEE 2012]**

Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are:

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

4. **[JEE 2011]**

Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then:

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) a focus of the hyperbola is (2, 0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

5. **[JEE 2009]**
An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the co-ordinate axes, then:

(A) equation of ellipse is $x^2 + 2y^2 = 2$
(B) the foci of ellipse are $(\pm 1, 0)$
(C) equation of ellipse is $x^2 + 2y^2 = 4$
(D) the foci of ellipse are $(\pm \sqrt{2}, 0)$

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension I

[JEE Advanced 2017]

Match the information given in the three columns of the following table appropriately. COLUMN 1, 2 and 3 contains conics, equations of tangents to the conics and points of contact, respectively.

COLUMN-I	COLUMN-II	COLUMN-III
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{1}{\sqrt{a^2m^2-1}}\right)$

1. For $a = \sqrt{2}$, if a tangents is drawn to a suitable conic (Column I) at the point of contact $(-1, 1)$, then which of the following options is the only correct combination for obtaining its equation?
(A) (I)(i)(P) (B) (I)(ii)(Q)
(C) (II)(ii)(Q) (D) (III)(i)(P)

2. If a tangent to a suitable conic (Column I) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only correct combination ?
(A) (I)(ii)(Q) (B) (II)(iv)(R)
(C) (III)(i)(P) (D) (III)(ii)(Q)

3. The tangent to a suitable conic (Column I) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only correct combination ?
(A) (IV)(iii)(S) (B) (IV)(iv)(S)
(C) (II)(iii)(R) (D) (II)(iv)(R)

[JEE 2010]

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

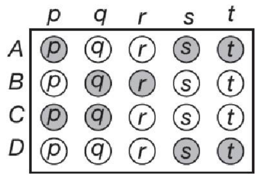
4. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is:
(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$
(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

5. Equation of the circle with AB as its diameter is:
(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$
(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. [JEE Advanced 2018]

Let $H: \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where, $a > b > 0$, be a hyperbola in the xy-plane, whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the ΔLMN be $4\sqrt{3}$ sq. units.

COLUMN-I	COLUMN-II
(P) The length of the conjugate axis of H is	(1) 8
(Q) The eccentricity of H is	(2) $\frac{4}{\sqrt{3}}$
(R) The distance between the foci of H is	(3) $\frac{2}{\sqrt{3}}$
(S) The length of the latus rectum of H is	(4) 4

Choose the correct option.

(A) $P \to 4; Q \to 2; R \to 1; S \to 3$

(B) $P \to 4; Q \to 3; R \to 1; S \to 2$

(C) $P \to 4; Q \to 1; R \to 3; S \to 2$

(D) $P \to 3; Q \to 4; R \to 2; S \to 1$

2. [JEE 2009]

Match the conics in **COLUMN-I** with the statements/expressions in **COLUMN-II.**

COLUMN-I	COLUMN-II
(A) Circle	(p) The locus of the point (h, k), for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x=\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y=\frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \lvert z\rvert^2 + 1$

3. [JEE 2007]

Match the statements in **COLUMN-I** with their properties in **COLUMN-II.**

COLUMN-I	COLUMN-II
(A) Two intersecting circles	(p) have a common tangent
(B) Two mutually external circles	(q) have a common normal
(C) Two circles, one strictly inside the other	(r) do not have a common tangent
(D) Two branches of a hyperbola	(s) do not have a common normal

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. [JEE 2010]

The line $2x + y = 1$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is:

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** B	**3.** C	**4.** C	**5.** B	**6.** D	**7.** C	**8.** A	**9.** B	**10.** B
11. A	**12.** C	**13.** C	**14.** B	**15.** C	**16.** A	**17.** A	**18.** B	**19.** D	**20.** B
21. C	**22.** D	**23.** B	**24.** D	**25.** A	**26.** C	**27.** A	**28.** A	**29.** B	**30.** B
31. B	**32.** B	**33.** B	**34.** B	**35.** C	**36.** C	**37.** D	**38.** B	**39.** D	**40.** B
41. C	**42.** A	**43.** D	**44.** C	**45.** A	**46.** A	**47.** B	**48.** B	**49.** A	**50.** C
51. A	**52.** D	**53.** B	**54.** A	**55.** D	**56.** A	**57.** B	**58.** C	**59.** B	**60.** D
61. C	**62.** B	**63.** A	**64.** C	**65.** D	**66.** D	**67.** D	**68.** B	**69.** C	**70.** B
71. C	**72.** A	**73.** D	**74.** A	**75.** B	**76.** B	**77.** A	**78.** B	**79.** B	**80.** C
81. B	**82.** B	**83.** C	**84.** C	**85.** B	**86.** D	**87.** B	**88.** B	**89.** C	

Multiple Correct Choice Type Questions

1. B, C, D	**2.** A, B, D	**3.** A, D	**4.** A, B, C, D	**5.** B, C
6. A, B, C, D	**7.** C, D	**8.** A, B, C	**9.** B, D	**10.** A, B, D
11. B, C	**12.** C, D	**13.** A, D	**14.** A, C	**15.** A, B, C, D
16. B, C	**17.** A, B, C	**18.** C, D	**19.** B, C	**20.** A, B, C
21. C, D	**22.** B, C	**23.** B, C, D	**24.** A, C	**25.** A, B
26. B, D	**27.** A, B, C	**28.** B, C	**29.** A, D	**30.** A, D
31. A, D				

Linked Comprehension Type Questions

1. C	**2.** B	**3.** C	**4.** C	**5.** D	**6.** D	**7.** B	**8.** A	**9.** B	**10.** D
11. A	**12.** A	**13.** D	**14.** C	**15.** D	**16.** B	**17.** A	**18.** C	**19.** B	**20.** C
21. A	**22.** A	**23.** B	**24.** C	**25.** A	**26.** A	**27.** C	**28.** B	**29.** A	**30.** C
31. C	**32.** C	**33.** B	**34.** D	**35.** B	**36.** A	**37.** A	**38.** C	**39.** D	**40.** B
41. C	**42.** C	**43.** A	**44.** B	**45.** D	**46.** B	**47.** C	**48.** C	**49.** D	**50.** B
51. B	**52.** A	**53.** B							

Matrix Match/Column Match Type Questions

1. A → (r);	B → (q);	C → (r);	D → (p, q, r, s)
2. A → (p);	B → (r);	C → (t);	D → (s)
3. A → (p);	B → (p, q);	C → (q);	D → (s)
4. A → (r);	B → (p);	C → (s);	D → (q)
5. A → (r);	B → (p);	C → (s);	D → (q)
6. A → (p);	B → (q);	C → (r);	D → (s)

7. A → (p);	B → (s);	C → (t);	D → (r)
8. A → (r);	B → (s);	C → (s);	D → (p)
9. A → (s);	B → (p);	C → (r);	D → (r)

Integer/Numerical Answer Type Questions

1. 3	**2.** 8	**3.** 1	**4.** 8	**5.** 0
6. 8	**7.** 9	**8.** 4	**9.** 8	**10.** 1
11. 4	**12.** 3	**13.** 4	**14.** 2	**15.** 4
16. 4	**17.** 1	**18.** 2	**19.** 5	**20.** 2
21. 5	**22.** 1	**23.** 2	**24.** 4	**25.** 2
26. 9	**27.** 7	**28.** 1	**29.** 8	**30.** 4
31. 0	**32.** 3			

ARCHIVE: JEE MAIN

1. 4	**2.** A	**3.** C	**4.** A	**5.** B	**6.** B	**7.** C	**8.** D	**9.** D	**10.** C
11. C	**12.** A	**13.** C	**14.** B	**15.** A	**16.** A	**17.** D	**18.** C	**19.** D	

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. B	**2.** B	**3.** B	**4.** A

Multiple Correct Choice Type Problems

1. A, B, C	**2.** A, B, D	**3.** A, B	**4.** B, D	**5.** A, B

Linked Comprehension Type Questions

1. B	**2.** C	**3.** D	**4.** B	**5.** A

Matrix Match/Column Match Type Questions

1. B			
2. A → (p);	B → (s, t);	C → (r);	D → (q, s)
3. A → (p, q);	B → (p, q);	C → (q, r);	D → (q, r)

Integer/Numerical Answer Type Questions

1. 2

This page is intentionally left blank

HINTS AND EXPLANATIONS

This page is intentionally left blank

CHAPTER 1: POINT AND CARTESIAN SYSTEM

Test Your Concepts-I

1. $A(x, 9)$ and $B(-2, 3)$

$AB = 10 \Rightarrow (x+2)^2 + 36 = 100$

$\Rightarrow \quad x + 2 = \pm 8$

$\Rightarrow \quad x = -10, +6$

2. Distance $= \sqrt{(a + r\cos\theta - a)^2 + (b + r\sin\theta - b)^2}$

$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

3. Let $A(3, 4)$ $\quad B(8, -6)$ $\quad C(13, 9)$

$AB = \sqrt{25 + 100} = 5\sqrt{5}$

$BC = \sqrt{25 + 225} = 5\sqrt{10}$

$CA = \sqrt{100 + 25} = 5\sqrt{5}$ $\quad [\therefore AB^2 + CA^2 = BC^2]$ $\angle A = 90°$

4. $OA^2 + OB^2 = a^2 + b^2 + c^2 + d^2$

$\angle AOB = 90°$

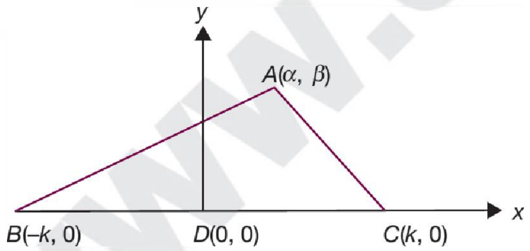
By Pythagoras theorem, $OA^2 + OB^2 = AB^2$

$\Rightarrow \quad a^2 + b^2 + c^2 + d^2 = (a-c)^2 + (b-d)^2$

$\Rightarrow \quad 2(ac + bd) = 0$

$\Rightarrow \quad ac + bd = 0$

5. We take D as the origin and DC and DY as the x and y-axes, respectively.



Let $BC = 2k$, then $B = (-k, 0)$, $C = (k, 0)$.

Let $A = (\alpha, \beta)$

Now LHS $= AB^2 + AC^2$

$= (\alpha + k)^2 + (\beta - 0)^2 + (\alpha - k)^2 + (\beta - 0)^2$

$= \alpha^2 + k^2 + 2\alpha k + \beta^2 + \alpha^2 + k^2 - 2\alpha k + \beta^2$

$= 2\alpha^2 + 2\beta^2 + 2k^2 = (\alpha^2 + \beta^2 + k^2)$

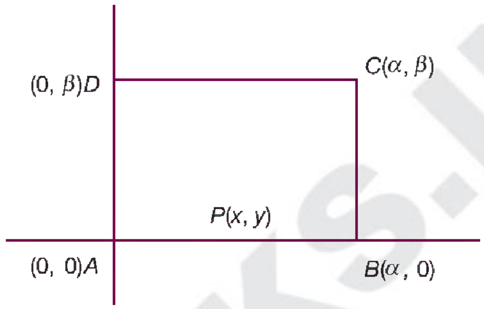
and RHS $= 2(AD^2 + BD^2)$

$= 2[(\alpha - 0)^2 + (\beta - 0)^2 + (0 + k)^2 + (0 - 0)^2]$

$= 2(\alpha^2 + \beta^2 + k^2)$

$\therefore \quad$ LHS = RHS

6.



$\therefore \quad \sigma(7)$

Let us consider a rectangle of length α and width β.

LHS $= PA^2 + PC^2$

$= x^2 + y^2 + (x - \alpha)^2 + (y - \beta)^2$

$= (x^2 + (y - \beta) + ((x - \alpha)^2 + y^2)$

$= PD^2 + PB^2 =$ RHS

7. $AB = BC = CD = DA = 5$

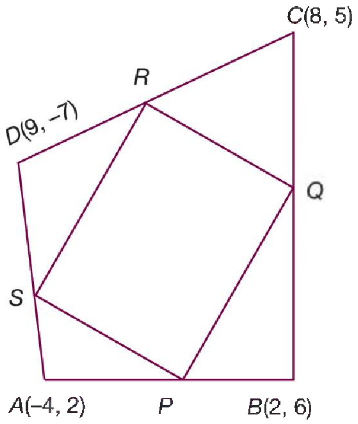
8. $A(6, 8)$; $B(3, 7)$; $C(-2, -2)$; $D(x, y)$

In a parallelogram, the diagonals bisect.

$\therefore \quad \left(\frac{6-2}{2}, \frac{8-2}{2}\right) = \left(\frac{3+x}{2}, \frac{7+y}{2}\right)$

$\Rightarrow \quad x = 1;\ y = -1$

9. Let A (–4, 2), B (2, 6), C (8, 5) and D (9, –7). Let P, Q, R, S be the midpoints of the sides AB, BC, CD and DA, respectively. Then



$P\left(\frac{-4+2}{2}, \frac{2+6}{2}\right) = (-1, 4)$

$Q\left(\frac{2+8}{2}, \frac{6+5}{2}\right) = \left(5, \frac{11}{2}\right)$

$$R\left(\frac{8+9}{2},\frac{5-7}{2}\right)=\left(\frac{17}{2},-1\right)$$

$$S\left(\frac{9+4}{2},\frac{-7+2}{2}\right)=\left(\frac{5}{2},-\frac{5}{2}\right)$$

By using distance formula we get, $PQ = RS = \frac{\sqrt{153}}{2}$

$$QR = PS = \frac{\sqrt{218}}{2}$$

As the diagonals also bisect, $PQRS$ is a parallelogram.

10. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC and P and Q be the midpoints of sides AB and AC, respectively.

$\therefore$ Slope of line PQ, $m_1 = \dfrac{\frac{y_1+y_2}{2}-\frac{y_1+y_3}{2}}{\frac{x_1+x_2}{2}-\frac{x_1+x_3}{2}} = \dfrac{y_2+y_3}{x_2+x_3}$

Also slope of line BC, $m_2 = \dfrac{y_2-y_3}{x_2-x_3}$

Since $m_1 = m_2$, PQ is parallel to BC.

Test Your Concepts-II

1. For the IVth quadrant point $(x, y) = (2, -3)$, we have,

$$\tan\theta = \frac{-3}{2}$$

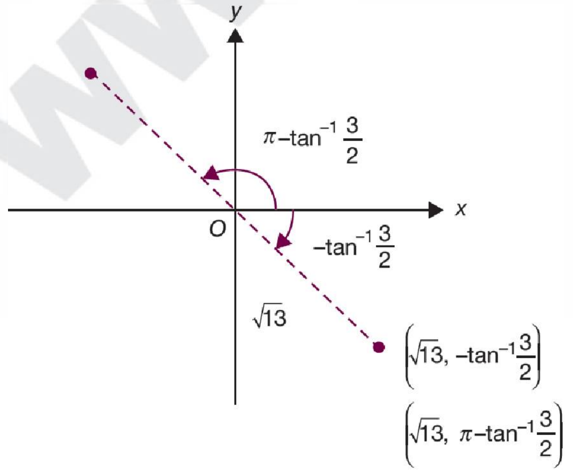
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{-3}{2}\right) = -\tan^{-1}\frac{3}{2}$$

Using positive r, we have,

$$r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(2)^2} = \sqrt{13}$$

So, one set of polar coordinates is,

$$(r, \theta) = \left(\sqrt{13},\ -\tan^{-1}\frac{3}{2}\right)$$


y
$\pi-\tan^{-1}\frac{3}{2}$
x
O
$-\tan^{-1}\frac{3}{2}$
$\sqrt{13}$
$\left(\sqrt{13}, -\tan^{-1}\frac{3}{2}\right)$
$\left(\sqrt{13}, \pi-\tan^{-1}\frac{3}{2}\right)$

Using negative r, we have the coordinates as $\left(-\sqrt{13}, \pi-\tan^{-1}\frac{3}{2}\right)$

2. $y = 10$

$\Rightarrow \quad r\sin\theta = 10$

$\Rightarrow \quad r = \dfrac{10}{\sin\theta}$

$\Rightarrow \quad r = 10\operatorname{cosec}\theta$

3. (i) $x = r\cos\theta = 2\cos\pi = -2$

$y = r\sin\theta = 2\sin\pi = 0$

$\therefore$ Point $(-2, 0)$

(ii) $x = r\cos\theta = \sqrt{3}\cos\dfrac{\pi}{6} = \dfrac{3}{2}$

$y = r\sin\theta = \sqrt{3}\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\therefore$ Point $\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$

4. (i) $x = 1; y = -1$

$r = \sqrt{1+1} = \sqrt{2}$ IVth quadrant

$\theta = -\tan^{-1}1 = -\dfrac{\pi}{4}$

$\therefore$ Point $\left(\sqrt{2}, -\dfrac{\pi}{4}\right)$ or $\left(\sqrt{2}, \dfrac{7\pi}{4}\right)$

(ii) $x = -3, y = -4$

$r = \sqrt{9+16} = 5$ IIIrd quadrant

$\theta = \pi + \tan^{-1}\dfrac{4}{3}$

$\therefore$ Point $\left(5, \pi+\tan^{-1}\dfrac{4}{3}\right)$

5. $x = r\cos\theta; y = r\sin\theta$

Given relation, $2x^2 + 3y^2 = 6$

$\Rightarrow \quad 2r^2\cos^2\theta + 3r^2\sin^2\theta = 6$

$\Rightarrow \quad 2r^2 + r^2\sin^2\theta = 6$

$\Rightarrow \quad r^2(2+\sin^2\theta) = 6$

6. $r = 4\tan\theta\sec\theta = \dfrac{4\sin\theta}{\cos^2\theta}$

Given relation, $2x^2 + 3y^2 = 6$

$\Rightarrow \quad 2r^2\cos^2\theta + 3r^2\sin^2\theta = 6$

$\Rightarrow \quad 2r^2 + r^2\sin^2\theta = 6$

$\Rightarrow \quad r^2(2+\sin^2\theta) = 6$

$\Rightarrow \quad r^2\cos^2\theta = 4r\sin\theta$

$(\because \quad x = r\cos\theta;\ y = r\sin\theta)$

$\therefore \quad x^2 = 4y$

7. $P(r\cos\theta, r\sin\theta)$

$3x + 4y = 10$

$\Rightarrow \quad r(3\cos\theta + 4\sin\theta) = 10$

$$r = \frac{10}{3\cos\theta + 4\sin\theta}$$

$(\because -5 \le 3\cos\theta + 4\sin\theta \le 5)$

$(3\cos\theta + 4\sin\theta)_{\max} = 5$

$$r_{\min} = \frac{10}{5} = 2$$

8. Given axes are rotated through an angle of 30°. To find the position of point $\left(4, -2\sqrt{3}\right)$ with respect to new axes, we know that, if the position of a point with respect to X and Y-axes is (x, y) and if the axis are rotated through an angle q, then the new coordinates of the point (x, y) will be (x', y'). Then,

$x' = x\cos\theta + y\sin\theta$

$y' = -x\cos\theta + y\cos\theta$

Here, $\theta = 30°$, $x = 4$ and $y = -2\sqrt{3}$

$\therefore \quad x' = 4\cos 30° + (-2\sqrt{3})\sin 30°$

$$= \frac{4\sqrt{3}}{2} - 2\sqrt{3} = \frac{1}{2} = 2\sqrt{3} - \sqrt{3} = \sqrt{3}$$

$y' = -4\sin 30° - 2\sqrt{3}\cos 30°$

$= -2 - 3 = -5$

Hence, the new coordinates of point $(4, -2\sqrt{3})$ is $(\sqrt{3}, -5)$.

9. If point $(2, -1)$ gets reflected about y-axis, then the coordinates of new point will be $(-2, -1)$. It is the position of point $(-2, -1)$, when the axes are rotated through an angle 45° in negative direction (i.e., in clockwise direction).

If X and Y are the new coordinate of the point $(-2, -1)$, then

$X = x\cos\theta - y\sin\theta$

$Y = x\sin\theta + y\cos\theta$

$\therefore \quad X = (-2)\cos 45° - (-1)\cos 45°$

$$= -2 \times \frac{1}{\sqrt{2}} + \frac{1}{2} = -\frac{1}{2}$$

$Y = (-2)\sin 45° + (-1)\cos 45°$

$$= \frac{-2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$$

Hence, the new coordinates of the point is $\left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$.

10. The origin is shifted to the point $(1, -2)$ without changing the direction of coordinate axes.

By replacing x by $(x + 1)$ and y by $(y - 2)$ in the given equation of curve, we get,

$(x + 1)^2 + (y - 2)^2 = 2$

$\Rightarrow \quad x^2 + y^2 + 2x - 4y + 5 = 2$

$\Rightarrow \quad x^2 + y^2 + 2x - 4y + 3 = 0$

Test Your Concepts-III

1. Let $A = (1, 5)$, $B = (2, 4)$ and $C = (3, 3)$

Now, $AB = \sqrt{(1-2)^2 + (5-4)^2} = \sqrt{2}$

$BC = \sqrt{(2-3)^2 + (4-3)^2} = \sqrt{2}$

and $AC = \sqrt{(1-3)^2 + (5-3)^2} = 2\sqrt{2}$

Here, the greatest of AB, BC and AC is AC.

and $AB + BC = \sqrt{2} + \sqrt{2} = 2\sqrt{2} = AC$.

Hence, the points A, B and C are collinear.

2. $y = f(x) = \sqrt{(x^2-2)^2 + (x-3)^2} - \sqrt{x^2 + (x^2-1)^2}$

Note that the first radical sign describes the distance between $P(x, x^2)$ and $A(3, 2)$, whereas the second radical sign describes the distance between $P(x, x^2)$ and $B(0, 1)$.

For possible positions of P, $PA - PB \le AB$..

Hence, P, A, B should be collinear.

$\therefore$ Required maximum value $AB = \sqrt{10}$

3. Given that, $A(\operatorname{cosec}^2\theta, 0)$, $B(0, \sec^2\theta)$, $C(1, 1)$

$AC = \sqrt{\cot^4\theta + 1}$; $AC \cdot BC = \sqrt{\cot^4\theta + \tan^4\theta + 2}$

$= \cot^2\theta + \tan^2\theta$

$BC = \sqrt{1 + \tan^4\theta}$

$AB = \sqrt{\operatorname{cosec}^4\theta + \sec^4\theta} = \sqrt{(1+\cot^2\theta)^2 + (1+\tan^2\theta)^2}$

$= \sqrt{1 + \cot^4\theta + 2\cot^2\theta + 1 + \tan^4\theta + 2\tan^2\theta}$

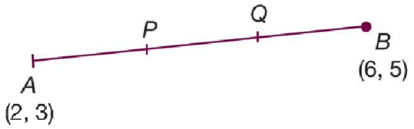
$= \sqrt{AC^2 + BC^2 + 2(\tan^2\theta + \cot^2\theta)}$

$= \sqrt{AC^2 + BC^2 + 2AC \cdot BC} = AC + BC$

$AB = AC + BC$

$\therefore$ A, B and C are collinear.

4. Given, $AP = 3AB$



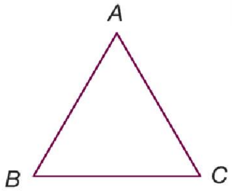
∴ $BP = AP - AB = 3AB - AB = 2AB$

∴ $\dfrac{AP}{BP} = \dfrac{3}{2}$

Hence, P divides AB externally in the ratio 3:2.

∴ $P = \left(\dfrac{3\times 2 - 2\times 1}{3-2}, \dfrac{3\times -3 - 2\times 1}{3-2}\right)$ or $P = (4, -11)$.

5. $\dfrac{\Delta DBC}{\Delta ABC} = \dfrac{1}{2} \Rightarrow \Delta DBC = \pm\dfrac{1}{2}\Delta ABC$



$$\Rightarrow \begin{vmatrix} x & 3x & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} 6 & 3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}(30 + 6 + 12 - 20 + 12 + 9)$$

$\Rightarrow\ 28x - 14 = \pm\dfrac{49}{2}$

$\Rightarrow\ 56x = 77;\ 56x = -21$

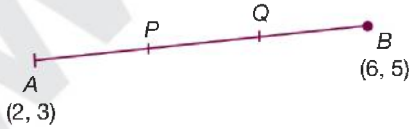
$x = \dfrac{11}{8},\ x = -\dfrac{3}{8}$

6. P divides AB in the ratio 1:2.

∴ $P\left(\dfrac{6+4}{1+2}, \dfrac{5+6}{1+2}\right) = \left(\dfrac{10}{3}, \dfrac{11}{3}\right)$

Q divides AB in the ratio 2:1.

∴ $Q\left(\dfrac{12+2}{2+1}, \dfrac{10+3}{2+1}\right) = \left(\dfrac{14}{3}, \dfrac{13}{3}\right)$



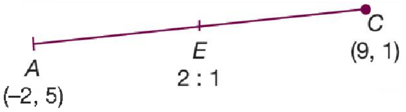
7. $AB = CD = 6\sqrt{2}$

$BC = AD = \sqrt{29}$

Midpoint of diagonals AC and BD is $\left(\dfrac{7}{2}, 3\right)$

∴ Diagonal bisects

∴ $ABCD$ is parallelogram



$E\left(\dfrac{18-2}{3}, \dfrac{2+5}{3}\right) = \left(\dfrac{16}{3}, \dfrac{7}{3}\right)$

Midpoint of BC, $F\left(\dfrac{13}{2}, 0\right)$

$$\text{Area of } \Delta DEF = \frac{1}{2}\begin{vmatrix} 3 & 7 \\ \frac{16}{3} & \frac{7}{3} \\ \frac{13}{2} & 0 \\ 3 & 7 \end{vmatrix} = 0$$

Hence, they are collinear.

8. Let P divides line joining $A(2, -3)$ and $B(5, 6)$ is λ:1

$P\left(\dfrac{5\lambda+2}{\lambda+1}, \dfrac{6\lambda-3}{\lambda+1}\right)$ lies on x-axis

$\dfrac{6\lambda-3}{\lambda+1} = 0 \Rightarrow \lambda = \dfrac{1}{2}$; 1:2 ratio internally.

9. $\text{Area} = \dfrac{1}{2}\begin{vmatrix} t & t-2 & 1 \\ t+3 & t & 1 \\ t+2 & t+2 & 1 \end{vmatrix}$

$$= \frac{1}{2}(t(t - t - 2) - (t-2)(t + 3 - t - 2) + (t^2 + 5t + 6 - t^2 - 2t))$$

$$= \frac{1}{2}(-2t - t + 2 + 3t + 6) = 4$$

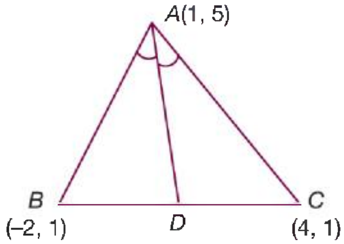
10. $\text{Area} = \dfrac{1}{2}\begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow ab - b - a = 0$

$\Rightarrow\ 1 = \dfrac{1}{a} + \dfrac{1}{b}$

Hence, proved.

Test Your Concepts-IV

1.



$AB = c = 5$

$AC = b = 5$

D divides BC in $C:B$ ratio

$\therefore \quad D(1, 1)$

Length of angle bisector $= AD = \sqrt{0 + 4^2} = 4$

2. Centroid of ΔABC $G\left(\frac{9+1-7}{3}, \frac{2+10-6}{3}\right) = (1, 2)$

Midpoint of $BC = P(-3, 2)$

Midpoint of $CA = Q(1, -2)$

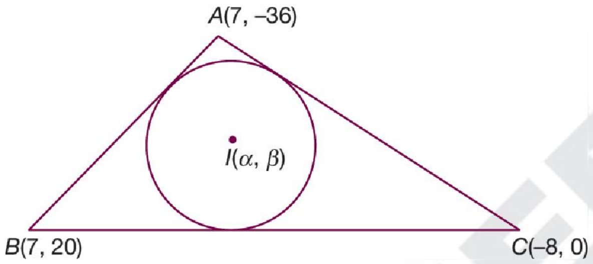
Midpoint of $AB = R(5, 6)$

Centroid of ΔPQR $G'\left(\frac{-3+1+5}{3}, \frac{2-2+6}{3}\right) = (1, 2)$

Yes, G and G' are same point.

3. Let A (7, –36), B (7, 20), C (–8, 0)

Let I (α, β) be the incentre of ΔABC



Now $a = BC = \sqrt{(7+8)^2 + (20-0)^2} = 25$

$b = AC = \sqrt{(7+8)^2 + (-36-0)^2} = 39$

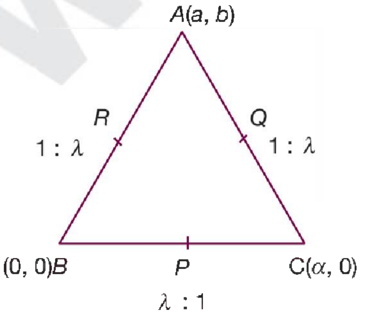
$c = AB = \sqrt{(7-7)^2 + (-36-20)^2} = 56$

$$\alpha = \frac{ax_1 + bx_2 + cx_3}{a+b+c} = \frac{7(25) + 39(7) + 56(-8)}{25+39+56} = 0$$

$$\beta = \frac{ay_1 + by_2 + cy_3}{a+b+c} = \frac{25(-36) + 39(20) + 56(0)}{25+39+56} = -1$$

Hence, $I = (0, -1)$

4.



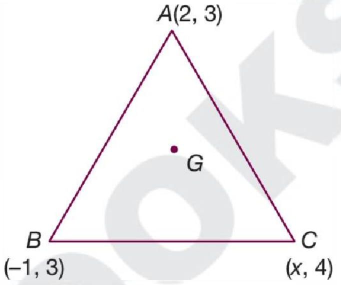
$$P\left(\frac{\lambda\alpha}{\lambda+1}, 0\right) Q\left(\frac{\alpha+\lambda a}{\lambda+1}, \frac{\lambda b}{\lambda+1}\right) R\left(\frac{a}{1+\lambda}, \frac{b}{1+\lambda}\right)$$

Centroid of $\Delta PQR = G_1\left(\frac{\lambda\alpha + \alpha + \lambda\alpha + a}{3(\lambda+1)}, \frac{b(\lambda+1)}{3(1+\lambda)}\right)$

$$G_1\left(\frac{\alpha+a}{3}, \frac{b}{3}\right)$$

$\therefore \quad G$ and G_1 coincide.

5.



$$G\left(\frac{x-1+2}{3}, \frac{y+3+3}{3}\right) = \left(\frac{2}{3}, 2\right)$$

$x = 1; y = 0 \qquad C\ (1, 0)$

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 3 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{2}(6 + 3 - 3 + 3) = \frac{9}{2}$$

6. Vertices are $A(a, 0)$, $B(-a, 0)$ and $C(b, c)$

$\therefore \quad$ Centroid is $G\left(\frac{b}{3}, \frac{c}{3}\right)$

$$\frac{AB^2 + BC^2 + CA^2}{GA^2 + GB^2 + GC^2}$$

$$= \frac{4a^2 + (a+b)^2 + c^2 + (a-b)^2 + c^2}{\left(\frac{b}{3} - a\right)^2 + \left(\frac{c}{3}\right)^2 + \left(\frac{b}{3} + a\right)^2 + \left(\frac{c}{3}\right)^2 + \left(\frac{2b}{3}\right)^2 + \left(\frac{2c}{3}\right)^2}$$

$$= \frac{4a^2 + 2c^2 + 2a^2 + 2b^2}{\frac{2b^2}{9} + 2a^2 + \frac{6c^2}{9} + \frac{4b^2}{9}} = 3$$

Test Your Concepts-V

1. Let the moving point be $P(x, y)$ and let $A = (0, 2)$ and $B = (0, -2)$.

Given, $PA + PB = 6$

$\therefore \quad \sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + (y+2)^2} = 6$

$\Rightarrow \quad \sqrt{x^2 + (y-2)^2} = 6 - \sqrt{x^2 + (y+2)^2}$

$\Rightarrow \quad x^2 + (y-2)^2 = 36 + x^2 + (y+2)^2 - 2(6)\sqrt{x^2 + (y+2)^2}$

$$\Rightarrow \quad 12\sqrt{x^2+(y+2)^2} = (y+2)^2-(y-2)^2+36$$

$$\Rightarrow \quad 12\sqrt{x^2+(y+2)^2} = 4(2y+9)$$

$$\Rightarrow \quad 3\sqrt{x^2+(y+2)^2} = 2y+9$$

$$\Rightarrow \quad 9[x^2+(y+2)^2] = (2y+9)^2$$

$$\Rightarrow \quad 9[x^2+y^2+y4+4] = 4y^2+36y+81$$

$$\Rightarrow \quad 9x^2+5y^2 = 45$$

$$\Rightarrow \quad \frac{x^2}{5}+\frac{y^2}{9} = 1$$

This is the required equation of locus.

2. Let $P = (x, y)$

According to the question,

$x = a\cos\theta$ ...(1)

$y = b\sin\theta$ ...(2)

From (1), $\cos^2\theta = \frac{x^2}{a^2}$ and from (2), $\sin^2\theta = \frac{y^2}{b^2}$

Now, $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \cos^2\theta+\sin^2\theta$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

This is the required locus of P.

3. Let $A = (x_1, y_1)$

Let the moving point be $P(\alpha, \beta)$

According to question, $A(x_1, y_1)$ lies on the locus $y^2 = 8x$

$\therefore \quad y_1^2 = 8x_1$ ...(1)

Also, $P(\alpha, \beta)$ is the midpoint of OA.

$\therefore \quad \alpha = \frac{0+x_1}{2}$ or $x_1 = 2\alpha$ ...(2)

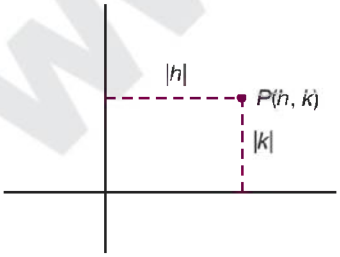
$\beta = \frac{0+y_1}{2}$ or $y_1 = 2\beta$ ...(3)

By substituting the values of α and β in (1), we get,

$4\beta^2 = (8)(2)\alpha$ or $\beta^2 = 4\alpha$

Thus, the locus of point $P(\alpha, \beta)$ is $y^2 = 4x$.

4.



$|h|^2 + |k|^2 = p^2$

$\Rightarrow \quad h^2+k^2 = p^2$

Locus $h = x; k = y$

$x^2+y^2 = p^2$

5. $x = \cos\theta+\sin\theta;\ y = \sin\theta-\cos\theta$

$$x^2+y^2 = (\cos^2\theta+\sin^2\theta+2\cos\theta+\sin\theta) + (\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta)$$

$$x^2+y^2 = 2$$

6. $2PA = 3PB$

$$\Rightarrow \quad 4((x+5)^2+(y-3)^2) = 9((x-2)^2+(y-4)^2)$$

$$\Rightarrow \quad 4((x^2+y^2+25+9+10x-6y) = 9(x^2+y^2-4x-8y+4+16)$$

$$\Rightarrow \quad 5x^2+5y^2-76x-48y+44 = 0$$

7. $\frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ -2 & 3 & 1 \end{vmatrix} = \pm 9$

$$\Rightarrow \quad x(-2)-y(3)+5 = \pm 18$$

$$\Rightarrow \quad -2x-3y+5 = \pm 18$$

$$2x+3y+13 = 0;\ 2x+3y-23 = 0$$

8. $m_{PA}\times m_{PB} = -1$

$$\Rightarrow \quad \left(\frac{y}{x-2}\right)\left(\frac{y}{x+2}\right) = -1$$

$$\Rightarrow \quad x^2+y^2 = 4$$

9. $P\left(\frac{9\cos\theta+2}{4}, \frac{6\sin\theta+3}{4}\right) = (x, y)$

$$\frac{4x-2}{9} = \cos\theta;\ \frac{4y-3}{6} = \sin\theta$$

$$\because \quad \cos^2\theta+\sin^2\theta = 1 \Rightarrow \left(\frac{4x-2}{9}\right)^2+\left(\frac{4y-3}{6}\right)^2 = 1$$

Single Correct Choice Type Questions

1. The given points are $A(-1, 11)$, $B(-9, -8)$ and $C(15, -2)$.

$$AB = \sqrt{64+361} = \sqrt{425}$$

$$BC = \sqrt{576+36} = \sqrt{612}$$

$$AC = \sqrt{256+169} = \sqrt{425}$$

Thus, $AB = AC \neq BC$

Therefore, triangle is isosceles and in an isosceles triangle, all centres are collinear.

Hence, the correct answer is (A).

2. Let origin be shifted at point (h, k) without changing the direction of coordinate axis.

Then, we replace x by $(x+h)$ and y by $(y+k)$ in the equation of given curve, then the transformed equation is $(x+h)^2+(y+k)^2-4(x+h)+6(y+k)-7 = 0$

$$\Rightarrow \quad x^2+h^2+2hx+y^2+k^2+2ky-4x-4h+6y+6k-7 = 0$$

$$\Rightarrow \quad x^2+y^2+x(2h-4)+y(2k+6)+h^2+k^2-4h+6k-7 = 0$$

Since, this equation is required to be free from the terms of first degree.

$\therefore$ $2h - 4 = 0$ and $2k + 6 = 0$

$\Rightarrow$ $h = 2$ and $k = -3$

Hence, the point to which the origin be shifted is (2, – 3).

Hence, the correct answer is (B).

3. If the coordinate axes are rotated through an angle θ, then replace x by $(x\cos\theta - y\sin\theta)$ and y by $(x\sin\theta + y\cos\theta)$ in the given equation of curve.

$$\therefore \quad 5(x\cos\theta - y\sin\theta)^2 + 8(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 5(x\sin\theta + y\cos\theta)^2 + 3(x\cos\theta - y\sin\theta) + 2(x\sin\theta + y\cos\theta) + 5 = 0$$

$$\therefore \quad x^2(8\sin\theta\cos\theta + 5\sin^2\theta + 5\cos^2\theta) + y^2(5\sin\theta + 5\cos^2\theta) + x(8\cos\theta + 2\sin\theta) + xy(8\cos^2\theta - 8\sin^2\theta) + x(8\cos\theta + 2\sin\theta) + y(2\cos\theta - 8\sin\theta) + 5 = 0$$

$$\Rightarrow \quad x^2(5 + 4\sin 2\theta) + y^2(5 - 4\sin 2\theta) + 8(\cos 2\theta).\,xy + x(8\cos\theta + 2\sin\theta) + y(2\cos\theta - 8\sin\theta + 2\sin\theta)$$

To remove xy term, put coefficient of xy equal to zero, i.e.,

$\Rightarrow$ $8\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0$

$\Rightarrow$ $2\theta = \dfrac{\pi}{2}$ $\qquad \therefore\ \theta = \dfrac{\pi}{4}$

Hence, the correct answer is (B).

4. Let the first term be A and the common difference be D of the corresponding A.P.

Given pth term of H.P. $= a$

$\therefore$ pth term of A.P. $= \dfrac{1}{a}$

$\therefore$ $A + (p-1)D = \dfrac{1}{a}$

Similarly, $A + (q-1)D = \dfrac{1}{b}$

$A + (r-1)D = \dfrac{1}{c}$

Subtracting, we get,

$$(p-q)D = \frac{1}{a} - \frac{1}{b}$$

and $(q-r)D = \dfrac{1}{b} - \dfrac{1}{c}$

Dividing, we get,

$$\frac{p-q}{q-r} = \frac{bc-ac}{ac-ab} \qquad \ldots(1)$$

The given points are $A(bc, p)$, $B(ca, q)$ and $C(ab, r)$.

Slope of $AB = \dfrac{q-p}{ca-ab}$

and Slope of $BC = \dfrac{r-q}{ab-ca} = \dfrac{q-p}{ca-bc}$ [Using (1)]

$\therefore$ Slope of BC = Slope of AB

$\therefore$ A, B and C are collinear.

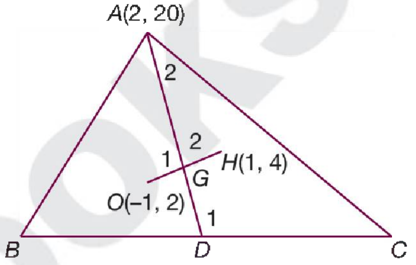
Hence, the correct answer is (A).

5. P is the centroid of ΔABC

$\therefore$ Area of $\Delta ABC = 6 \times 5 = 30$ sq. units.

Hence, the correct answer is (C).

6. Centroid $G = \left(-\dfrac{1}{3}, \dfrac{8}{3}\right)$



So, $D = \left(-\dfrac{3}{2}, -6\right)$

Hence, the correct answer is (C).

7. Let G be the centroid

Let $A'G = \alpha$ and $B'G = \beta \Rightarrow AG = 2\alpha$ and $BG = 2\beta$

$\Rightarrow$ $\alpha^2 + 4\beta^2 = \dfrac{9}{4}$ and $4\alpha^2 + \beta^2 = 4$

$\Rightarrow$ $AB = 2\sqrt{\alpha^2 + \beta^2} = \sqrt{5}$

Hence, the correct answer is (A).

8. $4 < k < 26$.

For a triangle to be obtuse, either $112 + 152 < k^2$

or $112 + k^2 < 152$

$\Rightarrow$ $k \in \{5, 6, 7, 8, 9, 10, 19, 20, 21, 22, 23, 24, 25\}$

Hence, the correct answer is (C).

9. Given triangle is equilateral with side $a = 2$

$\therefore$ In-radius $= r = \dfrac{a}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\therefore$ Area $= \pi r^2 = \dfrac{\pi}{3}$

Hence, the correct answer is (D).

10. Rhombus, $a^2\sin\theta = 25\sin\theta \le 25$

$\therefore$ Maximum area = 25

Hence, the correct answer is (B).

11. $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = 4 \text{ (Area of } \Delta ABC)^2$$

$$= 4\left(\frac{\sqrt{3}}{4}36\right)^2 = 972$$

Hence, the correct answer is (D).

12. $$f_3 \cdot f_2 \cdot f_1(x, y) = f_3 \cdot f_2(y, x) = f_3(y + 3x, x)$$

$$= \left(\frac{y + 3x - x}{2}, \frac{y + 3x + x}{2}\right)$$

$$= \left(\frac{2x + y}{2}, \frac{4x + y}{2}\right)$$

∴ $A(0, 0) \to A'(0, 0)$, $B(2, 0) \to B'(2, 4)$, $C(2, 2) \to C'(3, 5)$, $D(0, 2) \to D'(1, 1)$

∴ It is easily seen that $A'B' = D'C'$, $A'D' = B'C'$, $A'B' \neq B'C'$.

Hence, it is neither a square nor a rhombus.

Further, the slope of $A'B' \times A'D' = \frac{4}{2} \cdot \frac{1}{1} \neq -1$

Hence, AB is not perpendicular to AD.

⇒ It is not a rectangle.

∴ $ABCD$ is a parallelogram.

Hence, the correct answer is (C).

13. In ΔABC, where $A(0, 0)$, $B(a, 2)$ and $C(2, b)$ is equilateral.

$$\Rightarrow \quad a^2 + 4 = b^2 + 4 = (a - 2)^2 + (b - 2)^2$$

$$\Rightarrow \quad a^2 = b^2 \text{ and } b^2 - 4a - 4b + 4 = 0$$

If $a = -b$, $b^2 - 4a - 4b + 4 = 0 \Rightarrow b^2 + 4 = 0$

$$\Rightarrow \quad b^2 - 8b + 4 = 0 \Rightarrow b = 4 \pm 2\sqrt{3} = a$$

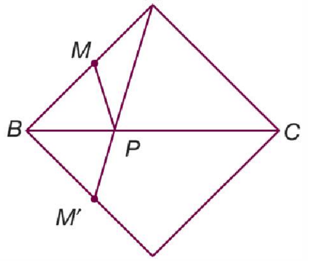
Let $a = b = 4 + 2\sqrt{3}$

$$\therefore \quad 4(a + b) - ab = 4(8 + 4\sqrt{3}) - (4 + 2\sqrt{3})^2 = 4$$

Similarly, $a = b = 4 - 2\sqrt{3}$ also $\Rightarrow 4(a + b) - ab = 4$

Hence, the correct answer is (A).

14.



Take the reflection of ΔABC in BC.

$PM = PM'$

$PA + PM = PA + PM'$ is minimum, when $M'PA$ lies in a line.

Now apply cosine rule in $\Delta ABM'$.

We will get $AP + PM = 10\sqrt{7}$

Hence, the correct answer is (A).

15. α can take 5 values: 0, 3, –3, 13, and –13

Hence, the correct answer is (A).

16. In any triangle, the centroid divides the line joining orthocentre and circumcentre internally in the ratio 2:1. So, the centroid is (2, 4).

Hence, the correct answer is (B).

17. R is centroid. S is circumcentre. $R = \left(3, \frac{4}{3}\right)$,

$$S = \left(3, \frac{7}{8}\right)$$

$$RS = \frac{11}{24}$$

Hence, the correct answer is (D).

18. $$\text{Centroid } G = \left(\frac{3x_1 + 3x_2 + 3x_3}{3}, \frac{3y_1 + 3y_2 + 3y_3}{3}\right)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3) = H$$

$$\therefore \quad \angle ABC = \frac{\pi}{3}$$

Hence, the correct answer is (B).

19. a, x_1, x_2 are in G.P. with C.R is 'r', b, y_1, y_2 are in G.P. with C.R is 's', $x_1 = ar$, $x_2 = r^2$, $y_1 = bs$, $x_2 = bs^2$

Hence, the correct answer is (C).

20. Let $OA = a$, $OB = b$, $AB = 3l$

$A = (a, 0)$, $b = (0, b)$

Let $G(x, y) = \left(\frac{a}{3}, \frac{b}{3}\right)$, $a = 3x$, $b = 3y$

$$a^2 + b^2 = 9l^2 \Rightarrow x^2 + y^2 = l^2$$

Hence, the correct answer is (B).

21. New origin $(x_1, y_1) = \left(\frac{-f}{h}, \frac{-g}{h}\right) = (-2, 1)$

Transformed equation of $xy - x + 2y + 6 = 0$ is $xy = 4$

⇒ $c = 4$

Hence, the correct answer is (A).

22. Let $A(h, k)$ then $\cos 60° = \frac{h}{2} \Rightarrow h = 1$

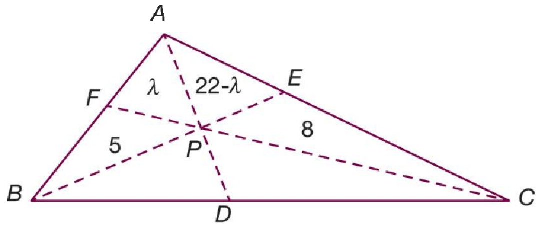
$$\sin 60° = \frac{k}{2} \Rightarrow k = \sqrt{3}$$

$\therefore \quad A(1, \sqrt{3})$

$\therefore \quad \text{Centroid} = \left(\frac{5}{3}, \frac{\sqrt{3}}{3}\right)$

Hence, the correct answer is (B).

23. Let area of $\Delta PBC = x$



$\Rightarrow \quad \frac{X}{5+\lambda} = \frac{8}{22-\lambda}$ and $\frac{X}{30-\lambda} = \frac{5}{\lambda}$

$\Rightarrow \quad \frac{\lambda+5}{30-\lambda} = \frac{(22-\lambda)5}{8\lambda}$

$\Rightarrow \quad 8\lambda^2 + 40\lambda = 5(\lambda^2 - 52\lambda + 660)$

$\Rightarrow \quad \lambda^2 + 100\lambda - 1100 = 0$

$\Rightarrow \quad (\lambda + 110)(\lambda - 10) = 0 \Rightarrow \lambda = 10$

$\Rightarrow \quad x = \frac{(30-\lambda)5}{\lambda} = \frac{(30-10)\times 5}{10} = 10$ sq. units

Hence, the correct answer is (C).

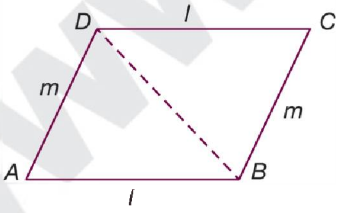
24. Let BD be integer and $l \geq m$

$2(l+m) = 40$

$\Rightarrow \quad l + m = 20$

Possible values of $m = 1, 2, 3, ..., 10$

Note: In any triplet of l, m, BD if atleast one is different parallelogram will be non-congruent.



Now $l - m < BD < l + m$ (triangle inequality)

$\Rightarrow \quad 20 - 2m < BD < 20$

$\Rightarrow$ Number of possible values of BD for a given 'm' is $20 - (20 - 2m) - 1 = 2m - 1$

$\Rightarrow$ Total number of non-congruent parallelogram

$$= \sum_{m=1}^{10} (2m-1) = 10^2 = 100$$

Hence, the correct answer is (D).

25. $r = \sqrt{x^2 + y^2} = \sqrt{9+3} = 2\sqrt{3}$

$\alpha = \tan^{-1}\frac{|y|}{|x|}$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Point $(-3, \sqrt{3})$ lies in the II^{nd} quadrant.

$\therefore \quad \theta = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

So, the polar coordinates are $\left(2\sqrt{3}, \frac{5\pi}{6}\right)$.

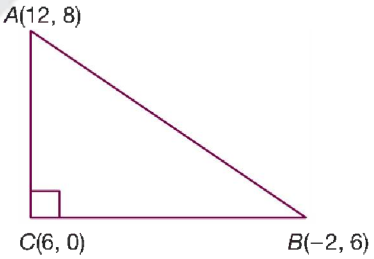
Hence, the correct answer is (D).

26. It divides in ratio of $t : 1 - t$ internally so $t > 0$ and $1 - t > 0 \Rightarrow t < 1$

$\therefore \quad 0 < t < 1$

Hence, the correct answer is (B).

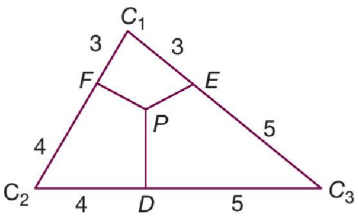
27.



ΔABC is right isosceles triangle with centroid as $\left(\frac{16}{3}, \frac{14}{3}\right)$, orthocentre as $(6, 0)$ and circumcentre as $(5, 7)$.

Hence, the correct answer is (D).

28.



$PD = PE = PF = r = \frac{\Delta}{s}$

$2s = 2(3 + 4 + 5) \Rightarrow s = 12$

$a = 7, b = 8, c = 9$

$\Rightarrow \quad \Delta = 12\sqrt{5}$

$$r = \frac{\Delta}{s} = \sqrt{5}$$

Hence, the correct answer is (A).

29. We know that the area of the triangle formed by joining the midpoints of any triangle is $\frac{1}{4}$th of that triangle.

$\therefore$ Required area $= 4\lambda$

Hence, the correct answer is (C).

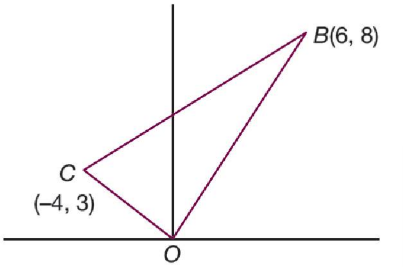
30. Transformed equation:

$$(x + 2)^2 + 2(y - 3) - 3 = 0$$

$$\Rightarrow \quad x^2 + 4x + 2y - 5 = 0$$

Hence, the correct answer is (C).

31.



Slope of $OB = \frac{8}{6}$; Slope of $OC = -\frac{3}{4}$

$$\therefore \quad \angle BOC = \frac{\pi}{2}$$

ΔOBC is right angled at O

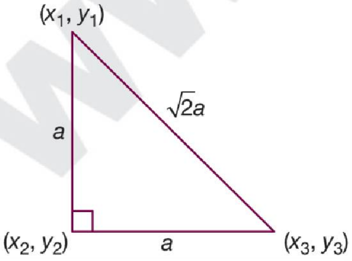
Circumcentre = Midpoint of hypotenuse = $\left(1, \frac{11}{2}\right)$

Orthocentre = Vertex $O(0, 0)$

Distance $= \sqrt{\left(1 + \frac{121}{4}\right)} = \frac{5\sqrt{5}}{2}$ unit

Hence, the correct answer is (C).

32.



Since a is an integer, $\sqrt{2}a$ is irrational.

$\therefore$ Incentre will be,

$$\left(\frac{ax_1 + ax_2 + \sqrt{2}ax_3}{2a + a\sqrt{2}}, \frac{ay_1 + ay_2 + \sqrt{2}ay_3}{2a + a\sqrt{2}}\right)$$

Clearly, it is an irrational point.

Hence, the correct answer is (B).

33. $A = \begin{vmatrix} 1 & p^2 & 1 \\ 0 & 1 & 1 \\ p & 0 & 1 \end{vmatrix} = \frac{1}{2}[1(1 - 0) + p(p^2 - 1)]$

$= \frac{1}{2}(p^3 - p + 1)$. Hence, $A = \frac{1}{2}\,|p^3 - p + 1|$

Now, the minimum value of modulus is zero. Since, $A(p)$ is a cubic it must vanish for some p other than given as (A), (B), (C) ⇒ (D).

Hence, the correct answer is (D).

34. We have $P = (1, 0)$, $Q = (-1, 0)$, $R = (2, 0)$. Let $S = (x, y)$.

Now, given that $SQ^2 + SR^2 = 2SP^2$.

Hence, $(x + 1)^2 + y^2 + (x - 2)^2 + y^2 = 2[(x - 1)^2 + y^2]$

$$\Rightarrow \quad 2x^2 + 2y^2 - 2x + 5 = 2x^2 + 2y^2 - 4x + 2$$

$$\Rightarrow \quad 2x + 3 = 0$$

$$\Rightarrow \quad x = \frac{-3}{2}$$

Which is a straight line parallel to the y-axis.

Hence, the correct answer is (D).

35. If I be the incentre of ΔOAB, inradius $= r$.

If P at I then $d(P, OA) = d(P, OB) = d(P, AB) = r$

However, $d(P, OA) < \min\{d(P, OB), d(P, AB)\}$

Which is possible only when P lies in the region, ΔOIA

$$\therefore \quad \tan 15^\circ = \frac{ID}{OD} = \frac{r}{1} \Rightarrow r = 2 - \sqrt{3}$$

Required area $= \frac{1}{2}(2)(r) = 2 - \sqrt{3}$ sq. units

Hence, the correct answer is (D).

36. Here, (0, 0) is the circumcentre.

If $G = (h, k)$ be the centroid of triangle,

$$3h = 3 + 5(\cos\theta + \sin\theta),\ 3k = 4 + 5(\sin\theta - \cos\theta)$$

If $H(\alpha, \beta)$ be the orthocentre, then $OG:GH = 1:2$

$$\Rightarrow \alpha = 3h,\ \beta = 3k$$

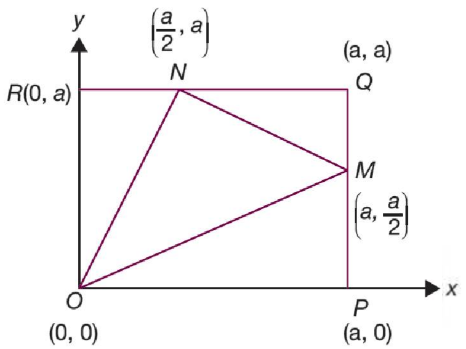
$$\cos\theta + \sin\theta = \frac{\alpha - 3}{5},\ \sin\theta - \cos\theta = \frac{\beta - 4}{5}$$

$$\therefore \quad \sin\theta = \frac{\alpha + \beta - 7}{10},\ \cos\theta = \frac{\alpha - \beta + 1}{10}$$

Locus of (α, β) is $(x + y - 7)^2 + (x - y + 1)^2 = 100$

Hence, the correct answer is (D).

37.



$$\text{Area of the } \Delta OMN = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a & \frac{a}{2} & 1 \\ \frac{a}{2} & a & 1 \end{vmatrix} = \frac{3a^2}{8}$$

Area of the square $= a^2$

$\therefore$ The required ratio = 8:3

Hence, the correct answer is (C).

38. Rectangle $ABCD$ has vertices $A(0, 0)$ and $D(4, 4)$.
Let B is (h, k).

$AB \perp BC$

$$\therefore \quad \frac{k}{h} \cdot \frac{k-4}{h-4} = -1$$

$\therefore$ The locus of the exterminates of the other diagonal is $(x - 0)(x - 4) + (y - 4)(y - 0) = 0$

or $x^2 + y^2 - 4x - 4y = 0$

Hence, the correct answer is (A).

39. Slope of line $= \dfrac{\sqrt{3} - 3}{3 - \sqrt{5}}$

Clearly numerator and denominator have different irrational number.

So, there will be no rational point on the line.

Hence, the correct answer is (A).

40. $\left(-2, \dfrac{2\pi}{3}\right)$

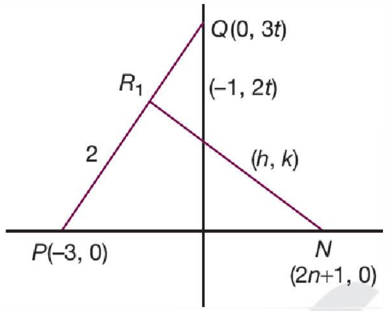
$x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore \quad x = (-2)\cos\frac{2\pi}{3} \text{ and } y = (-2)\sin\frac{2\pi}{3}$$

$$\therefore \quad x = (-2)\left(-\frac{1}{2}\right) \text{ and } y = (-2)\left(\frac{\sqrt{3}}{2}\right) \quad \therefore (1, -\sqrt{3})$$

Hence, the correct answer is (C).

41.



$P(-3, 0)$, $Q(0, 3t)$ and $R(-1, 2t)$

Let the midpoint of RN be (h, k) $\therefore$ $k = t$.

$$RN \perp PQ \Rightarrow \frac{2t}{-2h-2} \times \frac{3t}{3} = -1$$

$$2t^2 = 2h + 2$$

$$t^2 = h + 1;\ k^2 = h + 1$$

Locus of (h, k) is $y^2 = x + 1$.

Hence, the correct answer is (D).

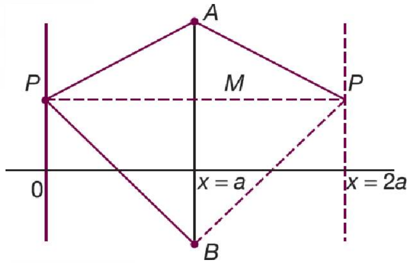
42. $\sin 10^\circ = \dfrac{a}{2b} \Rightarrow \sin 30^\circ = 3\sin 10^\circ - 4\sin^3 10^\circ$

$$\frac{1}{2} = \frac{3a}{2b} - \frac{4a^3}{8b^3}$$

$$1 = \frac{3a}{b} - \frac{a^3}{b^3} \Rightarrow a^3 + b^3 = 3ab^2$$

Hence, the correct answer is (B).

43.



$$\frac{1}{2}|AB.PM| = a^2$$

$$\Rightarrow \quad |PM| = \frac{2a^2}{2a} = a$$

$\therefore$ P lies on y-axis or on line $x = 2a$

Hence, the correct answer is (D).

44. $\tan B + \tan C = \dfrac{\sin A}{\cos B \cos C} = \dfrac{2R\sin A}{2R\cos B\cos C}$

$$= \frac{a}{2R\cos B\cos C} = k$$

$$\Rightarrow 2R\cos B\cos C = k$$

Hence, the locus of orthocentre is a line parallel to BC.

$$\left(\text{Slope} = \frac{-8}{11}\right)$$

Substitute $x = 10$

$$y = \frac{35}{11}$$

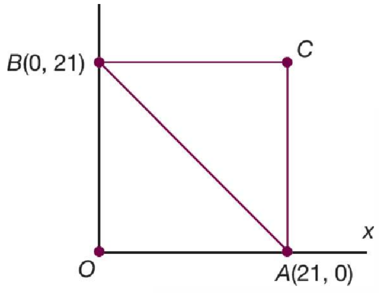
Hence, the correct answer is (A).

45. $x_2 = x_1 r,\ x_3 = x_1 r^2$ and so is $y_2 = y_2 r,\ y_3 = y_1 r^2$

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = r \cdot r^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0$$

Hence the points lie on a line, i.e., they are collinear.
Hence, the correct answer is (A).

46.



Total number of points within the square $OABC = 20 \times 20 = 400$. Total number of points on $\overline{AB} = 20$. The points are (1, 1), (2, 2), … (20, 20)

$\therefore$ Total number of points within OBC and ΔABC $= 400 - 20 = 380$

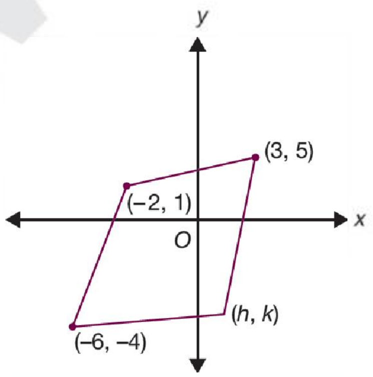
By symmetry, number of points within $\Delta AOB = \frac{380}{2} = 190$.

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. If the remaining vertex is (h, k), then

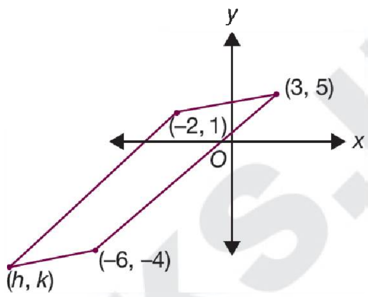
CASE-1:



$h - 2 = -6 + 3,\ k + 1 = 5 - 4$

$\Rightarrow\ h = -1,\ k = 0$

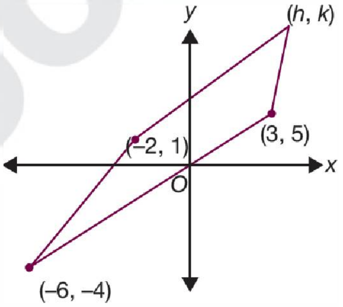
CASE-2:



$h - 2 = -6 - 2,\ k + 5 = -4 + 1$

$\Rightarrow\ h = -11,\ k - 8$

CASE-3:



$h - 6 = 3 - 2,\ k - 4 = 5 + 1$

$\Rightarrow\ h = 7,\ k = 10$

Hence, (B), (C) and (D) are correct.

2. Vertices $(a\cos\theta_1, a\sin\theta_1)$, $(a\cos\theta_2, a\sin\theta_2)$ and $(a\cos\theta_3, a\sin\theta_3)$ are equidistant from the origin (0, 0). Hence, the origin is the circumcentre (centroid) of circumcircle. Therefore, the coordinates of the centroid are
$$\left(\frac{a(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)}{3}, \frac{a(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)}{3}\right)$$
As the centroid is the origin (0, 0), we have,
$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$$
and $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$

Hence, (A) and (B) are correct.

3. Since $AB = AC = 1$, the triangle is right angled at point A, we have,
$$\tan\alpha \tan\beta = -1$$
or $\cos(\alpha - \beta) = 0$ or $\alpha - \beta = \pm\frac{\pi}{2}$

Hence, (A), (C) and (D) are correct.

4. When all the vertices are integral it cannot be equilateral but it can be right angle and isosceles triangle like (0, 0) (0, 1) and (1, 0).
Hence, (A) and (C) are correct.

5. As H (orthocentre), G (centroid), and C (Circumcentre) are collinear, we have,

$$\begin{vmatrix} 4 & b & 1 \\ b & 2b-8 & 1 \\ -4 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 4 & b & 1 \\ b-4 & b-8 & 0 \\ -(b+4) & 16-2b & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad (b-4)(16-2b) + (b+4)(b-8) = 0$

$\Rightarrow \quad 2(b-4)(8-b) + (b+4)(b-8) = 0$

$\Rightarrow \quad (8-b)[(2b-8)-(b+4)] = 0$

$\Rightarrow \quad (8-b)(b-12) = 0$

Hence, $b = 8$ or 12, which is wrong because collinearity does not explain centroid, orthocentre and circumcentre.

Now, H, G, and C are collinear and $\dfrac{HG}{GC} = 2$. Therefore,

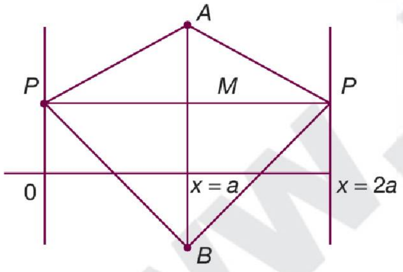
$$\frac{-8+4}{3} = b \text{ or } b = \frac{-4}{3}$$

and $\dfrac{16-b}{3} = 2b-8$ or $b = 8$

$\therefore$ No common value of b is possible.

Hence, (A), (B), (C) and (D) are correct.

6.



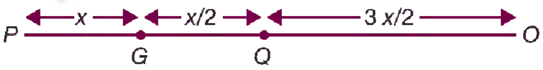
$$\frac{1}{2}|AB.PM| = a^2$$

$$\Rightarrow \quad |PM| = \frac{2a^2}{2a} = a$$

$\therefore$ P lies on y-axis or on line $x = 2a$

Hence, (B) and (D) are correct.

7.



G = centroid and O = orthocentre

Hence, (A), (B) and (C) are correct.

8. Lengths of sides of S_1, S_2, S_3 .... are

$10, \dfrac{10}{\sqrt{2}}, \dfrac{10}{\left(\sqrt{2}\right)^2}$, respectively

$\therefore \quad S_n$ has side $= 10\left(\dfrac{1}{\sqrt{2}}\right)^{n-1}$

Area of $S_n = 100\left(\dfrac{1}{2}\right)^{n-1}$; $\dfrac{200}{2^n} < 1 \Rightarrow 2^n > 200$

Hence, (B), (C) and (D) are correct.

9. Area of quadrilateral $ABCD$ is maximum, when area of ACD is maximum.

$\Rightarrow$ Distance D from AC is maximum.

i.e., $\cos\theta - \sin\theta$ is maximum.

$\Rightarrow \quad \sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right)$ is maximum $\Rightarrow \theta = \dfrac{7\pi}{4}$

and area $= \dfrac{6}{\sqrt{2}}(2\sqrt{2}) = 12$

(Since, $ABCD$ is a rectangle)

Hence, (B) and (D) are correct.

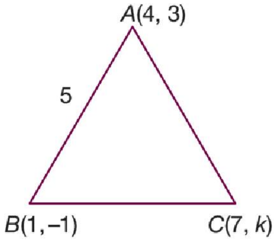
10. Point $A(x_1, y_1)$ is rational, slope of OA = rational $= \dfrac{y_1}{x_1}$

Point B is given as irrational, slope of OB = irrational

Angle between OA and OB is θ, $\tan\theta$ will be irrational, when $\theta = 30°, 15°, 60°$

Hence, (A), (B) and (C) are correct.

11. For $k = 7$, then points are collinear.



Make ΔABC isosceles, $C(7, k)$.

So either $AB = AC$

$\Rightarrow \quad 25 = 9 + (k-3)^2$

$\Rightarrow \quad k = 7, -1 \Rightarrow k = -1$

But $k \neq 7$

As, A becomes midpoint of B and C.

Also, $AC = BC$

$\Rightarrow \quad 36 + (k+1)^2 = 9 + (k-3)^2$

$\Rightarrow \quad k = \dfrac{-19}{8}$

Hence, (B) and (C) are correct.

12. Let the third vertex be (x, y) then,

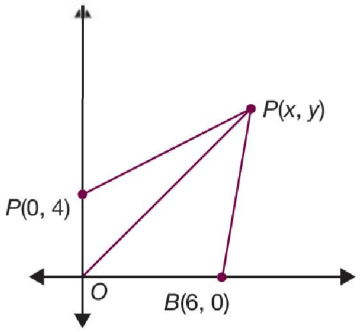
$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ -4 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \pm 4 \Rightarrow x + 5y + 4 = \pm 8 \Rightarrow x + 5y + 12 = 0$$

or $x + 5y - 4 = 0$

Hence, the third vertex lies on $x + 5y + 12 = 0$ or $x + 5y - 4 = 0$.

Hence, (C) and (D) are correct.

13. Let point P be (x, y)



$$\Rightarrow \quad \Delta POA = \frac{1}{2}(OA)\,|x| = 2\,|x|$$

$$\Rightarrow \quad \Delta POB = \frac{1}{2}(OB)\,|y| = 2\,|y|$$

Where $P = (x, y)$ or $2\,|x| = 6\,|y|$

$$\Rightarrow \quad |x| = 3\,|y| \text{ or } 3y - x = 0 \text{ or } 3y + x = 0$$

Hence, (A) and (D) are correct.

14. We have $PA = PB$, i.e., $(PA)^2 = (PB)^2$

$$\Rightarrow [x-(a+b)]^2+[y-(b-a)]^2=[x-(a-b)]^2+[y-(a+b)]^2$$

$$\Rightarrow [(x-a)-b]^2+[(y-b)+a]^2=[(x-a)+b]^2+[(y-a)-a]^2$$

$$\Rightarrow [(x-a)+b]^2+[(x+b)-b]^2=[(y-b)+a]^2-[(y-b)-a]^2$$

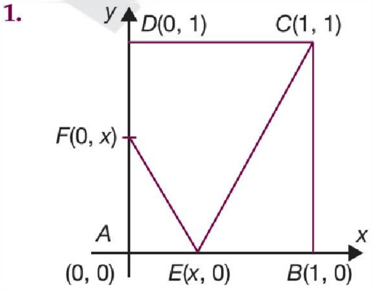
$4a(x - a) = 4a(y - b) \Rightarrow bx = ay$

$P(a, b)$ also satisfies it, so that P can be (a, b).

Hence, (B) and (D) are correct.

Linked Comprehension Type Questions

1.



Area of $CDFE$

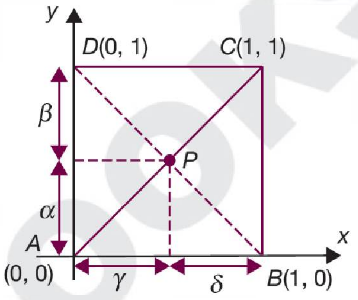
$$A = 1 - \frac{1}{2}x^2 - \frac{1}{2}(1 - x)$$

$$= \frac{2 - x^2 - 1 + x}{2} = \frac{1 + x - x^2}{2}$$

$$\therefore \quad A_{\max} = \frac{1 + \frac{1}{2} - \frac{1}{4}}{2} = \frac{5}{8} \text{ at } x = \frac{1}{2}$$

Hence, the correct answer is (C).

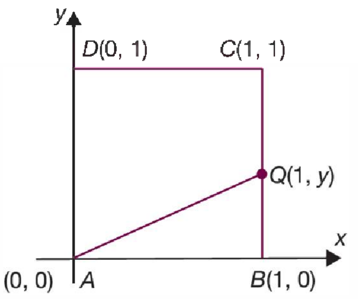
2.



$$(PA)^2 - (PB)^2 + (PC)^2 - (PD)^2$$
$$= (a^2 + g^2) - (a^2 + d^2) + (d^2 + b^2) - (g^2 + b^2) = 0$$

Hence, the correct answer is (D).

3.



$$\frac{1}{2}(y)(1) = \frac{1}{3}(1);\ y = \frac{2}{3}$$

$$L_{AQ} = \sqrt{(1)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{13}}{3}$$

Hence, the correct answer is (B).

4. Since, $\cos A = \dfrac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$

A is obtuse.

$\because \quad \cos A < 0$

$\therefore \quad AB^2 + AC^2 < BC^2$

Also, A, B and C are non-collinear.

$\Rightarrow \quad 10 > \beta^2 + 4 + 1 + (\beta - 1)^2$

$\Rightarrow \quad (\beta - 2)(\beta + 1) < 0$

$\Rightarrow \quad \beta \in (-1, 2)$

However, A, B and C are collinear for $\beta = \frac{2}{3}$

$\therefore$ Correct interval for β is $\left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right)$

Hence, the correct answer is (C).

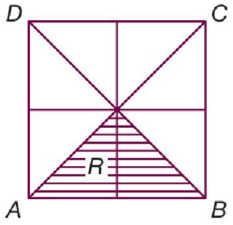
5. Whenever A is obtuse, it is the largest angle also. However, there may be more values of β for which A is largest but is not obtuse. Angle A will be largest if, $BC > AC$, $BC > AB$, A, B, C are not collinear.

$\Rightarrow$ $10 > 1 + (\beta - 1)^2$, $10 > 4 + \beta^2$, $\beta \neq \frac{2}{3}$

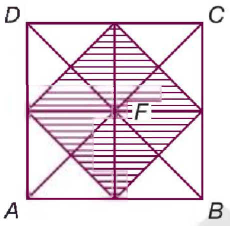
$\therefore$ Correct interval for β is $\left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \sqrt{6}\right)$

Hence, the correct answer is (C).

6. $A = \frac{1}{2}(2)(1) = 1$



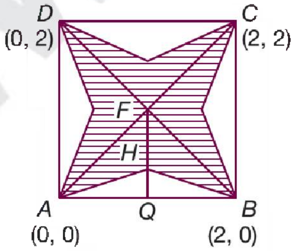
Hence, the correct answer is (B).

7.


$= \frac{1}{2} \times 4 = 2$

Hence, the correct answer is (C).

8. According to the given situation, region R is shown in figure.



$AF = \sqrt{2}$, $AQ = 1$

Since AH is the angle bisector,

$\therefore$ $FH{:}HQ = \sqrt{2}:1$

$\therefore$ $HQ = \frac{1}{1+\sqrt{2}}$

$\therefore$ Area of $AHB = \frac{1}{2} \times 2 \times \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$

Hence, the correct answer is (C).

9. P, Q and R are the vertices of a right angle triangle $P(-8, 5)$, $Q(-15, -19)$ and $R(1, -7)$

$\angle PRQ = 90°$

Orthocentre $= R\ (1, -7)$

Circumcentre $=$ Midpoint PQ

$S = \left(\frac{-23}{2}, -7\right)$, $RS = \left|1 + \frac{23}{2}\right| = \frac{23}{2}$

Circumdiameter $= PQ = 25$

Circumradius $= \frac{25}{2}$

Hence, the correct answer is (A).

10. Radius of nine point circle $= \frac{25/2}{2} = \frac{25}{4}$

$(\frac{1}{2}$ circumradius)

Hence, the correct answer is (C).

11. Let $BP{:}PC = \alpha{:}1$

$\therefore$ $P\left(\frac{\alpha x_3 + x_2}{\alpha+1}, \frac{\alpha y_3 + y_2}{\alpha+1}\right)$

P lies on $lx + my + n = 0$

$\therefore$ $l\left(\frac{\alpha x_3 + x_2}{\alpha+1}\right) + m\left(\frac{\alpha y_3 + y_2}{\alpha+1}\right) + n = 0$

$\therefore$ $\alpha = \frac{-(lx_2 + my_2 + n)}{(lx_3 + my_3 + n)}$

Similarly, if $CQ{:}QA = \beta{:}1$

$\therefore$ $\beta = \frac{-(lx_3 + my_3 + n)}{(lx_1 + my_1 + n)}$

and if $AR{:}RB = \gamma{:}1$

$\therefore$ $\gamma = \frac{-(lx_1 + my_1 + n)}{(lx_2 + my_2 + n)}$

$\therefore$ $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$

Hence, the correct answer is (A).

12. From the above result, $\frac{2}{1} \cdot \frac{1}{3} \cdot \frac{AR}{RB} = -1$

$\therefore$ $\frac{AR}{RB} = -\frac{3}{2}$

$\therefore$ R divides AB externally in the ratio 3:2.

Hence, the correct answer is (D).

Matrix Match/Column Match Type Questions

1. A → (q); B → (p); C → (s); D → (r)

(A) The lines by which triangle is formed are $x = 0$, $y = 0$ and $x + y = 1$.
Clearly, it is right angled triangle and we know that in a right angled triangle, orthocentre coincides with the vertex at which right angle is formed. Hence, the orthocentre is (0, 0).

(B) Hence, $AB = BC = CA = 2$. So, it is an equilateral triangle and the incentre with centroid. Therefore,

$$\text{centroid} = \left(\frac{0+1+2}{3}, \frac{0+0+\sqrt{3}}{3}\right) \Rightarrow \left(1, \frac{1}{\sqrt{3}}\right)$$

(C) $Ar(\Delta OPR) = Ar(\Delta PQR) = Ar(\Delta PQR)$

$\therefore$ By simple geometry, R should be the centroid of ΔPQO

$$\Rightarrow \quad R\left(\frac{3+6+0}{3}, \frac{4+0+0}{3}\right) = \left(3, \frac{4}{3}\right)$$

(D) $PQRS$ will represent a parallelogram iff the midpoint of PR is same as that of the midpoint of QS.

That is, iff $\frac{1+5}{2} = \frac{4+a}{2}$ and $\frac{2+7}{2} = \frac{6+b}{2} \Rightarrow a = 2$ and $b = 3$.

2. A → (s); B → (p, s); C → (q); D → (r)

Given vertices are $A(2a, 4a)$ and $B(2a, 6a)$.

(A) When C $(4a, 5a)$, then $AC^2 = 5a^2$, $BC^2 = 5a^2$, $BC^2 = 5a^2$ and $AB^2 = 4a^2$

$\Rightarrow$ ΔABC is isosceles.

$\Rightarrow$ (A) → (s)

(B) When $C(2 + \sqrt{3})a, 5a)$, then $AC^2 = 4a^2$, $BC^2 = 4a^2$ and $AB^2 = 4a^2$

$\Rightarrow$ ΔABC is equilateral and isosceles.

$\Rightarrow$ (B) → (p, s)

(C) When $C(6a, 4a)$, then $AC^2 = 16a^2$, $BC = 20a^2$ and $AB^2 = 4a^2$

$\Rightarrow$ ΔABC is right angled triangle.

$\Rightarrow$ (C) → (q)

(D) When $C(a, 3a)$, then $AC^2 = 2a^2$, $BC^2 = 10a^2$ and $AB^2 = 4a^2$

$\Rightarrow$ Observe that $BC^2 > AB^2 + AB^2$

$\Rightarrow$ $\angle A$ is obtuse.

$\Rightarrow$ (D) → (r)

3. A → (p); B → (r); C → (s); D → (q)

(A) Side = $9 + (h - 2)2 = 36$

$(h - 2) = \pm 3\sqrt{3}$

(B) $c = 5, a = \sqrt{68}, b = 13$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{63}{65}$$

(C) Length of median $BM = \sqrt{13}$, where M is midpoint of AC

$$BG = \frac{2}{3}BM = \frac{\sqrt{52}}{3}$$

(D) Circumcentre; only one point

4. A → (q); B → (p); C → (p); D → (p)

(A) A and B are on opposite sides of $y - 1 = 0$. Least of $AP + PB$ is equal to $AB = \sqrt{3^2 + 4^2} = 5$

(B) Centroid of the triangle

$$= \left(\frac{\cos\theta - \sin\theta}{3}, \frac{1 + \sin\theta + \cos\theta}{3}\right)$$

Circumcentre = (0, 0) O, G, S are collinear.

$\therefore$ Orthocentre $= (\cos\theta - \sin\theta,\ 1 + \sin\theta + \cos\theta)$

$= (x, y)$

Eliminate 'θ' from $x + y - 1 = 2\cos\theta$ and $x - y + 1 = -2\sin\theta$

$$\Rightarrow (x + y - 1)^2 + (x - y + 1)^2 = 4$$

(C) $\Delta PAB_{\max} = \frac{1}{2}.AB(\text{maximum height})$ max is the radius of the circle $= \sqrt{22}$

$$= \frac{1}{2}(2)(\sqrt{22})$$

$\therefore$ $\Delta PAB = 1 <$ maximum area

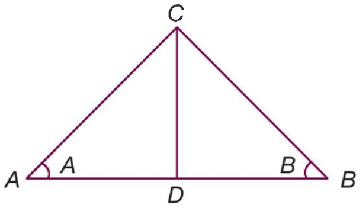
$\therefore$ Four positions of P gives area as '1' unit.

(D) $x^2 - y^2 = 0$ and $x^2 + y^2 - 1 = 0$

$$x^2 = \frac{1}{2}\ y^2 = \frac{1}{2}\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$$

Integer/Numerical Answer Type Questions

1.



$$\sin(A + B) = 2\sin A \sin B$$

$$\Rightarrow \quad \sin A \cos B + \cos A \sin B = 2\sin A \sin B$$

$$\Rightarrow \quad \cot A + \cot B = 2 \qquad \left(\because \cot B = \frac{DB}{DC}\right)$$

$$\Rightarrow \quad \cot A + \cot B = \frac{AD}{DC} + \frac{BD}{DC} = \frac{AD + DB}{DC} = \frac{AB}{DC}$$

$$\Rightarrow \quad DC = \frac{AB}{2} = \text{Constant}$$

$\therefore$ DC is constant. Hence, C will always move parallel to AB. Locus of C will be a straight line.

Hence, the maximum number of points of intersection will be 2.

Hence, the correct answer is (2).

2. $(h, k) = (t^2 - t + 1, t^2 + t + 1)$

$\Rightarrow \quad h = t^2 - t + 1$ and $k = t^2 + t + 1$

$\Rightarrow \quad k - h = 2t$

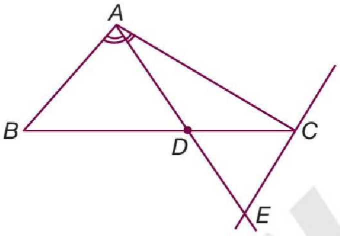
$$\Rightarrow \quad t = \frac{k-h}{2}$$

$$\Rightarrow \quad h = \left(\frac{k-h}{2}\right)^2 - \left(\frac{k-h}{2}\right) + 1$$

$\therefore$ The required locus is $x = \left(\frac{y-x}{2}\right)^2 - \left(\frac{y-x}{2}\right) + 1.$

Hence, the correct answer is (1).

3.



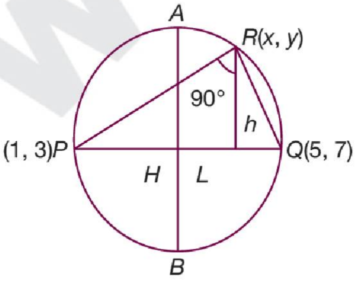
$$\frac{AD}{DE} = \frac{BC}{DC} = \frac{AB}{AC} \qquad \therefore \text{ Point } D\left(\frac{1}{3}, \frac{1}{3}\right) \text{ and } E\,(-2, 0)$$

Area of ΔACE + Area of $\Delta ABC = 2 \times$ Area of ΔACE

$$= 2 \times \frac{19}{2} = 19 \text{ sq.units}$$

Hence, the correct answer is (9).

4.



$$PQ = \sqrt{(5-1)^2 + (7-3)^2} = \sqrt{16+16} = 4\sqrt{2}$$

$$\text{Area of } \Delta PQR = \frac{1}{2}(PQ)(RL) = \frac{1}{2}4\left(\sqrt{2}\right)(RL)$$

$$2 = 2\sqrt{2}h \Rightarrow h = \frac{1}{\sqrt{2}} < \text{radius of the circle}$$

$\therefore \quad RL < AH$

Hence, there will be 4 such points R. Two are on PAQ and two are on PBC.

Hence, the correct answer is (4).

5. $AB^2 + BC^2 + CA^2 = \frac{4}{3}(AD^2 + BE^2 + CF^2)$

$$\Rightarrow \quad 16 = \frac{4}{3}\,(3k)$$

$\Rightarrow \quad k = 4$

Hence, the correct answer is (4).

6. Using section formula, $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of ΔABC is 2 sq. units. Therefore,

$$\frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ 7 & -2 & 1 \\ \frac{3k-5}{k+1} & \frac{5k+1}{k+1} & 1 \end{vmatrix} = \pm 2$$

Operating $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get,

$$\begin{vmatrix} 1 & 5 & 1 \\ 6 & -7 & 0 \\ \frac{3k-5}{k+1} - 1 & \frac{5k+1}{k+1} - 5 & 0 \end{vmatrix} = \pm 4$$

$$\Rightarrow \quad 6\left(\frac{5k+1-5k-5}{k+1}\right) + 7\left(\frac{3k-5-k-1}{k+1}\right) = \pm 4$$

$\Rightarrow \quad -24 + 7(2k - 6) = \pm 4(k + 1)$ i.e., $k = 7$

$$\Rightarrow \quad k = \frac{31}{9}$$

Hence, the correct answer is (7).

7. Given vertices of triangle are $O(0, 0)$, $B(6, 8)$, and $C(-4, 3)$

$$\text{Slope of } OB = \frac{8}{6} = \frac{4}{3}$$

$$\text{Slope of } OC = -\frac{3}{4}$$

$$\therefore \quad \angle BOC = \frac{\pi}{2}$$

ΔOBC is right angled at O.

Circumcentre = Midpoint of hypotenuse $BC = \left(1, \frac{11}{2}\right)$

Orthocentre = Vertex $O(0, 0)$

$$\therefore \text{ The required distance} = \sqrt{\left(1 + \frac{121}{4}\right)} = \frac{5\sqrt{5}}{2} \text{ units}$$

Hence, the correct answer is (5).

8. ($AP + PC$) is minimised when P lies on AC and $AP + PD$ is minimised when P lies on BD. Thus,

$$AP + PB + PC + PD = AC + BD = 2\sqrt{5}$$

Hence, the correct answer is (4).

9. We know that the area of the triangle formed by joining the midpoints of any triangle is $\frac{1}{4}$ th of that triangle.

$\therefore$ The required area is 8 sq. units.

Hence, the correct answer is (8).

10. $$G = \left(\frac{a+b+c}{3}, \frac{a+b+c}{3}\right) = (0,0)$$

$$a+b+c=0 \Rightarrow a^3+b^3+c^3 = 3abc$$

$\therefore \quad k = 3$

Hence, the correct answer is (3).

11. $P(5, 5)$, $Q(-2, 4)$, $R(7, -7)$

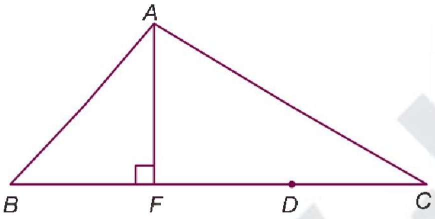
Area of ΔPQR = Area of ΔABC

$\therefore \quad k = 43$

$$\Rightarrow \frac{86}{k} = \frac{86}{43} = 2$$

Hence, the correct answer is (2).

12.



Drop the perpendicular from A to BC and let F be its foot. Furthermore, suppose $BF = x$ and $AF = y$.
Then, by Pythagorean theorem,

$$AB^2 = AF^2 + BF^2 = x^2 + y^2$$

$$AC^2 = CF^2 + AF^2 = (3-x)^2 + y^2$$

$$AD^2 = DF^2 + AF^2 = (2-x)^2 + y^2$$

$$\Rightarrow AB^2 + 2AC^2 - 3AD^2 = 6$$

Hence, the correct answer is (6).

13. Centroid is the obvious point, $G(0, 1)$.

Hence, the correct answer is (1).

14. In the graph, number of squares of dimension 1 × 1 is 4. Number of squares of dimension 2 × 2 is 1.

Each square forms 4 right angled isosceles triangles.

$\therefore$ The number of required triangles $= 5 \times 4 = 20$.

Hence, the correct answer is (5).

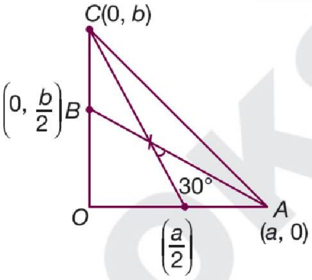
15. $\frac{1}{2}(ax+bz+cy)$ = Area of ΔABC

$$\Rightarrow \frac{a}{2}(2004) = \frac{\sqrt{3}}{4}a^2$$

$$\frac{\sqrt{3}}{2}a = \text{Length of altitude} = 2004$$

Hence, the correct answer is (1).

16.



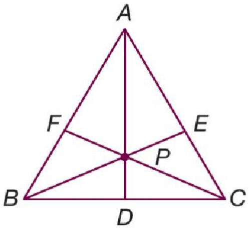
$$\text{Slope of } GC = \frac{-2b}{a}, \text{ slope of } AG = \frac{-b}{2a}$$

$$\tan 30° = \frac{3b/2a}{1+\frac{b^2}{a^2}} \text{ and } a^2+b^2 = 9$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{3ab}{2(a^2+b^2)} \Rightarrow \frac{1}{2}ab = \left(\frac{a^2+b^2}{3\sqrt{3}}\right) = \frac{9}{3\sqrt{3}} = \sqrt{3}$$

Hence, the correct answer is (3).

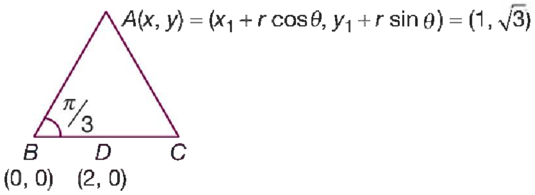
17.



There exist exactly 4 triangles ABC, APC, APB, BPC satisfying the condition of which three are obtuse.

Hence, the correct answer is (4).

18.



$$B(0, 0) \quad \angle ABC = \frac{\pi}{3}, AB = 2 = r$$

$$A(1, \sqrt{3})\ B(0, 0)\ C(4, 0)$$

$$G = \left(\frac{5}{3}, \frac{1}{\sqrt{3}}\right)$$

Hence, the correct answer is (6).

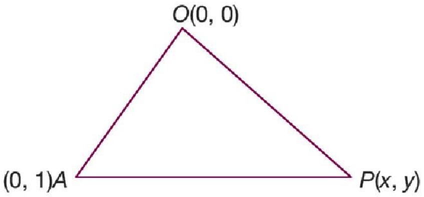
ARCHIVE: JEE MAIN

1. P is centroid of the triangle ABC $P = \left(\frac{17}{6}, \frac{8}{3}\right)$

$\Rightarrow \quad PQ = 5$

Hence, the correct answer is (5).

2.



Given that,

$$OA + AP + OP = 4$$

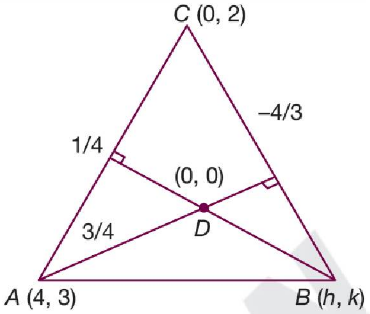
$$1 + \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} = 4$$

$$(\sqrt{x^2 + y^2})^2 = (3 - \sqrt{x^2 + (y-1)^2})^2$$

$$9x^2 + 8y^2 - 8y = 16$$

Hence, the correct answer is (B).

3.



BC and AD are perpendicular

$$\Rightarrow \quad \frac{(k-2)}{h} = -\frac{4}{3} \Rightarrow 3k - 6 = -4h \qquad \ldots(1)$$

BH is perpendicular to AC $\frac{k}{h} = -4 \Rightarrow k = -4h \qquad \ldots(2)$

$$\Rightarrow \quad -12h - 6 = -4h \Rightarrow -6 = 8h \Rightarrow h = -\frac{3}{4}$$

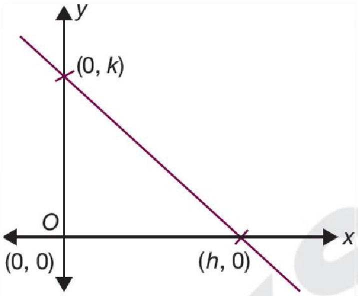
$$k = -4\left(\frac{-3}{4}\right) \quad \Rightarrow k = 3$$

$$B\left(\frac{-3}{4}, 3\right)$$

$\therefore$ The third vertex lies in II^{nd} quadrant.

Hence, the correct answer is (C).

4.



$$\text{Area} = \frac{1}{2}(h)(k) = 50$$

$$(h)(k) = 100 = (2^2)(5^2)$$

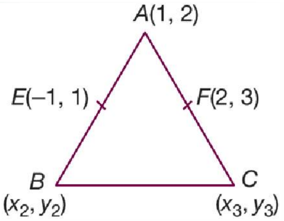
$$\text{Total divisors} = (2+1)(2+1) = 9$$

if $h > 0, k > 0$

However, $\left.\begin{matrix} h > 0, & k < 0 \\ h < 0, & k > 0 \\ h < 0, & k < 0 \end{matrix}\right\}$ all are possible so that the total number of positive cases $9 + 9 + 9 + 9 = 36$

Hence, the correct answer is (B).

5.



$$\frac{x_2 + 1}{2} = -1 \text{ and } \frac{y_2 + 2}{2} = 1$$

$$\frac{x_3 + 1}{2} = 2 \text{ and } \frac{y_3 + 2}{2} = 3$$

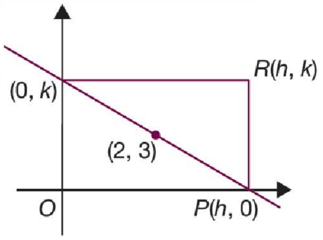
$x_2 = -3, y_2 = 0 \qquad x_3 = 3, y_3 = 4$

$B(-3, 0) \qquad C(3, 4)$

$$\text{Centroid}\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{1}{3}, 2\right)$$

Hence, the correct answer is (A).

6.



Equation of PQ is $\frac{x}{h}+\frac{y}{k}=1$ ...(1)

Since, (1) passes through the fixed point (2, 3), $\frac{2}{h}+\frac{3}{k}=1$

Then, the locus of R is $\frac{2}{h}+\frac{3}{k}=1$ or $3x+2y=xy$

Hence, the correct answer is (B).

7. Since the orthocentre of the triangle is $A(-3, 5)$ and centroid of the triangle is $B(3, 3)$, then

$$AB=\sqrt{40}=2\sqrt{10}$$

Centroid divides orthocentre and circumcentre of the triangle in the ratio 2:1.

$\therefore$ $AB:BC=2:1$

Now, $AB=\frac{2}{3}AC$

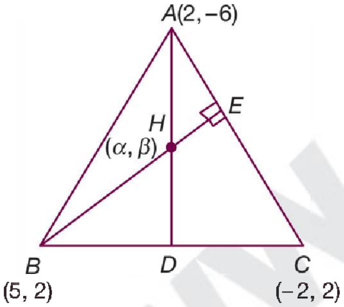
$$\Rightarrow AC=\frac{3}{2}AB=\frac{3}{2}(2\sqrt{10})\Rightarrow AC=3\sqrt{10}$$

$\therefore$ Radius of circle with AC as diameter

$$=\frac{AC}{2}=\frac{3}{2}\sqrt{10}=3\sqrt{\frac{5}{2}}$$

Hence, the correct answer is (B).

8.



We have,

$$\frac{1}{2}\begin{vmatrix} k & -3k & 1\\ 5 & k & 1\\ -k & 2 & 1\end{vmatrix}=\pm 28$$

$\Rightarrow$ $5k^2+13k-46=0$ or $5k^2+13k+66=0$ (no real root)

Now, $5k^2+13k-46=0$

$$\therefore \quad k=\frac{-23}{5}; k=2$$

Since k is an integer, $k=2$

For orthocentre,

$BH \perp AC$

$$\therefore \quad \left(\frac{\beta-2}{\alpha-5}\right)\left(-\frac{8}{4}\right)=-1 \Rightarrow \alpha-2\beta=1$$

Also $CH \perp AB$

$$\therefore \quad \left(\frac{\beta-2}{\alpha+2}\right)\left(\frac{8}{3}\right)=-1 \Rightarrow 3\alpha+8\beta=10$$

Solving (1) and (2), we get,

$$\alpha=2, \beta=\frac{1}{2}$$

$\therefore$ The orthocentre is $\left(2,\frac{1}{2}\right)$

Hence, the correct answer is (A).

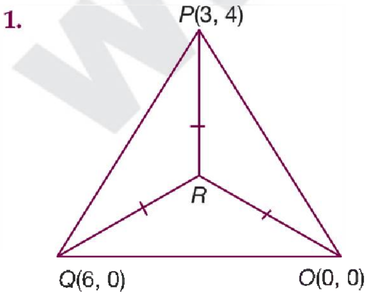
9. $P\left(\frac{6+2}{5},\frac{12+2}{5}\right)=\left(\frac{8}{5},\frac{14}{5}\right)$, now it lies on $2x+y=k$

$$\Rightarrow \quad k=\frac{16}{5}+\frac{14}{5}=6$$

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

1.



$\because$ $\text{Ar}(\Delta OPR)=\text{Ar}(\Delta PQR)=\text{Ar}(\Delta OQR)$

$\therefore$ By simple geometry, R should be the centroid of ΔPQO

$$\Rightarrow \quad R\left(\frac{3+6+0}{3},\frac{4+0+0}{3}\right)=\left(3,\frac{4}{3}\right)$$

Hence, the correct answer is (C).

CHAPTER 2: STRAIGHT LINE

Test Your Concepts-I

1. The equation of the required line will be $y-3=\frac{3+2}{-1-4}(x+1)$.

$\Rightarrow \quad x+y-2=0$

2. Here, $m=-1$ and $c=-4$.

So, the equation of the line is $y=mx+c$

$\Rightarrow \quad y=-x-4$

$\Rightarrow \quad x+y+4=0$

3. Given, (0, 0) is the one end of the diagonal. The intersection of the other two lines, $x+y=1$ and $6x+y=3$ is $\left(\frac{2}{5},\frac{3}{5}\right)$, which is the other end of the diagonal.

$\therefore$ The equation of the diagonal that passes through these two points is $3x-2y=0$.

4. Any point A on the 1st line is $(t, 5t-4)$.

Any point B on the 2nd line is $\left(r,\frac{3r-4}{4}\right)$.

Hence, $1=\frac{2r+t}{3}$ and $5=\frac{\frac{3r-4}{2}+5t-4}{3}$

$\Rightarrow \quad 2r+t=3$ and $3r+10t=42$

On solving, we get $t=\frac{75}{17}$.

Hence, the point A is $\left(\frac{75}{17},\frac{307}{17}\right)$.

5. Here, $\cos\theta=-\frac{1}{3}$ (a negative number) so that, $\frac{\pi}{2}<\theta<\pi$

$\Rightarrow \quad \tan\theta=-\sqrt{8}$ = slope of line.

Equation of the straight line passing through the point (x_1, y_1) having slope m is $y-y_1=m(x-x_1)$.

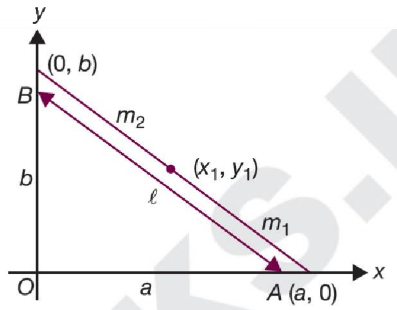
$\therefore$ The equation of the required line is

$y-2=-\sqrt{8}\,(x-1)$

$\Rightarrow \quad \sqrt{8}x+y-\sqrt{8}-2=0$

6. Let (x_1,y_1) be the point that divides the rod $AB=\ell$, in the ratio $m_1:m_2$ and $OA=a$, $OB=b$ (say)

$\therefore \quad a^2+b^2=\ell^2 \qquad \ldots(1)$



Now, $x_1=\left(\frac{m_2 a}{m_1+m_2}\right)\Rightarrow a=\left(\frac{m_1+m_2}{m_2}\right)x_1$

$y_1=\left(\frac{m_1 b}{m_1+m_2}\right)\Rightarrow b=\left(\frac{m_1+m_2}{m_1}\right)y_1$

Putting these values in (1), we get, $\frac{(m_1+m_2)^2}{m_2^2}x_1^2+\frac{(m_1+m_2)^2}{m_1^2}y_1^2=\ell^2$

$\therefore$ Locus of the point on the rod is $m_1^2x^2+m_2^2y^2=\left(\frac{m_1m_2\ell}{m_1+m_2}\right)^2$.

7. Let the line $x+y=5$ divides AB in the ratio k:1 at P.

$\therefore$ Co-ordinates of P are $\left(\frac{-3k+1}{k+1},\frac{4k+2}{k+1}\right)$

$\because$ P lies on $x+y-5=0$

$\therefore \quad \frac{-3k+1}{k+1}+\frac{4k+2}{k+1}-5=0$

$\Rightarrow \quad k=-\frac{1}{2}$

$\therefore$ The required ratio is 1:2 (externally).

Alternate

Let the ratio is m:n using direct formula

$\therefore \quad \frac{m}{n}=-\frac{(1\times1+1\times2-5)}{1\times(-3)+1\times4-5}=-\frac{1}{2}$

$\therefore$ The required ratio is 1:2 (externally).

8. Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1 \qquad \ldots(1)$

As this passes through (3, 4), then $\frac{3}{a}+\frac{4}{b}=1 \qquad \ldots(2)$

It is given that $a+b=14$

$\Rightarrow \quad b=14-a$

Putting $b = 14 - a$ in (2), we get,

$$\frac{3}{a} + \frac{4}{14-a} = 1$$

$\Rightarrow \quad a^2 - 13a + 42 = 0$

$\Rightarrow \quad (a-7)(a-6) = 0$

$\Rightarrow \quad a = 7, 6$

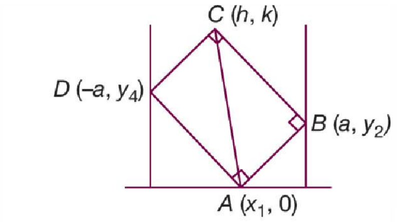
For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$.

Putting the values of a and b in (1), we get the equations of the lines as

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{y}{6} + \frac{y}{8} = 1.$$

$\Rightarrow \quad x + y = 7$ and $4x + 3y = 24$

9. Let A be $(x_1, 0)$, B be (a, y_2) and D be $(-a, y_4)$. We are given AB and AD with fixed directions and hence their slopes are constants, i.e., m and m_1 (say).



$\therefore \quad \frac{y_2}{a - x_1} = m$ and $\frac{y_4}{-a - x_1} = m_1$

Further, $mm_1 = -1$. ($\because$ $ABCD$ is a rectangle)

$$\frac{y_2}{a - x_1} = m \text{ and } \frac{y_4}{-a - x_1} = -\frac{1}{m}$$

The midpoint of BD is $\left(0, \frac{y_2 + y_4}{2}\right)$ and midpoint of AC is $\left(\frac{x_1 + h}{2}, \frac{k}{2}\right)$, where C is taken to be (h, k). This gives $h = -x_1$ and $k = y_2 + y_4$.

So, C is $(-x_1, y_2 + y_4)$.

Also, $\frac{y_2}{a - x_1} = m$ and $\frac{y_4}{a + x_1} = \frac{1}{m}$, which gives $m(k - y_2)$ $= a + x_1 = m(k - m(a - x_1)) = a + x_1$

$\Rightarrow \quad mk - m^2(a - x_1) = a + x_1$

$\Rightarrow \quad m^2(a + h) - mk + a - h = 0$

$\Rightarrow \quad (m^2 - 1)h - mk = -(m^2 + 1)a$

$\Rightarrow \quad (1 - m^2)h + mk = (m^2 + 1)a$

$\Rightarrow \quad (1 - m^2)x + my = (m^2 + 1)a$

$\therefore$ The locus of C is $(1 - m^2)x + my = (m^2 + 1)a$.

10. Equation of the line, having intercepts a and b on the x and the y axes respectively, is $\frac{x}{a} + \frac{y}{b} = 1$.

It passes through (1, 1) $\Rightarrow \frac{1}{a} + \frac{1}{b} = 1$

$\because$ The area of the triangle formed by the lines and the axes is A, then

$ab = 2A$.

From (1) and (2), we get $a + b = 2A$.

Here, a, b are the roots of $x^2 - (a+b)x + ab = 0$ or $x^2 - 2Ax + 2A = 0$.

11. Let $P(h, k)$ be the point of intersection of the given lines.

Then, $h\cos\alpha + k\sin\alpha = a$ …(1)

$h\sin\alpha - k\cos\alpha = b$ …(2)

Here α is a variable so we have to eliminate α.

Squaring and adding (1) and (2), we get,

$(h\cos\alpha + k\sin\alpha)^2 + (h\sin\alpha - k\cos\alpha)^2 = a^2 + b^2$

Hence, the locus of (h, k) is $x^2 + y^2 = a^2 + b^2$.

12. Let the required straight line be $\left(\frac{x}{a}\right) + \left(\frac{y}{b}\right) = 1$

Using the given conditions, $P\left(\frac{2a + (1)(0)}{2+1}, \frac{(2)(0) + (1)(b)}{2+1}\right)$ is the point, which divides $(a, 0)$ and $(0, b)$ internally in the ratio 1:2.

However, P is $(-5, 4)$.

Hence, $-5 = \frac{2a}{3}$ and $4 = \frac{b}{3}$.

$\Rightarrow \quad a = -\frac{15}{2}, b = 12$

$\therefore \quad$ The required equation is $-\frac{2x}{15} + \frac{y}{12} = 1$.

13. Let the equation of the variable line be $\frac{x}{a} + \frac{y}{b} = 1$. This meets the co-ordinate axes at $A(a, 0)$ and $B(0, b)$. Let $P(h, k)$ be the point, which divides AB in the ratio 2:1.

Then, the co-ordinates of P are $h = \frac{a}{3}$ and $k = \frac{2b}{3}$

$\Rightarrow \quad a = 3h$ and $b = \frac{3k}{2}$. Here, a and b are the variables.

$\because$ The line passes through $\left(\frac{a}{3}, \frac{2b}{3}\right)$, $\frac{6}{5a} + \frac{6}{5b} = 1$ …(1)

Putting the value of a and b in (1), we get $\frac{6}{15h}+\frac{12}{15k}=1 \Rightarrow \frac{2}{5h}+\frac{4}{5k}=1$

$\Rightarrow \quad 2k + 4h = 5\,hk$

Hence, the locus of (h, k) is $5\,xy = 2(2x + y)$.

14. Let the equation of the 3rd line be

$(ax+by+c)+\lambda(lx+my+n)=0$, where λ is a parameter.

It meets the x-axis at A, where $y = 0$ and $x=\frac{-(\lambda n+c)}{\lambda l+a}$.

Also, it meets the y-axis at B, where $x = 0$ and $y=-\frac{(\lambda n+c)}{\lambda m+b}$.

The ΔOAB is a right angled triangle. Its circumcentre is the midpoint of the hypotenuse.

Let it be (α, β)

$$\Rightarrow \quad 2\alpha=-\frac{\lambda n+c}{\lambda l+a} \text{ and } 2\beta=-\frac{\lambda n+c}{\lambda m+b}$$

$$\Rightarrow \quad \lambda=-\frac{2\alpha a+c}{2\alpha l+n}=-\frac{2\beta b+c}{2\beta m+n}$$

Hence, the locus of (α, β) is $\frac{c+2ax}{n+2lx}=\frac{c+2by}{n+2my}$

$$\Rightarrow \quad 2xy(ma-bl)+x(an-lc)+y(mc-bn)=0$$

Test Your Concepts-II

1. Slope of the line joining the points $(3, -2)$ and $(1, 4) = -3$

$\therefore$ The required straight line is $y = -3x + 3$.

2. The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$2x - 3y + \lambda = 0$ ...(1)

This passes through the point $(3, 4)$

$\therefore \quad 3\times 2-3\times 4+\lambda=0$

$\Rightarrow \quad \lambda=6$

Putting $\lambda = 6$ in (1), we get $2x - 3y + 6 = 0$.

3. Given line is $2x - 3y = 7$

$3y=2x-7 \Rightarrow y=\frac{2}{3}x-\frac{7}{3}$ ...(1)

$\therefore \quad$ Slope of (1) is $\frac{2}{3}$

The required line is parallel to (1), so its slope is also $\frac{2}{3}$ and the y-intercept of required line is 4.

$\therefore$ By using $y=mx+c$ form, the equation of the required line is $y=\frac{2}{3}x+4$.

$\therefore$ The required equation is $2x - 3y + 12 = 0$.

4. Since 'P' is the orthocentre, then $m_{OB}.m_{AP}=-1$, $m_{OP}.m_{AB}=-1$ and $m_{OA}.m_{BP}=-1$.

$$\Rightarrow \quad \frac{4}{3}.\frac{4-k}{1-h}=-1,\ \frac{4}{1}.\frac{4-k}{3-h}=-1,\ \frac{k}{h}.\frac{4-4}{3-1}=-1$$

$$\Rightarrow \quad 16-4k=3h-3,\ 16-4k=h-3,\ h=0$$

$$\Rightarrow \quad h=0,\ k=\frac{19}{4}$$

Hence, the point A is $\left(0, \frac{19}{4}\right)$.

5. $2(b'-b)y+2(a'-a)x=\left(b'^2-b^2+a'^2-a^2\right)$

Equation of AB is $y-b=\frac{b'-b}{a'-a}(x-a)$

i.e., $y(a'-a)-x(b'-b)=a'b-ab'$

Equation to the line perpendicular to AB is of the form $(b'-b)y+(a'-a)x+k=0$.

$\because \quad$ The midpoint of AB lies on (1),

$$(b'-b)\left(\frac{b+b'}{2}\right)+(a'-a)\left(\frac{a+a'}{2}\right)+k=0$$

Hence, the required equation of the straight line is

$$2(b'-b)y+2(a'-a)x=\left(b'^2-b^2+a'^2-a^2\right).$$

6. $y=3x-k,\ y=3x-\frac{k}{2}$ and $m_1=m_2$.

(Constant terms are also distinct. So, they cannot be coincident lines and hence they are parallel)

7. Slope of the line in the new position is $\frac{b}{a}$, since it is perpendicular to the line $ax + by + c = 0$ and it cuts the x-axis. Hence, the required line passes through $(2, 0)$ and its slope is $\frac{b}{a}$.

The required equation is $y-0=\frac{b}{a}(x-2)$

$\Rightarrow \quad ay = bx - 2b \Rightarrow ay - bx + 2b = 0$

Test Your Concepts-III

1. Let A, B and C are the vertices of the triangle.

$$A(-7, 5),\ B\left(\frac{5}{4}, \frac{7}{8}\right),\ C\left(\frac{1}{3}, \frac{1}{9}\right).$$

Sign of A w.r.t BC is negative. If P lies inside the ΔABC, then sign of P will be the same as the sign of A w.r.t the line BC.

$\Rightarrow \quad 5\alpha-6\alpha^2-1<0$ ...(1)

Similarly, $2\alpha + 3\alpha^2 - 1 > 0$ ...(2)

and $\alpha + 2\alpha^2 - 3 < 0$ ...(3)

Solving (1), (2) and (3) and then taking intersection,

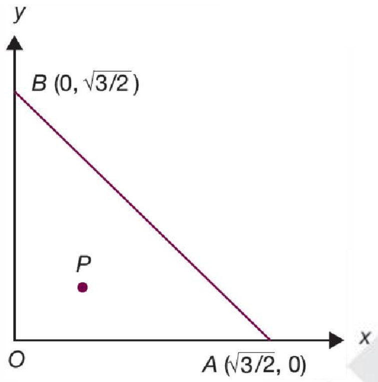
we get, $\alpha \in \left(\frac{1}{2}, 1\right) \cup \left(-\frac{3}{2}, -1\right)$.

2. At (1, 4), the value of $x + 3y + 7 = 1 + 3(4) + 7 = 20 > 0$.

At (0, –3), the value of $x + 3y + 7 = 0 + 3(-3) + 7 = -2 < 0$.

$\therefore$ The points (1, 4) and (0, – 3) are on the opposite sides of the given line.

3. Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{\frac{3}{2}}$, respectively.



Now, P (sinθ, cosθ) must be in 1st quadrant.

So, $\cos\theta > 0$ and $\sin\theta > 0$.

$$\therefore \quad 0 < \theta < \frac{\pi}{2}$$

Now, P and O lie on the same side of $x + y = \sqrt{\frac{3}{2}}$

$$\therefore \quad \sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$$

Hence, P will lie inside the ΔABC, if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

As both $\sin\theta$ and $\cos\theta$ are positive, $\theta \in \left(0, \frac{\pi}{2}\right)$. Now, $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$

$$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{4}\right) < \frac{\sqrt{3}}{2}$$

i.e., $0 < \theta + \frac{\pi}{4} < \frac{\pi}{3}$ or $\frac{2\pi}{3} < \theta + \frac{\pi}{4} < \pi$

$$\because \quad \sin\theta > 0 \text{ and } \cos\theta > 0,\ 0 < \theta < \frac{\pi}{12} \text{ or } \frac{5\pi}{12} < \theta < \frac{3\pi}{4}$$

4. Slope of the given line is $1 = \tan\frac{\pi}{4}$.

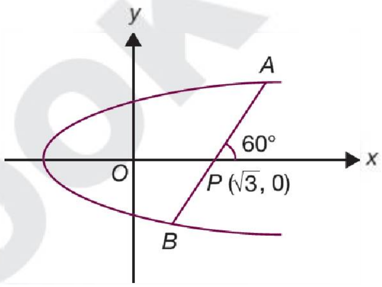
Equation of the straight line can be written as $y - 2 = x$

or $\quad \dfrac{y-2}{\frac{1}{\sqrt{2}}} = \dfrac{x}{\frac{1}{\sqrt{2}}} = r$.

Any point on the line is $\left(\frac{r}{\sqrt{2}}, 2 + \frac{r}{\sqrt{2}}\right)$.

The point (x, y) is at a distance r from the point (0, 2).

5. Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$.



If line makes an angle q with x-axis, then $\tan\theta = \sqrt{3}$

$$\therefore \quad \theta = 60^\circ$$

$$\frac{x - \sqrt{3}}{\cos 60^\circ} = \frac{y - 0}{\sin 60^\circ} = r$$

$\Rightarrow \quad \left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ be a point on the parabola $y^2 = x + 2$

Then, $\frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$$\therefore \quad PA.PB = r_1 r_2 = \left|\frac{-4(2+\sqrt{3})}{3}\right| = \frac{4(2+\sqrt{3})}{3}$$

6. The equation of a line through A and making an angle of 45° with the x-axis is $\dfrac{x-2}{\cos 45^\circ} = \dfrac{y-3}{\sin 45^\circ}$.

$$\Rightarrow \quad \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} \Rightarrow x - y + 1 = 0$$

Suppose if this line meets the line $x + y + 1 = 0$ at P, such that $AP = r$, then the co-ordinates of P are given by $\dfrac{x-2}{\cos 45^\circ} = \dfrac{y-3}{\sin 45^\circ} = r$.

$$\Rightarrow \quad x = 2 + r\cos 45^\circ,\ y = 3 + r\sin 45^\circ$$

$$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}} \text{ and } y = 3 + \frac{r}{\sqrt{2}}$$

Thus, the co-ordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$.

$\because$ P lies on $x+y+1=0$, so $2+\dfrac{r}{\sqrt{2}}+3+\dfrac{r}{\sqrt{2}}+1=0.$

$\Rightarrow \quad \sqrt{2}\,r=-6 \Rightarrow r=-3\sqrt{2}$

$\Rightarrow \quad$ Length $AP=|\,r\,|=3\sqrt{2}$

Thus, the length of the intercept $=3\sqrt{2}$ units

7. Slope of $AB(m)=1$

$\Rightarrow \quad m=\tan\theta=1 \Rightarrow \theta=45°$

So that, the slope of $AC=\tan(45°+15°)=\tan 60°$

$(\because$ Angle between AB and $AC=15°)$

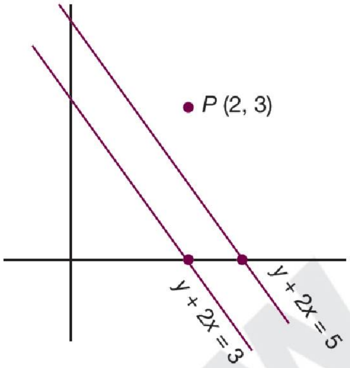
Also, $AB=AC=\sqrt{2}$ and A is (2, 0).

Hence, the equation of the line AC is $\dfrac{x-2}{\cos 60°}=\dfrac{y-0}{\sin 60°}$

$$\Rightarrow \quad \frac{x-2}{\frac{1}{2}}=\frac{y}{\frac{\sqrt{3}}{2}}=r=\sqrt{2}$$

$$\Rightarrow \quad C \text{ is } \left(2+\sqrt{2}.\frac{1}{2}, 0+\sqrt{2}.\frac{\sqrt{3}}{2}\right), \text{ i.e., } C \text{ is } \left(2+\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$$

8.



$y+2x=3$ ...(1)

$y+2x=5$ ...(2)

Equation of the line passing through $P(2, 3)$ is

$$\frac{x-2}{\cos\theta}=\frac{y-3}{\sin\theta}=r \qquad \text{...(3)}$$

The points of intersection of (3) with (2) and (1) are $A(2+r\cos\theta, 3+r\sin\theta)$ and $B(2+(r-2)\cos\theta, 3+(r-2)\sin\theta)$, respectively for some r.

$\therefore \quad (3+r\sin\theta)+2(2+r\cos\theta)=3$ ...(4)

$\quad (3+(r-2)\sin\theta)+2(2+(r-2)\cos\theta)=5$ ...(5)

By (4) and (5),

$\sin\theta+2\cos\theta=-1 \Rightarrow \sin\theta=-1-2\cos\theta$

$\sin^2\theta=(-1-2\cos\theta)^2 \Rightarrow \cos\theta(5\cos\theta+4)=0$

$\therefore \cos\theta=0$ or $\cos\theta=-\dfrac{4}{5}$ and accordingly $\sin\theta=1$ or $\dfrac{3}{5}$.

$\therefore$ Equation (3) gives the equations of required lines.

9. Clearly, symmetric or parametric form shall be most helpful in such questions.

$x=x_1+r\cos\alpha;\ y=y_1+r\sin\alpha$

$\alpha=\dfrac{2\pi}{3},\ x_1=1,\ y_1=3,\ r=\pm 3\sqrt{2}$

So points are

$$x=1\pm 3\sqrt{2}\cos\frac{2\pi}{3}=1\pm 3\sqrt{2}\left(-\frac{1}{2}\right)=1\mp\frac{3}{\sqrt{2}}$$

$$y=3\pm 3\sqrt{2}\sin\frac{2\pi}{3}=3\pm 3\sqrt{2}\left(\frac{\sqrt{3}}{2}\right)=3\pm\frac{3\sqrt{3}}{\sqrt{2}}$$

$\therefore \left(1+\dfrac{3}{\sqrt{2}}, 3-\dfrac{3\sqrt{3}}{\sqrt{2}}\right)$ and $\left(1+\dfrac{3}{\sqrt{2}}, 3-\dfrac{3\sqrt{3}}{\sqrt{2}}\right)$ are the required points.

10. It is obvious that the desired points lie on $3x+2y=5$ as well as on the bisectors of angles formed by the lines $4x+3y=7$ and $2y-5=0$.

Equations to the angle bisectors are

$$\frac{4x+3y-7}{5}=\pm\frac{2y-5}{2}$$

$\Rightarrow \quad 8x+6y-14=\pm(10y-25)$

$\Rightarrow \quad 8x+16y-39=0;\quad 8x-4y+11=0$

Solving $3x+2y=5$ and $8x+16y=39$,

we get $x=\dfrac{1}{16},\ y=\dfrac{77}{32}$

Solving $3x+2y=5$ and $8x-4y+11=0$,

we get $x=-\dfrac{1}{14},\ y=\dfrac{73}{28}$

Test Your Concepts-IV

1. Let Q be the point of intersection of the incident ray and the line mirror, then

$x_1-2y_1-3=0$ and $3x_1-2y_1-5=0.$

On solving these equations, we get, $x_1=1$ and $y_1=-1$.

$\because$ $P(-1, -2)$ be a point lies on the incident ray, so we can find the image of the point P on the reflected ray about the line mirror (by property of reflection).

Let $P'(h, k)$ be the image of point P about line mirror, then

$$\frac{h+1}{3}=\frac{k+2}{-2}=\frac{-2(-3+4-5)}{13}$$

⇒ $h=\dfrac{11}{13}$ and $k=\dfrac{-42}{13}$

So, $P'\left(\dfrac{11}{13},\dfrac{-42}{13}\right)$

Then, equation of reflected ray will be $(y+1)$

$$=\frac{\left(\frac{-42}{13}+1\right)(x-1)}{\left(\frac{11}{13}-1\right)}$$

∴ $2y-29x+31=0$ is the required equation of reflected ray.

2. Equation of line through (4, –5) with slope of m is $y+5=m(x-4)$.

⇒ $mx-y-4m-5=0$

Then, $\dfrac{|m(-2)-3-4m-5|}{\sqrt{m^2+1}}=12$

⇒ $|-6m-8|=12\sqrt{(m^2+1)}$

On squaring, $(6m+8)^2=144(m^2+1)$

⇒ $4(3m+4)^2=144(m^2+1)$

⇒ $(3m+4)^2=36(m^2+1)$

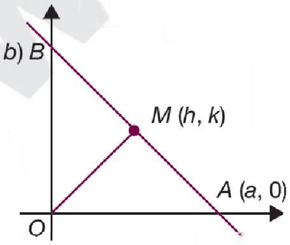
⇒ $27m^2-24m+20=0$ ...(1)

∵ The discriminant of (1) is $(-24)^2-4.27.20=-1584$, which is negative.

∴ There is no real value of m.

Hence, no such line is possible.

3. Let the equation of AB be $\dfrac{x}{a}+\dfrac{y}{b}=1$ ...(1)



Given, $a+b=K$ ...(2)

Now, $m_{AB}\times m_{OM}=-1$

⇒ $ah=bk$ ...(3)

From (2) and (3),

$$a=\frac{kK}{h+k}\text{ and }b=\frac{hK}{h+k}$$

We get $\dfrac{x(h+k)}{k.K}+\dfrac{y(h+k)}{h.K}=1$ ... from (1)

As it passes through (h, k),

$$\frac{h(h+k)}{k.K}+\frac{k(h+k)}{h.K}=1$$

⇒ $(h+k)(h^2+k^2)=Khk$

∴ Locus of (h, k) is $(x+y)(x^2+y^2)=Kxy$.

4. Let $x=t$ in the equation $x+y=4$ of the given line, we obtain $y=4-t$. So, the co-ordinates of an arbitrary point on the given line are $P(t, 4-t)$. Let, $P(t, 4-t)$ be the required point. Then, the distance of P from the line $4x+3y-10=0$ is unity.

⇒ $\left|\dfrac{4t+3(4-t)-10}{\sqrt{4^2+3^2}}\right|=1$

⇒ $|t+2|=5\Rightarrow t+2=\pm 5$

⇒ $t=-7$ or $t=3$

Hence, the required points are (–7, 11) and (3, 1).

5. Clearly, the length of the side of the square is equal to the distance between the parallel lines.

$x+y-1=0$ ...(1)

$x+y+2=0$ ...(2)

Putting $x=0$ in (1), we get $y=1$. So, (0, 1) is a point on the line (1).

Now, distance between the parallel lines

= length of the perpendicular from (0, 1) to $x+y+2$

$=0=\dfrac{|0+1+2|}{\sqrt{1^2+1^2}}=\dfrac{3}{\sqrt{2}}$

Thus, the length of the side of the square is $\dfrac{3}{\sqrt{2}}$ and hence its area $=\left(\dfrac{3}{\sqrt{2}}\right)^2=\dfrac{9}{2}$

6. Let the two perpendicular lines through the origin intersect $2x+y=a$ at A and B, so that the ΔOAB is isosceles.

OM = Length of perpendicular from O to AB,

$OM=\dfrac{a}{\sqrt{5}}$.

Also, $AM=MB=OM$

⇒ $AB=\dfrac{2a}{\sqrt{5}}$

Area of $\Delta OAB=\dfrac{1}{2}AB\cdot OM=\dfrac{1}{2}\cdot\dfrac{2a}{\sqrt{5}}\cdot\dfrac{a}{\sqrt{5}}=\dfrac{a^2}{5}$

7. $2x - y = 0; \left|\dfrac{2a}{\sqrt{5}}\right| = 2$

$\Rightarrow \quad a = \sqrt{5}$

8. Let the line be $lx + my + 1 = 0$. ...(1)

According to the given condition,

$$\frac{2l+1+2m+1+l+m+1}{\sqrt{l^2+m^2}} = 0$$

$\Rightarrow \quad l+m+1=0 \Rightarrow l(1)+m(1)+1=0$.

Comparing it with (1), we find that line (1) is passing through (1, 1).

9. Equation of the line BC is $x + y - 2 = 0$. Let, PG and AD be perpendiculars from P and A on BC.

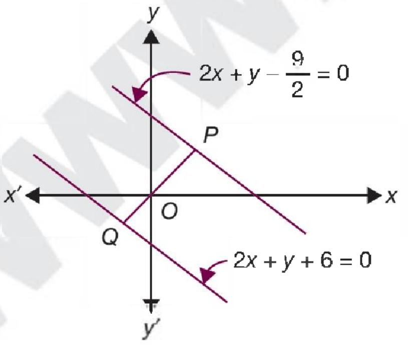
Ratio of areas of ΔPBC and ΔABC is $\dfrac{\frac{1}{2}.BC.PG}{\frac{1}{2}.BC.AD} = \dfrac{PG}{AD}$.

However, PG is the length of the perpendicular from $P(x, y)$ on $x + y - 2 = 0$ and AD is the length of the perpendicular from $A(6, 3)$ on $x + y - 2 = 0$.

$\Rightarrow \quad$ Ratio of the areas of ΔPBC and ΔABC

$$= \frac{\left|\dfrac{x+y-2}{\sqrt{2}}\right|}{\left|\dfrac{6+3-2}{\sqrt{2}}\right|} = \left|\frac{x+y-2}{7}\right|$$

10. Taking PQ in any direction, the ratio of OP and OQ will be the same, so taking $PQ \perp$ to the given parallel lines



OP = Perpendicular from (0, 0) on $2x + y - \dfrac{9}{2} = 0$

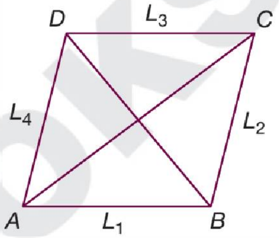
$$= \left|\frac{-\frac{9}{2}}{\sqrt{4+1}}\right| = \frac{9}{2\sqrt{5}}$$

OQ = Perpendicular from (0, 0) on $2x + y + 6 = 0$

$$= \left|\frac{6}{\sqrt{4+1}}\right| = \frac{6}{\sqrt{5}}$$

$$\therefore \quad \frac{OP}{OQ} = \frac{\frac{9}{2\sqrt{5}}}{\frac{6}{\sqrt{5}}} = \frac{9}{12} = \frac{3}{4}$$

11. As L_1 is not perpendicular to L_2, so it cannot be a square or a rectangle.



Diagonal (AC) $L_1L_2 - L_3L_4 = 0$

$$\left(\frac{x}{3}+\frac{y}{4}-1\right)\left(\frac{x}{4}+\frac{y}{3}-1\right)-\left(\frac{x}{3}+\frac{y}{4}-2\right)\left(\frac{x}{4}+\frac{y}{3}-2\right)=0$$

$\Rightarrow \quad 7x + 7y = 36$

Diagonal (BD) $L_1L_4 - L_2L_3 = 0$

$$\Rightarrow \quad \left(\frac{x}{3}+\frac{y}{4}-1\right)\left(\frac{x}{4}+\frac{y}{3}-2\right)-\left(\frac{x}{4}+\frac{y}{3}-1\right)\left(\frac{x}{3}+\frac{y}{4}-2\right)=0$$

$$\Rightarrow \quad -\left(\frac{x}{3}+\frac{y}{4}\right)+\left(\frac{x}{4}+\frac{y}{3}\right)=0$$

$\Rightarrow \quad x - y = 0$

$BD \perp AC$

As the diagonals are perpendicular, the lines form a rhombus.

Test Your Concepts-V

1. As the given lines are concurrent,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0 \qquad \text{...(1)}$$

As $ax + by + p = 0$ and include an angle $\dfrac{\pi}{4}$,

$$\pm\tan\frac{\pi}{4} = \frac{-\dfrac{a}{b}+\dfrac{\cos\alpha}{\sin\alpha}}{1+\dfrac{a\cos\alpha}{b\sin\alpha}}$$

$\Rightarrow \quad -a\sin\alpha + b\cos\alpha = \pm(b\sin\alpha + a\cos\alpha)$

$\Rightarrow \quad -a \sin \alpha + b \cos \alpha = \pm 1$ (from (1)) ...(2)

Squaring and adding (1) and (2), we get $a^2 + b^2 = 2$.

2. As the lines are concurrent, $-2 = m(a + m)$

$\Rightarrow \quad m^2 + am + 2 = 0$

$\because$ m is real, $a^2 \geq 8$ and $|a| \geq 2\sqrt{2}$.

3. The 4 sides of the rhombus are

$ax + by + c = 0, ax + by - c = 0, ax - by + c = 0$

and $ax - by - c = 0$.

On solving these equations, we get the vertices as

$$A\left(\frac{c}{a}, 0\right),\ B\left(0, \frac{c}{b}\right),\ C\left(\frac{-c}{a}, 0\right) \text{ and } D\left(0, \frac{-c}{b}\right).$$

The length of the diagonal AC is $\frac{2c}{a}$ and that of the diagonal BD is $\frac{2c}{b}$.

$\therefore$ The area of the rhombus is $\frac{1}{2}\left(\frac{2c}{a}\right)\left(\frac{2c}{b}\right) = \frac{2c^2}{ab}$.

4. The given equation can be written as $a(x + y - 2) + b(x - y) = 0$.

$\Rightarrow \quad (x + y - 2) + \frac{b}{a}(x - y) = 0$

This is a family of lines concurrent at $x - y = 0$ and $x + y - 2 = 0$.

On solving these two equations, we get (1, 1).

5. Slope of $AD = -t_3$

Equation of AD is $y - a(t_1 + t_2) = -t_3(x - at_1t_2)$ ...(1)

Equation of CF is $y - a(t_3 + t_1) = -t_2(x - at_3t_1)$ ...(2)

Subtracting (1) from (2), we get $x = -a$

$\Rightarrow \quad y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$.

Hence, the point of concurrency of the altitudes is $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

6. Let ABC be the triangle having vertices $(0, b)$, $(-a, 0)$ and $(a, 0)$.

Such that, $BE.CF = AB^2$

$$\Rightarrow \quad \frac{BE}{AB} = \frac{AB}{CF}$$

$$\Rightarrow \quad \frac{BE}{AB} = \frac{AC}{CF} = \frac{\lambda}{1}$$

Hence, the co-ordinates of E and F are $(-a(\lambda + 1), -\lambda b)$ and $\left[a\left(1 + \frac{1}{\lambda}\right), \frac{-b}{\lambda}\right]$.

Equation of line EF is

$$y + \lambda b = \frac{-\lambda b + \frac{b}{\lambda}}{-a(\lambda + 1) - \frac{a(\lambda + 1)}{\lambda}}[x + a(\lambda + 1)]$$

$$\Rightarrow \quad y + \lambda b = \frac{\frac{b}{\lambda}(1 - \lambda^2)}{-\frac{a(\lambda + 1)}{\lambda}(1 + \lambda)}(x + a[\lambda + 1])$$

$$\Rightarrow \quad y + \lambda b = \frac{b(\lambda - 1)}{a(\lambda + 1)}(x + a[\lambda + 1])$$

$$\Rightarrow \quad a(\lambda + 1)y + ab\lambda(\lambda + 1) = b(\lambda - 1)x + ab(\lambda^2 - 1)$$

$$\Rightarrow \quad (bx + ay + ab) - \lambda(bx - ay - ab) = 0,$$

which is the equation of a family of lines passing through the points of intersection of the lines $bx + ay + ab = 0$ and $bx - ay - ab = 0$, as the point of intersection being $(0, -b)$.

Hence, the line EF passes through a fixed point.

Test Your Concepts-VI

1. According to the question, the equation of the bisectors of the angle between the lines

$$x^2 - 2pxy - y^2 = 0 \quad \text{...(1)}$$

$$x^2 - 2qxy - y^2 = 0 \quad \text{...(2)}$$

$\therefore$ The equation of bisectors of the angle between the line (1) is $\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}$

$\Rightarrow \quad -px^2 - 2xy + py^2 = 0$

$\because$ (2) and (3) are identical, comparing (2) and (3),

we get $\frac{1}{-p} = \frac{-2q}{-2} = \frac{-1}{p}$

$\Rightarrow \quad pq = -1$

2. $\sqrt{3}y - 2px = 1$ is the given chord. By homogenising the equation of the curve, we get, $py^2 - 4x(\sqrt{3}y - 2px) + (\sqrt{3}y - 2px)^2 = 0$.

$\Rightarrow \quad (4p^2 + 8p)x^2 + (p + 3)y^2 - 4xy - 4\sqrt{3}pxy = 0$

Now, angle at the origin is 90°.

∴ Coefficient of x^2 + Coefficient of $y^2 = 0$

∴ $4p^2 + 8p + p + 3 = 0$

⇒ $4p^2 + 9p + 3 = 0$

∴ $p = \frac{-9 \pm \sqrt{81-48}}{8} = \frac{-9 \pm \sqrt{33}}{8}$.

3. Here, $a = \lambda, b = 12, c = -3, f = -8, g = \frac{5}{2}, h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have,

$$\lambda(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - \lambda(64)$$

$$-12\left(\frac{25}{4}\right) + 3(25) = 0$$

⇒ $-36\lambda + 200 - 64\lambda - 75 + 75 = 0$

⇒ $100\lambda = 200$

∴ $\lambda = 2$

4. Let the line be $\frac{x+2}{\cos\theta} = \frac{y+3}{\sin\theta} = r$

⇒ $x = r\cos\theta - 2, y = r\sin\theta - 3$ ...(1)

Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ ...(2)

Taking intersection of (1) with (2) and considering terms of r^2 and constant (as we need $PQ \cdot PR = r_1 \cdot r_2$ = product of the roots),

$$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta)$$

$$+ (\text{some terms})r + 80 = 0$$

∴ $r_1 . r_2 = PQ.\ PR = \frac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$

∴ $\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4$ $(\because PQ.PR = 20)$

∴ $\sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

⇒ $(\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

∴ $\tan\theta = 1, \tan\theta = 3$

Hence, the equation of the line is $y + 3 = 1(x + 2)$

⇒ $x - y = 1$ and $y + 3 = 3(x + 2)$

∴ The equation of the line is $3x - y + 3 = 0$.

5. The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we obtain $a = 6$, $b = 1$ and $2h = -5$.

∴ $h^2 - ab = \frac{25}{4} - 6 = \frac{1}{4} > 0$

⇒ $h^2 > ab$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now, $6x^2 - 5xy + y^2 = 0$.

⇒ $\left(\frac{y}{x}\right)^2 - 5\left(\frac{y}{x}\right) + 6 = 0$

⇒ $\left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2\left(\frac{y}{x}\right) + 6 = 0$

⇒ $\left(\frac{y}{x} - 3\right)\left(\frac{y}{x} - 2\right) = 0$

⇒ $\left(\frac{y}{x} - 3\right) = 0$ or $\left(\frac{y}{x} - 2\right) = 0$

⇒ $y - 3x = 0$ or $y - 2x = 0$

So, the given equation represents the straight lines $y - 3x = 0$ and $y - 2x = 0$.

6. We have $2x^2 - 7xy + 2y^2 = 0$

⇒ $2x^2 - 6xy - xy + 3y^2 = 0$

⇒ $2x(x - 3y) - y(x - 3y) = 0$

⇒ $(x - 3y)(2x - y) = 0$

⇒ $x - 3y = 0$ or $2x - y = 0$

Thus, the given equation represents the lines $x - 3y = 0$ and $2x - y = 0$. The equations of the lines passing through the origin and perpendicular to the given lines are $y - 0 = -3(x - 0)$ and $y - 0 = -\frac{1}{2}(x - 0)$.

($\because$ Slope of $(x - 3)(y = 0)$ is $\frac{1}{3}$ and slope of $2x - y = 0$ is 2)

∴ The required equations are $y + 3x = 0$ and $2y + x = 0$.

7. Given equation is $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$. Writing the equation (1) as a quadratic equation in x, we have,

$$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$$

∴ $x = \frac{-(5y+6) \pm \sqrt{(5y+6)^2 - (4)(2)(3y^2+7y+4)}}{4}$

$= \frac{-(5y+6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4}$

$= \frac{-(5y+6) \pm \sqrt{y^2 + 4y + 4}}{4} = \frac{-(5y+6) \pm (y+2)}{4}$

∴ $x = \frac{-5y - 6 + y + 2}{4}, \frac{-5y - 6 - y - 2}{4}$

$\Rightarrow$ $4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$

$\Rightarrow$ $x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence, equation (1) represents a pair of straight lines whose equations are

$$x + y + 1 = 0 \qquad \text{...(1)}$$

$$2x + 3y + 4 = 0 \qquad \text{...(2)}$$

Solving these two equations, we get the required point of intersection as (1, – 2).

8. Let $y = mx$ ...(1)

be a line through the origin making an angle or 60° with the line

$Ax + By + C = 0.$ ...(2)

Then, we have $\tan 60° = \pm\dfrac{m-\left(-\dfrac{A}{B}\right)}{1+m\left(-\dfrac{A}{B}\right)}.$

From (1), we have $m = \dfrac{y}{x}$

Substituting this value of m in the above result, i.e., $3\left(B - Am^2\right) = (mB + A)^2$, we have

$$\left(A^2 - 3B^2\right)x^2 + 8ABxy + \left(B^2 - 3A^2\right)y^2 = 0 \qquad \text{...(3)}$$

These are the straight lines through the origin making angles of 60° with (2), i.e., forming an equilateral triangle with (2).

Now, OL = length of perpendicular to $Ax + By + C = 0$ from (0, 0)

$$= \frac{C}{\sqrt{\left(A^2 + B^2\right)}}$$

$\Rightarrow$ Area of the triangle $= \dfrac{C^2}{\sqrt{3}\left(A^2 + B^2\right)}$

9. Given lines are $ax + y + 1 = 0$ ...(1)

$$x + by = 0 \qquad \text{...(2)}$$

$$ax + by = 1 \qquad \text{...(3)}$$

Joint equation of (1) and (2) is $(ax + y + 1)(x + by) = 0$

$$\Rightarrow\ ax^2 + by^2 + (ab+1)xy + x + by = 0 \qquad \text{...(4)}$$

Making (4) homogeneous with the help of (1) we have,

$$ax^2 + by^2 + (ab+1)xy + x(ax+by) + by(ax+by) = 0$$

Since the angle between these two lines is 90°, coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore 2a + b + b^2 = 0$ is the required condition.

Single Correct Choice Type Questions

1. $f(\alpha, 2+\alpha) = 5\alpha$

$$f\left(\frac{3\alpha}{2}, \alpha^2\right) = 3\alpha + 3\alpha^2 - 6$$

$$5\alpha\left(3\alpha + 3\alpha^2 - 6\right) < 0$$

$\Rightarrow$ $\alpha(\alpha + 2)(\alpha - 1) < 0$

Hence, the correct answer is (A).

2. $5x + 2y = 7$ is a line passing through the intersection of $x + y = 2$ and $3x + 4y = 7$, which is perpendicular to $2x - 5y = 8$.

Hence, the correct answer is (A).

3. $\because$ $(\lambda, 2)$ lies between $x - 2y + 3 = 0$ and $x - 2y + 4 = 0$, so $(\lambda - 1)(\lambda) < 0$

As $\lambda \in (0,1)$,

$$\log_\lambda\left(x^2 + x\right) > \log_\lambda(2x + 12)$$

$$x^2 + x < 2x + 12$$

$$x^2 - x - 12 < 0$$

$$(x + 3)(x - 4) < 0$$

$\therefore$ $x \in (-3, 4)$

Hence, the correct answer is (B).

4. Area of rectangle $BCDE$ is $4\lambda\mu$ sq. units. Area of ΔABE is $\dfrac{2\lambda(\lambda-\mu)}{2} = \lambda^2 - \lambda\mu$ sq. units.

$\therefore$ The area of pentagon is $4\lambda\mu + \lambda^2 - \lambda\mu = \lambda^2 + 3\lambda\mu$ sq. units.

Hence, the correct answer is (B).

5. $2x - 5y - 20 = 0$

$$3x + my - m = 0$$

$$\Rightarrow\ \frac{x}{25m} = \frac{y}{2(m-30)} = \frac{1}{2m+15}$$

$$\therefore\ x = \frac{25m}{2m+15},\ y = \frac{2(m-30)}{2m+15}$$

$$x > 0 \Rightarrow m\left(m+\frac{15}{2}\right) > 0$$

$$y > 0 \Rightarrow (m-30)\left(m+\frac{15}{2}\right) > 0$$

$$\Rightarrow \quad m > 30 \text{ or } m < -\frac{15}{2}$$

Hence, the correct answer is (D).

6. $\left(\frac{y-3}{x+2}\right)\left(\frac{y-0}{x-2}\right) = -1$

$$\Rightarrow \quad x^2+y^2-3y-4=0$$

Alternate Method

It is the circle described on (–2, 3) (2, 0) as the extremities of a diameter.

$$x^2+y^2-x(-2+2)-y(3+0)-4+0=0$$

$$x^2+y^2-3y-4=0$$

Hence, the correct answer is (A).

7. Solving the equation $5x + 3y = 87$, $y = mx + 1$ for x, we get $x = \frac{84}{5+3m}$.

The factors of 84 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 7, \pm 12, \pm 84$ out of only $5 + 3m = -1, 2, -4, -7$ gives the integral values for m.

Hence, the correct answer is (D).

8. Let $A(4, 3)$, $B(5\sin\theta, 5\cos\theta)$ and $C(5\cos\theta, -5\sin\theta)$ be the vertices and G, H be the centroid and orthocentre of ΔABC. If O is the origin, then, $OA^2 = 25 = OB^2 = OC^2$.

$\therefore$ O is the circumcentre of ΔABC. G divides OH in the ratio 1:2.

Hence, $\frac{4+5\sin\theta+5\cos\theta}{3} = \frac{x_1}{3}$ and

$$\frac{3+5\cos\theta-5\sin\theta}{3} = \frac{y_1}{3}$$

H is $(4+5\sin\theta+5\cos\theta,\ 3+5\cos\theta-5\sin\theta) = (x_1, y_1)$

H is $(x_1, y_1) \Rightarrow (x_1-4)^2+(y_1-3)^2 = 50$

$\therefore$ Locus of H is $(x-4)^2 + (y-3)^2 = 50$, which is a circle.

Hence, the correct answer is (B).

9. If the points of intersection of two lines with co-ordinate axes be concyclic, then the product of intercepts on x-axis is equal to the product of intercepts on y-axis by these lines. This is a geometry property. The intercepts on x-axis are q and p, whose product is pq. Also, the intercepts on y-axis are p and q, whose product is pq. Hence, the 4 points are concylic.

Hence, the correct answer is (C).

10. $A = (3, 0)$, $B = (4, 0)$

Let l be $y-2 = m(x-2)$

$$\therefore \quad C = \left(2-\frac{2}{m}, 0\right)$$

$$\therefore \quad 3\left(2-\frac{2}{m}\right) = 4^2 = 16$$

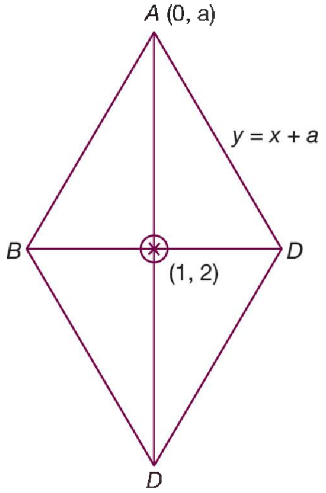
$$6m - 6 = 16m$$

$$\therefore \quad m = \frac{-3}{5}$$

$\therefore$ The line is $y-2 = -\frac{3}{5}(x-2) \Rightarrow 3x+5y-16=0$

Hence, the correct answer is (B).

11. We can suppose without the loss of generality, that the two sides passing through the vertex A (0, a) are $x-y+a=0$ and $7x-y+a=0$.



$\therefore$ Equations of the angular bisector of $\angle A$ are

$$\frac{x-y+a}{\sqrt{2}} = \pm\frac{7x-y+a}{5\sqrt{2}}$$

$\therefore$ These bisectors intersect at (1, 2).

$$\Rightarrow \quad a = 0, \frac{5}{2} \quad (\because a \neq 0)$$

$$\Rightarrow \quad a = \frac{5}{2}$$

Hence, the correct answer is (D).

12. Using symmetric form of straight line $\frac{x}{\cos 60^\circ} = \frac{y}{\sin 60^\circ} = r$, cuts the given curve

$$\left(\frac{r}{2}\right)^3+\left(\frac{r\sqrt{3}}{2}\right)^3+3\left(\frac{r}{2}\right)\left(\frac{r\sqrt{3}}{2}\right)+5\left(\frac{r}{2}\right)^2+3\left(\frac{r\sqrt{3}}{2}\right)^2$$

$$+4\left(\frac{r}{2}\right)+5\left(\frac{r\sqrt{3}}{2}\right)-1=0$$

$$\therefore \quad \left(\frac{3\sqrt{3}+1}{8}\right)r^3+\left(\frac{14+3\sqrt{3}}{4}\right)r^2+\left(\frac{4+5\sqrt{3}}{2}\right)r-1=0$$

is a cubic equation, whose product of roots is

$$OA.OB.OC = \frac{8}{3\sqrt{3}+1} = \frac{4\left(3\sqrt{3}-1\right)}{13}$$

Hence, the correct answer is (C).

13. $y = \dfrac{-(hx+f)\pm\sqrt{(h^2-ab)x^2+gx(hf-gb)+f^2-bc}}{b}$

to represent parallel lines $h^2 = ab$, $hf = gb$, $f^2 > bc$.

Hence, the correct answer is (C).

14. Let A' is an image of $A(2, 3)$ on $x + y = 0$.

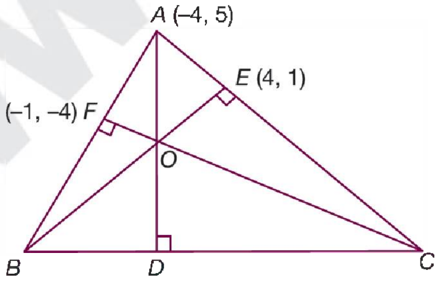
$\Rightarrow \quad A' \equiv (-3, -2)$

Now, B is the point of intersection of PA' with $x + y = 0$

$\Rightarrow \quad B \equiv \left(\frac{1}{13}, -\frac{1}{13}\right)$

Hence, the correct answer is (C).

15. Slope of $AF = \frac{5-(-4)}{-4+1} = \frac{9}{-3} = -3$



$\therefore \quad$ Slope of $CF = \frac{-1}{-3} = \frac{1}{3}$

Also, F lies on CF

$\therefore \quad$ Equation of CF is $x - 3y - 11 = 0$ ...(1)

Slope of $AE = \frac{5-1}{-4-4} = \frac{4}{-8} = -\frac{1}{2}$

Slope of $BE = \frac{-1}{-\frac{1}{2}} = 2$

Also, E lies on BE.

$\therefore$ Equation of BE is $2x - y - 7 = 0$. ...(2)

Solving (1) and (2), we get the point $O \equiv (2, -3)$

($\because$ CB is passing through intersection of AC and FC)

$\therefore$ Equation of $CB: x+2y-6+\lambda(x-3y-11)=0$

or $\quad x(\lambda+1)+y(2-\lambda)-(6+11\lambda)=0$ ...(3)

Slope of $CB = \frac{\lambda+1}{3\lambda-2}$

Also, slope of $AO = \frac{5-(-3)}{-4-2} = -\frac{4}{3}$

$\because \quad CB \perp AO; \frac{\lambda+1}{3\lambda-2} \times -\frac{4}{3} = -1$ or $\lambda = 2$

Equation of BC is $3x - 4y - 28 = 0$.

Hence, the correct answer is (A).

16. $\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ p & q & -r \end{vmatrix} = 0 \Rightarrow \quad p + 5q - 3r = 0$

So, $2px + 3qy + \frac{4}{3}(p + 5q) = 0$

Hence, the fixed point is $\left(-\frac{2}{3}, -\frac{20}{9}\right)$

$\therefore$ The slope of required line (perpendicular to line joining origin to fixed point) is $-\frac{3}{10}$.

Hence, the correct answer is (D).

17. $\frac{1}{a} + \frac{1}{b} = \frac{1}{4}$

$\Rightarrow \quad b = \frac{4a}{a-4}$

So, $\quad A = \frac{4a^2}{2(a-4)}$

$A_{\min} = 32 = c$

So, the last digit of $c^c = 32^{32} = (30 + 2)^{32} = 10k + 2^{32}$
$= 10k + 4\,(30 + 2)^6$

$= 10m + 4 \cdot 2^6 = 10m + 256$

$\therefore$ Last digit is 6.

Hence, the correct answer is (D).

18. Let the axes of system 1 and 2 are inclined at angle α and β respectively having the same origin, then

$$OA^2 = (x-0)^2 + (y-0)^2 + 2xy\cos\alpha \qquad \ldots(1)$$

and $$OA^2 = (x'-0)^2 + (y'-0)^2 + 2x'y'\cos\beta \qquad \ldots(2)$$

$$\therefore \quad x^2 + y^2 + 2xy\cos\alpha = (x')^2 + (y')^2 + 2x'y'\cos\beta$$

(From (1) and (2))

$\because \quad x = px' + qy'$ and $y = p'x' + q'y'$ (Given)

$$\Rightarrow (p^2 + p'^2)(x')^2 + (q^2 + q'^2)(y')^2 + 2pqx'y' + 2p'q'x'y' + \left(pp'(x')^2 + qq'(y')^2 + pq'x'y' + p'qx'y'\right)\cos\alpha$$

$$= x'^2 + y'^2 + 2x'y'\cos\beta$$

$$\Rightarrow (p^2 + p'^2 - 1)(x')^2 + (q^2 + q'^2)(y')^2 + \left\{(pp')(x')^2 + (qq')(y')^2\right\}\cos\alpha$$

$$= \left\{-2pq - 2p'q' + 2\cos\beta - (pq' + p'q)\cos\alpha\right\}x'y'$$

$$\therefore \quad p^2 + p'^2 - 1 = -pp'\cos\alpha \text{ and } q^2 + q'^2 - 1 = -qq'\cos\alpha$$

Hence, $\dfrac{p^2 + p'^2 - 1}{q^2 + q'^2 - 1} = \dfrac{pp'}{qq'}$

Hence, the correct answer is (C).

19. Put $y = 0 \Rightarrow 2x^2 - 4x - 6 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\therefore$ Length of intercept $= |-3-1| = 4$ units.

Hence, the correct answer is (B).

20. $\tan\theta = \sqrt{3} \Rightarrow \theta = 60^\circ \Rightarrow \angle POR = 120^\circ$

$\Rightarrow$ Bisector will have slope tan 120°

$\therefore$ Equation of bisector is $\sqrt{3}x + y = 0$.

Hence, the correct answer is (C).

21. The line $\dfrac{x}{p} + \dfrac{y}{q} = 1$, where '$p$' is a prime and '$q$' is a positive integer passes through (4, 3) only if

$$q = \frac{3p}{p-4} = 3 + \frac{12}{p-4}$$

$\because$ 'q' is a positive integer, $p > 4$ and $(p - 4)$ divides 12, there are only two such primes

$p = 5$ and $p = 7$

Hence, the correct answer is (B).

22. The reflection of (α, β) in the line $y = 2x$ is

$$(\alpha_1, \beta_1) = \left(\frac{4\beta - 3\alpha}{5}, \frac{4\alpha + 3\beta}{5}\right) = \alpha_1\beta_1 = 1$$

Hence, the correct answer is (A).

23. Area of ΔABC = Sum of areas of $\Delta OAB + \Delta OBC + \Delta OCA$

$$\Rightarrow \frac{\sqrt{3}}{4}a^2 = \frac{a}{2}\left(4\sqrt{3} + 5\sqrt{3} + 7\sqrt{3}\right)$$

Hence, the correct answer is (C).

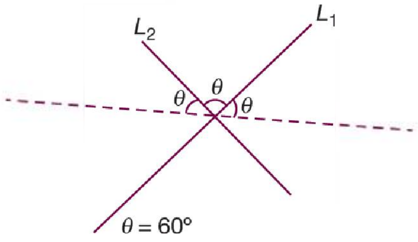
24. $m = \tan(45^\circ + \theta) = 3$, where $\tan\theta = \dfrac{1}{2}$

Point of trisection nearest to B (5,4)

$\therefore$ The required line is $3x - y - 11 = 0$.

Hence, the correct answer is (B).

25.



Hence, the correct answer is (C).

26. Let $OP = a', OQ = b' \Rightarrow a - a' = b - b'$ (Given)

Equation of $AQ : \dfrac{x}{a} + \dfrac{y}{b'} = 1$; of $PB : \dfrac{x}{a'} + \dfrac{y}{b} = 1 \Rightarrow \dfrac{x}{aa'} + \dfrac{y}{bb'} = 0$ (Using given)

Also, $\dfrac{x}{ab} + \dfrac{y}{b'} = \dfrac{1}{b}, \dfrac{x}{aa'} + \dfrac{y}{ab} = \dfrac{1}{a} \Rightarrow \dfrac{x+y}{ab} = \dfrac{a+b}{ab}$

Hence, the correct answer is (B).

27. $\alpha L_1 + L_2 = 0$

$a\alpha^2 + 2b\alpha + c = 0$

Hence, the correct answer is (B).

28. $ax^2+2hxy+by^2=0$ represents two sides of triangle (c, d) is orthocentre, then the equation of third side is $(a+b)(cx+dy)=ad^2-2hcd+bc^2$

$$\Rightarrow \quad 3(2x+y)=11-(-3)2+(2)(4)$$

$$\Rightarrow \quad 2x+y=5$$

Hence, the correct answer is (C).

29. Equation of angle bisector of pair of lines $9x^2+12xy+ky^2=0$ is $6(x^2-y^2)-xy(9-k)=0$

Bisector is parallel to $2x+3y=7$ and through origin

$$\Rightarrow \quad y=\frac{-2x}{3}$$

Pair of required lines $(2x+3y)(3x-2y)=0$

Coefficient of $xy=k-9=5$

$$\Rightarrow \quad k=14$$

Hence, the correct answer is (B).

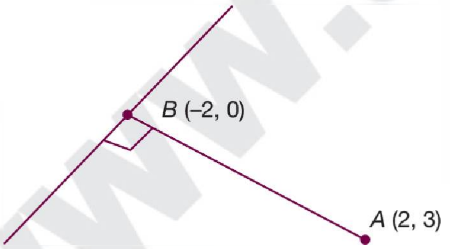
30. Angle bisectors are perpendicular.

Other bisectors $3x+2y+k=0$

Shifting origin to (–1, 3), we get the other line equation as $3x+2y=3$.

Hence, the correct answer is (D).

31. The family of lines $3x+4y+6+\lambda(x+y+2)=0$ are concurrent at the intersection of $3x+4y+6=0$ and $x+y+2=0$, i.e., at (–2, 0).



∴ The line through (–2, 0) lying at the greatest distance from A (2, 3) is the line through B (–2, 0) perpendicular to AB.

Hence, its equation is $y-0=-\frac{4}{3}(x+2)$, i.e., $4x+3y+8=0$.

Hence, the correct answer is (B).

32. Line always passes through the point $\left(2,\frac{8}{3}\right)$.

Hence, $6a+8b+6=0$

$$\Rightarrow \quad 3a+4b+3=0$$

$bx-ay+4=0$ and $3x+4y+5=0$ are concyclic.

So, $m_1m_2=1$

$$\frac{b}{a}\cdot-\frac{3}{4}=1 \Rightarrow 4a+3b=0$$

Solving, we get, $a=\frac{9}{7}, b=\frac{-12}{7}$

Hence, the correct answer is (C).

33. The joined equation of straight line $y=x-1$ and $y=-x+1$ is $(x-y-1)(x+y-1)=0$.

$$\Rightarrow \quad x^2-y^2-2x+1=0 \qquad (1)$$

Let equation of line passes through (2, 0) is

$$y=m(x-2) \qquad (2)$$

By homogenising equation (1) with the help of line (2), we get

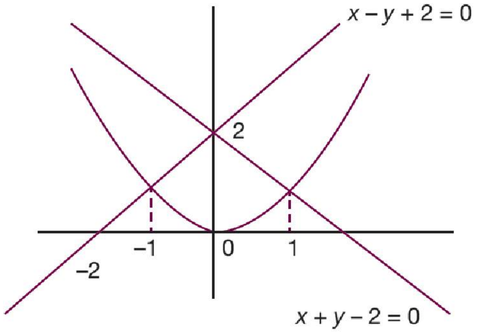
$$x^2-y^2-2x\left(\frac{mx-y}{2m}\right)+\left(\frac{mx-y}{2m}\right)^2=0$$

∵ Coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow \quad m=\pm\frac{1}{\sqrt{3}}$$

Hence, the correct answer is (C).

34. (a, a^2) lies of $y=x^2$



$a-a^2-2=0 \qquad a=-1, 2$

$a+a^2-2=0 \qquad a=1, -2$

Hence, the correct answer is (B).

35. Three non-parallel lines are concurrent, if $D = 0$.

$$\Rightarrow \begin{vmatrix} k & 2 & 2 \\ 2 & k & 3 \\ 3 & 3 & k \end{vmatrix} = 0$$

$\Rightarrow \quad k = 2, 3, -5$

However, for $k = 2$, first two lines are parallel.

Hence, the correct answer is (C).

36. As the line passes through the point of intersection of $x - 2y - 2 = 0$ and $2x - by - 6 = 0$, it can be represented as $\lambda(x - 2y - 2) + (2x - by - 6) = 0$.

As it passes through the origin.

$-2\lambda - 6 = 0$

$\lambda = -3$

$\therefore \quad$ Equation of the line is $-x + (6 - b)y = 0$

Its slope is $\dfrac{1}{6-b}$

As its angle with $y = 0$ is less than $\dfrac{\pi}{4}$

$\therefore \quad -1 < \dfrac{1}{6-b} < 1$

$\Rightarrow \quad 6 - b > 1$ or $< -1 \Rightarrow b < 5$ or $b > 7$

However, $b \neq 4$ (as the lines intersect)

$\therefore \quad B \in (-\infty, 4) \cup (4, 5) \cup (7, \infty)$

Hence, the correct answer is (D).

37. Since $3 \times 1 - 4 \times 2 + 1$ and $12 \times 1 + 5 \times 2 - 3$ are of the opposite sign, so the required angle bisector is given by

$$\frac{3x - 4y + 1}{5} = -\left(\frac{12x + 5y - 3}{13}\right)$$

Hence, the correct answer is (B).

38. Clearly, circumcentre of ΔABQ will lie on perpendicular bisector of line AB, which is $6x - 8y + 7 = 0$.

Hence, the correct answer is (A).

39. By solving the sides of the rhombus, the vertices are

$$\left(0, \frac{-n}{m}\right), \left(\frac{-n}{\ell}, 0\right)\left(0, \frac{n}{m}\right), \left(\frac{n}{\ell}, 0\right)$$

$\therefore \quad$ The area $= \dfrac{1}{2}\left(\dfrac{2n}{m}\right)\left(\dfrac{2n}{\ell}\right) = 2$

$\Rightarrow \quad n^2 = \ell$ m sq. units

Hence, the correct answer is (B).

40. Image of orthocenter of ΔABC w.r.t. BC lies on the circumcircle $= H'(2, 8)$

Now, radius $= OH' = 5$

Hence, the correct answer is (B).

41. $x^2 - (y - 1)^2 = 0$ is the given pair of lines

Vertices are $(0, 1), (0, 3), (2, 1)$

Angular bisector is $x(y - 1) = 0$

$\therefore \quad$ Area = 2 sq.units

Hence, the correct answer is (D).

42. Third vertex 'A' lies on $x - y = 0$ and in III quadrant

Perpendicular distance from $(0, 0)$ to $x + y = 1$ is $\dfrac{1}{\sqrt{2}}$ units.

$\therefore \quad AO = \sqrt{2}$

$\Rightarrow \quad A(-1, -1)$

If m is the slope of other side, then

$$\tan 60^\circ = \left|\frac{m+1}{1-m}\right|$$

$\Rightarrow \quad m = 2 \pm \sqrt{3}$

Hence, the correct answer is (A).

43. Circumcentre of the triangle formed by the given lines is given by

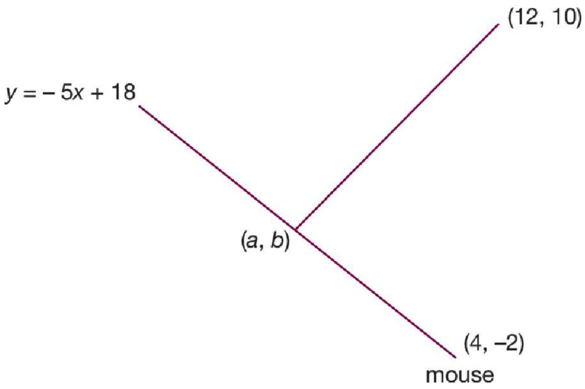
$$\left(\frac{\ell}{\ell^2 - m^2}, \frac{m}{\ell^2 - m^2}\right)$$

Hence, the locus of this point is $(x^2 - y^2)^2 = x^2 + y^2$.

Hence, the correct answer is (A).

44. $a = 2, b = 8$

$a + b = 10$



Hence, the correct answer is (B).

45. $a = x$-intercept, $b = y$-intercept

$2h = a + b, k^2 = ab$

$\frac{x}{a} + \frac{y}{b} = 1$, substitute (1, 1)

$\frac{1}{a} + \frac{1}{b} = 1$

$a + b = ab$

$2h = k^2$

$\Rightarrow \quad y^2 = 2x$

Hence, the correct answer is (A).

46. Equation of the line $\frac{x}{a} + \frac{y}{b} = 1$

Transformed equation is

$$\frac{1}{a}(x \cos a - y \sin a) + \frac{1}{b}(x \sin a + y \cos a) = 1$$

Intercepts are equal

x-coefficient + y-coefficient

$\therefore \quad \tan a = \frac{a - b}{a + b}$

Hence, the correct answer is (B).

47. Given $|x| + |y| = 3$

$\therefore \quad$ Required area $= \frac{2c^2}{|ab|} = 9 \times 2 = 18$ sq. units

Hence, the correct answer is (A).

48. Origin and (a, a) lies on the same side of the given lines

$\therefore \quad |a| < 1$

Hence, the correct answer is (C).

49. The line 'l' can be any one of the bisectors of the angles between the lines $3x - 4y = 5$ and $5x + 12y = 13$.

$\therefore \quad$ Angular bisectors, $\frac{3x - 4y - 5}{5} = \pm\left(\frac{5x + 12y - 13}{13}\right)$

$\Rightarrow \quad x - 8y = 0, 32x + 4y - 65 = 0$

Hence, the correct answer is (B).

50. Solve the equations $x = \frac{25m}{2m + 15}, y = \frac{2m - 60}{2m + 15}$

However, $x > 0, y > 0 \Leftrightarrow 25m > 0, 2m + 15 > 0, 2m - 60 > 0$

$\Leftrightarrow m > 30$ or $m < \frac{-15}{2}$

Hence, the correct answer is (D).

51. We have, $OA_n = n \cdot OA_{n-1} = n(n-1) \cdot OA_{n-2} =$ ------

$\therefore \quad OA_n = \frac{n!}{\sqrt{2}}$

$\Rightarrow \quad \sqrt{2}\,(2520\sqrt{2}) = n! \Rightarrow n! = 5040$

$\Rightarrow \quad n = 7$

Hence, the correct answer is (C).

52. $L_1 \equiv 2x + 2y = 0$

$L_1(1, 3) > 0$

So, $\quad a$ is wrong

$L_2 \equiv 2x + y + 1 = 0$

$L_2(1, 3) > 0, L_2(5, 6) > 0 \quad \Rightarrow b$ is true

$L_3(-1, 2) > 0$

Hence, the correct answer is (B).

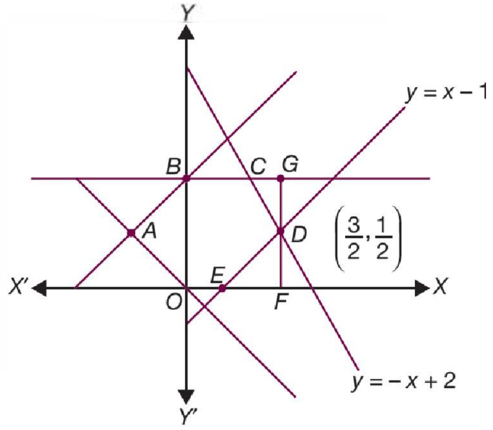
53. Here, S must be the centroid of ΔPQR

$\Rightarrow \quad \frac{1 + 2 + \alpha}{3} = 2$ and $\frac{1 + 4 + \beta}{3} = 2$

$\Rightarrow \quad \alpha = 3$ and $\beta = 1$

Hence, the correct answer is (C).

54. Lines are $y = 1, y = 0, y = -x, y = -x + 2, y = x + 1, y = x - 1$



Area of $OABCDE$

$= $ Area of $OBGF$

$= \frac{3}{2} \times 1 = \frac{3}{2}$ sq. units.

Hence, the correct answer is (C).

55. The required altitude will be the bisector of obtuse angle between the 2 given altitudes.

Hence, the correct answer is (B).

56. The required line is at a distance of 3 units from given line and parallel to it . Hence, it is

$x + 2y - 4 - 3\sqrt{5} = 0$, cuts x-axis at $C(4 + 3\sqrt{5}, 0)$

with slope $\tan q = \frac{-1}{2}$. After rotation about C by

30°, the slope becomes $m = \tan(q - 30°) = \frac{-(2+\sqrt{3})}{2\sqrt{3}-1}$

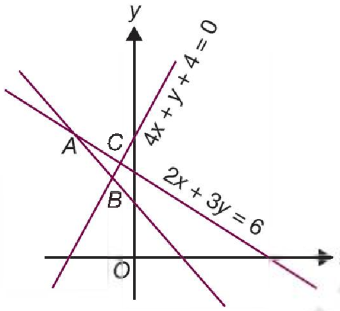
$= \frac{-(4+3\sqrt{3})}{11} \Rightarrow [m] = -1$

Hence, the correct answer is (A).

57. Co-ordinates of A and B are (–3, 4) and $\left(-\frac{3}{5}, \frac{8}{5}\right)$, if orthocentre $p\ (h, k)$

Then, (slope of PA) × (slope of BC) = –1

$$\frac{k-4}{h+3} \times 4 = -1$$



$\Rightarrow \quad 4k - 16 = -h - 3$

$\Rightarrow \quad h + 4k = 13 \qquad \ldots(1)$

Slope of PB × slope of AC = – 1

$$\Rightarrow \quad \frac{k - \frac{8}{5}}{k + \frac{3}{5}} \times -\frac{2}{3} = -1$$

$\Rightarrow \quad 3h - 2k + 5 = 0 \qquad \ldots(2)$

Solving equations (1) and (2), we get, $h = \frac{3}{7}, k = \frac{22}{7}$

Hence, orthocentre lies in Ist quadrant.

Hence, the correct answer is (A).

58. Since, the given equation represents a pair of parallel lines, we have $h^2 = ab$

$\Rightarrow \quad h = \pm 6$

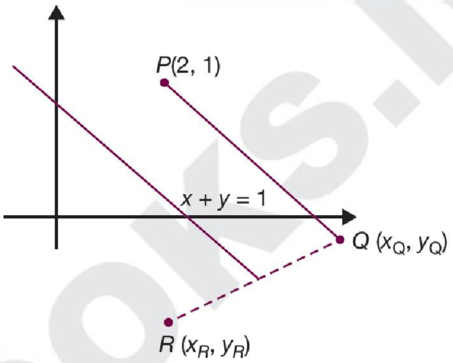
$$\text{Condition for pair of lines} \begin{vmatrix} 9 & h & 3 \\ h & 4 & f \\ 3 & f & -3 \end{vmatrix} = 0$$

$\Rightarrow \quad 108 \pm 36f - 9f^2 - 144 = 0$

$\Rightarrow \quad f = 2, h = 6$ and $f = -2, h = -6$

Hence, the correct answer is (A).

59. $PQ = 3\sqrt{2}$



$\Rightarrow \quad x_Q = 2 + \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 5$

$y_Q = 1 - 3\sqrt{2} \cdot \frac{1}{\sqrt{2}} = -2$

$\Rightarrow \quad Q \equiv (5, -2)$

Distance of Q from the line $x + y = 1$ is

$$\frac{|5 - 2 - 1|}{\sqrt{2}} = \sqrt{2}$$

$\Rightarrow \quad QR = 2\sqrt{2}$

$\Rightarrow \quad x_R = 5 - 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 3$ and $y_R - 2 - 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = -4$

Thus, $R \equiv (3, -4)$

Hence, the correct answer is (A).

60. $A(\alpha, 2\alpha + 3)$, $BC = 1$ unit

Equation of BC is $y - 3 = 0$

p = Distance of A from $BC = |2\alpha + 3 - 3|$

Area of $\Delta ABC = \Delta = |\alpha|; 5 \le \Delta < 6 \Rightarrow 5 \le |\alpha| < 6$

Hence, the correct answer is (B).

61. The image of $P(a, b)$ on $y = -x$ is $Q(-b, -a)$ (interchange and change signs) and the image of $Q(-b, -a)$ on $y = x$ is $R(-a, -b)$ (merely interchange)

∴ The midpoint of PR is (0, 0).

Hence, the correct answer is (D).

62. The equation of the line L be $y - 2 = m(x - 8)$, $m < 0$ co-ordinates of P and Q are $P\left(8 - \frac{2}{m}, 0\right)$ and $Q(0, 2 - 8m)$.

So, $OP + OQ = 8 - \frac{2}{m} + 2 - 8m = 10 + \frac{2}{(-m)} + 8(-m)$

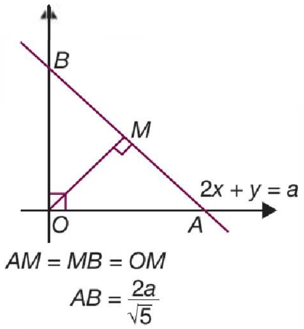
$$\geq 10 + 2\sqrt{\frac{2}{(-m)} \times 8(-m)} \geq 18$$

So, absolute minimum value of $OP + OQ = 18$.

Hence, the correct answer is (B).

63. Let the two perpendiculars through the origin intersect $2x + y = a$ at A and B so that the triangle OAB is isosceles.

OM = Length of perpendicular from O to AB



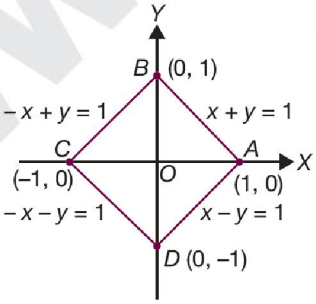
$$OM = \frac{a}{\sqrt{5}}$$

Area of $\Delta OAB = \frac{1}{2} \cdot AB \cdot OM$

$$= \frac{1}{2} \cdot \frac{2a}{\sqrt{5}} \cdot \frac{a}{\sqrt{5}} = \frac{a^2}{5} \text{ sq.units}$$

Hence, the correct answer is (C).

64. $d(x, y) = 1 \Rightarrow |x| + |y| = 1$



The graph of which is shown in the figure

The graph is a square $AB – BC = CD = DA = \sqrt{2}$

Where, $OA = 1 = OB$, $\angle ABC = 90°$

Area $= AB.AD = \sqrt{2} \times \sqrt{2} = 2$ sq.units

Hence, the correct answer is (C).

65. As $(-1, 1)$ is a point on $3x - 4y + 7 = 0$, the rotation is possible.

Slope of the given line $= \frac{3}{4}$

Slope of the line in its new position $= \dfrac{\frac{3}{4} - 1}{1 + \frac{3}{4}} = -\frac{1}{7}$

The required equation is $y - 1 = -\frac{1}{7}(x + 1)$ or $7y + x - 6 = 0$

Hence, the correct answer is (A).

66. Solving the given equations, we get,

$$x = \frac{5}{3 + 4m}$$

x is an integer, if $3 + 4m = \pm 1, \pm 5$

$$\therefore \quad m = -\frac{2}{4}, -\frac{4}{4}, \frac{2}{4}, -\frac{8}{4}$$

So, m has two integral values.

Hence, the correct answer is (A).

67. Given equations, $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ and $mx + 2y + 6 = 0$ are concurrent for all real values of a and b.

Then, $\dfrac{2a + b}{m} = \dfrac{(a + 3b)}{2} = \dfrac{b - 3a}{6}$

Taking last two ratios, $\dfrac{a + 3b}{2} = \dfrac{-3a + b}{6}$

$$\Rightarrow \quad b = -\frac{3}{4}a$$

Taking first two ratios, $m = \dfrac{2(2a + b)}{a + 3b} = \dfrac{2(2a - (3/4))a}{a + 3(-3/4)a}$

$$\Rightarrow \quad \frac{10}{-5} = -2$$

Hence, the correct answer is (A).

68. A' is the image of A w.r.t x-axis.

$$\Delta PA^1C \sim \Delta CQB$$

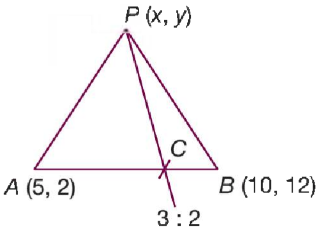
$$\frac{x - a}{c - x} = \frac{b}{d}$$

$$\Rightarrow \quad dx - ad = bc - bx$$

$$\Rightarrow \quad x = \frac{bc+ad}{b+d}$$

Hence, the correct answer is (D).

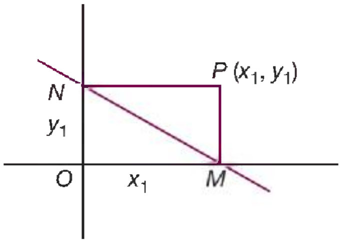
69. C divides AB in the ratio 3:2.



C(8, 8)

Hence, the correct answer is (B).

70.



Hence, the correct answer is (C).

71. Let $A(\alpha, 0)$

Slope of reflected ray $= \dfrac{3-0}{5-\alpha} = \tan\theta$ ...(1)

Slope of incident ray $= \dfrac{2-0}{1-\alpha} = \tan(\pi-\theta)$ ...(2)

(1) and (2) $\Rightarrow \alpha = \dfrac{13}{5}$

Hence, the correct answer is (A).

72. $2x - y + 4 = 0, -x + 2y + 1 = 0$

$a_1a_2 + b_1b_2 = 2(-1) + (-1)(2) = -4 < 0$

Equation of acute angle by vector is

$$\frac{2x-y+4}{\sqrt{5}} = +\frac{(-x+2y+1)}{\sqrt{5}}$$

$$\Rightarrow \quad x - y + 1 = 0$$

Hence, the correct answer is (B).

73. $x^2 - y^2 + 2y - 1 = 0 \quad \Rightarrow x - y + 1 = 0$ and $x + y - 1 = 0$

Angle bisectors are $x = 0, y = 1$ and the line is $x + y = 3$

Required area enclosed is $= \dfrac{1}{2}(2)(2) = 2$ sq. units.

Hence, the correct answer is (A).

74. $P_1 = |2a\sin\alpha\cos\alpha| \qquad P_2 = |a\cos 2\alpha|$

$$\left(\frac{P_1}{P_2} + \frac{P_2}{P_1}\right)^2 = \frac{\left(P_1^2 + P_2^2\right)^2}{(P_1P_2)^2} = \frac{a^4}{a^4\sin^2 2\alpha\cos^2 2\alpha}$$
$$= 4\operatorname{cosec}^2 4\alpha$$

Hence, the correct answer is (C).

75. Reflection of H about BC lies on circumcircle

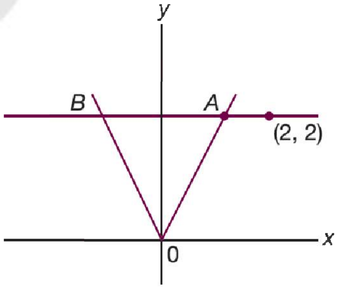
Let $P(h,k)$ we have

$$\frac{(h-1)\sqrt{5}}{-2} = \frac{(k-2)\sqrt{5}}{1} = \frac{-6}{\sqrt{5}} \Rightarrow P \equiv \left[\frac{17}{5}, \frac{4}{5}\right]$$

$$\Rightarrow \quad CP = \sqrt{\left(\frac{17}{5}-2\right)^2 + \left(\frac{4}{5}-4\right)^2} = \sqrt{\frac{305}{5}} = \sqrt{\frac{61}{5}} \text{ units}$$

Hence, the correct answer is (C).

76. The length of an altitude of $\Delta OAB = 2$ units.



$\Rightarrow$ The length of a side of a $\Delta OAB = \dfrac{4}{\sqrt{3}}$ units

$\Rightarrow$ Area $\Delta OAB = \dfrac{\sqrt{3}}{4}\left(\dfrac{4}{\sqrt{3}}\right)^2$ sq. units

Hence, the correct answer is (D).

77. Equation of the line in normal form is $x\cos\alpha + y\sin\alpha = p$; $l = \cos\alpha, m = \sin\alpha, n = -p$

Hence, the correct answer is (C).

78. Let the co-ordinates of the centre of mean position of the points

$A_i, i = 1,2,3,---n$ be (x,y), then

$$x = \frac{x_1 + x_2 + --- + x_n}{n}, \quad y = \frac{y_1 + y_2 + --- + y_n}{n}$$

$$x = \frac{np + a(1+2+---+n)}{n} = p + \frac{n(n+1)}{2n}a$$
$$= p + \frac{n+1}{2}a$$

$$y=\frac{nq+b(1+2+---+n)}{n}=q+\frac{n(n+1)}{2n}\cdot b$$

$$=q+\frac{n+1}{2}b$$

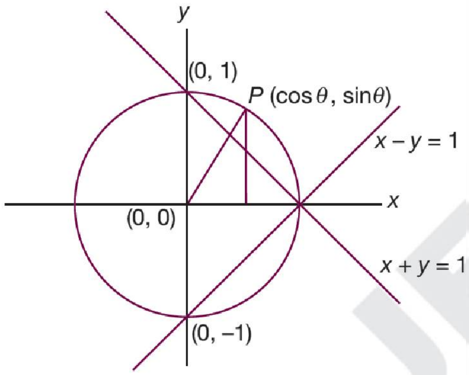
$$\frac{2(x-p)}{a}=\frac{2(y-q)}{b}\Rightarrow bx-ay=bp-aq$$

Hence, the correct answer is (C).

79. Put $\left(1+\frac{r}{\sqrt{2}},2+\frac{r}{\sqrt{2}}\right)$ in $x^2+4xy+y^2=0$, we get $3r^2+9\sqrt{2}\,r+13=0$ for which product of the roots is $PA.PB=\frac{13}{3}$

Hence, the correct answer is (A).

80. Lines $y=|x-1|$, so $y>0$

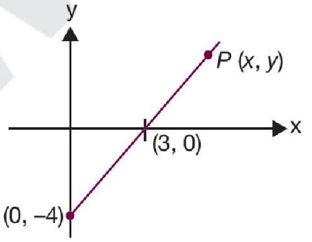


Point lies on $x^2+y^2=1$

From figure, θ can vary from 0 to $\frac{\pi}{2}$.

Hence, the correct answer is (C).

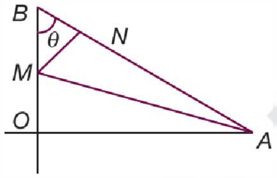
81. The distance between the two points (3, 0) and (0, –4) is 5 units.



Locus of P is the part of line $\frac{x}{3}-\frac{y}{4}=1$.

Hence, the correct answer is (D).

82. Let $\frac{AN}{BN}=\frac{\lambda}{1}$, then $AN=\left(\frac{\lambda}{\lambda+1}\right)\sqrt{2}a$



$$tan\frac{\pi}{4}=\frac{MN}{BN}\Rightarrow MN=BN=\frac{1}{\lambda+1}\sqrt{2}a$$

Now, area of $\Delta\,AMN=\frac{3}{8}OAB$

$$\Rightarrow\quad MN.AN=\frac{3}{8}\cdot a^2$$

$$\Rightarrow\quad\frac{\lambda}{(\lambda+1)^2}\cdot 2a^2=\frac{3}{8}a^2$$

$$\Rightarrow\quad\frac{\lambda}{(\lambda+1)^2}=\frac{3}{16}$$

$$\Rightarrow\quad\lambda=3$$

Hence, the correct answer is (A).

83. The common line is $3x+4y=0$

Required parallel line is $3x+4y+\lambda=0$

Distance between parallel lines $=\frac{|\lambda-0|}{5}=7$

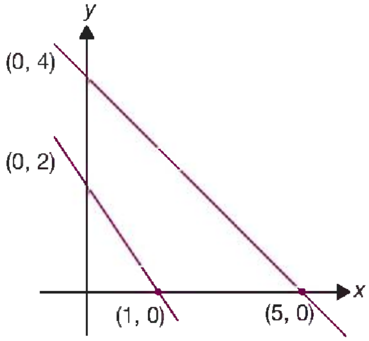
$|\lambda|=35$

Hence, the correct answer is (C).

84. Area of required triangle $=\frac{2abc\times\text{area of }\Delta ABC}{(a+b)(b+c)(c+a)}$

Hence, the correct answer is (C).

85.



Possible $P(m, n)$ are (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (3, 1)

$\Rightarrow$ 6 points

Hence, the correct answer is (D).

86. $(a-b)^2-c^2=0$

$\Rightarrow\ (a-b-c)(a-b+c)=0$

If $a-b=c$

$\Rightarrow\ ax+by+(a-b)=0$

$\Rightarrow\ a(x+1)+b(y-1)=0$

$\Rightarrow\ x=-1,\ y=1$

If $a-b=-c$

$\Rightarrow\ ax+by+(b-a)=0$

$\Rightarrow\ a(x-1)+b(y+1)=0$

$\Rightarrow\ x=1,\ y=-1$

Hence, the correct answer is (C).

87. $x^3-3x+2=\frac{2-x}{3}$

$\Rightarrow\ 3x^3-8x+4=0$

$\Rightarrow\ \Sigma y_i=\sum_{i=1}^{3}\left(\frac{2}{3}-\frac{1}{3}x_i\right)=2$

$P(0,2)$

Hence, the correct answer is (D).

88. $m_1m_2=-1\Rightarrow\left(\frac{-1}{\lambda^2-\lambda+1}\right)\left(\frac{-1}{2}\right)=-1$

$\Rightarrow\ 2\lambda^2-2\lambda+3=0\Rightarrow$ no real λ

Hence, the correct answer is (A).

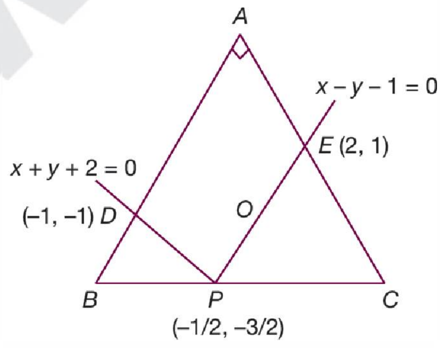
89. Line $(2hy+2g)\left(x+\frac{f}{h}\right)=0,\ if\ c=\frac{2gf}{h}$

$$P=\frac{-f}{h}\times\frac{-g}{h}=\frac{-fg}{h^2}=\frac{-c}{2h}$$

Hence, the correct answer is (B).

90. PA is the circumradius of ΔABC

$PA^2=PD^2+PE^2$



$$=\left(-1+\frac{1}{2}\right)^2+\left(-1+\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2+\left(\frac{5}{2}\right)^2$$

$$=\frac{1}{4}+\frac{1}{4}+\frac{25}{4}+\frac{25}{4}=13$$

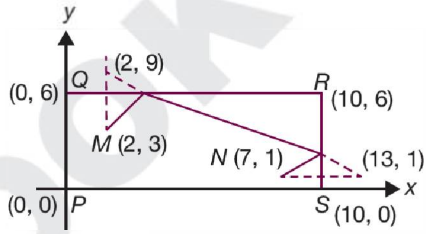
$PA=\sqrt{13}$.

Hence, the correct answer is (C).

91. Equation through (2,9) and (13, 1) in $8x+11y=115$

Now $x=10$

Required point is $(10, 35/11)$



Hence, the correct answer is (B).

92. Let the co-ordinates are $A(x_1, x_1)$ and $B(x_2, mx_2)$.

Clearly, $(x_1-x_2)^2+(x_1-mx_2)^2=16$

Let midpoint of $P(h, k)$

$\Rightarrow\ x_1+x_2=2h$ and $x_1+mx_2=2k$

$\Rightarrow\ (x_1-x_2)^2+4x_1x_2=4h^2$

and $(x_1-mx_2)^2+4mx_1x_2=4k^2$

$(x_1-x_2)^2+(x_1-mx_2)^2=4h^2+4k^2=16$ (When $m=-1$)

Hence, the correct answer is (B).

93. $x^2-5xy+6y^2=0$ represents a pair of straight lines given by $x-3y=0$ and $x-2y=0$. Also, $ax^2+by^2+c=0$ will represent a circle, if $a=b$ and c is of sign opposite to that of a.

Hence, the correct answer is (B).

94. The triangle is formed by the lines

$AB:(1+p)x-py+p(1+p)=0$

$AC:(1+q)x-qy+q(1+q)=0$

BC: $y=0$ (Diagram)

So that, the vertices are $A(pq,(p+1)(q+1)),b(-p,0),$ $C(-q,0)$

Let $H(h,\ k)$ be the orthocentre of ΔABC . Then as $AH\perp BC$ and passes through $A(pq,(p+1)(q+1))$.

Then, equation of AH is $x = pq$

$\therefore \quad h = pq \qquad \ldots(1)$

Also, BH is perpendicular to AC

$\therefore \quad m_1 m_2 = -1$

$\Rightarrow \quad \frac{k-0}{h+p} \times \frac{1+q}{q} = -1$

$\Rightarrow \quad \frac{k}{pq+p} \times \frac{1+q}{q} = -1$ (Using equation (1))

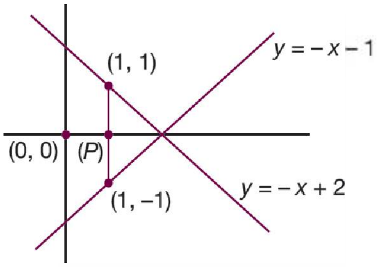
$\Rightarrow \quad k = -pq \qquad \ldots(2)$

From (1) and (2), we observe $h + k = 0$

$\therefore$ Locus of (h, k) is $x + y = 0$, which is a straight line.

Hence, the correct answer is (D).

95. For non-vertical lines through (1, 0),



$y = m(x-1); x^2 - y^2 - 4x + 4 = 0$ is the pair of lines

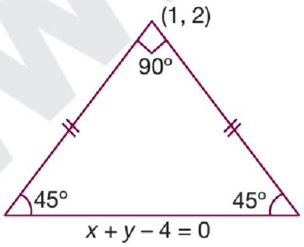
Homogenise and then take coefficient x^2 + coefficient $y^2 = 0$ to get $\frac{1}{m^2} = 0$, which is not possible.

For vertical lines

A vertical line through (1, 0) makes an intercept between the lines which subtends a right angle at (0, 0).

Hence, the correct answer is (C).

96. Let (x, y) be any point on the required lines.



$m_1 = \frac{y-2}{x-1}, m_2 = \pm$ Slope of $x + y - 4 = 0$ is -1

$$\tan 45° = \left| \frac{\frac{y-2}{x-1} - (-1)}{\left|1 - \frac{y-2}{x-1}\right|} \right| = 1$$

$$|x + y - 3| = |x - y + 1|$$

$$(x+y)^2 - 6(x+y) + 9 = (x-y)^2 + 2(x-y) + 1$$

$$4xy - 8x - 4y + 8 = 0$$

$$xy - 2x - y + 2 = 0$$

Hence, the correct answer is (A).

97. ABC is the pedal triangle of the $\Delta I_1 I_2 I_3$.

The orthocentre of the $\Delta I_1 I_2 I_3$ is

$I = (2, 3)$ and the circumcentre is $(4, 7)$

$\therefore$ The centre of the 9 point circle is

N = the midpoint = (3, 5), which is the circumcentre of the ΔABC.

Hence, the correct answer is (A).

98. R is centroid (G) of ΔOPQ

Hence, the correct answer is (C).

99. Lines can be written as $\frac{4}{b}x + y\frac{3}{b} + 1 - y = 0$

$$\frac{1}{b}(4x + 3y) + 1 - y = 0$$

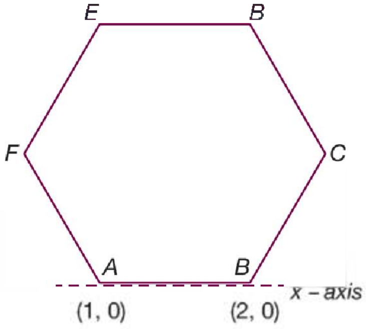
Solving, $4x + 3y = 0$ and $1 - y = 0$

$\therefore \quad y = 1$ and $x = -\frac{3}{4}$

Hence, the required line is $y - x = \frac{7}{4}$

Hence, the correct answer is (A).

100.



$C = (2 + \cos 60°, 0 + \sin 60°)$

$= \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$

$E = (1, \sqrt{3})$

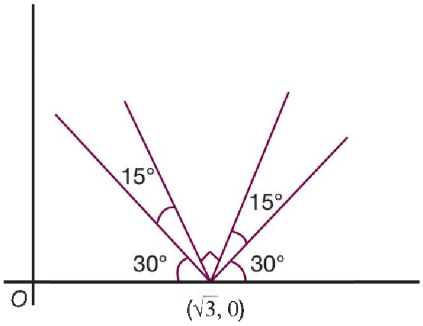
$\therefore$ Equation of CE is $x + \sqrt{3}y = 4$.

Hence, the correct answer is (B).

101. $ax + y = 0$ and $x + by = 0$ are perpendicular $\Rightarrow a + b = 0$

Hence, the correct answer is (B).

102.



Given lines are $y = (\tan 30°)x - 1, y = \tan(150°)x + 1$

After rotation, the lines are

$$y = (\tan 45°)(x - \sqrt{3}), y = (\tan 135°)(x - \sqrt{3})$$

$$\therefore \quad (y - x + \sqrt{3})(y + x - \sqrt{3}) = 0$$

Hence, the correct answer is (A).

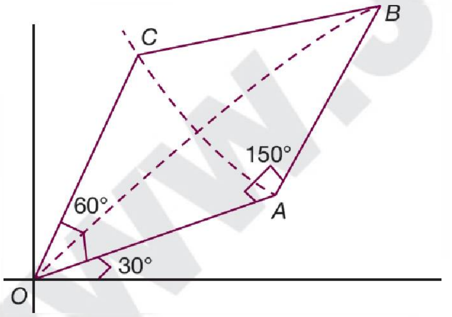
103. For $x \geq 0, (y + 20)(y + 2x - 20) = 0$

For $x \leq 0, (y - 20)(y + 2x + 20) = 0$

Area $= 20 \times 40 = 800$ sq. units

Hence, the correct answer is (B).

104. Let the length of the sides of the rhombus be R.



$= \text{Area } (OA)^2 \sin 30° = 2 \Rightarrow OA = 2$ sq. units

$$OB^2 = OA^2 + AB^2 - 2(OA)(AB)\cos 150°$$

$$= 4 + 4 - 2(4)\left(\frac{-\sqrt{3}}{2}\right) = 4\left(2 + \sqrt{3}\right)$$

$$OB = 2\sqrt{2 + \sqrt{3}} \Rightarrow B = \left(\sqrt{4 + 2\sqrt{3}}, 4 + 2\sqrt{3}\right)$$

$$= \left(1 + \sqrt{3}, 1 + \sqrt{3}\right)$$

Hence, the correct answer is (A).

105. Let, AB and AC be the equal sides of the isosceles triangle ABC.

The bisectors are

$$\frac{3x - 4y + 9}{5} = \pm \frac{4x + 3y - 13}{5}$$

i.e., $x + 7y - 22 = 0$ and $7x - y - 4 = 0$.

$\therefore$ Equation of the base is $y - 3 = 7(x - 1)$, i.e.,
$7x - y - 4 = 0$.

Hence, the correct answer is (A).

106. $(2, -1)$ reflected about $x = 4$ is $(6, -1)$ $(6, -1)$ is translated in the direction $y = x$ by 2 units, then the new position is $\left(6 + \sqrt{2}, -1 + \sqrt{2}\right)$

$\left(6 + \sqrt{2}, -1 + \sqrt{2}\right)$ reflected about $y = 4$ is

$$\left(6 + \sqrt{2}, 9 - \sqrt{2}\right)$$

Hence, the correct answer is (B).

107. $\alpha + \beta = 6p_1; \alpha\beta = 2$

$\beta + \gamma = 6p_2; \beta\gamma = 3$

$\gamma + \alpha = 6p_3; \gamma\alpha = 6$

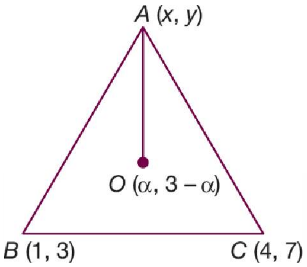
$\alpha\beta\gamma = 6$

$\Rightarrow \quad \alpha = 1, \beta = 2, \gamma = 3$

$$\text{Centroid} = \left(\frac{\alpha + \beta + \gamma}{3}, \frac{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}{3}\right) = \left(2, \frac{11}{18}\right)$$

Hence, the correct answer is (C).

108. $BO \perp AC$



$$\frac{y - 7}{x - 4} \cdot \frac{\alpha}{1 - \alpha} = -1 \Rightarrow \frac{\alpha - 1}{\alpha} = \frac{y - 7}{x - 4}$$

$$1 - \frac{1}{\alpha} = \frac{y - 7}{x - 4} \Rightarrow \frac{1}{\alpha} = 1 - \frac{y - 7}{x - 4} \Rightarrow \alpha = \frac{x - 4}{x - y + 3}$$

$AO \perp BC$

$$\frac{y - 3 + \alpha}{x - \alpha} \times \frac{4}{3} = -1 \Rightarrow 4y - 12 + 4\alpha = -3x + 3\alpha$$

$$\alpha = -3x - 4y + 12$$

$\therefore$ The locus is $\dfrac{x-4}{x-y+3} = -3x - 4y + 12$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. The given points are

$$P(-\sin(\beta-\alpha), -\cos\beta), Q(\cos(\beta-\alpha), \sin\beta),$$
$$R(\cos(\beta-\alpha+\theta), \sin(\beta-\theta)), \text{ where}$$

$$0 < \alpha, \beta, \theta < \frac{\pi}{4}$$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ -\sin(\beta-\alpha) & \cos(\beta-\alpha) & \cos(\beta-\alpha+\theta) \\ -\cos\beta & \sin\beta & \sin(\beta-\theta) \end{vmatrix}$$

Operating $C_3 - C_1 \sin\theta - C_2 \cos\theta$ we get,

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & 1-\sin\theta-\cos\theta \\ -\sin(\beta-\alpha) & \cos(\beta-\alpha) & 0 \\ -\cos\beta & \sin\beta & 0 \end{vmatrix}$$

$$= (1 - \sin\theta - \cos\theta)$$
$$(\cos\beta\cos[\beta-\alpha] - \sin\beta\sin[\beta-\alpha])$$

$$\Rightarrow \quad \Delta = (-[\sin\theta + \cos\theta])\cos(2\beta - \alpha)$$

$$\therefore \quad 0 < \alpha, \beta, \theta < \frac{\pi}{4} \quad \therefore \sin\theta + \cos\theta \neq 1$$

Also, $2\beta - \alpha < \dfrac{\pi}{4}$

$$\Rightarrow \quad \cos(2\beta - \alpha) \neq 0$$

$$\therefore \quad \Delta \neq 0$$

$\Rightarrow$ The three points are non collinear.

Hence, (A), (B) and (C) are correct.

2. Given equation becomes

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2+x_3 & y_2+y_3 & 2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \dfrac{x_2+x_3}{2} & \dfrac{y_2+y_3}{2} & 1 \end{vmatrix} = 0$$

It is the equation of the line passing through A and midpoint of BC.

Hence, (C) and (D) are correct.

3. For concurrency of three lines

$$px + qy + r = 0, \; qx + ry + p = 0 \text{ and } rx + py + q = 0$$

We must have, $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$$\Rightarrow \quad C_1 \to C_1 + C_2 + C_3$$

$$\begin{vmatrix} p+q+r & q & r \\ p+q+r & r & p \\ p+q+r & p & q \end{vmatrix} = 0$$

$$\Rightarrow \quad R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$$

$$\Rightarrow \quad (p+q+r)\begin{vmatrix} 0 & q-r & r-p \\ 0 & r-p & p-q \\ 1 & p & q \end{vmatrix} = 0$$

$$\Rightarrow \quad (p+q+r)(p - q^2 - rp + rq - r^2 + pr + pr - p^2) = 0$$

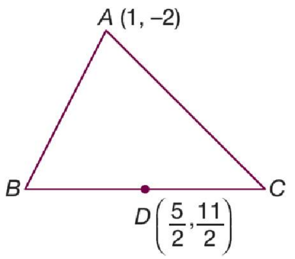
$$\Rightarrow \quad (p+q+r)(p^2 + q^2 + r^2 - pq - pr - rq) = 0$$

$$\Rightarrow \quad p^3 + q^3 + r^3 - 3pqr = 0$$

It is clear that A, B and C are correct options.

Hence, (A), (B) and (C) are correct.

4.



$$A = (1, -2), B = (k, -2k), C = (l, l-3)$$

$$\therefore \quad \frac{k+l+1}{3} = 2, \quad \frac{-2-2k+l-3}{3} = 3$$

$$\Rightarrow \quad k = -3, \ell = 8$$

$$\therefore \quad B(-3, 6), C(8, 5)$$

BC equation: $x + 11y = 63$

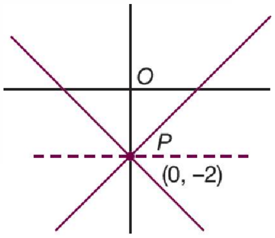
$$\Rightarrow \quad p + q = 74$$

Radius through A is $5x - y - 7 = 0$.

Attitude through s is $11x - y - 13 = 0$.

Hence, (A), (C) and (D) are correct.

5.



$x^2 - y^2 - 4y - 4 = 0$

$\Rightarrow \quad x^2 - (y + 2)^2 = 0$

$\therefore$ lines are

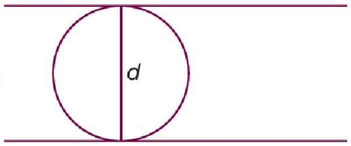
$x + y + 2 = 0$

$x - y - 2 = 0$

$\therefore$ angle bisectors are $x = 0$ and $y = -2$

Hence, (A), (B) and (D) are correct.

6.



$(3x + 4y)2 - 25 = 0$

$\Rightarrow \quad 3x + 4y + 5 = 0$

$3x + 4y - 5 = 0$

are parallel to tangents to the circle distance between parallel tangents

$= d =$ diameter of the circle

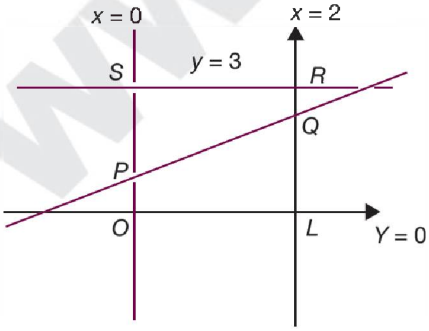
$\therefore \quad d = \frac{10}{5} = 2$

$\therefore \quad r = 1$

$\therefore$ area of circle $= \pi$ sq. units

Hence, (B) and (D) are correct.

7.



Let $OP = h, LQ = h_1$

Area of $OLQP =$ Area of $PQRS$

$\Rightarrow \quad \frac{1}{2}.2(h + h_1) = \frac{1}{2}.2(3 - h + 3 - h_1)$

$\Rightarrow \quad h + h_1 = 3$

$(0, h), (2, h_1)$ lies on $x - 4y + k = 0$

$\Rightarrow \quad k = 5$

Hence, the correct answer is (B).

8. $x + ky + k_1 = 0$, will be parallel to either $x + y + 4 = 0$ or $x - y + 6 = 0$. Hence,

$k = 1$ and $k = -1$.

Hence, (A) and (B) are correct.

9. $L_1(0,0) = 0 - 0 - 6 = -6 < 0$

$L_1(1,1) = 1 - 1 - 6 = -6 < 0$

$L_2(0,0) = 0 + 0 + 4 = 4 > 0$

$L_2(1,1) = 1 + 1 + 4 = 6 > 0$

$L_1(-3,-3) = -3 + 3 - 6 = -6 < 0$

$L_2(-3,-3) = -3 - 3 + 4 = -2 < 0$

$\Rightarrow \quad (1,1) \rightarrow 1^{st}$

$\Rightarrow \quad (-3,-3) \rightarrow 4^{th}$

Hence, (A) and (C) are correct.

10. Let the slope of $u = 0$ be m. Then, slope of $v = 0$ is $\frac{9m}{2}$.

$$\therefore \quad \frac{7}{9} = \left| \frac{m - \frac{9m}{2}}{1 + m \times \frac{9m}{2}} \right| = \left| \frac{-7m}{2 + 9m^2} \right|$$

$\Rightarrow \quad 9m^2 - 9m + 2 = 0$ or $9m^2 + 9m + 2 = 0$

$$m = \frac{9 \pm \sqrt{81 - 72}}{18} = \frac{9 \pm 3}{18} = \frac{2}{3}, \frac{1}{3}$$

or $\quad m = \frac{-9 \pm 3}{18} = -\frac{2}{3}, -\frac{1}{3}$

Therefore, equations of lines are

i. $3y = x$ and $2y = 3x$

ii. $3y = 2x$ and $y = 3x$

iii. $x + 3y = 0$ and $3x + 2y = 0$

iv. $2x + 3y = 0$ and $3x + y = 0$

Hence, (A), (B), (C) and (D) are correct.

11. The separate equations of the sides are $x-y=0, x+y=0$ and $2x+3y-6=0$. The point $(-2, a)$ moves on the line $x=-2$ and $(b, 1)$ moves on the line $y=1$. From the figure, the y-co-ordinates of points of intersection of $x=-2$ with $y=-x$ and $2x+3y=6$ give the range of values of a. The x-co-ordinates of the points of intersection of $y=1$ with $y=-x$ and $y=x$ give the range of values of b.

Hence, (A) and (D) are correct.

12. Here, $\underline{|COA} = 30°$

Area of rhombus $= 2.\frac{1}{2}.OA.OC \sin 30°$

$$\Rightarrow \quad 2=\frac{1}{2}x^2 \Rightarrow OA=OC=2$$

Also, $\underline{|OAB} = 150°$

$$\Rightarrow \quad \cos 150° = \frac{OA^2+AB^2-OB^2}{2OA\times AB}$$

$$OB^2 = 8+4\sqrt{3} \Rightarrow OB=\sqrt{2}\left(\sqrt{3}+1\right)$$

Hence, the co-ordinates of B are

$$\left(\pm\sqrt{2}\left(\sqrt{3}+1\right)\right)\cos 45°, \pm\sqrt{2}(\sqrt{3}+1)\sin 45°)$$

Hence, (A) and (B) are correct.

13. Let P(x, y) be the moving point. Then the area of ΔPOA is $\frac{1}{2}|x\cdot 4-y\cdot 0| = |2x|$ and that of ΔPOB is $\frac{1}{2}|x\cdot 0-y\cdot 6|=|-3y|$, because one vertex of the triangle is at the origin.

∴ From the given condition, we have

$$|2x|=2|-3y| \text{ or } x=\pm(3y)$$

$$\Rightarrow \quad x-3y=0 \text{ or } x+3y=0$$

Hence, (A) and (B) are correct.

14. Two lines will be identical, if their exists some real number k such that

$b^3-c^3=k(b-c), c^3-a^3=k(c-a)$ and $a^3-b^3=k(a-b)$

$\Rightarrow \quad b-c=0$ or $b^2+c^2+bc=k$

$c-a=0$ or $c^2+a^2+ca=k$

and $a-b=0$ or $a^2+b^2+ab=k$

$\Rightarrow \quad a=b$ or $b=c$ or $c=a$

or $\quad a=b$ or $a+b+c=0$

Hence, (A), (B), (C) and (D) are correct.

15. As the third vertex lies on the line $y=x+3$, its co-ordinates are of the form $(x, x+3)$. The area of the triangle with vertices $(2, 1)$, $(3, -2)$ and $(x, x+3)$ is given by

$$\frac{1}{2}\begin{vmatrix} x & x+3 & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix}$$

$$=|2x-2|=5 \quad \text{(Given)}$$

$$\therefore \quad 2x-2=\pm 5$$

$$\Rightarrow \quad x=\frac{-3}{2}, \frac{7}{2}$$

Thus, the co-ordinates of the third vertex are $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(\frac{-3}{2}, \frac{3}{2}\right)$.

Hence, (A) and (C) are correct.

16.
$$\begin{vmatrix} 1 & -2 & -6 \\ 3 & 1 & -4 \\ \lambda & 4 & \lambda^2 \end{vmatrix}=0$$

$$\Rightarrow \quad \lambda^2+2\lambda-8=0 \;\; (\therefore (\lambda+4)(\lambda-2)=0)$$

$$\Rightarrow \lambda=-4, 2$$

Hence, (A) and (D) are correct.

17. Equation of any line through the point of intersection of the given lines is $(3x+y-5)+\lambda(x-y+1)=0$.

∵ This line is perpendicular to one of the given lines

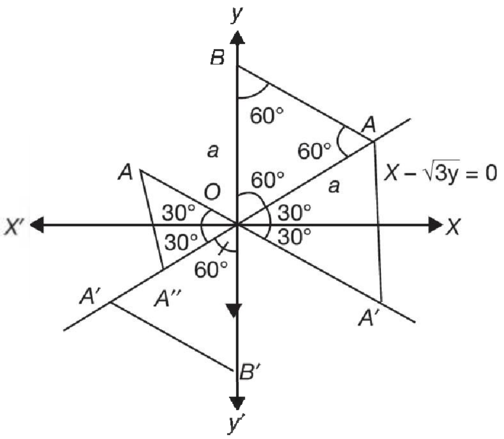
$$\frac{3+\lambda}{\lambda-1}=-1 \text{ or } \frac{1}{3}$$

$$\Rightarrow \quad \lambda=-1 \text{ or } -5$$

∴ The required straight line is $x+y-3=0$ or $x-3y+5=0$.

Hence, (A) and (D) are correct.

18. If B lies on y-axis, then co-ordinates of B are $(0, a)$ or $(0, -a)$.



If third vertex in IVth quadrant or in IInd quadrant, then its co-ordinates are $(a\cos 30°, -a\sin 30°)$,

i.e., $\left(\frac{a\sqrt{3}}{2}, -\frac{a}{2}\right)$ and $\left(-\frac{a\sqrt{3}}{2}, \frac{a}{2}\right)$.

Hence, (A), (B), (C) and (D) are correct.

19. $\because$ $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow$ $3abc - a^3 - b^3 - c^3 = 0$

$\Rightarrow$ $-(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

As, $a + b + c \neq 0$

$\therefore$ $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$$\frac{1}{2}\left\{(a-b)^2 + (b-c)^2 + (c-a)^2\right\} = 0$$

As a, b and c are real numbers.

$\therefore$ $b - c = 0, c - a = 0, a - b = 0$

$\Rightarrow$ $a = b = c$

Hence, (A), (C) and (D) are correct.

20. We know that the bisectors are mutually perpendicular to each other.

Let, $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = (ax^2 + pxy - ay^2)(x^2 + qxy - y^2)$

By comparing $b = aq + p$, $c = pq - 2a$, $d = -p - ap$

$b + d = 0 \qquad c + 2a = pq$

However, given that bisectors of one pair are given by the other,

$$\frac{x^2-y^2}{a-(-a)} = \frac{xy}{p/2} \Rightarrow \frac{x^2-y^2}{4xy} = \frac{a}{p} \quad \text{(is same as the other)}$$

$\therefore$ $\frac{x^2-y^2}{4xy} = \frac{-q}{4}$ $\qquad$ $\therefore$ $\frac{a}{p} = \frac{-q}{4} \Rightarrow pq = -4a$

$\Rightarrow$ $c + 2a = -4a$

$\Rightarrow$ $c + 6a = 0$

Hence, (A) and (C) are correct.

21. Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$r^2\left(5\cos^2\theta - 8\sin\theta\cos\theta + 5\sin^2\theta\right) = 4$$

$\Rightarrow$ $\frac{8}{r^2} = 10 - 8\sin 2\theta$

$\therefore$ $OP = r = \sqrt{\frac{4}{5 - 4\sin 2\theta}}$

For maximum value of $\theta = \frac{\pi}{4}$ and maximum value $= 2$

For minimum value of $\theta = \frac{3\pi}{4}$ and minimum value $= \frac{2}{3}$

Hence, (B) and (C) are correct.

22. $h^2 = ab \Rightarrow h = \pm 6$

$af^2 = bg^2 \Rightarrow f = \pm 2$

$\Delta = 0 \Rightarrow h^2 - 3f^2 + 2fh = 48$

$\therefore$ $h = 6, f = 2$ and $h = -6, f = -2$

Hence, (A) and (B) are correct.

23. Slope of $AC = \frac{1}{3} = \tan\theta$

Slope of $AB = \tan\left(\theta - \frac{\pi}{4}\right) = \frac{\frac{1}{3} - 1}{1 + \frac{1}{3}.1} = -\frac{1}{2}$

Equation of AB is $x + 2y - 7 = 0$ and AD is $2x - y + 1 = 0$.

Hence, (A) and (D) are correct.

24. Equation represents a parallelogram and coefficient of x^2 + coefficient of $y^2 = 0$ (Rectangle)
Given lines are $3x + 2y = 0$, $2x - 3y = 0$, $3x + 2y - 1 = 0$, $2x - 3y + 1 = 0$. They form squares.

Hence, (A), (B), (C) and (D) are correct.

25. The required lines are parallel to the angle bisectors at the given lines.
The angle bisectors of the given lines are

$$\frac{3x - 4y - 7}{\sqrt{9 + 16}} = \pm\left(\frac{12x - 5y + 6}{\sqrt{144 + 25}}\right)$$

$\Rightarrow$ $13(3x - 4y - 7) = \pm 5(12x - 5y + 6)$

$\Rightarrow$ $21x + 27y + 121 = 0$ or $99x - 77y - 61 = 0$

Equation of the lines passing through (4, 5) and parallel to these lines are

$21(x - 4) + 27(y - 5) = 0$

$\Rightarrow$ $7x + 9y - 73 = 0$

$99(x - 4) - 77(y - 5) = 0$

$\Rightarrow$ $9x - 7y - 1 = 0$

Hence, (A) and (C) are correct.

26. Vertices are $A(2, -2)$, $B(-2, 2)$ and $C(1, 1)$.

Clearly, $AC = BC$ and $\angle ACB > 90°$

Hence, (A) and (D) are correct.

27. Note that the lines $u = 0$, $v = 0$ are perpendicular. Make the co-ordinate axes coincide with $u = 0$, $v = 0$. Now, the lines $L_1 \equiv 0$, $L_2 \equiv 0$, are equally inclined with $u - v$ axes.

$\therefore$ $u = 0$, $v = 0$ are bisectors.

Hence, (A) and (D) are correct.

28. Inclinations of two lines are θ and ϕ

$\therefore$ Inclination of angle bisector is $\dfrac{\theta+\phi}{2}$

$$\alpha = \frac{\theta+\phi}{2} \text{ and } \tan\alpha \times \frac{\gamma}{\beta} = -1$$

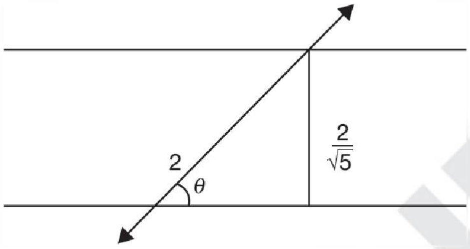
$$\Rightarrow \quad \tan\alpha = \frac{-\beta}{\gamma}$$

$$\Rightarrow \quad \therefore \beta = -\sin\alpha,\ \gamma = \cos\alpha$$

$$\Rightarrow \quad \beta^2 + \gamma^2 = 1$$

Hence, (A), (B) and (C) are correct.

29.



$$\tan\theta = \frac{1}{2}$$

Slope of required line $= m$

Slope of given lines $= -2$

$$\left|\frac{m+2}{1+2m}\right| = \frac{1}{2} \Rightarrow m = \frac{-5}{4} \text{ or } \infty$$

The lines are $3x + 4y - 18 = 0$, $x - 2 = 0$

Hence, (B) and (C) are correct.

30. $(x-3)(y+4) = 0, (x+4)(y-3) = 0$

The vertices are $A = (-4,-4), B = (-4,3), C = (3,3)$, $D = (-4,3)$

Diagonal AC is $x = y$ and diagonal BD is $x + y + 1 = 0$.

Hence, (A) and (B) are correct.

31. $x\cos\alpha + y\sin\alpha = P$

Axis rotated through angle 'θ'

Transformed equation is $\cos\alpha(x\cos\theta - y\sin\theta) + \sin\alpha (x\sin\theta + y\cos\theta) = P$

$$x\cos(\alpha-\theta) + y\sin\alpha(\alpha+\theta) = P \quad \Rightarrow x\cos\beta + y\sin\beta = P$$

$$\cos\beta = \cos(\alpha-\theta), \quad \sin\beta = \sin(\alpha-\theta)$$

Hence, (A) and (C) are correct.

32. (3, 2) (–4, 1) (–5, 8) form a right angle triangle at (–4, 1)

Orthocentre is (–4, 1), circumcentre is midpoint of (3, 2) (–5, 8). i.e., (–1, 5).

Hence, (B) and (C) are correct.

33. Dividing point of $P(-5, 1)$, $Q(3, 5)$ in the ratio $k : 1$ is

$$A = \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right), B = (1, 5), C = (7, -2)$$

Area of $\Delta ABC = 2$ sq. units

$$\therefore \quad 9k^2 - 94k + 217 = 0$$

$$(k-7)(9k-31) = 0$$

$$k = 7, \frac{31}{9}$$

Hence, (A) and (D) are correct.

34. $ax^2 + 2hxy + by^2 = 0$

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

Now we get

$$\alpha\left(x'^2\right) + \beta\left(y'^2\right) + x'y'\left(2h\cos 2\theta + (b-a)\sin 2\theta\right) = 0$$

Now

$$2h\cos 2\theta = (a-b)\sin 2\theta$$

$$\tan 2\theta = \frac{2h}{a-b}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4\sqrt{3}}{-4}\right) = \tan^{-1}(-\sqrt{3})$$

$$\Rightarrow \quad \theta = \frac{1}{2}\left(\pi - \frac{\pi}{3}\right), \frac{1}{2}\left(-\frac{\pi}{3}\right)$$

$$\therefore \quad \theta = \frac{\pi}{3}, -\frac{\pi}{6}$$

Hence, (B) and (D) are correct.

35. It is clear that diagonals of the rhombus will be parallel to the bisectors of the given lines and will pass through (1, 3). Equations of bisectors of the given lines are

$$\frac{x+y-1}{\sqrt{2}} = \pm\left(\frac{7x-y-5}{5\sqrt{2}}\right)$$

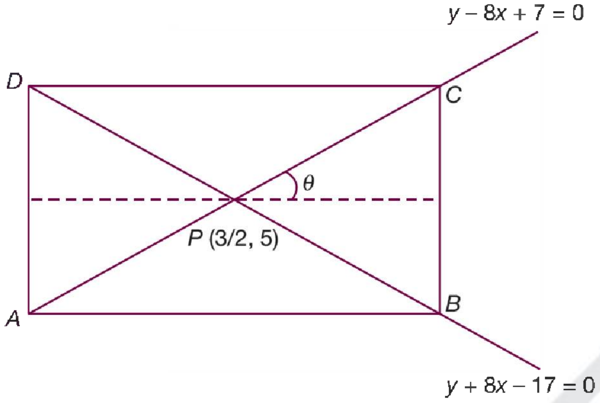
Or $2x - 6y = 0$, $6x + 2y = 5$

$\therefore$ The equations of diagonals are $x - 3y + 8 = 0$ and $3x + y - 6 = 0$. Thus, the required vertex will be the point where these lines meet the line $y = 2x$. Solving these lines we get possible coordinates as $\left(\frac{8}{5}, \frac{16}{5}\right)$ and $\left(\frac{6}{5}, \frac{12}{5}\right)$.

Hence, (A) and (C) are correct.

36. The intersection point of the given diagonals is $P \equiv \left(\frac{3}{2}, 5\right)$

Equation of angular bisectors of the diagonals are



$$\frac{y+8x-17}{\sqrt{65}} = \pm\frac{y-8x+7}{\sqrt{65}}$$

$\Rightarrow \quad x = \frac{3}{2}$ and $y = 5$

Let length of BC be a and that of CD be b

Then, $\tan\theta = \frac{a/2}{b/2} = \frac{a}{b} = 8.$

Also, $ab = 8$

$\Rightarrow \quad a = 8, b = 1.$

So, equations of sides are $y = 1, y = 9, x = 1$ and $x = 2$.

Hence, (A) and (C) are correct.

37. $6a^2 - 7a - 3 < 0 \Rightarrow a \in \left(-\frac{1}{3}, \frac{3}{2}\right)$

Hence, (A) and (B) are correct.

38. Equation of the curve passing through all four points A, B, C, D can be written as

$$(3x + 4y - 24)(4x + 3y + -24) + \lambda xy = 0$$

Now, for different values of λ we will get different curves.

Hence, (A), (B), (C) and (D) are correct.

39. Let $x^2 + 2hxy + y^2 = 0$ be given by $y = m_1x$ and $y = m_2x$.

$$m_1 + m_2 = -2h$$

Slope of $y + x = -1$

$$\tan\alpha = \left|\frac{m_1+1}{1-m_1}\right|, \ \tan\alpha = \left|\frac{m_2+1}{1-m_2}\right|;$$

$$\tan\alpha = \frac{m_1+1}{1-m_1}, \ \tan\alpha = -\left(\frac{m_2+1}{1-m_2}\right)$$

(For +ve signs, in both gives the same value but $m_1 \neq m_2$).

$$\Rightarrow \quad m_1 = \frac{\tan\alpha - 1}{\tan\alpha + 1}, m_2 = \frac{\tan\alpha + 1}{\tan\alpha - 1}$$

$$m_1 + m_2 = -2\sec 2\alpha$$

$$\Rightarrow \quad h = \sec 2\alpha$$

$$\Rightarrow \quad \cos 2\alpha = \frac{1}{h} \Rightarrow 2\cos^2\alpha - 1 = \frac{1}{h}$$

$$\Rightarrow \quad \cos\alpha = \left(\frac{1+h}{2h}\right)^{\frac{1}{2}} \Rightarrow \cot\alpha = \sqrt{\frac{h+1}{h-1}}$$

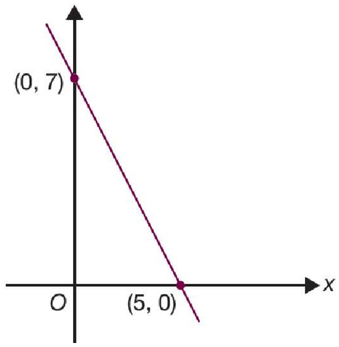
Hence, (A), (B), (C) and (D) are correct.

40. $(2a + 3b - c)(3a - b + c) = 0$

$\Rightarrow \quad -2a - 3b + c = 0$ or $3a - b + c = 0$

Hence, (A) and (B) are correct.

41.



Other diagonal passes through midpoint of opposite vertices and perpendicular to the given diagonal

$\Rightarrow \quad 7y - 5x = 12$

Hence, (A) and (C) are correct.

42. $\because$ C lies on $x = 2a$, the coordinates of C be $(2a, k)$, $(k > 0)$

If $\quad AC = AB \Rightarrow k = a\sqrt{5}$, if $AC = BC$

$$\Rightarrow \quad K = \frac{5a}{2}$$

If $\quad AB = BC \Rightarrow k = 2a$

Hence, the equation of BC is $y - a = \frac{K - a}{2a}x$

Hence, (A), (B) and (C) are correct.

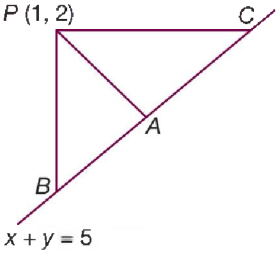
43. General point on the line $(3 + r\cos\theta, 4 + r\sin\theta)$

$$3 + r\cos\theta = 6 \Rightarrow r = \frac{3}{\cos\theta} = PR \text{ and } 4 + r\sin\theta = 8$$

$$\Rightarrow r = \frac{4}{\sin\theta} = PS$$

Hence, (A), (B) and (D) are correct.

44.



$$x - y + 1 + \lambda(2x - y) = 0$$

All the lines pass through (1, 2)

$$PA = \sqrt{2}$$

PB and PC make angle $\frac{\pi}{6}$ with the line PA: $x - y + 1 = 0$

The equations are $y - 2 = (2 + \sqrt{3})(x - 1)$ and $y - 2 = (2 - \sqrt{3})(x - 1)$.

Hence, (A) and (B) are correct.

45. Fixed point = (2, 4)

Locus in a circle on (0, 0) (2, 4) as diameter $\Rightarrow \quad x^2 + y^2 - 2x - 4y = 0$

Line at maximum distance $x + 2y - 10 = 0$

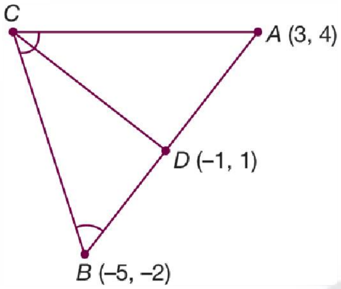
Line at minimum distance $2x - y = 0$

Maximum distance = distance between (0, 0), (2, 4) $= \sqrt{20} < 5$

$\therefore$ Number of lines = 0

Hence, (A) and (B) are correct.

46.



D is the midpoint of AB. Move D at right angles to AB by $BD \cot\frac{C}{2} = 5.\cot\frac{C}{2}$ units.

$$\tan C = 2 \Rightarrow \cot\frac{C}{2} = \frac{\sqrt{5}+1}{2}$$

$$CD = 5\left(\frac{\sqrt{5}+1}{2}\right) = \frac{5}{2}\left(\sqrt{5}+1\right)$$

$$\text{Slope of CD} = \frac{-4}{3}$$

$$\frac{x+1}{-\frac{3}{5}} = \frac{y-1}{\frac{4}{5}} = \pm\frac{5}{2}\left(\sqrt{5}+1\right)$$

$$\left(-\frac{5+3\sqrt{5}}{2}, 2\sqrt{5}+3\right) \text{ and } \left(\frac{3\sqrt{5}+1}{2}, -\left(2\sqrt{5}+1\right)\right)$$

Hence, (A) and (B) are correct.

47. If θ is the angle made by the required line with are of the given lines,

then $\sin\theta = \frac{12}{13} \Rightarrow \tan\theta = \frac{12}{5}$

$$m_1 = \frac{-5}{12}, m_2 = \frac{m_1 + \tan\theta}{1 - m_1\tan\theta} \text{ and } \frac{m_1 - \tan\theta}{1 + m_1\tan\theta}$$

$$\Rightarrow \quad m_2 = \frac{119}{120} \text{ or undefined}$$

$\therefore$ Equations of the required line $x = 0$ and $119x - 120y = 0$.

Hence, (B) and (C) are correct.

48. Substituting the co-ordinates of the points (1, 3), (5, 0) and (–1, 2) in $3x + 2y$, we obtain the values 8, 15 and 1, which are all positive.

$\therefore$ All the points lying inside the Δ formed by given points satisfy $3x + 2y = 0$.

Hence, (A) is the correct answer.

Substituting the co-ordinates of the given points in $2x + 3y - 13$, we find the values –8, –3 and –13, which are all negative.

So, (B) is not correct.

Again substituting the given points in $2x - 3y - 12$ we get –19, –2, –20, which are all negative.

It follows that all points lying inside the Δ formed by given points satisfy $2x - 3y - 12 \geq 0$.

So, (C) is the correct answer.

Finally substituting the co-ordinates of the given points in $-2x + y$, we get 1, –10 and 4, which are not all positive.

So, (D) is not correct.

Hence, (A) and (C) are the correct answers.

Hence, (A) and (C) are correct.

49. $x_2 = x_1 r, x_3 = x_1 r^2$ and so is $y_2 = y_1 r, y_3 = y_1 r^2$

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = r.r^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & \frac{1}{r} \\ x_1 & y_1 & \frac{1}{r^2} \end{vmatrix} = 0$$

Hence, the points lie on a line, i.e., they are collinear.

Hence, the correct answer is (A).

50. $\log_5 \frac{(k+4)}{k} = \frac{1}{2} \Rightarrow k = 1 + \sqrt{5}$

Hence, (B) and (C) are correct.

51. $\frac{x-4}{2-3m} = \frac{y-3}{1+4m} = \frac{-2(4(2-3m)+3(1+4m)+2-m)}{(2-3m)^2+(1+4m)^2}$

Now, find the image and then eliminate m.

Hence, the correct answer is (D).

52. Find the angular bisector of $x - \sqrt{3}y + 2\sqrt{3} = 0$ and $\sqrt{3}x - y + 3 = 0$. This will be parallel to the diagonal which is passing through $(1, 2)$. Diagonals are $x + y = 3$ and $x - y + 1 = 0$.

$A(3, 0)$ $C(-1, 4)$

$A(-1, 0)$ $C(3, 4)$

Hence, (A) and (C) are correct.

Linked Comprehension Type Questions

1. Image of A with respect to the given internal angle bisectors lies on the side BC.

Image of A in $x - y - 1 = 0 \Rightarrow (3, 0)\, A_1$

Image of A in $y + 2x - 9 = 0 \Rightarrow (5, 4)\, A_2$

Equation of A_1A_2 or BC is $y = 2(x - 3)$

Hence, the correct answer is (A).

2. The equation of altitude from A on BC is $2y + x = 5$.

Hence, the correct answer is (C).

3. Intersection of BC with the given angle bisectors, we get $B(5, 4)$ and $C\left(\frac{15}{4}, \frac{3}{2}\right)$ as the co-ordinates of triangle.

Hence, the correct answer is (B).

4. $A(4,0)$, $B(-4, 0)$ are two point.

Let, $P = (x, y)$

$PA + PB = 10 = 2a$

$\Rightarrow$ $a = 5$

$(h, k) = (4, 0), (-h, k) = (-4, 0)$

$\therefore$ The equation of locus is $\frac{x^2}{a^2} + \frac{(y-k)^2}{a^2 - h^2} = 1$.

$\frac{x^2}{25} + \frac{y}{25-16} = 1,\ \frac{x^2}{25} + \frac{y^2}{9} = 1$

Hence, the correct answer is (B).

5. $PA = PB$

$\Rightarrow$ $PA^2 = PB^2$

$(x - y)^2 + y^2 = (x + y)^2 + y^2$

$\therefore$ $x = 0$

Hence, the correct answer is (A).

6. Area of ΔPAB = 4 sq. units

$\Rightarrow$ $\frac{1}{2}|x(0) + 4(0 - y) - 4(y - 0)| = 4$

$\Rightarrow$ $|-4y - 4y| = 8$

$\Rightarrow$ $8y = 8$

$\therefore$ $y = 1$

$\Rightarrow$ $y - 1 = 0$

Hence, the correct answer is (C).

7. Let the ΔABC has vertices $A(0, 4)$, $B(-3, 0)$ and $C(3, 0)$. Let the point P be (α, β).

Equation of line AC is 4; $x + 3y - 12 = 0$ and the equation of line AB is $4x - 3y + 12 = 0$

$$\Rightarrow \quad |\beta| = \sqrt{\frac{|(4\alpha + 3\beta - 12)(-4\alpha + 3\beta - 12)|}{\sqrt{25} \times \sqrt{25}}}$$

$$\Rightarrow \quad 2(\alpha^2 + \beta^2) + 9\beta - 18 = 0 \qquad \ldots(1)$$

$\because$ Point P lies inside the ΔABC, its locus is the minor arc of a circle.

Hence, the correct answer is (B).

8. The circle (1) cuts the y-axis at $R\left(0, \frac{3}{2}\right)$ and $S(0, -6)$.

Hence, the minimum distance of A from the locus OS,

$$AR = 4 - \frac{3}{2} = \frac{5}{2}$$

Hence, the correct answer is (A).

9. Equation of the tangents to (1) at C is obtained by $T = 0$.

$\Rightarrow$ The tangent is $2(3x + 0y) + \frac{9}{2}y - 18 = 0$

$\Rightarrow$ $4x + 3y - 12 = 0$, which is same as the line AC.

Hence, the tangents at B and C intersect at A.

$\Rightarrow$ Area of the ΔPBC is $\frac{1}{2}$ (6)(4) = 12.

Hence, the correct answer is (B).

10. $$\frac{x - x_1}{1} = \frac{y - y_1}{1} = \frac{-1(x_1 + y_1 + 2)}{2}$$

$$x = x_1 + \frac{-2(x_1 + y_1 + 2)}{2}$$

$$y = y_1 + \frac{-2(x_1 + y_1 + 2)}{2}$$

$$\therefore \quad y - \frac{2(x_1 + y_1 + 2)}{2} = 1 + \frac{1}{\left(x - \frac{2(x_1 + y_1 + 2)}{2}\right)^2}$$

$$\Rightarrow \quad (x + 3)(y + 2)^2 + 1 = 0$$

Hence, the correct answer is (B).

11. $$\frac{x - x_1}{3} = \frac{y - y_1}{2} = \frac{-2(3x_1 + 2y_1 + 1)}{13}$$

$$x = x_1 + \frac{-6(3x_1 + 2y_1 + 1)}{13}$$

$$y = y_1 + \frac{-4(3x_1 + 2y_1 + 1)}{13}$$

$$\therefore \quad 2\left\{x - 6\left(\frac{3x + 2y + 1}{13}\right)\right\} + 3\left\{y - 4\left(\frac{3x + 2y + 1}{13}\right)\right\} = 1$$

$$\Rightarrow \quad 46x + 9y + 37 = 0$$

Hence, the correct answer is (A).

12. $$\frac{x - x_1}{1} = \frac{y - y_1}{1} = \frac{-2(x_1 + y_1 + 2)}{2}$$

$$x = x_1 \frac{-2(x_1 + y_1 + 2)}{2}$$

$$y = y_1 + \frac{-2(x_1 + y_1 + 2)}{2}$$

$$\therefore \quad \left\{x - 2\frac{(x + y - 2)}{2}\right]^2 + 2\left\{x - \frac{2(x + yz - 2)}{2}\right\}$$

$$\left\{y - \frac{2(x + y - 2)1}{2}\right\} + 2\left\{y - \frac{2(x + y - 2)}{2}\right]^2 = 1$$

$$\Rightarrow \quad 2x^2 + 2xy + y^2 - 12x - 8y + 19 = 0$$

Hence, the correct answer is (C).

13. $w = 60°$, $m = 2$

$$\tan\theta = \frac{m \sin w}{1 + m \cos w} = \frac{\sqrt{3}}{2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Hence, the correct answer is (B).

14. $w = 60°$, $m_1 = 2$, $m_2 = \frac{-1}{2}$

$$\tan\theta_1 = \frac{m_1 \sin w}{1 + \cos w} = \frac{\sqrt{3}}{2}$$

$$\tan\theta_2 = -\frac{1}{\sqrt{3}}$$

angle between the lines $\tan\phi = \left|\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 + \tan\theta_2}\right|$

$$= \tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$$

Hence, the correct answer is (D).

15. $$m = \frac{\sin 60°}{\sin(30° - 60°)} = -\sqrt{3}$$

$\therefore$ equation of the line

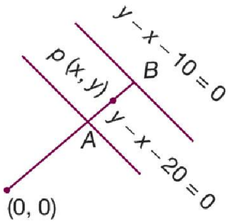
$$y - 0 = \sqrt{3}(x - 2)$$

$$y + \sqrt{3}x = 2\sqrt{3}$$

Hence, the correct answer is (C).

16. Let $OA = r_1$, $OP = r$, $OB = r_2$

$$A = (r_1\cos\theta, r_1\sin\theta)\,B = (r_2\cos\theta, r_2\sin\theta)$$



$$P = (x, y) = (r\cos\theta, r\sin\theta)$$

A lie on $y - x - 20 = 0$

$$\therefore\quad r_1\sin\theta - r_1\cos\theta - 20 = 0$$

$$\frac{20}{r_1} = \sin\theta - \cos\theta$$

$$\frac{10}{r_2} = \sin\theta - \cos\theta$$

$$\frac{2}{OP} = \frac{1}{OA} + \frac{1}{OB}$$

$$\Rightarrow\quad \frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{\sin\theta - \cos\theta}{20} + \frac{\sin\theta - \cos\theta}{10}$$

$$\Rightarrow\quad \frac{2}{r} = \frac{3\sin\theta - 3\cos\theta}{20}$$

$$\Rightarrow\quad 3(r\sin\theta) - 3(r\cos\theta) = 40$$

$$\Rightarrow\quad 3y - 3x = 40 \Rightarrow 3x - 3y + 40 = 0$$

Hence, the correct answer is (D).

17. $OP^2 = OA \cdot OB$

$$r^2 = r_1 r_2 \Rightarrow r^2 = \frac{20}{\sin\theta - \cos\theta} \cdot \frac{10}{\sin\theta - \cos\theta}$$

$$(r\sin\theta - r\cos\theta)^2 = 200$$

$$(y - x)^2 = 200$$

Hence, the correct answer is (C).

18. $r = r_1 + r_2$

$$r = \frac{20}{\sin\theta - \cos\theta} + \frac{10}{\sin\theta - \cos\theta}$$

$$\Rightarrow\quad r(\sin\theta - \cos\theta) = 30$$

$$y - x = 30$$

Hence, the correct answer is (C).

19. $m_1 + m_2 = 2\operatorname{cosec}\theta\sec\theta$, $m_1 m_2 = \operatorname{cosec}^2\theta\sec^2\theta - 1$,

$$m_1 - m_2 = \sqrt{(m_1 + m_2)^2 - 4m_1 m_2}$$

Hence, the correct answer is (D).

20. Equation of bisectors of lines $x^2 - 2hxy - y^2 = 0$ $\Rightarrow hx^2 + 2xy - hy^2 = 0$ which are identical to $x^2 - 2gxy - y^2 = 0 \Rightarrow gh = -1$

Hence, the correct answer is (B).

21. $\sum m_1 = -\frac{c}{d}$, $\sum m_1 m_2 = \frac{b}{d}$, $m_1 m_2 m_3 = -\frac{a}{d}$; given $m_1 m_2 = 1 \Rightarrow m_3 = -\frac{a}{d}$, which satisfies the given equation.

Hence, the correct answer is (C).

22. Let BC slope be 'm'.

$$AB = AC \Rightarrow \angle ABC = \angle ACB$$

$$\Rightarrow\quad m = \tan\theta = \left(\frac{7 - m}{1 + 7m}\right),\ \tan\theta = \frac{-1 - m}{1 - m}$$

$$\Rightarrow\quad \frac{7 - m}{1 + 7m} = \frac{m + 1}{1 - m} \Rightarrow -3, \frac{1}{3}$$

$$BC \text{ slope} = \frac{7x_1 + x - 8}{x_1 - x} = -3 \text{ or } \frac{1}{3}$$

$$\Rightarrow\quad 5x_1 - x = 4 \text{ or } x + 5x_1 = 6$$

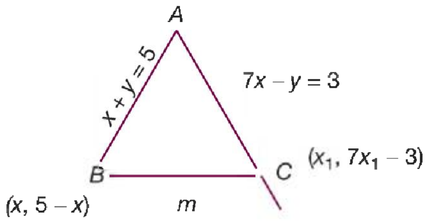
Hence, the correct answer is (D).

23. $x = \frac{7}{2} \Rightarrow x_1 = \frac{4 + \frac{7}{2}}{5} = \frac{3}{2}$ or $m = \frac{6 - 7/2}{5} = \frac{1}{2}$

'C' can't be $\left(\frac{-3}{2}, \frac{13}{2}\right)$

Hence, the correct answer is (C).

24.

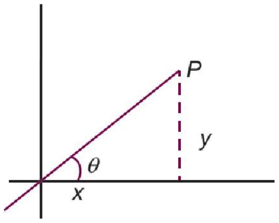


Area = 5 $\Rightarrow$ we get two equation

and then two equation we get form positive value of x and x_1

Hence, the correct answer is (B).

25.



Clearly,

$$|y|<|x|, \text{ if } \theta<\frac{\pi}{4},$$

$$|y|>|x|, \text{ if } \theta>\frac{\pi}{4} \text{ and}$$

$$|y|=|x|, \text{ if } \theta=\frac{\pi}{4}$$

Hence, the correct answer is (D).

26. We have maximum $\{|x-1|, |y-2|\}=4$.

If $|x-1| \geq |y-2|$, then $|x-1|=4$,

then $x=-3$ or 5.

If $|y-2| \geq |x-1|$, then $|y-2|=4$,

then $y=-2$ or 6

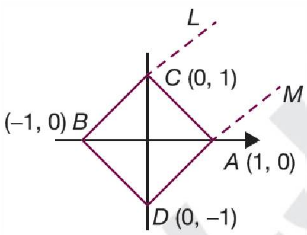
So, the locus of P bounds a square and the equation of whose sides are

$$x=-3, x=5, y=-2, y=6$$

Thus, the area is $(8)^2=64$ sq. units.

Hence, the correct answer is (A).

27. Consider a square with vertices at (–1, 0), (1, 0), (0, 1) and (0, –1) (see figure)



If we select a point P_1 in the region $ACLM$, then $d(B, P)=1+x$

$$D(A, P)=y$$

$$\therefore \quad d(A, P)+d(B, P)=1+x+y=2$$

However, for points P_2 and P_3 lying respectively on the line BC and below the line BC, $d(P, A)+d(P, B)=2$

The same argument holds for other quadrants also.

Hence, the $d(P, A)+d(P, B)=2$ represents the region lying inside the square $ABCD$.

Hence, the correct answer is (C).

28. $\because \quad OR=AR$

$$\Rightarrow \quad |x-0|+|y-0|=|x-1|+|y-2|$$

$$\Rightarrow \quad |x|+|y|=|x-1|+|y-2|$$

$$\because \quad 0 \leq x<1 \text{ and } 0 \leq y<2$$

$$\therefore \quad x+y=-(x-1)-(y-2)$$

$$\Rightarrow \quad 2x+2y=3$$

Hence, the correct answer is (D).

29. $\because \quad OS=BS$

$$\Rightarrow \quad |x-0|+|y-0|=|x-2|+|y-3|$$

$$\Rightarrow \quad |x|+|y|=|x-2|+|y-3|$$

$$\because \quad x \geq 2 \text{ and } 0 \leq y<3$$

$$\therefore \quad x+y=x-2+3-y$$

$$\Rightarrow \quad 2y=1$$

$$\therefore \quad y=\frac{1}{2}$$

Hence, the correct answer is (D).

30. Co-ordinate of a triangle are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and (1, 0)

$$\text{Centroid is } x=\frac{a\cos t+b\sin t+1}{3}$$

$$\Rightarrow \quad 3x-1=a\cos t+b\sin t \qquad \ldots(1)$$

$$\text{and} \quad y=\frac{a\sin t-b\cos t+0}{3}$$

$$\Rightarrow \quad 3x=a\cos t-b\cos t \qquad \ldots(2)$$

Squaring and adding equations (1) and (2), we get,

$$(3x-1)^2+(3y)^2=(a\cos t+b\sin t)^2+(a\sin t-b\cos t)^2$$

$$\Rightarrow \quad (3x-1)^2+(3y)^2=a^2+b^2$$

Hence, the correct answer is (B).

31. Let (x, y) be the co-ordinate of vertex C and (x_1, y_1) be the co-ordinate of centroid of the triangle

$$\therefore \quad x_1=\frac{x+2-2}{3} \text{ and } y_1=\frac{y-3+1}{3}$$

$$\Rightarrow \quad x_1=\frac{x}{3} \text{ and } y_1=\frac{y-2}{3}$$

$$\because \quad 2x_1+3y_1=1$$

$$\Rightarrow \quad 2\left(\frac{x}{3}\right)+3\left(\frac{y-2}{3}\right)=1$$

$\Rightarrow 2x+3y=9$, which is the required locus of the vertex C.

Hence, the correct answer is (A).

32. Midpoint of (a_1, b_1) and (a_2, b_2) is $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$,

which lies on $(a_1-a_2)x+(b_1-b_2)y+c=0$

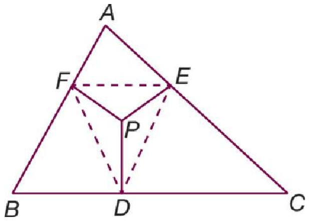
$$\Rightarrow \quad (a_1-a_2)\frac{(a_1+a_2)}{2}+(b_1-b_2)\frac{(b_1-b_2)}{2}+c=0$$

$$\therefore \quad c=\frac{1}{2}\left(a_2^2-a_1^2+b_1^2-b_1^2\right)$$

$$=\frac{1}{2}\left(a_2^2+b_2^2-a_1^2-b_1^2\right)$$

Hence, the correct answer is (A).

33. Let $DE = DF$, then $BDPF$ is a cyclic quadrilateral with BP as diameter, In ΔDBF,



$$\frac{DF}{\sin B} = BP \Rightarrow DF = BP \sin B$$

Similarly, $DE = CP \sin C$

Now for $DE = DF$, we have

$$\frac{BP}{PC} = \frac{\sin C}{\sin B} = \frac{c}{b}$$

Thus, $BP{:}PC$ is a constant ratio $(\neq 1)$. Thus, the locus of P is a circle., the locus of P for $\Delta\ DEF$ to be isosceles consists of three arcs of circles that lie in the interior of the triangle.

Hence, the correct answer is (C).

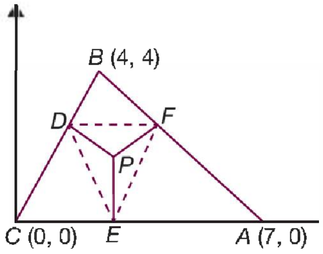
34. $a = 4\sqrt{2}, b = 7, c = 5$

As solved in previous for $DE = DF$

$$\frac{PB}{PC} = \frac{c}{b} = \frac{5}{7}$$

Points dividing BC internally and externally in the ratio $c{:}b$ = 5:7 are $K\left(\frac{7}{3}, \frac{7}{3}\right)$ and L (14, 14), respectively.

Clearly, P will lie on the circle with KL as diameter.



Hence, the correct answer is (B).

35. Clearly, for $DE = EF = DF$, P is the point at which 3 arcs for ΔDEF to be isosceles intersect.

Hence, the correct answer is (D).

36. ΔOAB is clearly an equilateral triangle.

$$\Delta OAB = \Delta OPA + \Delta APB + \Delta OPB = \frac{\sqrt{3}}{4} \times 4$$

$$= \frac{1}{2} \cdot 2\left(d(P,OA)+d(P,AB)+d(P,OB)\right)$$

$$\therefore \quad d(P,OA)+d(P,AB)+d(P,OB)=\sqrt{3}$$

Hence, the correct answer is (C).

37. We must have $d(P,OA) \leq d(P,OB)$ as well as $\left(d(P,OA) \leq d(P,AB)\right)$, then P lies either on or below the angle bisector of $\angle BOA$ and $\angle BAO$.

$$\text{Area } = \frac{1}{3}\Delta OAB = \frac{1}{\sqrt{3}} \text{ sq. units}$$

Hence, the correct answer is (C).

38. We must have $d(P,OA) \geq d(P,OB)$ as well as $d(P,OA) \geq d(P,AB)$, then P must be above bisector of $\angle BOA$ and $\angle BAO$

$$\text{Area of } \Delta = \frac{1}{3}\ \Delta OAB = \frac{1}{\sqrt{3}} \text{ sq. units}$$

Hence, the correct answer is (C).

39. Fixed point is (1, –1).

If $\lambda = \frac{7}{5}$, then A, C and B are collinear.

$$\therefore \quad \lambda \neq \frac{7}{5}$$

Hence, the correct answer is (A).

40. $h = \frac{1+3+\lambda}{3}, k = \frac{4-0+0}{3} = 1 \Rightarrow y = 1$

Hence, the correct answer is (A).

41. $x^2 = 4a(y-k)$

Locus is $y = 1$

Solving $y = 1$, we get,

$$x^2 = 4a(1-k)$$

$$\Rightarrow \quad 4a(1-k) < 0 \Rightarrow a < 0, k < 1$$

Hence, the correct answer is (B).

42. On eliminating β, we get,

$$y = x + 1$$

Hence, the correct answer is (A).

43. Equation of BC is $x - 8y + 2 = 0$.

If M and A lies on the same side of BC, then

$$\Rightarrow \quad -22\left(\beta - 8(1+p) + 2\right) > 0$$

$$\Rightarrow \quad \beta > \frac{-6}{7}$$

Hence, the correct answer is (C).

44. M will lie inside ΔABC. If M and A lie on the same side of BC, M and B lie on same side of AC and M and C lie on the same side of AB.

$$\therefore \quad \beta > -\frac{6}{7}$$

$$\Rightarrow \quad \beta < \frac{3}{2} \text{ and } \beta > -4$$

On taking the intersection of three in equations we get,

$$\frac{-6}{7} < \beta < \frac{3}{2}$$

Hence, the correct answer is (C).

45. Let a variable point on $x - y = 0$ be (h, h)

Then minimum of $|PQ + QR + PR|$ is minimum of

$$\sqrt{(h-5)^2 + (h+2)^2} + \sqrt{(h-5)^2 + (h-2)^2}$$

Differentiating and putting derivative = 0, minimum comes for $h = \frac{29}{10}$.

Hence, the correct answer is (B).

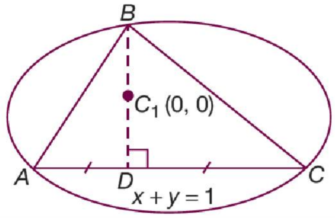
46. R is the point of intersection of the line joining $\left(\frac{29}{10}, \frac{29}{10}\right)$ and $(5, -2)$ with x-axis

Hence, the correct answer is (C).

47. Substituting $h = \frac{29}{10}$, $PQ + QR + PR = \frac{7\sqrt{58} + 3\sqrt{58}}{10}$

Hence, the correct answer is (B).

48. Perpendicular distance from C_1 to $AC = \frac{|0+0-1|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$



D is the foot of the perpendicular of C_1 onto AC

D is also the midpoint of AC

Co-ordinates of D are $\left(\frac{1}{2}, \frac{1}{2}\right)$

Co-ordinates of B are $(-1, -1)$

Assume co-ordinates of $A(h, 1-h)$

$BD = 3C_1D = \frac{3}{\sqrt{2}}$ In an equilateral triangle, height

$= BD = \frac{\sqrt{3}}{2}a$ ($\because$ a is the side of the triangle)

$$\Rightarrow \quad a = \sqrt{6}$$

$$BA = \sqrt{(h+1)^2 + (1-h+1)^2} = \sqrt{6}$$

Solving, we have $h = \frac{1-\sqrt{3}}{2}$ *or* $\frac{1+\sqrt{3}}{2}$

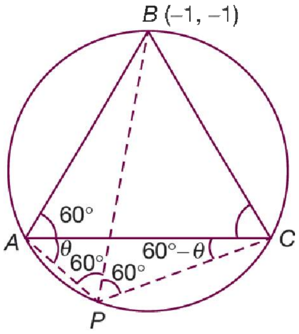
$$x_1x_2 + x_2x_3 + x_3x_1$$

$$= \left(\frac{1-\sqrt{3}}{2}\right)\left(1 + \frac{\sqrt{3}}{2}\right) + -1\left(\frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}\right)$$

$$= \frac{1-3}{4} - 1 = -\frac{3}{2}$$

Hence, the correct answer is (B).

49. $\angle BCP = \angle BPA = 60°$ (Angles in the same segment)



Let, $\angle CAP = \theta$ and circumradius of $\Delta ABC = R$.

$PA = 2R\sin(60 - \theta)$ (sine rule in ΔAPC)

$PC = 2R\sin\theta$ (sine rule in ΔAPC)

$PB = 2R\sin(60 + \theta)$ (sine rule in ΔAPB)

We have $PB = PA + PC$

$$\Rightarrow \quad (PA + PB + PC)_{\max} = (2PB)_{\max} = 2 \times 2R = 4R$$

$$= 4 \times \frac{\sqrt{6}}{2\sin 60} = 4\sqrt{2}$$

PB is maximum, when it is the diameter.

Hence, P is (1, 1).

Hence, the correct answer is (A).

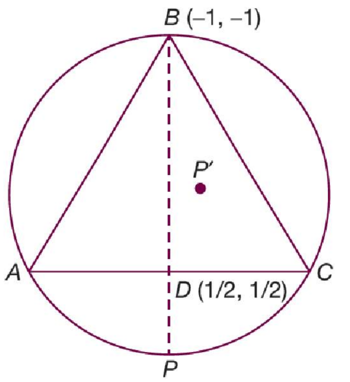
50. Observe that $P'A + P'C \geq AC$ (Triangle inequality)

$$P'B + P'P \geq BP$$

So, $P'A + P'B + P'C + P'P \geq AC + BP$

$$\geq \sqrt{6} + 2\sqrt{2}$$

$$\geq \sqrt{6} + \sqrt{8}$$



Hence if, $P'A + P'B + P'C + P'P = \sqrt{6} + \sqrt{8}$

P' can only be $\left(\frac{1}{2}, \frac{1}{2}\right)$

Hence, the correct answer is (C).

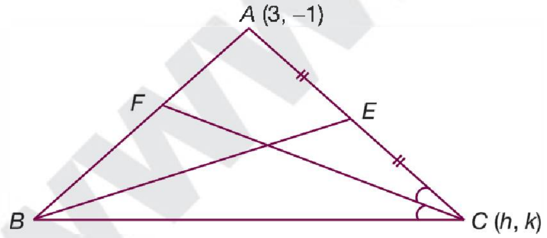
51. Let C be (h, k), then

$$h - 4k + 10 = 0 \quad \ldots(1)$$

and $E\left(\frac{h+3}{2}, \frac{k-1}{2}\right)$

lies on $6x + 10y - 59 = 0$

$$\Rightarrow \quad 3h + 5k = 55 \quad \ldots(2)$$



From (1) and (2), $h = 10, k = 5$

$\therefore$ C is (10, 5)

Slope of $AC = \frac{6}{7}$

Slope of BC, $m = \frac{6}{7}, \frac{-2}{9}$

$\because$ $\frac{6}{7}$ represents slope, we has slope $BC = \frac{-2}{9}$

Hence, the correct answer is (B).

52. Equation of BC is $2x + 9y - 65 = 0$

Now, the point B can be found by solving BC and BE.

Hence, equation of AB can be determine as $18x + 13y = 41$.

Hence, the correct answer is (C).

53. In above question, C has been found as (10, 5).

$\Rightarrow \quad AC = \sqrt{85}$

Hence, the correct answer is (C).

54.

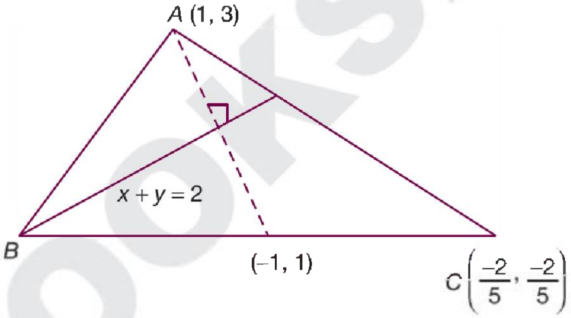


Image of $A(1, 3)$ in $x + y = 2$ is (–1, 1).

So, BC passes through (–1, 1) and $c\left(\frac{-2}{5}, \frac{-2}{5}\right)$

$\therefore$ Equation of BC is $7x + 3y + 4 = 0$.

Hence, the correct answer is (B).

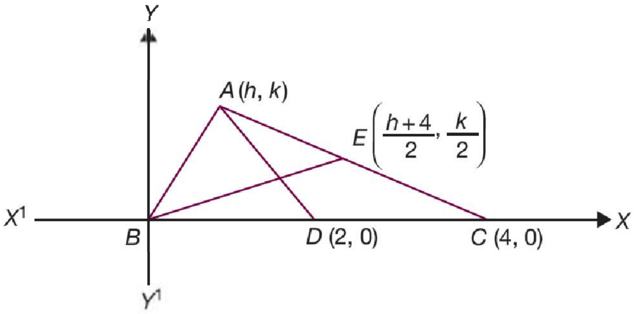
55. Vertex is point of intersection of $7x + 3y + 4 = 0$ and $x + y = 2$, i.e., $\left(\frac{-5}{2}, \frac{9}{2}\right)$

Hence, the correct answer is (C).

56. Line AB is $3x + 7y = 24$.

Hence, the correct answer is (A).

57. Take B as origin, BC as x-axis and take A as (h, k), C is (4, 0).



Area of $\Delta ABC = \frac{1}{2} \times (4)(k) = 2k$

AD and BE are perpendicular $m_{AD} \cdot m_{BE} = -1$

$$k^2 + (h+4)(h-2) = 0 \qquad \ldots(2)$$

$$AC = 3 \Rightarrow (h-4)^2 + k^2 = 9 \qquad \ldots(3)$$

Solving, (2) and (3), we get $h = \dfrac{3}{2}, k = \dfrac{\sqrt{11}}{2}$

Hence, the correct answer is (B).

58. Take B as origin and BC as x-axis and let A be (h, k) since AD is the bisector.

$$\frac{AB}{AC} = \frac{BD}{DC} \Rightarrow c = 2b$$

$$b + c > a \Rightarrow b + c > 6 \quad \text{(since } a = 6 \text{ given)}$$

$$\Rightarrow \quad b + 2b > 6 \Rightarrow b > 2$$

Again $\dfrac{b^2 + 4b^2 - 36}{4b^2} < 1 \Rightarrow b < 6$

Hence, the correct answer is (C).

59. Let A be (h, k) then altitude of the triangle is k.

We have $\dfrac{c}{b} = \dfrac{4}{2} = c^2 = 4b^2$

Now, $c^2 = h^2 + k^2$

$$b^2 = (h-6)^2 + k^2$$

$$c^2 - b^2 = 12h - 36$$

$$\Rightarrow \quad h = \frac{b^2 + 12}{4}$$

Now, it is given that $k^2 > 10$.

$$\Rightarrow \quad c^2 - h^2 > 10$$

$$\Rightarrow \quad 4b^2 - \left(\frac{b^2 + 12}{4}\right)^2 > 10$$

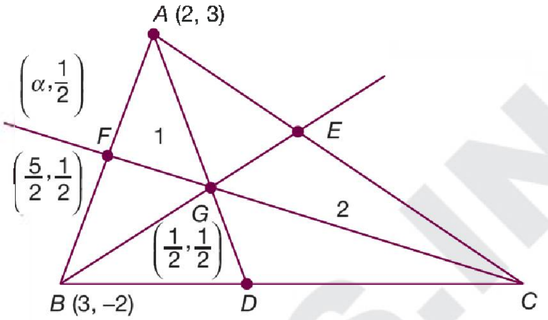
Hence, the correct answer is (C).

60. Centroid is point of intersection of $x + y - 1 = 0$;

$2y - 1 = 0;\ y = \dfrac{1}{2}$ and $x = \dfrac{1}{2}$

Centroid $G\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

Equation of GA $5x - 3y - 1 = 0$


A (2, 3)
$\left(\alpha, \frac{1}{2}\right)$
F
1
E
$\left(\frac{5}{2}, \frac{1}{2}\right)$
G
2
$\left(\frac{1}{2}, \frac{1}{2}\right)$
B (3, –2)
D
C

Hence, the correct answer is (B).

61. A point on $2y - 1 = 0;\ y = \dfrac{1}{2}$ is $F\left(\alpha, \dfrac{1}{2}\right)$.

Let this be the midpoint of AB.

$\therefore B$ is $(2\alpha - 2, -2)$ lies on $x + y - 1 = 0$

$$2\alpha - 2 - 2 - 1 = 0$$

$$\therefore \quad \alpha = \frac{5}{2} \qquad\qquad \therefore B(3, -2)$$

G divides CF in 2:1

$$\therefore \quad C = \left(\frac{-7}{2}, \frac{1}{2}\right)$$

Equation of BC $\dfrac{y+2}{\frac{1}{2}+2} = \dfrac{x-3}{\frac{-7}{2}-3}$ $\dfrac{y+2}{5} = \dfrac{x-3}{-13}$

$$\Rightarrow \quad 5x + 13y + 11 = 0$$

Hence, the correct answer is (A).

62. Altitude from A to BC $= \dfrac{|5(2) + 13(3) + 11|}{\sqrt{5^2 + 13^2}} = \dfrac{60}{\sqrt{194}}$

Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions

1. A → (r); B → (p), C → (q); D → (p)

(A) By using the given data, the vertex B is $(-1, 1)$ and the slope of AC is -3, the equation of altitude through vertex B is $x - 3y + 4 = 0$.

(B) The point of intersection of $x - 2y = 1$ and $x + y = 0$ is $\left(\dfrac{1}{3}, \dfrac{-1}{3}\right)$ and consider a point on the line $x - 2y = 1$ as $(1, 0)$ and its image with respect to $x + y = 0$ as $(0, -1)$.

The equation of the line passing through $\left(\frac{1}{3}, \frac{-1}{3}\right)$ and (0, –1) is $2x - y = 1$, which is the image of the given line.

(C) By the given data in ΔABC, the vertex C is the 4th vertex of the rectangle formed by $AOBC$ and the locus is OC. If C is (x, y), then $x{:}y = 4{:}3$

$\Rightarrow$ The locus of C is $3x = 4y$.

(D) Let (α, β) be the centroid of the ΔABC, where A (1, 2) and B (–1, 5). Then the vertex

$C(3\alpha, 3\beta - 7)$

$\therefore$ The vertex C lies on $2x + y - 2 = 0$ and it follows that $2\alpha + \beta - 3 = 0$, the equation of the locus is $2x + y = 3$.

2. A → (p, q, r); B → (q, r); C → (s); D → (q, r)

(A) $a = b \neq 0 \Rightarrow |x| + |y| = \frac{6}{|a|}$ represents a square (4 straight lines)

(B) $ab \neq 0, a \neq b \Rightarrow a|x| + b|y| = 6$ represents a rhombus (4 straight lines)

(C) $|a| + |b| = 0 \Rightarrow a = b = 0$

$\therefore$ Equation is meaningless.

(D) $a^2 + b^2 = 13,$

$ab = 6 \Rightarrow a = 3, b = 2$

$a > b, a, b > 0$

$\therefore$ It represents a rhombus (4 straight lines).

3. A → (p, q); B → (s); C → (t); D → (t)

(A) $x = 0; y = 0$

$x = \frac{1}{2}; y = \frac{1}{2}$

$0 \leq \sin^2\theta \leq \frac{1}{2} \Rightarrow 0 \leq \sin\theta \leq \frac{1}{2};$

correct $\theta = \frac{\pi}{6}, 3$

(B) Equation

$\Rightarrow \left(\cot(x+y) - \tan(x+y)\right)^2 + (y+1)^2 = 0$

$\Rightarrow y = -1, x + y = \pm\frac{\pi}{4}$

$PQ = \left|\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right| = \frac{\pi}{2}$

(C) $|(1,1)(1,2)(1,3)(2,1)(2,2)(3,1)|$ 6 points

(D) $\text{Area} = 2 \cdot 1/2 \cdot \sqrt{6} \cdot \sqrt{6} = 6$

4. A → (p, q); B → (p); C → (p); D → (t)

(A) $(t, 2-t) \Rightarrow |4(t) + 3(2-t) - 12| = 5 \Rightarrow t = 11, 1$

(B) Let shifted to $(a, b) \Rightarrow (x+a)^2 + (x+a)(y+b) - 3(x+a) - (y+b) + 2 = 0$ for given is true it $a = 1$ and $b = 1$

(C) Line passed through the centroid

(D) $P(h, k) \Rightarrow h^2 = 2k$. Let, (α, β) be the locus the point then $h = 2\alpha, k = 2\beta \therefore y^2 = 2x$

5. A → (s); B → (p, q); C → (r, s); D → (p)

(A) $2b = a + c \Rightarrow a - 2b + c = 0 \Rightarrow (1, -2)$

(B) $4a^2 + 9b^2 + 12ab = c^2 \Rightarrow c = \pm(2a + 3b)$

$\Rightarrow a(x \pm 2) + b(y \pm 3) = 0 \Rightarrow (2,3)$ and $(-2,-3)$

(C) $a^2 + 4b^2 - 4ab = (a - 2b)^2 = c^2 \Rightarrow c = \pm(a - 2b)$

$\Rightarrow a(x \pm 1) + b(y \mp 2) = 0 \Rightarrow (-1,2)$ and $(1,-2)$

(D) $2a + 3b - c = 0 \Rightarrow -2a - 3b + c = 0 \Rightarrow (-2,-3)$

6. A → (p); B → (s); C → (q); D → (r)

(A) $(x + 7y)^2 + 4\sqrt{2}(x + 7y) - 42 = 0$

$\Rightarrow (x + 7y)^2 + 7\sqrt{2}(x + y) - 3\sqrt{2}(x + y) - 42 = 0$

$\Rightarrow (x + y)\left[x + 7y + 7\sqrt{2}\right] - 3\sqrt{2}\left(x - 7y + 7\sqrt{2}\right) = 0$

$\Rightarrow \left(x + 7y + 7\sqrt{2}\right)\left(x + 7y - 3\sqrt{2}\right) = 0$

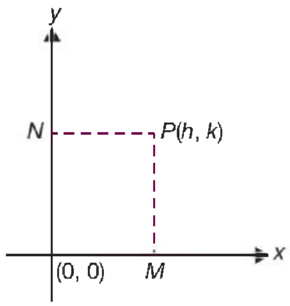
$\Rightarrow x + 7y + 7\sqrt{2} = 0$ and $x + 7y - 3\sqrt{2} = 0$

$\Rightarrow d = \left|\frac{7\sqrt{2} + 3\sqrt{2}}{\sqrt{1 + 49}}\right| = \frac{10\sqrt{2}}{\sqrt{50}} = 2$

(B) Let the two perpendicular lines are co-ordinate axes.

Then, $PM + PN = 1 \Rightarrow h + k = 1$

Hence, the locus is $x + y = 1$.



However, if the point lies in other quadrants also, then $|x| + |y| = 1$.

Hence, the value of k is 1.

(C) Angle bisector between the lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$ is

$$\frac{x+2y+4}{\sqrt{1+4}} = \pm\frac{(-4x+2y+1)}{\sqrt{16+4}}$$

$\Rightarrow \quad 2(x+2y+4) = \pm(-4x-2y+1)$

Since $AA' + BB' < 0$, so +ve sign gives acute angle bisector. Hence,

$2x+4y+8 = -4x-2y+1$

$\Rightarrow \quad 6x+6y+7=0$

$\Rightarrow \quad m=7$

(D) We have,

$y^2 - 9xy + 18x^2 = 0$ or

$y^2 - 6xy - 3xy + 18x^2 = 0$

$\Rightarrow \quad y(y-6x)-3x(y-6x) = 0$

$\Rightarrow \quad (y-3x) = 0$ and $y - 6x = 0$

The 3rd line is $y = 6$.

∴ The area of the triangle formed by these lines,

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 6 & 1 \\ 2 & 6 & 1 \end{vmatrix} = \frac{1}{2}|6-12|$$

$= 3$ sq. units

7. A → (s); B → (r); C → (p, q); D → (p, q, r, s)

(A) The equation represents pair of lines, if $(4)(k)(-9)-(-9)(4)^2 = 0$

$\Rightarrow \quad k = 4$

(B) $m_1 + m_2 = 4m_1m_2 \Rightarrow \frac{-2h}{b} = \frac{4a}{b} \Rightarrow \frac{-2(-C)}{-7} = \frac{4\times 1}{-7}$

$\Rightarrow \quad C = 2$

(C) $2m_2 + m_2 = \frac{-h}{2}, 2m_2^2 = \frac{1}{2} \Rightarrow 2\left(\frac{-h}{6}\right)^2 = \frac{1}{2} \Rightarrow h^2 = 9$

$\Rightarrow \quad h = \pm 3$

(D) The angular bisectors of $ax^2 + 2hxy + by^2 + \lambda(x^2+y^2) = 0$ is $h(x^2-y^2)-(a-b)xy = 0$, which are angular bisectors of $ax^2+2hxy+by^2 = 0$. The two pairs are equally inclined for any λ.

8. A → (p, t); B → (q); C → (s); D → (r)

$\sin A = \frac{4}{5}, AB = AC = 5\sqrt{2}$

So, point B will be (0, 10), (10, 0) and point C will be (4, – 2), (6, 12).

Centroid will be $\left(3, \frac{13}{3}\right)$, ...

Circumcentre will be $\left(\frac{-5}{2}, \frac{5}{2}\right)$

9. A → (r); B → (q); C → (s); D → (p)

Solving the equation $2m^3 - 3m^2 - 3m + 2 = 0$ we get, $2(m^3+1) - 3m(m+1) = 0$

$\Rightarrow \quad (m+1)(2m^2 - 5m + 2) = 0$

$\Rightarrow \quad (m+1)(2m-1)(m-2) = 0 \Rightarrow m = -1, \frac{1}{2}$ or 2.

Equation of the given lines can be written as $m_i^2 x - m_i y = -1$.

(A) Algebraic sum of the intercepts made by the lines on x-axis

$$= -\sum\frac{1}{m_i^2} = -\left[1+\frac{1}{4}+4\right] = -\frac{21}{4}$$

(B) Algebraic sum of the intercepts made by the lines on y-axis

$$= \sum\frac{1}{m_i} = -1 + 2 + \frac{1}{2} = \frac{3}{2}$$

(C) Let, p_i denote the perpendicular distance of the line from the origin, then $p_i = \left|\frac{-1/m_i}{\sqrt{1+m_i^2}}\right|$

$$\Rightarrow \quad \sum p_i = \frac{1}{\sqrt{1+1}} + \frac{2}{\sqrt{1+(1/4)}} + \frac{1/2}{\sqrt{1+4}}$$

$$= \frac{1}{\sqrt{2}} + \frac{4}{\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{1}{10}\left(5\sqrt{2}+9\sqrt{5}\right)$$

(D) l_i = Length of the line intercepted between the co-ordinates are

$$= \sqrt{\left(\frac{1}{m_i^2}\right)^2 + \left(\frac{1}{m_i}\right)^2}$$

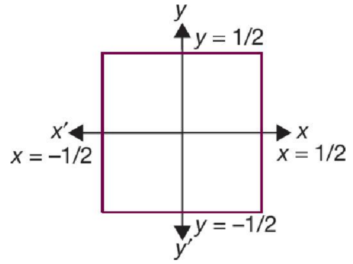
$$\sum l_i = \sqrt{1+1} + \sqrt{16+4} + \sqrt{\frac{1}{16}+\frac{1}{4}}$$

$$= \sqrt{2} + 2\sqrt{5} + \frac{\sqrt{5}}{4} = \left(4\sqrt{2}+9\sqrt{5}\right)/4$$

10. A → (q); B → (p, q, r); C → (r); D → (p, s)

(A) ∵ Maximum $\{|x|, |y|\} = 1/2$

$$\begin{cases} |x| = 1/2, & \text{if } \ |y| < 1/2 \\ |y| = 1/2, & \text{if } \ |x| < 1/2 \end{cases}$$



∴ Required area = 1 × 1 = 1 sq. units

(B) The line $y = x$ cuts the lines $|x + y| = 6$, i.e.,

$x + y = +6$

at $x = +3, y = \pm 3$

or $(-3, -3)$ and $(3, 3)$

Then, $-3 < a < 3$

∴ $0 < |a| < 3$ (∴ $[|a|] = 0, 1, 2$)

(C) ∵ $(0, 0)$ and $(1, 1)$ lie on the same side, $a^2 + ab + 1 > 0$.

∵ Coefficient of a^2 is > 0

∴ $D < 0$

$b^2 - 4 < 0$

or $-2 < b < 2 \Rightarrow b = -1, 0, 1$

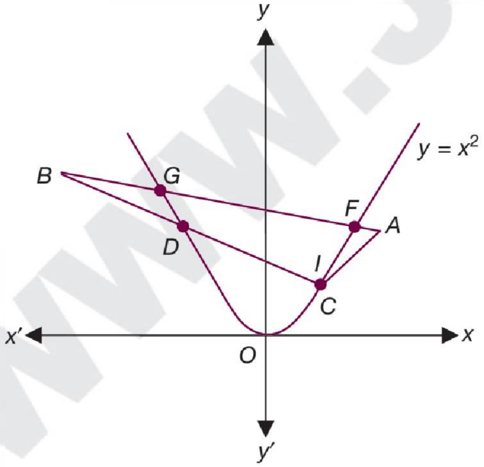
∴ Number of non-zero values of b are 2 ($b = -1$ and $b = 1$)

(D) Let $P(\alpha, \alpha^2)$ first draw the exact diagram of ΔABC.

The point $P(\alpha, \alpha^2)$ move on the curve $y = x^2$ for all α.

Now intersection of $y = x^2$ and $2x + 3y - 1 = 0$

or $2x + 3x^2 - 1 = 0$



∴ $x = -1, x = \dfrac{1}{3}$

Let intersection points $D(-1, 1)$ and $E\left(\dfrac{1}{3}, \dfrac{1}{9}\right)$

Intersection of $y = x^2$ and $x + 2y - 3 = 0$

or $x + 2x^2 - 3 = 0$

∴ $x = 1, x = \dfrac{-3}{2}$

Let the intersection points $F(1, 1)$

and $G\left(-\dfrac{3}{2}, \dfrac{9}{4}\right)$ and intersection of $y = x^2$

and $5x - 6y - 1 = 0$

or $5x - 6x^2 - 1 = 0$

∴ $x = \dfrac{1}{3}, x = \dfrac{1}{2}$

Let intersection points be

$H\left(\dfrac{1}{3}, \dfrac{1}{9}\right)$ and $I\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$

Thus, the points on the curve $y = x^2$ whose x-co-ordinates lies between $\dfrac{-3}{2}$ and -1 and $\dfrac{1}{2}$ and 1 lies within the ΔABC.

Hence, $-\dfrac{3}{2} < \alpha < -1$ and $\dfrac{1}{2} < \alpha < 1$,

i.e., $\alpha \in \left(-\dfrac{3}{2}, -1\right) \cup \left(\dfrac{1}{2}, 1\right)$

∴ $[\alpha] = -2, 0$

11. A → (t); B → (s); C → (p); D → (q)

(A) For point (a, a^2) lies inside the triangle $a > 0$,

$a^2 > 0, a + 2a^2 - 3 < 0$

$a \in (0, 1)$

(B) ∵ $\angle BCA = 90°$,

points A, O, B and C are concyclic.

Let, $\angle AOC = \theta$

$\angle BOC = \angle BAC$

$\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{5}{12}$

$\dfrac{x}{y} = \dfrac{5}{12}$

$\Rightarrow 12x - 5y = 0$

(C) Slope of line joining $(t - 1, 2t + 2)$ its image is $(2t + 1, t)$ is -1

Hence, the slope of the required line is 1.

(D) Image of the point $(1, 2)$ in the bisector of B and C lie on the line BC.

Image of A in $x = y$ is $(2, 1)$ image of A in $y = 0$ is $(1, -2)$

The equation of BC is $y = 3x - 5$

$$d(A,B,C) = \frac{4}{\sqrt{10}}, \ \sqrt{10}d(A,B,C) = 4$$

12. A → (r); B → (p); C → (q); D → (s)

(A) Put $y = 0 \Rightarrow 6x^2 - 24x + \beta = 0$. It must be perfect square $\Rightarrow \beta = 24$

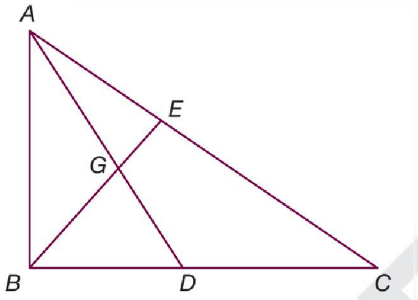
$$\text{Also, } 6\times - 3\times\beta + 2\times\frac{3}{2}\times - 12\times - \frac{\alpha}{2} - 6\times\frac{9}{4}$$

$$+ 3\times144 - \beta\times\frac{\alpha^2}{4} = 0 \Rightarrow \alpha = \frac{3}{2}.$$

(B) The required distance $= \sqrt{R^2 - 2rR} = 40$

(C) AD and BE are perpendicular $AD = 6$

$\Rightarrow \ GD = 2$



$\text{BE} = 11$

$$\Rightarrow \ BG = \frac{22}{3}$$

$$\text{Area of } \Delta BGD = \frac{1}{2} \text{ Area of } \Delta BGC$$

$$= \frac{1}{2}\times\frac{1}{3} \text{ (Area of } \Delta ABC)$$

$$\therefore \text{Area}(\Delta ABC) = 6\,\text{Area}(\Delta BGD) = 6 \times \frac{1}{2} \times 2 \times \frac{22}{3}$$

$= 44 \text{ cm}^2$.

(D) Line joining points $(6t_1, 6/t_1)$ and $(6t_2, 6/t_2)$ on the hyperbola is $x + yt_1t_2 - 6(t_1 + t_2) = 0$ it is parallel to $2x + y = 0$, if $2t_1t_2 = 1$.

Line joining points $(6t_1, 6/t_1)$ and $(6t_3, 6/t_3)$ is parallel to $6x + y = 0$, if $6t_1t_3 = 1$

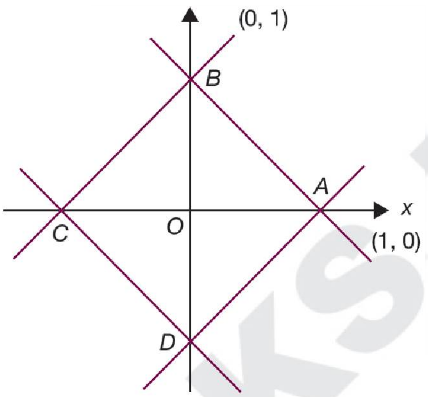
we get $t_2 = 3t_3$ ...(1)

Now, the line joining t_2 and t_3 is $x + yt_2t_3 - 6(t_2 + t_3) = 0$ from (1)

$x + 3yt_3^2 - 24t_3 = 0$, touches $xy = 48$ for all values of t_3.

Integer/Numerical Answer Type Questions

1.



$$|x| + |y| = 1$$

The curve represents four lines:

$x + y = 1$, $x - y = 1$, $-x + y = 1$ and $-x - y = 1$, which enclose a square of side = distance between opposite sides $x + y = 1$ and $x + y = -1 = \frac{1+1}{\sqrt{1+1}} = \sqrt{2}$

Using $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| = p$, (perpendicular distance between two parallel lines)

$$ax + by + c_1 = 0, ax + by + c_2 = 0$$

$\therefore$ Required area = (sides)^2 = 2 sq. units

Alternate

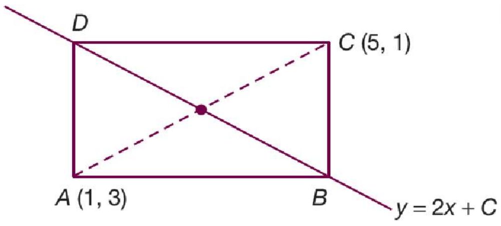
Required area = 4(ΔOAB)

$$= 4\left(\frac{1}{2}\times1\times1\right)$$

$$= 2$$

Hence, the correct answer is (2).

2. Let $ABCD$ be a rectangle and co-ordinates of its opposite vertices A and C are (1, 3) and (5, 1), respectively. Given that other two vertices B and D lie on $y = 2x + c$.



$\therefore$ Equation of diagonal BD is $2x - y + c = 0$

$\therefore$ Midpoint of AC will lie on BD

$$M\left(\frac{1+5}{2}, \frac{3+1}{2}\right) = (3, 2)$$

As (3, 2) lies on $2x - y + c = 0$

$\therefore\ 2 \times 3 - 2 + c = 0$

$\Rightarrow\ c = -4$

Hence, the correct answer is (4).

3. The given lines are

$$2x + y = \frac{9}{2} \qquad \ldots(1)$$

$$2x + y = -6 \qquad \ldots(2)$$

In particular, PQ can be taken perpendicular to the two parallel lines.

Then, required ratio = ratio of perpendicular distance of lines from origin = $OP{:}OQ$

$$= \left|\frac{9}{\sqrt{20}}\right| : \left|\frac{-6}{\sqrt{5}}\right|$$

$\Rightarrow\ 3 : 4$

Hence, the correct answer is (7).

4. Angle bisector of lines is equidistant for given bisector

$$24x + 7y - 20 = 4x + 3y - 2$$

$\Rightarrow\ 2x - 4y - 5 = 0$

$\Rightarrow\ |a + b + c| = 7$

Hence, the correct answer is (7).

5. $y = -2$

$7x - 2y = -10$

$7x + 2y = 10$

$\Rightarrow\ A(-2\ -2),\quad B(0,\ 5),\quad C(2, -2)$

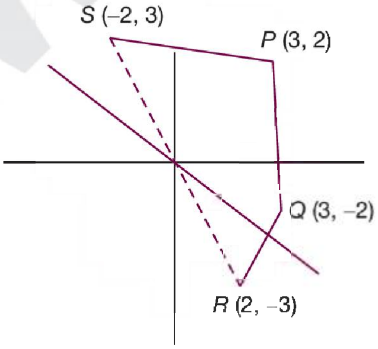
$$Ar\ (\Delta ABC) = \frac{1}{2}.4.7 = 14 = 2k$$

$\Rightarrow\ k = 7$

Hence, the correct answer is (7).

6. $P(3,2)\ Q(3,-2)$

$R(2,-3)\ S(-2,3)$



$$\text{Area of } PQRS = \frac{1}{2}\begin{vmatrix} 3 & 2 \\ -2 & 3 \\ 2 & -3 \\ 3 & -2 \\ 3 & 2 \end{vmatrix}$$

$$= \frac{1}{2}(13 + 6 - 6 - 4 + 9 + 6 + 6) = 15$$

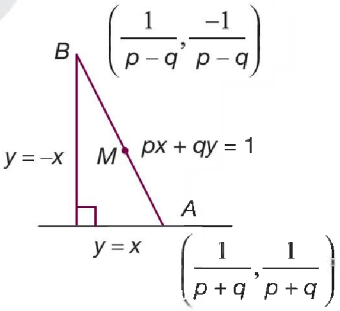
$15 = 5k$

$\Rightarrow\ k = 3$

Hence, the correct answer is (3).

7. $$M = \left(\frac{\frac{1}{p-q} + \frac{1}{p+q}}{2}, \frac{\frac{1}{p+q} - \frac{1}{p-q}}{2}\right)$$

$$= \left(\frac{p}{p^2 - q^2}, \frac{-q}{p^2 - q^2}\right) = (x, y)$$


B
$\left(\frac{1}{p-q}, \frac{-1}{p-q}\right)$
y = –x
M
px + qy = 1
A
y = x
$\left(\frac{1}{p+q}, \frac{1}{p+q}\right)$

$$x^2 - y^2 = \frac{1}{p^2 - q^2}$$

$$x^2 + y^2 = \frac{p^2 + q^2}{(p^2 - q^2)^2} = k(x^2 + y^2)^2$$

$k = 1$

$\Rightarrow\ k - 1 = 0$

Hence, the correct answer is (0).

8. The equation of angular bisectors of

$$x^2 - 2pxy - y^2 = 0 \ \text{ is } \ x^2 + \frac{2}{p}xy - y^2 = 0$$

Equivalent to $x^2 - 2qxy - y^2 = 0$

$\Rightarrow\ |pq| = 1$

Hence, the correct answer is (1).

9. $g^2 \geq 4,\ g < 8$

$2 \leq g < 8$

Hence, the correct answer is (7).

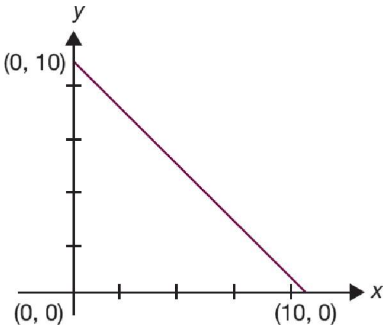
10. $m\,t^3 + tl - tx - ky = 0$ ...(1)

a, b and c are the roots of (1)

$a + b + c = 0$

Hence, the correct answer is (0).

11. For interior portion, the point must satisfy $x + y < 10, x > 0, y > 0$.



If $x = 1$, then $y = 1, 2, 3, 4, 5, 6, 7, 8$

Similarly, $x = 2, y = 1, 2, 3, 4, 5, 6, 7$

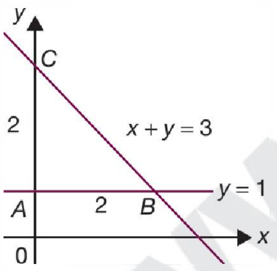
Total 7 points and so on.

Hence, total points $= 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = \dfrac{8 \times 9}{2}$ $= 36$.

$$k = 36 \Rightarrow \frac{k}{36} = 1$$

Hence, the correct answer is (1).

12.



$$x^2 - y^2 + 2y = 1 \Rightarrow x^2 = (y-1)^2$$

Equation of angle bisectors are $x = 0$ and $y = 1$

$$\text{Area} = \frac{1}{2}(2)(2) = 2 \text{ sq. units.}$$

Hence, the correct answer is (2).

13. Let the line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$

It passes through (h, k)

$$\frac{h}{a} + \frac{k}{b} = 1$$

$$\Delta = \frac{1}{2}ab = \frac{a}{2}\frac{ak}{(a-h)}$$

$$\frac{d\Delta}{da} = \frac{k}{2}\left(\frac{(a-h)2a - a^2}{(a-h)^2}\right) = 0$$

$$\Rightarrow a = 2h;\ b = 2k$$

$$\Delta_{\min} = 2hk \quad \left(\therefore \frac{2\Delta}{hk} = 4\right)$$

Hence, the correct answer is (4).

14. The line $y = \sqrt{3}x$ can be written as $x = \dfrac{r}{2}, y = \dfrac{r\sqrt{3}}{2}$.

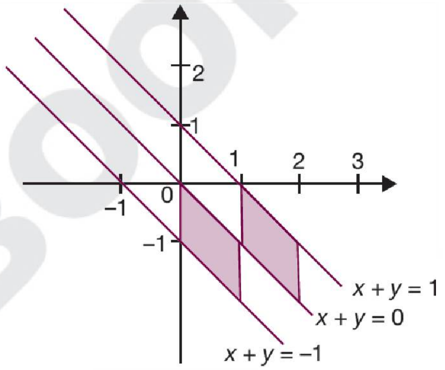
If this line cuts the given curve, then

$$\frac{r^4}{16} + \frac{ar^3\sqrt{3}}{8} + \frac{br^2\sqrt{3}}{4} + \frac{cr}{2} + \frac{dr\sqrt{3}}{2} + 6 = 0$$

$$\therefore \quad \frac{1}{12}OA \cdot OB \cdot OC \cdot OD = \frac{1}{12}\left|r_1 r_2 r_3 r_4\right| = 8$$

Hence, the correct answer is (8).

15.



If $x \in (0, 1)$

Then, $-1 \le x + y < 0$

If, $x \in (1, 2)$

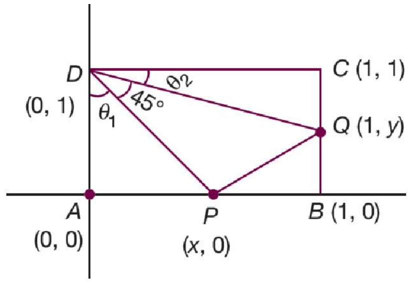
$0 \le x + y < 1$

$$\text{Required area} = 4\left(\frac{1}{2} \cdot 1 \cdot \sqrt{2}\sin\frac{\pi}{4}\right) = 2 \text{ sq. units}$$

Hence, the correct answer is (2).

16. $\tan\theta_1 = x$ and $\tan\theta_2 = 1 - y$

$\because \quad \theta_1 + \theta_2 = 45°$



$$\Rightarrow \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = 1$$

$$\Rightarrow \frac{x + (1-y)}{1 - x(1-y)} = 1$$

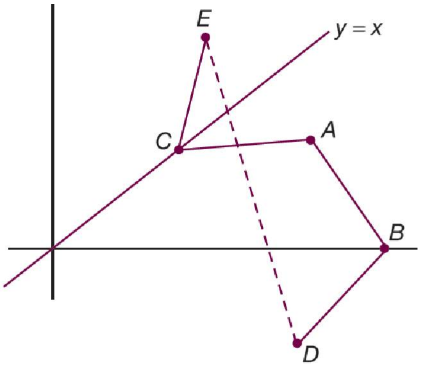
$$\Rightarrow y = \frac{2x}{1+x} \qquad \text{...(1)}$$

Now, perimeter = $1 - x + y + \sqrt{(1-x)^2 + y^2}$

By using (1), we get, perimeter = 2 units.

Hence, the correct answer is (2).

17.



Let, $D = (2, -1)$ be the reflection of A in x-axis, and let $E = (1, 2)$ be the reflection in the line $y = x$. Then $AB = BD$ and $AC = CE$, so the perimeter of ABC is $DB + BC + CE \geq DE = \sqrt{1+9} = \sqrt{10}$.

Hence, the correct answer is (3).

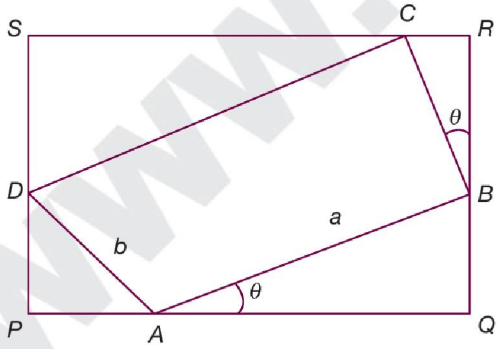
18. $2(a + b) = 0$ (a is constant)

Area of $PQRS = (b \sin\theta + \alpha\cos\theta)(\alpha\sin\theta + b\cos\theta)$

$$= ab + \frac{a^2+b^2}{2} \cdot \sin 2\theta \leq \frac{(\alpha+b)^2}{2} + \frac{x^2}{8}$$

$$\therefore \quad \frac{x^2}{8} = 32$$

$$\Rightarrow \quad x = 16$$



Hence, the correct answer is (4).

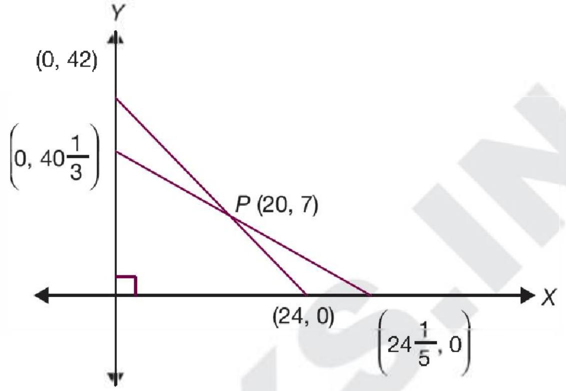
19. $\sec^2\theta + 2\operatorname{cosec}^2\theta \geq (\sqrt{2}+1)^2$

Perpendicular distance from $(1,3)$ to the line $3x + 4y = 5$ is 2.

$\therefore$ Number of points = 2

Hence, the correct answer is (2).

20. The given lines $7x + 4y = 168$ and $5x + 3y = 121$ intersect at $P(20, 7)$


Y
(0, 42)
$\left(0, 40\frac{1}{3}\right)$
P (20, 7)
X
(24, 0)
$\left(24\frac{1}{5}, 0\right)$

$\therefore$ Area of shaded region

$$A = \frac{1}{2}\left(42 - 40\frac{1}{3}\right)20$$

$$= \frac{1}{2}\left(\frac{5}{3}\right)20 = \frac{50}{3} \text{ sq. units}$$

Hence, the correct answer is (5).

21. Let B (a, b), C (c, b), A (a, d)

Then D (midpoint of BC) is $\left(\frac{a+c}{2}, b\right)$

E (midpoint of AB) is $\left(a, \frac{b+d}{2}\right)$

Given slope of $CE = 1$

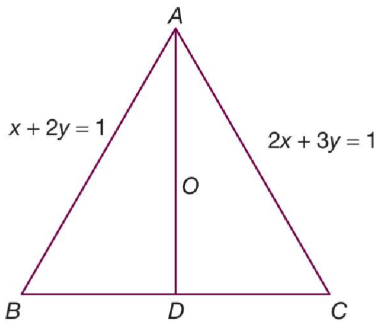
$$\Rightarrow \quad \frac{b - \frac{b+d}{2}}{c-a} = 1$$

$$\Rightarrow \quad \frac{(b-d)}{c-a} = 2$$

$$\text{Slope of } AD = \frac{b-d}{\frac{a+c}{2} - a} = 2\frac{(b-d)}{c-a} = 4$$

Hence, the correct answer is (4).

22. Solving $2x + 3y = 1, x + 2y = 1, A = (-1, 1)$



Orthocentre = (0, 0)

$\Rightarrow$ Slope if altitude $AD = -1$

Equation of BC is $x - y = k$

Solving $x-y=k$ and $x+2y=1$, we get,

$$B=\left(\frac{1+2k}{3},\frac{1-k}{3}\right)$$

Slope of $OB=\frac{1-k}{1+2k}$, slope of $AC=\frac{-2}{3}$

$$\therefore\quad \frac{1-k}{1+2k}=\frac{3}{2}\Rightarrow k=\frac{-1}{8}$$

Equation of BC is $x-y+\frac{1}{8}=0$

$\Rightarrow\quad -8x+8y-1=0$

$\Rightarrow\quad a=-8, b=8$

Hence, the correct answer is (4).

23. Let AC and BD intersect at P.

$$AP=\frac{12+6+2}{\sqrt{16+9}}=4$$

Area if $\Delta ABD=AP\times BP=\frac{24}{2}=12$

$BP=3$

$AB=\sqrt{AP^2+BP^2}=5$

Hence, the correct answer is (5).

24. Solving the altitudes AM and CL

Orthocenter $=\left(\frac{1}{2},\frac{1}{2}\right)$ lies on $x+y-k=0$

$k=1$

Hence, the correct answer is (1).

25. $x=X\cos\theta-Y\sin\theta$, $y=X\sin\theta+Y\cos\theta$

Substitute x and y in the equation

$a=2\cos^2\theta+3\cos\theta\sin\theta$, $b=2\sin^2\theta-3\cos\theta\sin\theta$, we get, $a+b=2$.

Hence, the correct answer is (2).

26. $A=(0,4)=(h,k)$, $B=(0,-4)=(h,-k)$

$|PA-PB|=6=2a$

$a=3$

Equation of locus is $\frac{(x-h)^2}{(a^2-k^2)}+\frac{y^2}{a^2}=1$

$\Rightarrow\quad \frac{x^2}{9-16}+\frac{y^2}{9}=1$

$\Rightarrow\quad 7y^2-9x^2=63$

$\Rightarrow\quad 9k=63$

$\Rightarrow\quad k=7$

Hence, the correct answer is (7).

27. $P(x,y)$, $A(3,4)$, $B(-4,3)$

Area of $\Delta PAB=0$

$$\frac{1}{2}|x(4-3)+3(3-y)-4(y-4)|=0$$

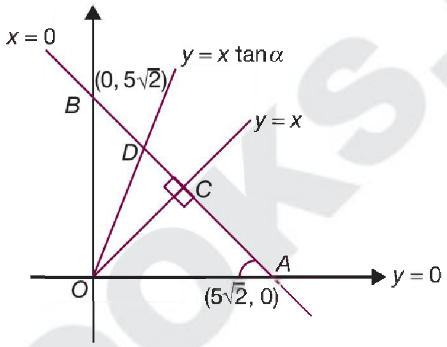
$x+9-3y-4y+16=0$

$x-7y+25=0\quad\Rightarrow ax+by+c=0$

$a+3b+c=1-21+25=5$

Hence, the correct answer is (5).

28.



Since AC, CD and DB are in A.P. and let their lengths be taken as $a-d$, a and $a+d$, respectively.

$\therefore\quad a-d+a+a+d=AB=10$

$\therefore\quad a=\frac{10}{3}$

In ΔOAC, $OA^2=OC^2+AC^2$

$\Rightarrow\quad 50=OC^2+OC^2$

$\therefore\quad OC=5$

In ΔOCD,

$$\tan(\alpha-45)=\frac{CD}{OC}$$

$$\frac{\tan\alpha-1}{\tan\alpha+1}=\frac{2}{3}$$

$\Rightarrow\quad \alpha=\tan^{-1}5$

Hence, the correct answer is (5).

29. $\because$ The lines are concurrent,

$$\Rightarrow\begin{vmatrix} a & b & p\\ \cos\alpha & \sin\alpha & -P\\ \sin\alpha & -\cos\alpha & 0\end{vmatrix}=0$$

$\Rightarrow\quad 1+a\cos\alpha+b\sin\alpha=0$

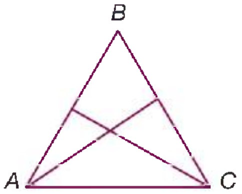
$$\cos\frac{\pi}{4}=\frac{|a\cos\alpha+b\sin\alpha|}{\sqrt{a^2+b^2}}\Rightarrow\sqrt{a^2+b^2}=\sqrt{2}\Rightarrow a^2+b^2=2$$

Hence, the correct answer is (2).

30. AC will be parallel to the bisector which contains the point $\left(-\sqrt{2},\sqrt{2}\right)$

Hence, slope $=0$.

Hence, the correct answer is (0).



31. Any point on the line is $\left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$, where r is the distance from the origin.

$\Rightarrow \quad \frac{r^2}{8}+\frac{3\sqrt{3}r^3}{8}=\frac{\sqrt{3}}{4}r^2+2$

$\Rightarrow \quad \frac{\left(1+3\sqrt{3}\right)}{8}r^3-\frac{\sqrt{3}}{4}r^2-2=0$

$\Rightarrow \quad r_1r_2r_3=\frac{2\times 8}{1+3\sqrt{3}}=\frac{16}{1+3\sqrt{3}}$

$\Rightarrow \quad [r_1r_2r_3]=2$

Hence, the correct answer is (2).

32. Given lines are $(x+5)(y+2)=0$ and $(y-2)(x+7)=0$

The sides are $2+2=4$ and $7-5=2$

Area of the rectangular 8 sq. units

Hence, the correct answer is (8).

33. Required rectangle has sides

$x+my=0, x+my=3+m$

$y=m(x-3), y=mx+1$

$\therefore \quad \Delta=\frac{(3+m)(1+3m)}{1+m^2}\le 8$

Max. area = 8

Hence, the correct answer is (8).

34. $A=(0,5)B=(-2,2)C=(2,-2)$

$AB=\sqrt{53}=AC$, $BC=4$ BC is smallest side

The largest Altitude 'l' is from A to BC

$l=\left|\frac{5+2}{\sqrt{0+1}}\right|=7$

Hence, the correct answer is (7).

35. Here A and B lies on the opposite sides of the given line. The image of 'B' is C(3, 0)

The maximum value of $|PA-PB|$ is $AC=\sqrt{8}$

Hence, the correct answer is (8).

36. Let the other two vertices be on $2x-y+k=0$.

Length of the side $=\frac{|K+17|}{\sqrt{5}}=4\sqrt{\frac{1}{4}(5)(1+k)}$ (length of chord).

$\Rightarrow \quad (K+17)^2=100(1+K)$

$\Rightarrow \quad K^2-66K+189=0\Rightarrow (K-3)(K-63)=0$

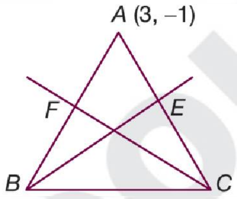
$\Rightarrow \quad \frac{L}{l}=\frac{80}{20}=4$

Hence, the correct answer is (4).

37. Area of rhombus $=\frac{1}{2}d_1d_2=\frac{1}{2}\times 2\times 6=6\,\text{sq.units}$

Hence, the correct answer is (6).

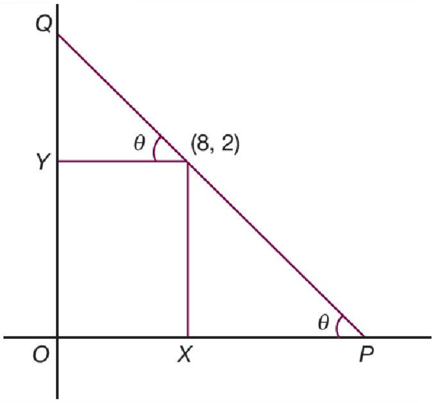
38.



Equation of BE is $x-y-2=0$. Let E be $(h,h-2)$ then, C is $(2h-3, 2h-3)$. Now, substitute this in equation CF. Find the image of A w.r.t. CF, which lies on BC.

Hence, the correct answer is (2).

39.



$OP+OQ=OX+XP+OY+YQ$

$=8+2\cot\theta+2+8\tan\theta$

$=10+8\tan\theta+2\cot\theta$

≥ 18

Hence, the correct answer is (9).

ARCHIVE: JEE MAIN

1. $x^2+2xy-3y^2=0$

M_N = slope of normal drawn to curve at (2, 2) is – 1

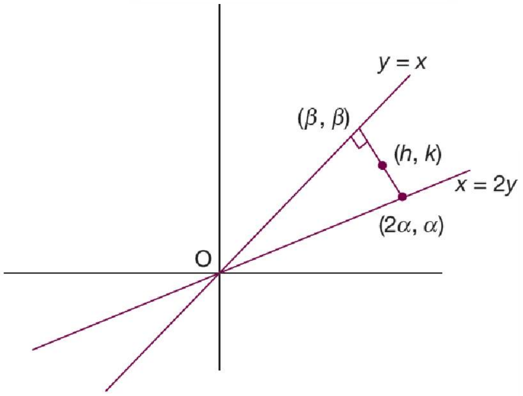
$L: x+y=4$.

Perpendicular distance of L from (0, 0) $=\frac{|0+0-4|}{\sqrt{2}}$

$=2\sqrt{2}$

Hence, the correct answer is (D).

2. Slope of perpendicular $= \dfrac{\alpha-\beta}{2\alpha-\beta} = -1$ or $3\alpha = 2\beta$



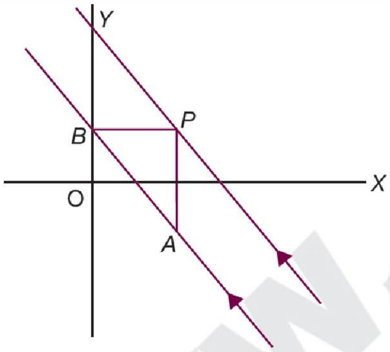
$$h = \frac{2\alpha+\beta}{2} \text{ or } 2h = \frac{7\alpha}{2}$$

$$k = \frac{\alpha+\beta}{2} \text{ or } 2k = \frac{5\alpha}{2}$$

$\dfrac{h}{k} = \dfrac{7}{5}$ hence the locus is $5x = 7y$

Hence, the correct answer is (B).

3.



$AB: 3x + y - 2 = 0$

Also, $\dfrac{1}{2} \times \sqrt{10} \times h = 5 \Rightarrow h = \sqrt{10}$

$$\Rightarrow \frac{|4\lambda-2|}{\sqrt{10}} = \sqrt{10} \Rightarrow \lambda = 3, -2$$

Hence, the correct answer is (D).

4. Centroid of $\Delta = (2, 2)$

line passing through intersection of $x + 3y - 1 = 0$ and $3x - y + 1 = 0$, be given by

$$(x + 3y - 1) + \lambda(3x - y + 1) = 0$$

$\because$ It passes through (2, 2)

$$\Rightarrow \quad 7 + 5\lambda = 0 \Rightarrow \lambda = -\frac{7}{5}$$

$\therefore$ Required line is $8x - 11y + 6 = 0$

$\because$ (–9, –6) satisfies this equation.

Hence, the correct answer is (B).

5. 3x + 5y = 15 equation of straight line

$x = y$	$x = -y$
$8x = 15$	$-2x = 15$
$x = 15/8$	$x = -15/2$
$y = 15/8$	$y = 15/2$
$P\left(\dfrac{15}{8}, \dfrac{15}{8}\right)$	$Q\left(-\dfrac{15}{2}, \dfrac{15}{2}\right)$
I Quadrant	II Quadrant

Hence, the correct answer is (B).

6. Equation of line passes through (1, 2) and (–3, 4)

$$\Rightarrow \quad (y - 2) = \frac{4-2}{-3-1}(x - 1)$$

$$\Rightarrow \quad x + 2y = 5 \quad \text{...(1)}$$

L_2 line $2x - y = \lambda$ it passes through (4, 3)

$$\lambda = 5$$

$$\Rightarrow \quad 2x - y = 5 \quad \text{...(2)}$$

Intersection point of line L_1 and line L_2 is $(3, 1) \equiv (h, k)$

$$\frac{k}{h} = \frac{1}{3}$$

Hence, the correct answer is (B).

7. Two lines are perpendicular.

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad \left(\frac{-1}{a-1}\right)\left(\frac{-2}{a^2}\right) = -1$$

$$\Rightarrow \quad a^3 - a^2 + 2 = 0$$

$$\Rightarrow \quad (a + 1)(a^2 - 2a + 2) = 0$$

$$\therefore \quad a = -1$$

So, the lines are L_1: $x - 2y + 1 = 0$ and L_2: $2x + y - 1 = 0$. Solving these equations, we get the point of intersection $P\left(\dfrac{1}{5}, \dfrac{3}{5}\right)$.

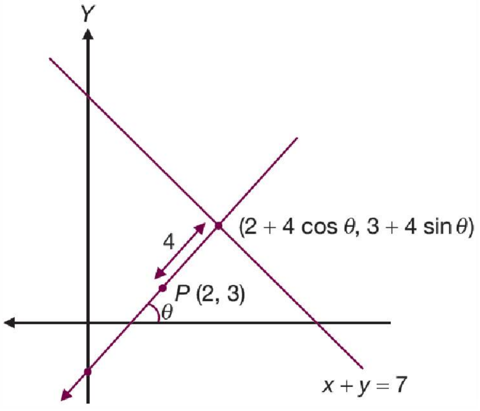
Hence, the correct answer is (B).

8. Let any point on the line is $P(2 \pm 4\cos\theta, 3 \pm 4\sin\theta)$. It also lies on the line $x + y = 7$.

$$\therefore \quad (2 \pm 4\cos\theta) + (3 \pm 4\sin\theta) = 7$$

$$\Rightarrow \quad (\sin\theta + \cos\theta) = \pm\frac{1}{2}$$

$$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = \frac{1}{4}$$



$$\Rightarrow \quad I + \sin 2\theta = \frac{1}{4}$$

$$\Rightarrow \quad \sin 2\theta = -\frac{3}{4}$$

$$\Rightarrow \quad \frac{2\tan\theta}{1+\tan^2\theta} = -\frac{3}{4}$$

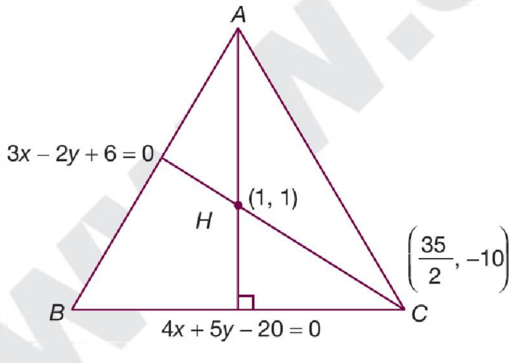
$$\Rightarrow \quad 3\tan^2\theta + 8\tan\theta + 3 = 0$$

$$\Rightarrow \quad \tan\theta = \frac{-8 \pm 2\sqrt{7}}{6}$$

$$\Rightarrow \quad \tan\theta = \frac{1-\sqrt{7}}{1+\sqrt{7}}, \frac{1+\sqrt{7}}{1-\sqrt{7}}$$

Hence, the correct answer is (C).

9.



$4x + 5y - 20 = 0$ …(1)

$3x - 2y + 6 = 0$ …(2)

Orthocentre is $(1, 1)$

AH : line perpendicular to BC

$4x + 5y - 20 = 0$ and passes through $(1, 1)$ is

$$(y-1) = \frac{5}{4}(x-1)$$

$\Rightarrow \quad 5x - 4y - 1 = 0$ …(3)

Solving (1) and (4), we get $C\left(\frac{35}{2}, -10\right)$

Solving (2) and (3), we get $A\left(-13, \frac{-33}{2}\right)$

AB: $3x - 2y + 6 = 0$

CH: $y - 1 = \frac{-2}{3}(x - 1)$

$\Rightarrow \quad 2x + 3y - 5 = 0$ …(4)

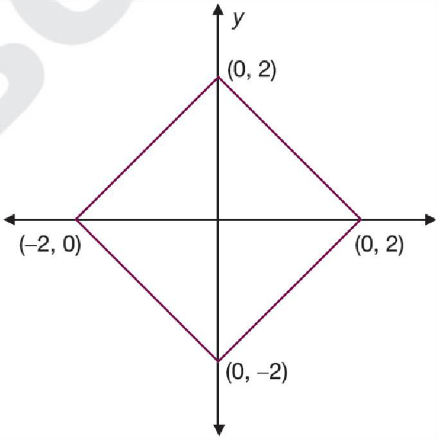
Side BC is $y + 10 = \dfrac{\frac{-33}{2}+10}{-13-\frac{35}{2}}\left(x, -\frac{35}{2}\right)$

$$\Rightarrow \quad y + 10 = \frac{13}{61}\left(x, -\frac{35}{2}\right)$$

$$\Rightarrow \quad 26x - 122y - 1675 = 0$$

Hence, the correct answer is (A).

10.



Shown figure is square with side length $2\sqrt{2}$ units.

Hence, the correct answer is (B).

11. Line parallel to $4x - 3y + 2 = 0$

is given as $4x - 3y + 1 = 0$.

Distance from origin is $\left|\frac{\lambda}{5}\right| = \frac{3}{5}$

$\Rightarrow \quad \lambda = \pm 3$

∴ Required lines are $4x - 3y + 3 = 0$ and $4x - 3y - 3 = 0$.

Now, check the options.

Hence, the correct answer is (B).

12. $P\left(h, \frac{2h+4}{3}\right)$

$Q(1, 4)$ $Q(3, -2)$

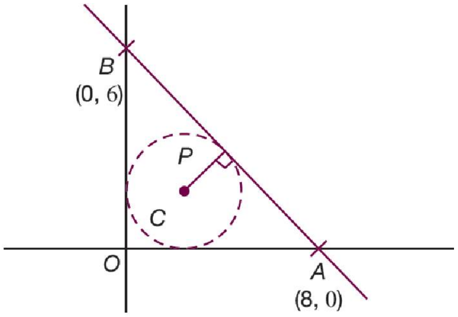
Centroid $G(\alpha, \beta) \equiv \left(\frac{h+4}{3}, \frac{2h+4+12-6}{9}\right)$

$$\alpha = \frac{h+4}{3};\ \beta = \frac{2h+10}{9};\ \beta = \frac{2(3\alpha-4)+10}{9};\ 9\beta = 6\alpha + 2$$

Locus $\alpha = x$; $\beta = y$; $y = \frac{2}{3}x + \frac{2}{9}$; slope $= \frac{2}{3}$

Hence, the correct answer is (B).

13. Incentre $\left(\frac{0.10+8.6+6.0}{10+8+6}, \frac{0.10+0.8+6.8}{10+8+6}\right) = (2, 2)$



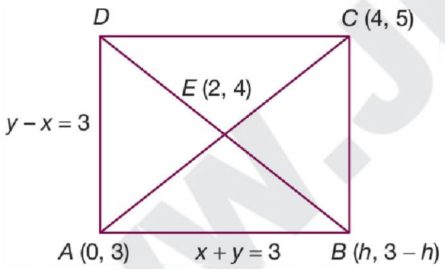
Hence, the correct answer is (B).

14. Slope of $BC = 1$

$\Rightarrow \quad \frac{3-h-5}{h-4} = 1$

$\Rightarrow \quad h = 1$

$\Rightarrow \quad B(1, 2)$



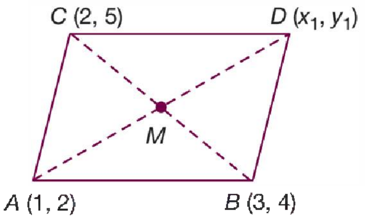
Now, E is the midpoint of BD.

So, $D(3, 6)$.

Hence, the correct answer is (D).

15. $M\left(\frac{3+2}{2}, \frac{4+5}{2}\right) \equiv \left(\frac{1+x}{2}, \frac{2+y}{2}\right)$

$\Rightarrow \quad x_1 = 4;\ y_1 = 7\ D(4, 7)$



Co-ordinates of point D are $(4, 7)$

Equation of line AD is $5x - 3y + 1 = 0$.

Hence, the correct answer is (B).

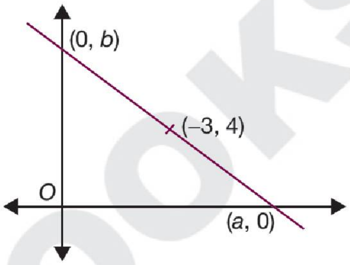
16. $\therefore m_1 m_2 = -1$

$$\left(\frac{17-\beta}{-8}\right) \times \left(\frac{2}{3}\right) = -1$$

$\beta = 5$

Hence, the correct answer is (D).

17.



$\frac{a}{2} = -3 \Rightarrow a = -6$

$\frac{b}{2} = 4 \Rightarrow b = 8$

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{-6} + \frac{y}{8} = 1$

$4x - 3y = -24$

$4x - 3y + 24 = 0$

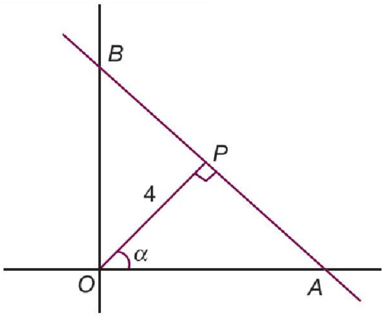
Hence, the correct answer is (B).

18. $OP = 4$

Given OP makes 60° with $x + y = 0$

Let slope of $OP = m = \tan a$

$\Rightarrow \quad \tan 60° = \left|\frac{m+1}{1-m}\right|$



$\Rightarrow \quad \frac{m+1}{1-m} = \sqrt{3}$ or $-\sqrt{3}$

$\Rightarrow \quad \tan\alpha = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $\tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$\Rightarrow$ Equation of line $x \cos \alpha + y \sin \alpha = P$

$\Rightarrow$ $(\sqrt{3}+1)x + (\sqrt{3}-1)y = 8\sqrt{2}$

or $(\sqrt{3}-1)x + (\sqrt{3}+1)y = 8\sqrt{2}$

Hence, (C) and (D) are correct.

19. Given lines are $px + qy + r = 0$ ...(1)

and $3p + 2q + 4r = 0$ ...(2)

$\therefore$ (2) can be written as

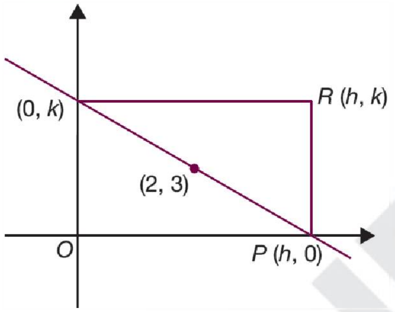
$$\frac{3}{4}p + \frac{q}{2} + r = 0 \quad \text{...(3)}$$

From (1) and (3),

$$x = \frac{3}{4} \text{ and } y = \frac{1}{2}$$

Hence, the correct answer is (B).

20. Equation of PQ is $\frac{x}{h} + \frac{y}{k} = 1$ ...(1)



$\because$ Equation (1) passes through the fixed point (2, 3),

then $\frac{2}{h} + \frac{3}{k} = 1$

$\therefore$ The locus of R is $\frac{2}{h} + \frac{3}{k} = 1$ or $3x + 2y = xy$.

Hence, the correct answer is (B).

21. Let other two sides of a rhombus are

$x - y + l = 0$

and $7x - y + m = 0$

Then, O is equidistant from AB and DC and from AD and BC.

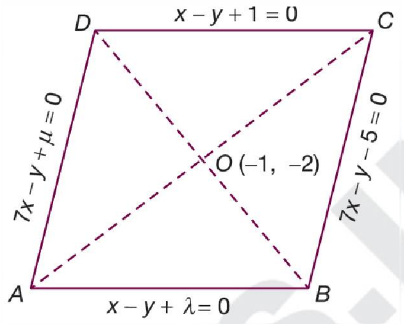
$\therefore$ $|-1 + 2 + 1| = |-2 + 2 + l|$

$\Rightarrow$ $l = -3$

and $|-7 + 2 - 5| = |-7 + 2 + m|$

$\Rightarrow$ $m = 15$

$\therefore$ Other two sides are $x - y - 3 = 0$ and $7x - y + 15 = 0$



On solving the equations of sides pairwise, we get the vertices as

$$\left(\frac{1}{3}, \frac{-8}{3}\right), (1, 2), \left(\frac{-7}{3}, \frac{-4}{3}\right), (-3, -6)$$

Hence, the correct answer is (D).

22. The given lines are

$4ax + 2ay + c = 0$

$5bx + 2by + d = 0$

The point of intersection will be

$$\frac{x}{2ad - 2bc} = \frac{-y}{4ad - 5bc} = \frac{1}{8ab - 10ab}$$

$$\Rightarrow \quad x = \frac{2(ad - bc)}{-2ab} = \frac{bc - ad}{ab}$$

$$\Rightarrow \quad y = \frac{5bc - 4ad}{-2ab} = \frac{4ad - 5bc}{2ab}$$

$\because$ Point of intersection is in IV$^{\text{th}}$ quadrant.

So, x is positive and y is negative.

Also, distance from axes is same.

So, $x = -y$ ($\because$ Distance from x-axis is $-y$, as y is negative)

$$\frac{bc - ad}{ab} = \frac{5bc - 4ad}{2ab}$$

$\Rightarrow$ $3bc - 2ad = 0$

Hence, the correct answer is (C).

23. Let P, Q and R be the vertices of DPQR

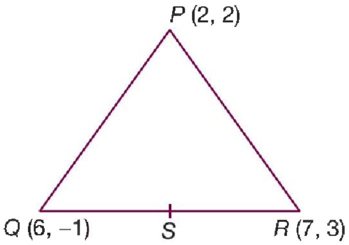
$\because$ PS is the median, S is the midpoint of QR

$$\text{So, } S = \left(\frac{7+6}{2}, \frac{3-1}{2}\right) = \left(\frac{13}{2}, 1\right)$$

$$\text{Now, slope of } PS = \frac{2-1}{2 - \frac{13}{2}} = -\frac{2}{9}$$

$\because$ The required line is parallel to PS.

$\therefore$ Slope of required line = Slope of PS



Now, equation of line passing through (1, –1) and having slope $-\frac{2}{9}$ is

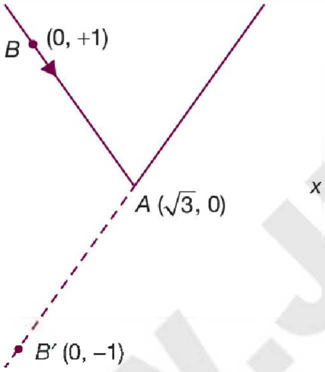
$$y-(-1)=-\frac{2}{9}(x-1)$$

$$9y+9=-2x+2$$

$$\Rightarrow\quad 2x+9y+7=0$$

Hence, the correct answer is (B).

24. Suppose $B(0, 1)$ be any point on given line and co-ordinate of A is $(\sqrt{3}, 0)$.

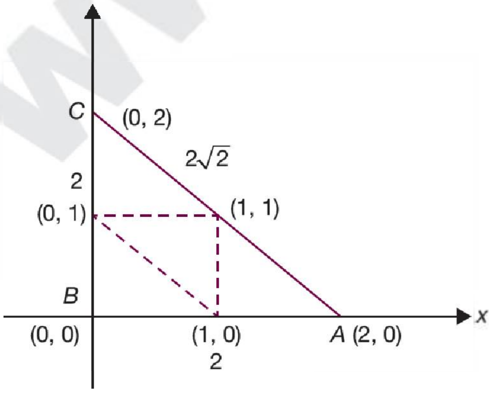


So, equation of reflected ray is $\frac{-1-0}{0-\sqrt{3}}=\frac{y-0}{x-\sqrt{3}}$

$$\Rightarrow\quad \sqrt{3}y=x-\sqrt{3}$$

Hence, the correct answer is (B).

25.



From the figure, we have

$$a=2,\ b=2\sqrt{2},\ c=2$$

$$x_1=0,\ x_2=0,\ x_3=2$$

Now, x-co-ordinate of incentre is given as

$$\frac{ax_1+bx_2+cx_3}{a+b+c}$$

$$\Rightarrow\quad x\text{-co-ordinate of incentre}=\frac{2\times 0+2\sqrt{2}.0+2.2}{2+2+2\sqrt{2}}$$

$$=\frac{2}{2+\sqrt{2}}=2-\sqrt{2}$$

Hence, the correct answer is (B).

26. $y-2=m(x-1)$

$$\Rightarrow\quad mx-y=m-2$$

$$\frac{x}{\left(\frac{m-2}{m}\right)}+\frac{y}{2-m}=1$$

$$\text{Area of }\Delta OPQ=\frac{1}{2}\left(\frac{m-2}{m}\right)(2-m)=\frac{1}{2}\left(2-m-\frac{4}{m}+2\right)$$

$$=\frac{1}{2}\left(4-m-\frac{4}{m}\right)\text{ sq. units}$$

From options if we see slope is negative.

Now, using $AM\geq GM$, we get $\dfrac{(-m)+\dfrac{4}{(-m)}}{2}\geq\sqrt{\dfrac{(-m)4}{(-m)}}$

$$\Rightarrow\quad -m+\frac{4}{(-m)}\geq 4$$

Least area is for $m=-2$.

Hence, the correct answer is (D).

27. Point of intersection $\left(\frac{|a|+1}{a+1},\frac{a|a|-1}{a+1}\right)$;

Point is in I$^{\text{st}}$ quadrant, x-co-ordinate $=\frac{|a|+1}{a+1}>0$

$$\Rightarrow\quad \frac{1}{a+1}>0$$

$$\Rightarrow\quad a\in(-1,\infty)\qquad\qquad \ldots(1)$$

y- co-ordinate $=\frac{a|a|-1}{a+1}>0$

If $a>0$, $\qquad a-1>0\Rightarrow a>1$

If $a<0$, $\qquad -\frac{(a^2+1)}{a+1}>0$

$\Rightarrow \quad a < -1$, we get $a \in ((-\infty, -1) \cup (1, \infty)$ ...(2)

$\therefore \quad a \in (1, \infty)$

Hence, the correct answer is (C).

28. Slope of line $L = -\dfrac{b}{5}$

Slope of line $K = -\dfrac{3}{c}$

Line L is parallel to line k.

$\Rightarrow \quad \dfrac{b}{5} = \dfrac{3}{c}$

$\Rightarrow \quad bc = 15$

(13, 32) is a point on L.

$\therefore \quad \dfrac{13}{5} + \dfrac{32}{b} = 1$

$\Rightarrow \quad \dfrac{32}{b} = -\dfrac{8}{5}$

$\Rightarrow \quad b = -20$ and $c = -\dfrac{3}{4}$

Equation of K is $y - 4x = 3 \Rightarrow 4x - y + 3 = 0$ and equation L is $4x - y = 20$.

Distance between L and $K = \dfrac{|20+3|}{\sqrt{17}} = \dfrac{23}{\sqrt{17}}$ units.

Hence, the correct answer is (D).

29. If the lines $p(p^2 + 1)\,x - y + q = 0$ and $(p^2 + 1)^2\,x + (p^2 + 1)\,y + 2q = 0$

are perpendicular to a common line, then these lines must be parallel to each other.

$\therefore \quad m_1 = m_2 \Rightarrow \; -\dfrac{p(p^2+1)}{-1} = -\dfrac{(p^2+1)^2}{p^2+1}$

$\Rightarrow \quad (p^2 + 1)\,(p + 1) = 0$

$\Rightarrow \quad p = -1$

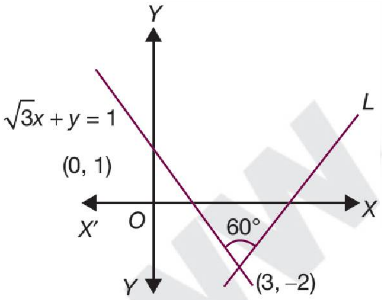
$\therefore \quad p$ can have exactly one value.

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1.



Let the slope of line L be m.

Then $\left|\dfrac{m+\sqrt{3}}{1-\sqrt{3}m}\right| = \sqrt{3}$

$\Rightarrow \quad m + \sqrt{3} = \pm(\sqrt{3} - 3m)$

$\Rightarrow \quad 4m = 0$ or $2m = 2\sqrt{3}$

$\Rightarrow \quad m = 0$ or $m = \sqrt{3}$

$\because \quad L$ intersects x-axis $\quad (\therefore m = \sqrt{3})$

$\therefore \quad$ Equation of L is $y + 1 = \sqrt{3}\,(x - 3)$

Or $\quad \sqrt{3}\,x - y - (2 + 3\sqrt{3}) = 0$

Hence, the correct answer is (B).

2. The intersection point of $y = 0$ with the Ist line is $B(-p, 0)$.

The intersection point of $y = 0$ with the IInd line is $A(-q, 0)$.

The intersection point of the 2 lines is $C(pq, (p + 1)(q + 1))$

The altitude from C to AB is $x = pq$.

The altitude from B to AC is

$$y = -\frac{q}{1+q}\,(x + p)$$

Solving these two, we get $x = pq$ and $y = -pq$.

$\therefore$ The locus of the orthocenter is $x + y = 0$.

Hence, the correct answer is (D).

Multiple Correct Choice Type Problems

1. The intersection point of two lines is $\left(\frac{-c}{a+b}, \frac{-c}{a+b}\right)$

Distance between (1, 1) and $\left(\frac{-c}{a+b}, \frac{-c}{a+b}\right) < 2\sqrt{2}$

$$\Rightarrow \quad 2\left(1+\frac{c}{a+b}\right)^2 < 8 \Rightarrow 1+\frac{c}{a+b} < 2$$

$$\Rightarrow \quad a+b-c > 0$$

Note

As given, $a > b > c > 0$ and $c > 0$

Add $a - b + c > 0$

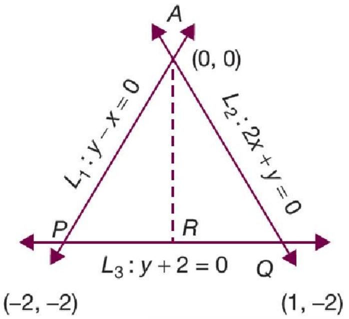
Option (C) is always correct.

In actual exam, it was asked as a single correct question, later they awarded marks for both options *A* or *C* or *AC*.

Hence, the correct answer is (A) and (C).

Reasoning Based Questions

1.



Point of intersection of L_1 and L_2 is $A(0, 0)$.

Also, $P(-2, -2)$, $Q(1, -2)$.

$\because AR$ is the bisector of $\angle PAQ$, R divides PQ in the same ratio as $AP{:}AQ$.

Thus, $PR{:}RQ = AP{:}AQ = 2\sqrt{2} : \sqrt{5}$

$\therefore$ Statement 1 is true and Statement 2 is definitely false.

Hence, the correct answer is (C).

Matrix Match/Column Match Type Questions

1. A → (s); B → (p, q); C → (r); D → (p, q, s)

$x + 3y - 5 = 0$ and $5x + 2y - 12 = 0$ intersect at (2, 1)

Hence, $6 - k - 1 = 0$

$$\Rightarrow \quad k = 5$$

For L_1, L_2 to be parallel,

$$\frac{1}{3} = \frac{3}{-k}$$

$$\Rightarrow \quad k = -9$$

For L_2, L_3 to be parallel,

$$\frac{3}{5} = \frac{-k}{2}$$

$$\Rightarrow \quad k = \frac{-6}{5}$$

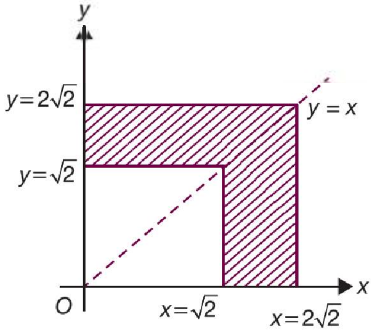
For $k \neq 5, -9, \frac{-6}{5}$ they form triangle.

For $k = 5, k = -9, \frac{-6}{5}$, which do not form a triangle.

Integer/Numerical Answer Type Questions

1. Let the point P be (x, y)

Then, $d_1(P) = \left|\frac{x-y}{\sqrt{2}}\right|$ and $d_2(P) = \left|\frac{x+y}{\sqrt{2}}\right|$



For P lying in Ist quadrant, $x > 0$ and $y > 0$.

Also, $2 \le d_1(P) + d_2(P) \le 4$

$$\Rightarrow \quad 2 \le \left|\frac{x-y}{\sqrt{2}}\right| + \left|\frac{x+y}{\sqrt{2}}\right| \le 4$$

If $x > y$, then $2 \le \frac{x-y+x+y}{\sqrt{2}} \le 4$

Or $\quad \sqrt{2} \le x \le 2\sqrt{2}$

$$2 \le \frac{y-x+x+y}{\sqrt{2}} \le 4 \text{ or } \sqrt{2} \le y \le 2\sqrt{2}$$

The required region is the shaded region in the figure given below.

$\therefore$ Required area $= \left(2\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2 = 8 - 2 = 6$ sq. units

Hence, the correct answer is (6).

CHAPTER 3: CIRCLE

Test Your Concepts-I

1. The equation of the circle is $(x - 1)^2 + (y - (-2))^2 = 4^2$

$\Rightarrow \quad (x - 1)^2 + (y + 2)^2 = 16$

$\Rightarrow \quad x^2 + y^2 - 2x + 4y - 11 = 0$

2. **(i)** Comparing it with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$2g = -4 \Rightarrow g = -2$$

$$2f = 6 \Rightarrow f = 3$$

and $\quad c = 12$

$\therefore \quad$ Centre is $(-g, -f)$, i.e., $(2, -3)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 - 12} = 1$

(ii) We rewrite the given equation as $x^2 + y^2 - \frac{8}{3}x - \frac{10}{3}y + 1 = 0$

$\Rightarrow \quad g = \frac{-4}{3}, f = \frac{-5}{3}, c = 1$

Hence, the centre is $\left(\frac{4}{3}, \frac{5}{3}\right)$ and the radius is $\sqrt{\frac{16}{9} + \frac{25}{9} - 1} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3}$

3. Equation of the circle, the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here, $x_1 = -1$, $x_2 = 4$, $y_1 = 2$ and $y_2 = -3$

So, the equation of the required circle is

$$(x + 1)(x - 4) + (y - 2)(y + 3) = 0$$

$\Rightarrow \quad x^2 + y^2 - 3x + y - 10 = 0$

4. Let the centre be $(3, k)$ and the radius be r.

The equation of circle is

$$(x - 3)^2 + (y - k)^2 = r^2 \quad \ldots(1)$$

$\because$ Circle passes through (3, 0), we have,

$$(3 - 3)^2 + (y - k^2) = r^2$$

$\Rightarrow \quad k^2 = r^2 \Rightarrow k = -r$

($\because$ Centre lies in the IVth quadrant)

Now, equation (1), becomes $(x - 3)^2 + (y + r)^2 = r^2$

$\because$ Circle also passes through (1, – 2), we have

$$4 + (r - 2)^2 = r^2 \Rightarrow 4 + r^2 + 4 - 4r = r^2 \Rightarrow r = 2$$

$\therefore \quad k = -2$

Now, equation of circle is $(x - 3)^2 + (y + 2)^2 = 4$

5. Let P be the point of intersection of the lines whose equations are respectively,

$$3x - 2y - 1 = 0 \quad \ldots(1)$$

$$4x + y - 27 = 0 \quad \ldots(2)$$

Solving (1) and (2), we get $x = 5$, $y = 7$. So, the co-ordinates of P are (5, 7). Let $C(2, -3)$ be the centre of the circle. $\because$ The circle passes through P,

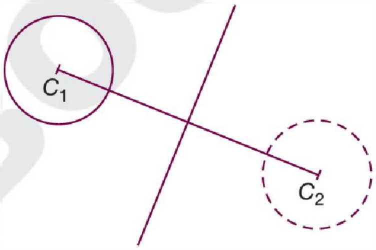
$$CP = \text{radius} \Rightarrow \sqrt{(5-2)^2 + (7+3)^2} = \text{radius}$$

$\Rightarrow \quad$ Radius $= \sqrt{109}$

Hence, the equation of the required circle is

$$(x - 2)^2 + (y + 3)^2 = \left(\sqrt{109}\right)^2$$

6.



The given circle has centre (–8, 12) and radius 5.

The given line is $4x + 7y + 13 = 0$

The required circle while (h, k), 5, where (h, k) is the reflection of centre (–8, 12) in the line.

$$4x + 7y + 13 = 0$$

Now, $\frac{h+8}{4} = \frac{k-12}{7} = -2\,\frac{(-32+84+13)}{16+49} = -2$

$\therefore \quad h = -16, k = -2$

$\therefore \quad$ Required circle is $(x + 16)^2 + (y + 2)^2 = 5^2$

$\Rightarrow \quad x^2 + y^2 + 32x + 4y + 235 = 0$

7. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(1)$$

$\because$ Centre $(-g, -f)$ lies on the line, $3x + 4y = 7$

$\therefore \quad -3g - 4f = 7 \quad \ldots(2)$

Again since circle (1) passes through the points (1, – 2) and (4, – 3)

$\therefore \quad 5 + 2g - 4f + c = 0$

$\Rightarrow \quad 2g - 4f + c = -5 \quad \ldots(3)$

and $\quad 25 + 8g - 6f + c = 0$

$\Rightarrow \quad 8g - 6f + c = -25 \quad \ldots(4)$

Subtracting equation (3) and (4), we get

$$-6g + 2f = 20 \text{ or } -3g + f = 10 \quad \ldots(5)$$

Solving we get

$$f = \frac{3}{5}, g = -\frac{47}{15}, c = \frac{11}{3}$$

Hence, the required circle is

$$x^2 + y^2 - \frac{94}{15}x + \frac{6}{5}y + \frac{11}{3} = 0$$

$$\Rightarrow \quad 15x^2 + 15y^2 - 94x + 18y + 55 = 0$$

8. Given equation is $px^2 + (2 - q)\,xy + 3y^2 - 6qx + 30y + 6q = 0$

$\because$ It represents an equation of circle.

$\therefore$ Coefficient of x^2 = Coefficient of y^2

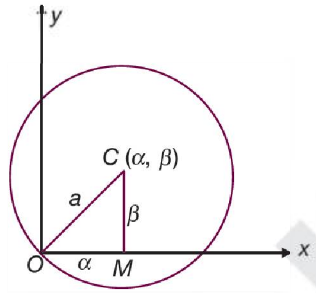
$\Rightarrow \quad p = 3$

Also, for a circle, the coefficient of $xy = 0$

$\Rightarrow \quad (2 - q) = 0 \Rightarrow q = 2$

Hence, the values of p and q are 3 and 2, respectively.

9. Let O be the origin and the centre is $C\,(\alpha, \beta)$, draw a perpendicular CM perpendicular to OX.



In ΔOMC,

$$OC^2 = CM^2 + OM^2$$

$$a^2 = \beta^2 + \alpha^2$$

Then, the equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2$$

$$\Rightarrow \quad x^2 - 2x\alpha + \alpha^2 - 2y\beta + \beta^2 = \alpha^2 + \beta^2$$

$$\Rightarrow \quad x^2 + y^2 - 2\alpha x - 2\beta y = 0$$

10. Equation of circle is $x^2 + y^2 + 2\,x\cos\theta + 2\,y\sin\theta = 8$

$$\Rightarrow \quad x^2 + y^2 + 2x\cos\theta + 2\,y\sin\theta - 8 = 0$$

Radius = $\sqrt{g^2 + f^2 - c}$

In the given equation,

$$g = \cos\theta, f = \sin\theta, c = -8$$

$$\text{Radius} = \sqrt{\cos^2\theta + \sin^2\theta + 8}$$

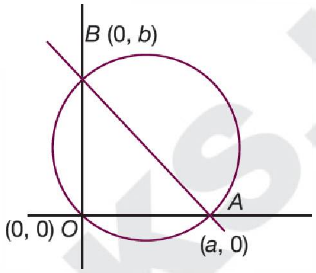
$$= \sqrt{1 + 8} \qquad (\because \cos^2\theta + \sin^2\theta = 1)$$

$$= \sqrt{9} = 3 \text{ units}$$

Hence, the radius of circle is 3 units.

11. The circle is passing through the lines $x = 0$, $y = 0$ and $\frac{x}{a} + \frac{y}{b} = 1$.

From the given equations of lines, the circle is passing through the point $O(0, 0)$ $A\,(a, 0)$ and $B(0, b)$.



The equations of the circle, $A(a, 0)$ and $B(0, b)$ as extremities of diameter is

$$(x - a)(x - 0) + (y - 0)(y - b) = 0$$

$$\Rightarrow \quad x^2 + y^2 - ax - by = 0$$

12. The equation of two diameters are $3x + y = 14$ and $2x + 5y = 18$.

The intersecting point of the two diameters is centre (4, 2).

If the circle is passing through origin, then the equation of circle is

$$x^2 + y^2 - 2ax - 2by = 0$$

Where, (α, β) or (4, 2) are co-ordinates of centre.

So, the equation of the circle is $x^2 + y^2 - 8x - 4y = 0$.

13. For smallest radius, these given points should be extremities of diameter

$$(x - 1)\,(x - 0) + (y - 0)\,(y - 1) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - y = 0$$

14. The centre of the circle lies at the intersection of $2x - 3y = 5$ and $3x - 4y = 7$.

So, the co-ordinates of centre are (1, – 1).

Area of circle = $\pi r^2 = 154$ sq. units

$$\Rightarrow \quad r^2 = \frac{154}{\pi} \Rightarrow r^2 = \frac{154 \times 7}{22} \Rightarrow r = 7$$

Hence, the required equation of circle is $(x - 1)^2 + (y + 1)^2 = 7^2 \Rightarrow x^2 + y^2 - 2x + 2y = 47$.

15. The intersection of the two diameters is the centre of the circle.

So, centre (1, – 1).

Here, $2\pi r = 10\pi$

$$\Rightarrow \quad r = \frac{10\pi}{2\pi} = 5 \text{ units}$$

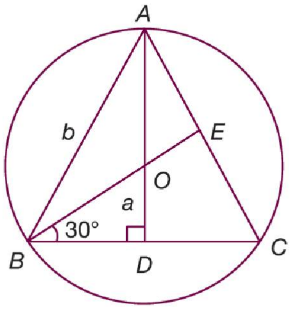
So, the equation of the circle is

$$(x - 1)^2 + (y + 1)^2 = 5^2 \Rightarrow x^2 + y^2 - 2x + 2y - 23 = 0$$

16. Here, the given equation of circle is

$$x^2 + y^2 = a^2$$

So, the radius of the circle = $OB = a$



In equilateral ΔABC, let the side is b.

The median AB is the altitude of the triangle.

So, $BD = DC = \frac{b}{2}$

In ΔBOD, $\angle ODB = 90°$ $[\because AD \perp BC]$

BE is a median, so $\angle ABE = \angle EBD = 30°$ ($\because$ ΔABC is equilateral)

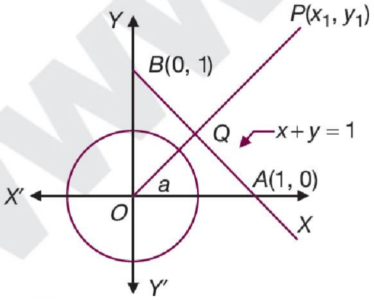
In, ΔBOD, $\cos 30° = \frac{\left(\frac{b}{2}\right)}{a}$

$$\Rightarrow \frac{\sqrt{3}}{2}a = \frac{b}{2} \Rightarrow b = \sqrt{3}\,a \text{ units}$$

17. Let P be image of the origin in the line $x + y = 1$.

$\because$ $OA = OB$, Q is the midpoint of AB.

$\therefore$ Co-ordinates of Q are $\left(\frac{1}{2}, \frac{1}{2}\right)$



Let the co-ordinates of P be (x_1, y_1) ($\because$ Q is the midpoint of OP)

$$\therefore \quad \frac{0 + x_1}{2} = \frac{1}{2} \text{ and } \frac{0 + y_1}{2} = \frac{1}{2}$$

$\Rightarrow x_1 = 1$ and $y_1 = 1$

$\therefore$ The co-ordinates of P are (1, 1).

18. The equations of the lines of square are $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$

So, on solving these equations, sides of the square are

$$x = 2, x = 6, y = 5, \text{ and } y = 9$$

The co-ordinates of the both ends of the diameter are (2, 5) and (6, 9).

$\therefore$ The co-ordinates of the centre $\left(\frac{2+6}{2}, \frac{5+9}{2}\right) = (4,7)$

19. Let $x = a, x = b, y = c$ and $y = d$ be the side of the square. The length of each side of the square is l. Then, diagonal of square = diameter of circle.

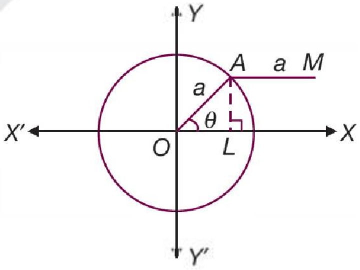
$$\Rightarrow \quad 2l^2 = (14\sqrt{2})^2 \Rightarrow l = 14$$

$\therefore$ Each side of the square is at a distance of 7 units from the centre (1, – 2) of the given circle.

This implies $a = -6$, $b = 8$, $c = -9$ and $d = 5$.

Hence, the vertices of the square are (– 6, – 9), (– 6, 5), (8, – 9) and (8, 5).

20.



Let $\angle AOL = \theta$

$\therefore$ $A = (a\cos\theta, a\sin\theta)$

$M = (a + a\cos\theta, a\sin\theta)$

$\Rightarrow$ $x = a + a\cos\theta$ and $y = a\sin\theta$

Eliminating θ, $(x - a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 = 2ax$

21. Given, equation of circle is $(x + 1)^2 + (y + 2)^2 = \left(2\sqrt{2}\right)^2$

Let the co-ordinates of Q be Q (α, β)

$\because$ Centre of the circle is midpoint of the line joining end points of diameter.

Midpoint of P (1, 0) and Q (α, β) is the centre of circle (–1,–2).

Co-ordinates of centre,

$$\frac{1+\alpha}{2} = -1 \text{ and } \frac{0+\beta}{2} = -2 \Rightarrow \alpha = -3 \text{ and } \beta = -4$$

So, the required point is (– 3, – 4).

22. The equation of the circle touching Y-axis, i.e.,

$$x = 0 \text{ at } (0, 2) \text{ is } (x-0)^2 + (y-2)^2 + \lambda x = 0 \quad \ldots(1)$$

It passes through (– 1, 0).

$$\therefore \ (-1-0)^2 + (0-2)^2 + (-1)\lambda = 0 \Rightarrow 1+4-\lambda = 0 \Rightarrow \lambda = 5$$

Putting $\lambda = 5$ in equation (1), we get

$$x^2 + (y-2)^2 + 5x = 0 \Rightarrow x^2 + y^2 + 5x - 4y + 4 = 0$$

Test Your Concepts-II

1. Given, $x^2 + y^2 - 4x - 2y + 1 = 0$

$$\Rightarrow (x^2 - 4x) + (y^2 - 2y) = -1$$

$$\Rightarrow (x-2)^2 + (y-1)^2 = 2^2$$

So, the parametric equations of this circle are $x = 2 + 2\cos\theta$, $y = 1 + 2\sin\theta$

2. Given, $x = a + c\cos\theta$, $y = b + c\sin\theta$

$$\Rightarrow \cos\theta = \frac{x-a}{c}, \ \sin\theta = \frac{y-b}{c}$$

$$\Rightarrow \left(\frac{x-a}{c}\right)^2 + \left(\frac{y-b}{c}\right)^2 = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow (x-a)^2 + (y-b)^2 = c^2$$

Clearly, it is a circle with centre at (a, b) and radius c.

3. **(i)** For point (1, 2),

$$S_1 = 1 + 4 - 4 + 4 - 11 = -6 < 0$$

⇒ It lies inside.

(ii) For point (6, 0),

$$S_1 = 36 + 0 - 24 + 0 - 11 = 1 > 0$$

⇒ It lies outside the circle.

4. The diameter of the circle is the perpendicular distance between the parallel lines (tangents) $3x - 4y + 4 = 0$ and $3x - 4y - \frac{7}{2} = 0$ and so it is equal to

$$\frac{4 + 7/2}{\sqrt{9+16}} = \frac{3}{2} = \text{diameter}$$

Hence, the radius is $\frac{3}{4}$

5. Equation of tangent is

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y+(-1)}{2}\right) + 109 = 0$$

$$\Rightarrow 4x - y - 15x - 60 + 3y - 3 + 109 = 0$$

$$\Rightarrow -11x + 2y + 46 = 0$$

$$\Rightarrow 11x - 2y - 46 = 0$$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$.

6. The tangent parallel to the line is $4x + 3y + k = 0$.

Centre $(3, -2)$, radius $= \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4 + 12} = 5$

For tangency, $p = r$

$$\left|\frac{12 - 6 + k}{\sqrt{16+9}}\right| = 5 \Rightarrow |6 + k| = 25 \Rightarrow k = 31, -19$$

∴ The equations of tangents are $4x + 3y + 31 = 0$ and $4x + 3y - 19 = 0$.

7. ∵ (1, 0) lies inside the circle, $S_1 < 0$ then $1 - 0 - 2c < 0$

$$\Rightarrow c > \frac{1}{2}$$

The given equation of line is $x + y - 2 = 0$, centre $(c, 0)$ radius $|c|$.

If the line touches the circle, then $p = r$.

$$\left|\frac{(1)(c) - 0 - 2}{\sqrt{1+1}}\right| = c \Rightarrow \left|\frac{c-2}{\sqrt{2}}\right| = c \Rightarrow c - 2 = \pm c\sqrt{2}$$

$$\Rightarrow c = \frac{2}{1+\sqrt{2}}, \ \frac{2}{1-\sqrt{2}}$$

$$\Rightarrow c = 2(\sqrt{2} - 1), -2(\sqrt{2} + 1) \text{ but } c > \frac{1}{2}$$

Hence, the required value of $c = 2(\sqrt{2} - 1)$.

Test Your Concepts-III

1. $PA \cdot PB = PT^2 = S_1 = 64 + 9 - 64 - 30 + 26 = 5$

2. Given circles are $x^2 + y^2 + x + y - 4 = 0$ and $x^2 + y^2 - \frac{x}{3} - \frac{y}{3} - \frac{\lambda}{3} = 0$.

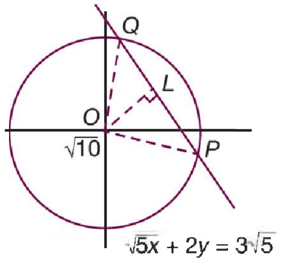
Now, we can use the formula for calculating the length of tangent.

$$\frac{\sqrt{S_1}}{\sqrt{S_2}} = \frac{\sqrt{1+4+1+2-4}}{\sqrt{1+4-\frac{1}{3}-\frac{2}{3}-\frac{\lambda}{3}}} = \frac{4}{3} \Rightarrow (4)(9)$$

$$= 16\left(4 - \frac{\lambda}{3}\right)$$

$$\Rightarrow \lambda = \frac{27}{4}$$

3. Length of perpendicular from the origin to the line $x\sqrt{5}+2y=3\sqrt{5}$ is



$$OL=\frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2+2^2}}=\frac{3\sqrt{5}}{\sqrt{9}}=\sqrt{5}$$

Radius of the given circle = $\sqrt{10}=OQ=OP$

$PQ=2QL=2\sqrt{OQ^2-OL^2}=2\sqrt{10-5}=2\sqrt{5}$

Thus, area of $\Delta OPQ=\frac{1}{2}\times PQ\times OL=\frac{1}{2}\times 2\sqrt{5}\times\sqrt{5}=5$

4. Equations of two circles are $x^2+y^2-5x-3=0$ and $3x^2+3y^2+2x+4y-6=0$.

$$\Rightarrow\quad x^2+y^2+\frac{2}{3}x+\frac{4}{3}y-2=0$$

$\because$ The lengths of the tangents are equal, $\sqrt{S_1}=\sqrt{S_2}$

$$\Rightarrow\quad S_1=S_2$$

$$\Rightarrow\quad x_1^2+y_1^2-5x_1-3=x_1^2+y_1^2+\frac{2}{3}x_1+\frac{4}{3}y_1-2$$

$$\Rightarrow\quad -5x_1-3-\frac{2}{3}x_1-\frac{4}{3}y_1+2=0$$

$$\Rightarrow\quad 17x_1+4y_1+3=0$$

$\therefore$ Locus of point $17x+4y+3=0$

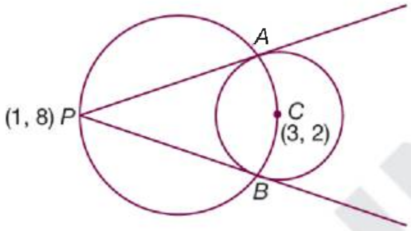
5. The line $y=mx\pm 2\sqrt{m^2+1}$ is tangent to the given circle for all m. It passes through A (3, 2).

So that, $2=3m\pm 2\sqrt{m^2+1}$

$$\Rightarrow\quad 5m^2-12m=0\Rightarrow m=0\quad\text{or}\quad m=\frac{12}{5}$$

Hence, the required tangents are $y=2$ and $5y=12x-26$. Obviously, the angle between this pair of tangents is $\tan^{-1}\left(\frac{12}{5}\right)$.

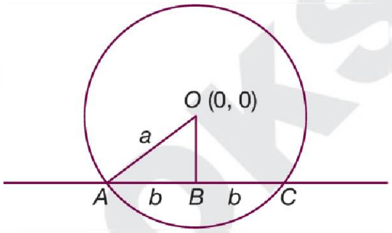
6. For required circle, $P(1, 8)$ and $O(3, 2)$ will be the end points of its diameter.



$$\therefore\quad (x-1)(x-3)+(y-8)(y-2)=0$$

$$\Rightarrow\quad x^2+y^2-4x-10y+19=0$$

7.



The length of the chord is $2b$ and the equation of line is $y=mx+c$.

The radius of circle is a and the centre is (0, 0).

In ΔOAB, $OB=\sqrt{a^2-b^2}$ (using Pythagoras theorem).

The equation of line OB is given by the distance between O and B.

$$\frac{c}{\sqrt{m^2+1}}=\sqrt{a^2-b^2}\ \Rightarrow c^2=(a^2-b^2)(m^2+1)$$

8. Let (p, q) be the point on $x^2+y^2+2gx+2fy+\alpha=0$.

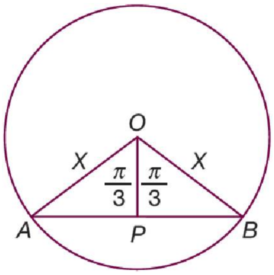
$$\therefore\quad p^2+q^2+2gx+2fq=-\alpha\qquad\ldots(1)$$

Now, the length of the tangent from (p, q) to $x^2+y^2+2gx+2fy+\beta=0$ is given by

$$\ell=\sqrt{p^2+q^2+2gp+2fq+\beta}=\sqrt{\beta-\alpha}$$

(Using result (1))

9. Here, the length of the chord $AB=2\sqrt{3}$ units.



The perpendicular drawn from the centre to the circle divides the chord into two equal parts.

So, $AP=PB=\sqrt{3}$

In ΔOAP, $\angle OAP=30°$

$$x\cos 30^\circ=\sqrt{3}\Rightarrow x=2$$

Area of circle $=\pi r^2=4\pi$ sq. units.

10. The line $5x - 2y + 6 = 0$ cuts y-axis at $Q(0, 3)$. Clearly, PQ is the length of the tangent drawn from Q on the circle $x^2 + y^2 + 6x + 6y - 2 = 0$.

$\therefore \quad PQ = \sqrt{0 + 9 + 6 \times 0 + 6 \times 3 - 3} = 5$ units

Test Your Concepts-IV

1. Normal of circle always passes through the centre of the circle, i.e., (20, –5).
However, it also passes through (4, –1).

$\therefore \quad$ Equation of normal is $y + 1 = \dfrac{-5+1}{20-4}(x - 4)$

$\Rightarrow \quad 4(y + 1) = -(x - 4)$

$\Rightarrow \quad x + 4y = 0$

2. The equation of the tangent to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at (5, 6) is

$$5x + 6y - 5\left(\frac{x+5}{2}\right) + 2\left(\frac{x+6}{2}\right) - 48 = 0.$$

$\Rightarrow \quad 10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$

$\Rightarrow \quad 5x + 14y - 109 = 0$

$\therefore \quad$ Slope of the tangent $= -\dfrac{5}{14}$

$\Rightarrow \quad$ Slope of the normal $= \dfrac{14}{5}$

Hence, the equation of the normal at (5, 6) is $y - 6 = \left(\dfrac{14}{5}\right)(x - 5)$

$\Rightarrow \quad 14x - 5y - 40 = 0$

3. Equation of the given circle is $S = x^2 + y^2 + 6x + 8y - 11 = 0$.
Let $L = (1, -1)$
For point $L(1, -1)$, $S_1 = 1^2 + (-1)^2 + (6)(1) + 8(-1) - 11 = -11$ and $T = (x)(1) + (y)(-1) + 3(x + 1) + 4(y - 1) - 11$, i.e., $T = 4x + 3y - 12$
Now, equation of the chord of circle, whose midpoint is $L(1, -1)$ is $T = S_1$

$\Rightarrow \quad 4x + 3y - 12 = -11$

$\Rightarrow \quad 4x + 3y - 1 = 0$

4. Let (h, k) be the co-ordinates of the midpoint.
Equation of the chord, with midpoint (h, k) is

$xh + yk - (x + h) - 3(y + k) - 10 = h^2 + k^2 - 2h - 6k - 10$ (Using $T = S_1$)

i.e., $h^2 + k^2 - h(x + 2) - k(6 + y) + x + h + 3y + 3k = 0$. This chord passes through the origin (0, 0).

Hence, $h^2 + k^2 - h - 3k = 0$. The required locus is $x^2 + y^2 - x - 3y = 0$.

5. Given, $x^2 + y^2 - 2x = 3$.
$\therefore$ Centre is (1, 0) and radius is 2.
Given, $x^2 + y^2 - 4y = 6$.
$\therefore$ Centre is (0, 2) and radius is $\sqrt{10}$

($\because$ The line $ax + by = 2$ touches the first circle)

$\therefore \quad \dfrac{|a(1) + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2$

$\Rightarrow \quad |(a - 2)| = \left(2\sqrt{a^2 + b^2}\right) \qquad \ldots(1)$

Also, the given line is normal to the second circle. Hence, it will pass through the centre of the second circle.

$\therefore \quad a(0) + b(2) = 2 \Rightarrow b = 1$

Putting this value in equation (1),

we get $|a - 2| = 2\sqrt{a^2 + 1^2}$

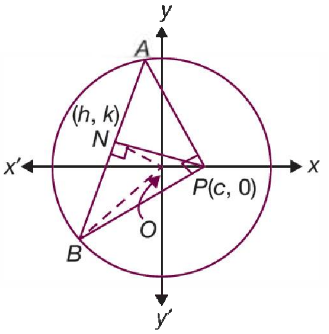
$\Rightarrow \quad (a - 2)^2 = 4(a^2 + 1)$

$\Rightarrow \quad a(3a + 4) = 0$

$\Rightarrow \quad a = 0, (a \neq 0)$

$\therefore \quad$ Values of a and b are $\left(-\dfrac{4}{3}, 1\right)$.

6.



Let $N(h, k)$ be the midpoint of any chord AB, which subtends a right angle at $P(c, 0)$.

$\because \angle APB = 90°, NA = NB = NP$

($\because$ Distance of the vertices from midpoint of the hypotenuse are equal)

$\Rightarrow \quad (NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2 \qquad \ldots(1)$

Also, $\angle BNO = 90°$

$\therefore \quad (OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow \quad -(NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow \quad -[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

$\Rightarrow \quad 2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore \quad$ Locus of $N(h, k)$ is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$

7. The line $x + 2y = 5$ is the tangent to the circle $x^2 + y^2 = 5$.

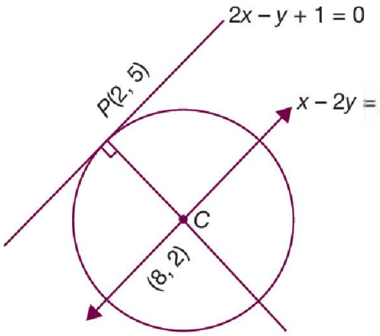
The equation of the line $x + 2y = 5$ has slope $m_1 = -\frac{1}{2}$

The normal is perpendicular to the tangent. So, the slope of the normal is 2 ($m_1 m_2 = -1$).

Equation of the normal, $(y - 0) = 2(x - 0)$

$\Rightarrow \quad y = 2x$

8. Clearly, the centre of the circles is the point of intersection of the line $x - 2y = 4$ and normal at P, i.e., a line passing through P and perpendicular to the tangent at P.



The normal equation of CP is $(y - 5) = -\frac{1}{2}(x - 2)$

$\Rightarrow \quad x + 2y = 12 \qquad \ldots(1)$

By solving this equation with $x - 2y = 4$, we obtain the co-ordinates of centre as C(8, 2).

$$\text{Radius} = CP = \sqrt{(8-2)^2 + (2-5)^2} = \sqrt{(36+9)} = \sqrt{45}$$

$$= 3\sqrt{5} \text{ units}$$

Test Your Concepts-V

1. The given equations of the circles are $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$.

Centres and radii of these circles are $C_1(0, 0)$, $C_2(4, 0)$ and $r_1 = 2$, $r_2 = \sqrt{16 - 12} = 2$

Now, $C_1C_2 = \sqrt{(4-0)^2 + (0-0)^2} = 4$

and $r_1 + r_2 = 2 + 2 = 4$

$\because C_1C_2 = r_1 + r_2$

It means that the two circles touch each other externally. So, the number of common tangents is 3.

2. The centres of the two circles are $C_1(2, 1)$, $C_2(6, 4)$ and radius is

$$r_1 = \sqrt{4+1+4} = 3 \text{ and } r_2 = \sqrt{36+16+36} = \sqrt{88} = 9.38$$

$$C_1C_2 = \sqrt{(4)^2 + (3)^2} = 5$$

$$r_1 + r_2 = 12.38$$

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$

$\therefore$ The two circles do not touch but they intersect each other.

3. Centre and radii of these circles are C_1 (0, 0), $C_2 = (3, 4)$ and $r_1 = 2$, $r_2 = \sqrt{9 + 16 + 24} = 7$

$$C_1C_2 = \sqrt{9+16} = 5$$

$$r_2 - r_1 = 5$$

$$|C_1C_2| = |r_1 - r_2| \text{ (Internal touch)}$$

$\therefore$ There is one common tangent.

4. The given equations of the circles are

$$x^2 + y^2 + 2hx - \left(r_1^2 - h^2\right) = 0 \qquad \ldots(1)$$

$$x^2 + y^2 + 2kx - \left(r_2^2 - k^2\right) = 0 \qquad \ldots(2)$$

The centres and radii of the circles are $C_1(-h, 0)$ and $(-k, 0)$.

$$R_1 = \sqrt{h^2 + r_1^2 - h^2} = r_1 \text{ and } R_2 = \sqrt{k^2 + r_2^2 - k^2} = r_2$$

$\therefore$ They touch externally.

$$|C_1C_2| = R_1 + R_2$$

$$\Rightarrow \quad \sqrt{h^2 + k^2} = r_1 + r_2$$

$$\Rightarrow \quad h^2 + k^2 = (r_1 + r_2)^2$$

5. The centres and the radii of the circles are

$C_1(1, 2), C_2(6, 8), r_1 = \sqrt{1+4-1} = 2$ and $r_2 = \sqrt{36+64-91} = 3$

$$|C_1C_2| = \sqrt{(6-1)^2 + (8-2)^2} = \sqrt{25+36} = 7.81$$

$$r_1 + r_2 = 2 + 3 = 5$$

$$|C_1C_2| > r_1 + r_2$$

$\therefore$ There are two direct and two transverse common tangents. So, there are a total of four tangents.

6. The tangents at the origin to C_1 and C_2 are $x + y = 0$ and $3x - 4y = 0$, respectively. Slope of the tangents are -1, $\frac{3}{4}$ respectively. Thus, if $-1 < m < \frac{3}{4}$, then the origin divides AB internally.

7. Given circles are $x^2 + y^2 + 2ax + c^2 = 0 \qquad \ldots(1)$

$$x^2 + y^2 + 2by + c^2 = 0 \qquad \ldots(2)$$

Let C_1 and C_2 be the centres of circles (1) and (2), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0)$, $C_2 = (0, -b)$, $r_1 = \sqrt{a^2 - c^2}$, $r_2 = \sqrt{b^2 - c^2}$

Here, we find that the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

$$\Rightarrow \quad \sqrt{(a^2+b^2)} = \left|\sqrt{(a^2-c^2)} \pm \sqrt{(b^2-c^2)}\right|$$

On squaring,

$$a^2+b^2 = a^2-c^2+b^2-c^2 \pm 2\sqrt{(a^2-c^2)}\sqrt{(b^2-c^2)}$$

$$\Rightarrow \quad c^2 = \pm\sqrt{a^2b^2-c^2(a^2+b^2)+c^4}$$

Again squaring, $c^4 = a^2b^2 - c^2(a^2+b^2) + c^4$

$$\Rightarrow \quad c^2(a^2+b^2) = a^2b^2$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$$

Test Your Concepts-VI

1. If two circles touch each other orthogonally, then

$$2(g_1g_2+f_1f_2) = C_1 + C_2$$

Here, $(-g_1, -f_1)$ is (3, 4) and $(-g_2, -f_2)$ is (7, 8)

Then, $2(21+32) = C_1 + C_2$

$$\Rightarrow \quad C_1 + C_2 = 106 \qquad \ldots(1)$$

$\because$ Radii of both circles are equal.

So, $r_1 = \sqrt{g_1^2+f_1^2-c_1}$ and $r_2 = \sqrt{g_2^2+f_2^2-c_2}$ $(\because r_1 = r_2)$

$$\Rightarrow \quad r_1^2 = r_2^2 \Rightarrow 9+16-C_1 = 49+64-C_2$$

$$\Rightarrow \quad -C_1 + C_2 = 88 \qquad \ldots(2)$$

On solving equations (1) and (2), we get

$$C_1 = 9 \text{ and } C_2 = 97$$

Hence, $r_1 = \sqrt{9+16-9} \Rightarrow r_1 = 4$ units

2. Required locus is radical axis.

$\therefore \quad S_1 - S_2 = 0$

$\Rightarrow \quad 9x - 10y + 11 = 0$

3. The given equations of the circles are

$x^2+y^2-2px=0$, $x^2+y^2-2qx=0$ and $x^2+y^2=p^2+q^2$

Radical axes of 1st and 3rd circles, $2px = p^2+q^2$

$$\Rightarrow x = \frac{p^2+q^2}{2p}$$

Radical axes of 1st and 3rd circles, $2qy = p^2+q^2$

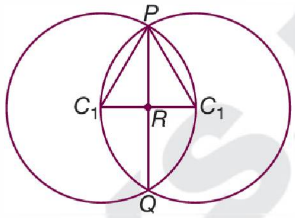
$$\Rightarrow y = \frac{p^2+q^2}{2q}$$

$\therefore$ Co-ordinates of radical centre $\left(\dfrac{p^2+q^2}{2q}, \dfrac{p^2+q^2}{2q}\right)$

4. The radical axis of two circles is always perpendicular to the line joining the centres of the circles.

5. Both the circles have same radius,

i.e. , $C_1P = C_2P = r$, where, C_1 and C_2 are their centres.



PQ is the common chord of both the circles and $C_1C_2 = r$.

ΔC_1PC_2 is an equilateral triangle, where $PR = r \sin 60°$

$$\Rightarrow \quad PR = \frac{\sqrt{3}r}{2}$$

$$\therefore \quad PQ = 2PR = 2 \times \frac{\sqrt{3}r}{2} = \sqrt{3}r$$

6. If a circle S_1 bisects the circumference of another circle S_2, then it will be the diameter of the circle S_2. Then, their radial axis passes through the centre of S_2.

7. The equation the common chord is $S_1 - S_2 = 0$

$$(x^2+y^2+2gx+2fy+c)-(x^2+y^2+2ax+2by+d)=0$$

$$\Rightarrow \quad 2gx+2fy+c-2ax-2by-d=0$$

It passes through $(-a, -b)$, i.e., the centre of the second circle.

$$\therefore \quad 2g(-a)+2f(-b)+c-2a(-a)-2b(-b)-d=0$$

$$\Rightarrow \quad -2ag-2fb+c+2a^2+2b^2-d=0$$

$$\Rightarrow \quad 2a(-g+a)+2b(b-f)+(c-d)=0$$

$$\Rightarrow \quad 2a(g-a)+2b(f-b)=(c-d)$$

8. Radical axes of 1st and 2nd circles,

$$c^2 - 2xc = 0$$

$$\Rightarrow \quad x = \frac{c}{2}$$

Radical axes of 1st and 3rd circle,

$$b^2 - 2by = 0 \Rightarrow y = \frac{b}{2}$$

So, the co-ordinates of centre are $\left(\dfrac{c}{2}, \dfrac{b}{2}\right)$.

9. The required point is the radical centre of the three given circles.

The radical axis of these three circles in pairs are

$$3x - 24 = 0,\ 16y + 120 = 0 \text{ and } -3x + 16y + 144 = 0$$

By solving any two of these three equations, we get,

$$x = 8 \text{ and } y = -\frac{15}{2}$$

$\therefore$ The required point is $\left(8, \frac{-15}{2}\right)$.

Test Your Concepts-VII

1. The equal of the sides of quadrilateral are

$L_1 = x - a = 0;$ $L_2 = y - a = 0$

$L_3 = x - 2a = 0;$ $L_4 = y + a = 0$

The equation of the curve passing through the vertices of the quadrilateral is

$$L_1L_3 + \lambda L_2L_2 = 0$$

$\Rightarrow$ $(x - a)(x - 2a) + \lambda(y - a)(y + a) = 0$

$\Rightarrow$ $x^2 - ax - 2ax + 2a^2 + \lambda(y^2 - a^2) = 0$

$\Rightarrow$ $x^2 - 3ax + 2a^2 + ly^2 - la^2 = 0$ …(1)

It will represent a circle, if coefficient of x^2 = coefficient of y^2.

$\Rightarrow$ $\lambda = 1$

On substituting the value of l in equation (1), we get

$$x^2 - 3a\,x + 2a^2 + y^2 - a^2 = 0$$

Thus, the circumcircle of the quadrilateral formed by the given line is

$$x^2 + y^2 - 3ax + a^2 = 0$$

2. Any circle through the intersection of given circles is $S_1 + \lambda S_2 = 0$

$\Rightarrow$ $(x^2 + y^2 - 2x - 4y - 4) + l(x^2 + y^2 - 10x - 12y + 40) = 0$

$\Rightarrow$ $(x^2 + y^2) - 2\dfrac{(1+5\lambda)}{1+\lambda}x - 2\dfrac{(2+6\lambda)}{1+\lambda}y + \dfrac{40\lambda - 4}{1+\lambda} = 0$ …(1)

It is given that, $r = \sqrt{g^2 + f^2 - c} = 4$

$\therefore$ $16 = \dfrac{(1+5\lambda)^2}{(1+\lambda)^2} + \dfrac{(2+6\lambda)^2}{(1+\lambda)^2} - \dfrac{40\lambda - 4}{1+\lambda}$

$16(1 + 2\lambda + \lambda^2) = 1 + 10\lambda + 25\lambda^2 + 4 + 24\lambda + 36\lambda^2 - 40\lambda^2$
$- 40\lambda + 4 + 4\lambda$

$\Rightarrow$ $16 + 32\lambda + 16\lambda^2 = 21\lambda^2 - 2\lambda + 9$

$\Rightarrow$ $5\lambda^2 - 34\lambda - 7 = 0$

$\therefore$ $(\lambda - 7)(5\lambda + 1) = 0$

$\therefore$ $\lambda = 7, -1/5$

Putting the values of λ in (1), the required circles are

$$2x^2 + 2y^2 - 18x - 22y + 69 = 0$$

and $x^2 + y^2 - 2y - 15 = 0$

3. The required circle by $S + \lambda P = 0$ is

$$x^2 + y^2 - 2x + 1 + \lambda(x + 2y - 1) = 0$$

$\Rightarrow$ $x^2 + y^2 - x(2 - \lambda) + 2ly + (1 - \lambda) = 0$

Centre$(-g, -f)$ is $\left(\dfrac{(2-\lambda)}{2}, -\lambda\right)$

$$r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{\frac{(2-\lambda)^2}{4 + \lambda^2 - (1-\lambda)}}$$

$$= \frac{1}{2}\sqrt{5\lambda^2} = \left(\frac{\lambda}{2}\right)\sqrt{5}$$

$\because$ The circle touches the line $2x - y + 3 = 0$, the perpendicular from centre is equal to radius

$$\frac{2.[(2-\lambda)/2] - (-\lambda) + 3}{\pm\sqrt{5}} = \frac{\lambda}{2}\sqrt{5}.$$

$\Rightarrow$ $5 = \pm\dfrac{\lambda}{2}\cdot 5$

$\therefore$ $\lambda = \pm 2$

Putting the values of λ in (1), the required circles are

$$x^2 + y^2 + 4y - 1 = 0 \text{ and}$$

$$x^2 + y^2 - 4x - 4y + 3 = 0$$

4. Equation of AB is $x - y = 0$

$\therefore$ Equation of circle is

$$(x - 1)(x - 2) + (y - 1)(y - 2) + \lambda(x - y) = 0$$

$\Rightarrow$ $x^2 + y^2 + (\lambda - 3)x - (\lambda + 3)y + 4 = 0$

However, radius = 1 (given)

$\therefore$ $\sqrt{\dfrac{(\lambda-3)^2}{4} + \dfrac{(\lambda+3)^2}{4} - 4} = 1$

$\Rightarrow$ $(\lambda - 3)^2 + (\lambda + 3)^2 - 16 = 4 \Rightarrow \lambda = \pm 1$

$\therefore$ Equations of circle are

$$x^2 + y^2 - 2x - 4y + 4 = 0$$

and $x^2 + y^2 - 4x - 2y + 4 = 0$

5. Equation of circle is $(x - 3)^2 + (y + 1)^2 + \lambda(x + 2y - 1) = 0$

$\because$ It passes through the point (2, 1),

$$1 + 4 + \lambda(2 + 2 - 1) = 0$$

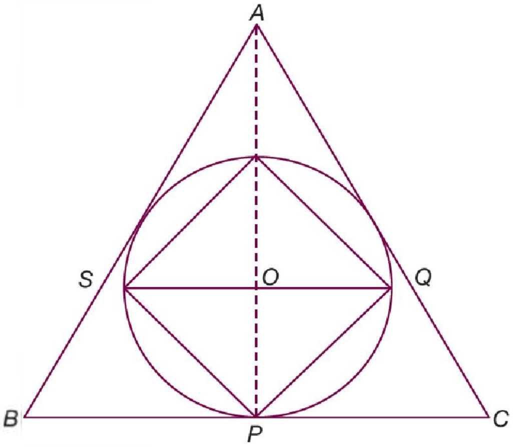
$\Rightarrow$ $\lambda = -\dfrac{5}{3}$

$\therefore$ Circle is $(x - 3)^2 + (y + 1)^2 - \dfrac{5}{3}(x + 2y - 1) = 0$

$\Rightarrow$ $3x^2 + 3y^2 - 23x - 4y + 35 = 0$

Single Correct Choice Type Questions

1.



Let a be the length of each sides of an equilateral triangle ABC. Then r the radius of the incircle $=\left(\frac{1}{3}\right)AP$ (the altitude, median and the angle bisector of angle)

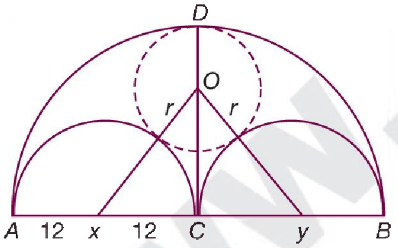
$$\Rightarrow \quad r=\frac{1}{3}\sqrt{a^2-\frac{a^2}{4}}=\frac{a}{2\sqrt{3}}$$

Area of the square $PQRS$ inscribed in this circle

$$=PQ^2=OQ^2=2r^2=2\times\frac{a^2}{4\times 3}=\frac{a^2}{6}$$

Hence, the correct answer is (D).

2.



ΔXCO

$$OX^2=XC^2+OC^2$$

$$(12+r)^2=12^2+(24-r)^2$$

$$r=8$$

Hence, the correct answer is (C).

3. Equation of line AE is $x+y=3$. Point C lies on AE such that midpoint of AC lies on BD. Let C is (α,β)

$$\Rightarrow \quad \alpha+\beta=3 \text{ and } 4\left(\frac{\alpha+1}{2}\right)+8\left(\frac{\beta+2}{2}\right)-7=0$$

$$\Rightarrow \quad \alpha=\frac{15}{2},\ \beta=-\frac{9}{2}$$

Hence, the correct answer is (A).

4. Let $O=(0,0)$, $P=(6,8)$ and $Q=(12,16)$

The shortest route consists of tangent OT, minor arc TR and tangent RQ.

$\because$ $OP=10$, $PT=5$ and $\angle OTP=90°$, it follows that $\angle OPT=60°$ and $OT=5\sqrt{3}$.

By similar reasoning, $\angle QPR=60°$ and $QR=5\sqrt{3}$, as O, P and Q are collinear $\angle RPT=60°$. So, arc TR is of length $\frac{5\pi}{3}$.

Hence, the length of the shortest route is $2\left(5\sqrt{3}\right)+\frac{5\pi}{3}$.

Hence, the correct answer is (C).

5.

$$\left.\begin{aligned}\frac{1}{\sqrt{p}}&=\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{q}}&=\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{6}}\end{aligned}\right\} H.M.=\frac{2}{\frac{1}{p}+\frac{1}{q}}=\frac{2}{2\left(\frac{1}{4}+\frac{1}{6}\right)}=\frac{12}{5}$$

Hence, the correct answer is (D).

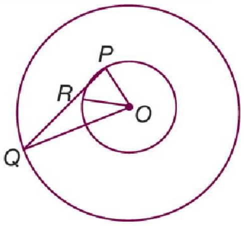
6. Using Apollonius theorem,

$$CP^2+CQ^2=2\left(CR^2+RQ^2\right)$$

$$\Rightarrow \quad 16+36=2\left(CR^2+\left(\frac{4}{2}\right)^2\right)$$

$$\Rightarrow \quad 52=2\left(CR^2+4\right)$$

$$\Rightarrow \quad CR=\sqrt{22}$$

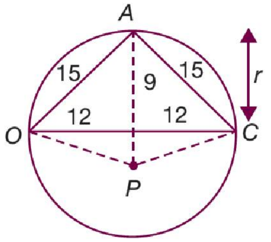


Hence, the correct answer is (B).

7. $(r-9)^2+12^2=r^2$

$$\Rightarrow \quad 81-18r+144=0$$

$$\Rightarrow \quad r=\frac{255}{18}=\frac{25}{2}$$



Locus is a concentric circle with radius $=PG$

$$=(r-9)+\frac{1}{3}(9)$$

$$=r-6=\frac{25}{2}-6=\frac{13}{2}$$

Locus is $(x+1)^2+(y-1)^2=\left(\frac{13}{2}\right)^2$

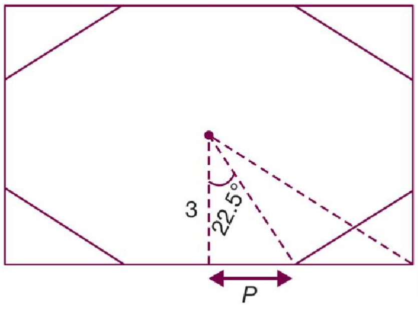
$$\Rightarrow \quad 4(x^2+y^2)+8x-8y-161=0$$

Hence, the correct answer is (D).

8. The given line is a diameter to the circle.

Hence, the correct answer is (C).

9.



Side of square = 6 $\Rightarrow r_1 = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

$$\tan 22\frac{1^\circ}{2} = \frac{p}{3} = \sqrt{2} - 1$$

$$\Rightarrow \quad p = 3\left(\sqrt{2} - 1\right)$$

$$r_2 = \text{Radius of } C_2 = \sqrt{3^2 + \left(3\left(\sqrt{2}-1\right)\right)^2}$$

$$= 3\sqrt{1 + \left(3 - 2\sqrt{2}\right)} = 3\sqrt{4 - 2\sqrt{2}}$$

$\therefore$ Required radius

$$= \sqrt{r_1^2 + r_2^2} = \sqrt{\left(3\sqrt{2}\right)^2 + \left(3\sqrt{4-2\sqrt{2}}\right)^2}$$

$$= 3\sqrt{2 + 4 - 2\sqrt{2}} \quad = 3\sqrt{6 - 2\sqrt{2}}$$

Hence, the correct answer is (A).

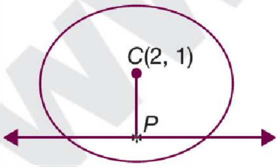
10. (3, 0) and (0, 4) are extremities of a diameter.

Hence, the correct answer is (D).

11. Equation circle passing through (1, 0) (0, 1) and (0, 0) is $x^2 + y^2 - x - y = 0$.

Hence, the correct answer is (D).

12.



Given, $x - 2y + 3 = 0$

P is the foot of perpendicular of C (2, 1) on $\left(x - 2y + 3 = 0\right)$.

$$\therefore \quad P\left(\frac{7}{5}, \frac{11}{5}\right)$$

Hence, the correct answer is (C).

13. $2\sqrt{r^2 - d^2}$, where $r = \sqrt{\frac{a^2 + b^2 - 4c}{4}}$

and $d = \frac{|a - b|}{2\sqrt{2}}$

Hence, the correct answer is (A).

14. r = Perpendicular distance from (4, 1) to $3x + 4y - 1 = 0$.

Hence, the correct answer is (B).

15. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

$$\Rightarrow \quad c = 40 \text{ and } \begin{array}{l} f + 2g + 15 = 0 \\ f = -2g - 15 \end{array}$$

$$\text{Radius} = \sqrt{g^2 + \left(2g + 15\right)^2 - 40}$$

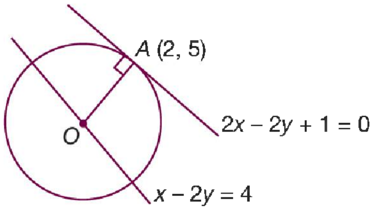
$g^2 + \left(2g + 15\right)^2 - 40$ to be minimum $g = -6 \Rightarrow f = -3$

Hence, the correct answer is (A).

16. $2x - y + 1 = 0$ is a tangent

Slope of line $OA = -\frac{1}{2}$

Equation $(y - 5) = -\frac{1}{2}(x - 2)$



$$2y - 10 = -x + 2$$

$$x + 2y = 12$$

$\therefore$ Intersection with $x - 2y = y$ will give the co-ordinates of centre.

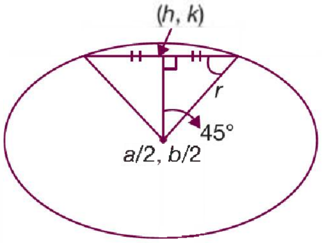
$$C = (8, 2)$$

$$\text{Distance } OA = \sqrt{(8-2)^2 + (2-5)^2}$$

$$= 3\sqrt{5}$$

Hence, the correct answer is (A).

17.



$$r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

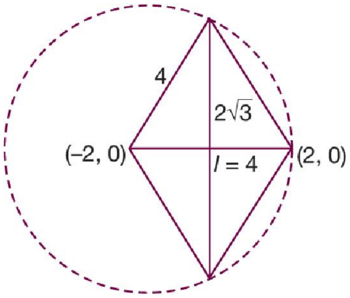
$$\sin 45^\circ = \frac{\sqrt{\left(h-\frac{a}{2}\right)^2+\left(k-\frac{b}{2}\right)^2}}{\frac{\sqrt{a^2+b^2}}{2}}$$

$$\Rightarrow \quad \frac{1}{2} = 4\left[\frac{\frac{(2h-a)^2}{4}+\frac{(2k-b)^2}{4}}{a^2+b^2}\right]$$

Simplify to get locus $x^2 + y^2 - ax - by + \frac{a^2+b^2}{8} = 0$.

Hence, the correct answer is (C).

18.



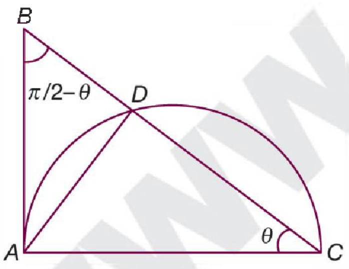
Circles with centre (2, 0) and (–2, 0) each with radius 4 units.

$\Rightarrow$ y-axis is their common chord.

The inscribed rhombus has its diagonals equal to 4 and 4 $\therefore A = \frac{d_1 d_2}{2} = 8\sqrt{3}$

Hence, the correct answer is (A).

19.



$$\therefore \frac{AC}{AD} = \frac{1}{\sin\theta}$$

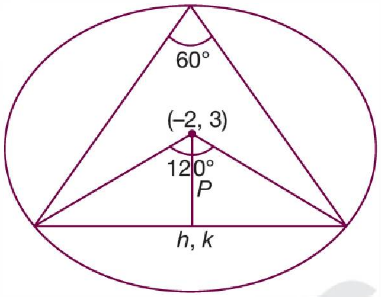
$$\Rightarrow \quad \frac{AD}{AB} = \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \Rightarrow \frac{AD^2}{AC^2}+\frac{AD^2}{AB^2} = 1$$

Aliter: $\Delta = \frac{1}{2}\cdot\sqrt{x^2+l^2}\cdot y \ = x^2l^2$

$$\Rightarrow \quad l = \frac{xy}{\sqrt{x^2-y^2}} = \frac{AB{\cdot}AD}{\sqrt{AB^2-AD^2}}$$

Hence, the correct answer is (D).

20.

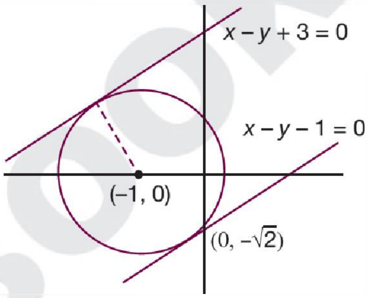


Radius = 5 $\Rightarrow p = 5\cos 60^\circ = 2.5$

$\Rightarrow$ Locus is $(h+2)^2 + (k-3)^2 = 6.25$

Hence, the correct answer is (B).

21.



$x^2 + y^2 + 2x - 1 = 0$, centre (–1, 0) and radius = $\sqrt{2}$

Line $x - y + c = 0$

$$\left|\frac{-1+c}{\sqrt{2}}\right| = \sqrt{2}\,;\ |c-1| = 2;\ c-1 = \pm 2$$

$\Rightarrow$ $c = 3$ or -1, $c = -1$ only acceptable

Hence, the correct answer is (D).

22. Equation of the two circles be $(x-r)^2 + (y-r)^2 = r^2$

i.e., $x^2 + y^2 - 2rx - 2ry + r^2 = 0$, where $r = r_1$ and r_2. Condition of orthogonally gives

$2\,r_1r_2 + 2\,r_1r_2 = r_1{}^2 + r_2{}^2 \Rightarrow 4\,r_1r_2 = r_1{}^2 + r_2{}^2$. Circle passes through (a, b)

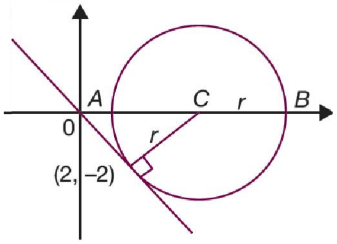
$\Rightarrow$ $a^2 + b^2 - 2ra - 2rb + r_2 = 0$,

i.e., $r^2 - 2r(a + (B) + a^2 + b^2 = 0$

$r_1 + r_2 = 2(a, b)$ and $r_1\,r_2 = a^2 + b^2$

Hence, the correct answer is (C).

23.



Equation of $O \Rightarrow (x-2)^2 + (y+2)^2 + k(x+y) = 0$

Use y coordinate of circle $= 0 \Rightarrow k = -4$

Centre of circle $\Rightarrow (4, 0)$

Radius $= 2\sqrt{2}$

Maximum value of a

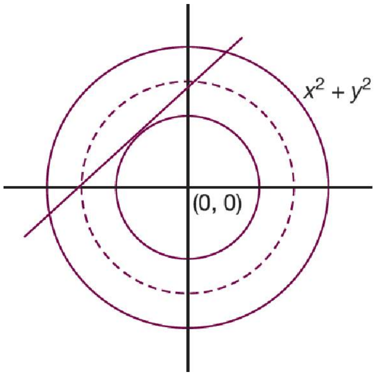
$\Rightarrow$ Co-ordinate of B

$\Rightarrow$ $OC + CB$ (CB = radius)

$\Rightarrow$ $4 + 2\sqrt{2}$

Hence, the correct answer is (C).

24.



Radius of circle are r_1, r_2 and 1.

Line $y = x + 1$

Perpendicular from (0, 0) on line $y = x + 1 = \frac{1}{\sqrt{2}}$

Now, $r_1 > \frac{1}{\sqrt{2}} \Rightarrow r_1 = 1 - 2d \Rightarrow \frac{1-r_1}{2} = d$

$\therefore d = \frac{\sqrt{2}-1}{2\sqrt{2}}$

Alternate:

Equations of circle are

$x^2 + y^2 = 1$

$x^2 + y^2 = (1 - d)$

$x^2 + y^2 = (1 - 2d^2)$

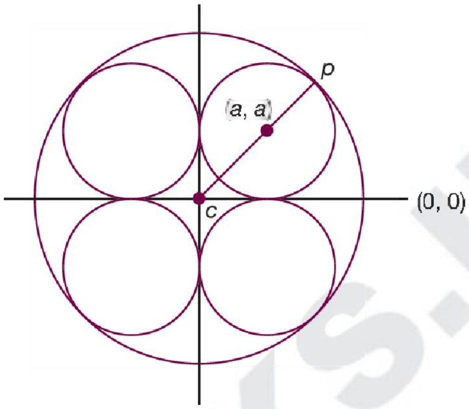
$\Rightarrow$ Solve any of circle with line $y = x + 1$,

i.e., $x^2 + y^2 = (1 - (D)^2 \Rightarrow 2x^2 + 2x + 2d - d^2 = 0$ cuts are real and distinct point. Hence, $\Delta > 0$.

$\Rightarrow$ $2d^2 - 4d + 1 > 0 \Rightarrow d = \frac{2 \pm \sqrt{2}}{4}$

Hence, the correct answer is (C).

25.

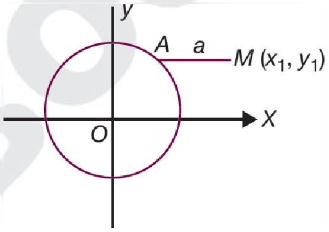


$r = cp = cc_1 + c_1p$

$= \sqrt{2}a + a = a\left(\sqrt{2}+1\right)$

Hence, the correct answer is (C).

26.



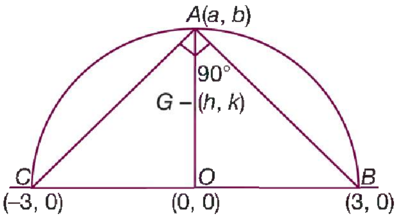
$A\left(x_1 - a, y_1\right)$ lies on $x^2 + y^2 = a^2$

$\Rightarrow$ $\left(x_1, -a\right)^2 + y_1^2 = a^2$

Locus in $x^2 + y^2 - 2ax = 0$

Hence, the correct answer is (B).

27.



Let $A(a, b)$ and $G(h, k)$.

Now, A, G and O are collinear.

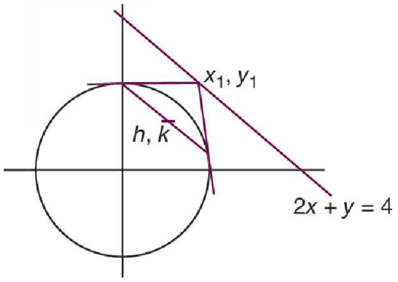
$\Rightarrow$ $h = \frac{2.0 + a}{3} \Rightarrow a = 3h$

and similarly $b = 3k$

Now, (a, b) lies on the circle $x^2 + y^2 = 9$.

Hence, the correct answer is (A).

28.



(x_1, y_1) lies on $2x + y = 4$

$\Rightarrow \quad 2x_1 + y_1 = 4 \qquad \ldots(1)$

Chord of contact w.r.t. (x_1, y_1)

$xx_1 + yy_1 = 1$

Also, equation of chord whose midpoint is (h, k) is $h^2 + k^2 = hx + ky$.

$$\therefore \quad \frac{x_1}{h} = \frac{y_1}{k} = \frac{1}{h^2+k^2} \Rightarrow x_1 = \frac{h}{h^2+k^2} \text{ and } y_1 = \frac{k}{h^2+k^2}$$

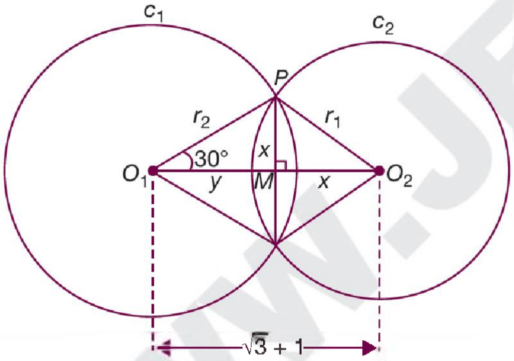
Substitute in (1),

$$2 \cdot \frac{h}{h^2+k^2} + \frac{k}{h^2+k^2} = 4$$

Locus $4(x^2 + y^2) = 2x + y$

Hence, the correct answer is (C).

29.



$$\frac{y}{\sin 60^\circ} = \frac{x}{\sin 30^\circ} \Rightarrow y = \sqrt{3}\,x$$

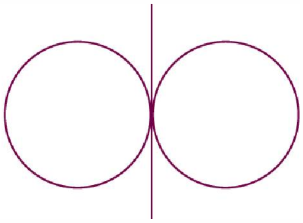
and $x\left(\sqrt{3}+1\right) = \sqrt{3} + 1$

$\Rightarrow \quad x = 1$

$\Rightarrow \quad r_1 = x\sqrt{2} = \sqrt{2}$ and $r_2 = 2$

Hence, the correct answer is (C).

30.



$C_1C_2 = r_1 + r_2$

$C_1 = (0, 0)$, $C_2 = (3\sqrt{3}, 3)$ and $r_1 = 2$, $r_2 = 4$

$\Rightarrow$ Circles touch each other

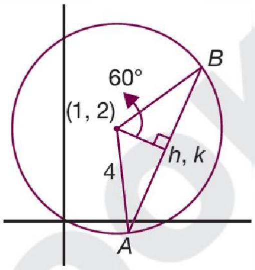
Transverse common tangent $= \sqrt{3}\,x + y - 4 = 0$

Comparing with $x\cos\theta + y\sin\theta = 2$

$$\theta = \frac{\pi}{6}$$

Hence, the correct answer is (D).

31.



(h, k) are midpoints of chord.

$$\frac{\sqrt{(1-h)^2 + (2-k)^2}}{4} = \cos 30^\circ$$

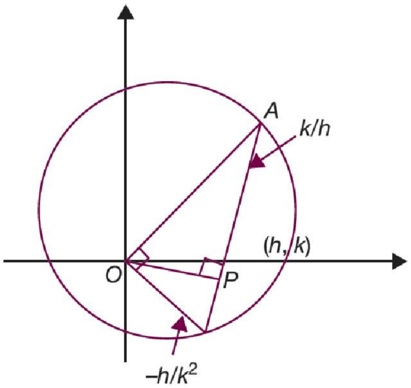
$$\Rightarrow \quad (1-h)^2 + (2-k)^2 = \left(\frac{\sqrt{3}}{2} \times 4\right)^2$$

$\Rightarrow \quad h^2 + 1 - 2h + k^2 + 4 - 4k = 12$

Hence, locus is $x^2 + y^2 - 2x - 4y - 7 = 0$.

Hence, the correct answer is (C).

32.



Let the point be $P\,(h, k)$.

Slope of perpendicular OP is $\frac{k}{h}$

Slope of chord AB is $-\left(\frac{h}{k}\right)$

Equation of line is $(y - k) = \frac{h}{k}(x - h)$

$ky - k^2 + hx - h^2 = 0$

$$\frac{ky+hx}{h^2+k^2}=1 \qquad \ldots(1)$$

Homogenise the equation of circle by (1),

$$x^2+y^2-4x\left(\frac{ky+hx}{h^2+k^2}\right)-6y$$

$$\left(\frac{ky+hx}{h^2+k^2}\right)-3\left(\frac{ky+hx}{h^2+k^2}\right)=0$$

$\therefore$ It subtends a right angle at the origin.

$\therefore$ Coefficient of x^2 + coefficient of $y^2=0$

$$\left(1-\frac{4h}{h^2+k^2}-\frac{3h^2}{(h^2-k^2)^2}\right)$$

$$+\left(1-\frac{6k}{h^2+k^2}-\frac{3k^2}{(h^2+k^2)^2}\right)=0$$

$$\Rightarrow \quad 2-\frac{2(2h+3k)}{h^2+k^2}-\frac{3}{(h^2+k^2)}=0;$$

$$\Rightarrow \quad 2(h^2+k^2)-4h-6k-3=0$$

So, locus is $2x^2+2y^2-4x-6y-3=0$.

Hence, the correct answer is (D).

33. Equation of line $y-1=m(x-7)$

$$mx-y+1-7m=0$$

Perpendicular distance from $(0, 0)=\frac{r}{2}$

$$p=\frac{r}{2} \text{ or } (7m-1)^2=25(1+m^2)$$

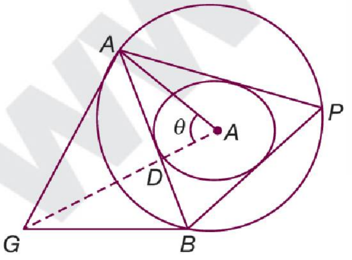
$$49m^2-14m+1=25+25m^2$$

$$24m^2-14m-24=0$$

$$\Rightarrow \quad m_1m_2=-1$$

Hence, the correct answer is (A).

34.



Let the equation of the line through (1, 2) be $y-2=m(x-1)$.

If p denotes the length of the perpendicular from (3, 1) on this line, then

$$p=\left|\frac{2m+1}{\sqrt{m^2+1}}\right|$$

$$\Rightarrow \quad p^2=\frac{4m^2+4m+1}{m^2+1}=4+\frac{4m-3}{m^2+1}=s(\text{say})$$

$$s=4+\frac{4m-3}{m^2+1}$$

$$\frac{ds}{dm}=\frac{-2(2m+1)(m-2)}{(m^2+1)^2}$$

$$\frac{ds}{dm}=0 \Rightarrow m=\frac{-1}{2},2$$

Also, $\frac{ds}{dm}<0$ if $m<\frac{1}{2}$

$$>0, \text{ if } \frac{-1}{2}<m<\frac{1}{2}$$

$$<0 \; m>2$$

So, s is greatest for $m=2$.

Thus, the equation of required line is $y=2x$.

Hence, the correct answer is (A).

35. We have, $OD=1$, $OA=2$

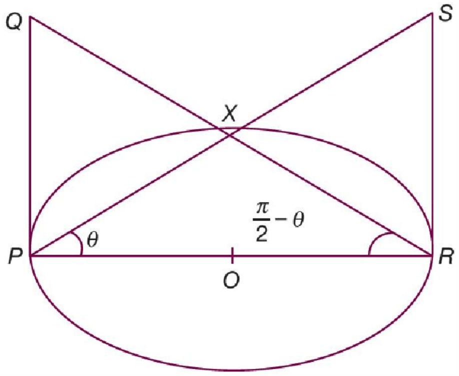
If $\angle AOD=\theta$, then $\cos\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$

$$\Rightarrow \quad OA\cdot\sec\frac{\pi}{3}=2.0\text{A}=4$$

$\therefore$ The locus of q is $x^2+y^2=16$.

Hence, the correct answer is (B).

36.



In ΔPQR, $\tan\left(\frac{\pi}{2}-\theta\right)=\frac{PQ}{PR}$

$$\Rightarrow \quad \cot\theta=\frac{PQ}{PR}$$

In ΔPSR, $\tan\theta=\frac{RS}{PR}$

$$\cot\theta\cdot\tan\theta=\left(\frac{PQ}{PR}\right)\left(\frac{RS}{PR}\right)$$

$$1=\left(\frac{PQ}{PR}\right)\left(\frac{RS}{PR}\right)\Rightarrow\left(PR^2\right)=(PQ)(RS)$$

$$\Rightarrow \quad 2l=\sqrt{PQ\cdot RS}$$

Hence, the correct answer is (A).

37. The radii are 5 units each. Let S be the area common to the circles

$$\frac{25\pi}{4}+\frac{25\pi}{4}-S=25$$

Hence, the correct answer is (B).

38. $AP=\sqrt{(1-\cos\alpha)^2+\sin^2\alpha}=2\left|\sin\frac{\alpha}{2}\right|=2\sin\frac{\alpha}{2}$

$$\left(\because \sin\frac{\alpha}{2}>0\right)$$

Similarly, $AQ=2\sin\frac{\beta}{2}$ and $AR=2\sin\frac{\gamma}{2}$

Now AP, AQ and AR are in G.P.

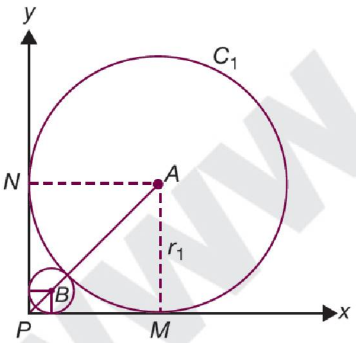
$\Rightarrow \quad \sin\frac{\alpha}{2}$, $\sin\frac{\beta}{2}$ and $\sin\frac{\gamma}{2}$ are in G.P.

$$\therefore \quad \frac{\sin\frac{\alpha}{2}+\sin\frac{\gamma}{2}}{2}\geq\sin\frac{\beta}{2}\Rightarrow\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{4}\geq\sin\frac{\beta}{2}$$

Also, $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\leq\sin\frac{\beta}{2}$.

Hence, the correct answer is (D).

39.



Let $r_1>r_2$ and A and B be the centres of circles C_1 and C_1, then $PM=PN=r_1\Rightarrow PA=r_1\sqrt{2}$.

Also, $PB=r_2\sqrt{2}$

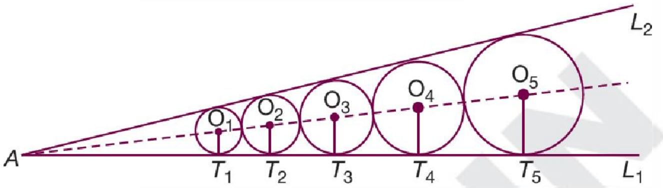
Now, $AB=r_1+r_2=PA-PB\Rightarrow r_1+r_2=(r_1-r_2)\sqrt{2}$

$$\Rightarrow \quad \frac{r_1}{r_2}=\frac{\sqrt{2}+1}{\sqrt{2}-1}=3+2\sqrt{2}\ .$$

Clearly, if $r_1<r_2$ then $\frac{r_1}{r_2}=3-2\sqrt{2}$

Hence, the correct answer is (D).

40.



If the lines L_1 and L_2 meet at A and the centres of the circles be O_1, O_2, O_3, O_4 and O_5 respectively, then ΔAO_1T_1, ΔAO_2T_2, ΔAO_3T_3, ΔAO_4T_4 and ΔAO_5T_5 are similar.

$\therefore \quad \frac{r_2}{r_1}=\frac{r_3}{r_2}=\frac{r_4}{r_3}=\frac{r_5}{r_4}\Rightarrow r_1, r_2, r_3, r_4, r_5$ are in G.P.

$$\Rightarrow \quad r_1r_5=r_2r_3=r_3^2\Rightarrow r_3^2=2\times32\Rightarrow r_3=8$$

Hence, the correct answer is (A).

41. Let x be the distance between the centre and C_1 and y be the distance between C_1 and C_2.

So, $r^2=x^2+10^2, r^2=8^2+(x+y)^2$ and $r^2=4^2+(x+2y)^2$

On solving these equations we get,

$$y=\sqrt{6}, x=\frac{15}{\sqrt{6}} \text{ and } r=5\frac{\sqrt{22}}{2}$$

Hence, the correct answer is (B).

42. Clearly, tangent $xx_1+yy_1-\alpha(x+x_1)=0$ at $Q(x_1,y_1)$

On the circle $(x-\alpha)^2+y^2-\alpha^2=0$ is also a tangent to $x^2+y^2-1=0$

$$\Rightarrow \quad \frac{|\alpha x_1|}{\sqrt{(x_1-\alpha)^2+y_1^2}}=1\Rightarrow|x_1|=1$$

Hence, the correct answer is (A).

43. Let $AO=3x$, $BO=2x$

$OC=\sqrt{13}$

$OF=5+\sqrt{13}$

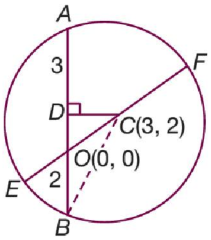
$OE=5-\sqrt{13}$

Now, $AO\cdot BO=OE\cdot OF\Rightarrow3x\cdot2x=12$

$x^2=2$

$x^2=\sqrt{2}$

Now, D is the midpoint of the chord AB.



$$AD = DB = \frac{5}{\sqrt{2}}$$

Hence, $CD = \frac{5}{\sqrt{2}}$

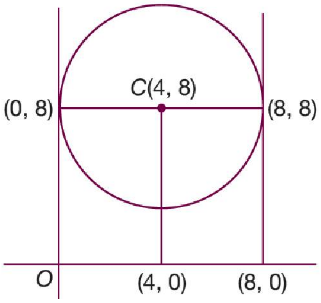
Equation of AB is $y = mx$

$$\Rightarrow \frac{|3m-2|}{\sqrt{1+m^2}} = \frac{5}{\sqrt{2}} \Rightarrow m = -1, -\frac{17}{7}.$$

Equation of AB is $y = -\frac{17}{7}x$, $y = -x$

Hence, the correct answer is (A).

44.



From figure, it is clear that the point is (8, 8).

Hence, the correct answer is (A).

45. $AB \cdot BX = (AX)^2$

$(y + x) \cdot x = y^2$

$$\cos\theta = \frac{2(x+y)^2 - y^2}{2(x+y)^2}$$

$$\cos\theta = \frac{2y^2 - x^2}{2y^2} = \frac{(y-x)(y+x)+y^2}{2y^2}$$

$$= \frac{(y-x)\frac{y^2}{x^2}+y^2}{2y^2} = \frac{y}{2x}$$

$$y^2 - xy - x^2 = 0$$

$$y = \frac{x \pm \sqrt{5}x}{2}$$

$$\cos\theta = \frac{\sqrt{5}+1}{4} \Rightarrow \theta = \frac{\pi}{5}$$

Hence, the correct answer is (C).

46. Let the perpendicular lines be co-ordinate axis and let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

$$10 = 2\sqrt{g^2 - c}, 6 = 2\sqrt{f^2 - c}$$

Squaring and subtracting, $g^2 - f^2 = \frac{1}{4}(100 - 36) = 16$

$\therefore$ Locus is $x^2 - y^2 = 16$ (Hyperbola)

Hence, the correct answer is (D).

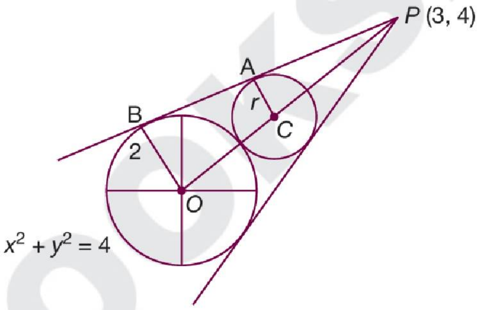
47. $\left|x-1-x^2\right| \le \left|x^2-3x+4\right|$

On solving, $x \le \frac{3}{2}$

$\therefore$ Number of integral point is 20.

Hence, the correct answer is (A).

48.

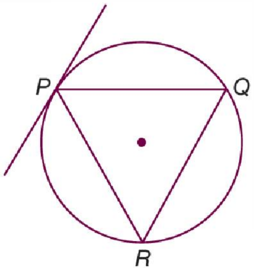


ΔPAC and ΔPBO are similar triangles.

$$\frac{3-r}{5} = \frac{r}{2} \Rightarrow r = \frac{6}{7}$$

Hence, the correct answer is (D).

49.

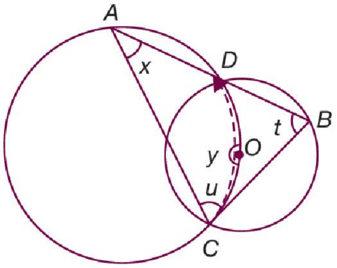


Maximum area is possible when the ΔPQR is an equilateral triangle of side $\sqrt{3}r$

$$\text{Area} = \frac{\sqrt{3}}{4}\left(3r^2\right) \text{sq. units.}$$

Hence, the correct answer is (B).

50.



Clearly, $S_1 = 0$ passes through the centre of $S_2 = 0$.

$x + y = \pi$ as $ADOC$ are concylic

$-u + x + t = \pi \qquad u = \pi - (x + t)$

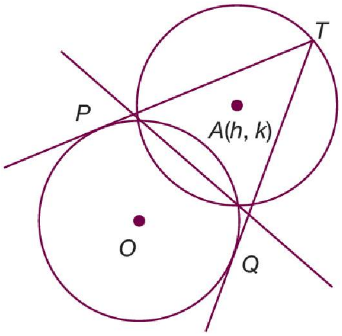
$\because \quad y = 2t$

$\Rightarrow \quad u = \pi - x - t = y - t = t \Rightarrow u = t$

$\Rightarrow \quad AC = AB = \pi$

Hence, the correct answer is (C).

51.



Let $T(x_1, y_1)$ be any point on line $px + qy = r$.

$px_1 + qy_1 = r$

Equation PQ, chord of contact of T of the circle

$x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$.

Equation of circle passing through P and Q is

$$\left(x^2 + y^2 - a^2\right) + \lambda\left(xx_1 + yy_1 - a^2\right) = 0,$$

which passes through (x_1, y_1) so $\lambda = -1$.

Equation of the circle ΔTPQ

$\Rightarrow \quad x^2 + y^2 - x_1x - y_1y = 0$

Let (h, k) be the centre of angle $h = \frac{x_1}{2}, k = \frac{y_1}{2}$

$x_1 = 2h,\ y_1 = 2k$

$2ph + 2qk = r$

$\therefore \quad$ Locus $= 2px + 2qy = r$

Hence, the correct answer is (B).

52. Common solutions of $a^2 + 10a + 16 > 0$ and $a^2 - 12a + 36 > 0$.

Hence, the correct answer is (A).

53. Let r and R be radius of required and given circle respectively and let centre is (h, k).

By given condition, $\sqrt{(h-1)^2 + (k-2)^2} = R - r$

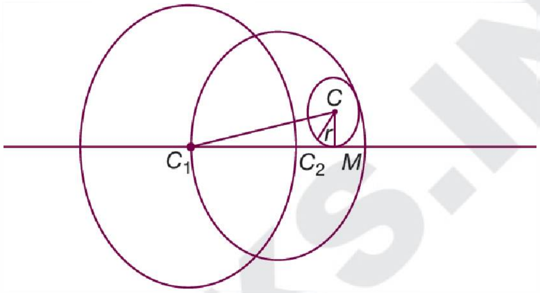
$$\frac{r}{AB} = \tan 30^\circ$$

$$r = \frac{R-r}{\sqrt{3}} \Rightarrow AB = R - r \quad \Rightarrow \quad R = 1 + \sqrt{3}$$

$$\sqrt{(h-1)^2 + (k-2)^2} = \sqrt{3} \Rightarrow (x-1)^2 + (y-2)^2 = 3$$

Hence, the correct answer is (C).

54.



Let the centres of the first two circles are at C_1 and C_2 and that of the 3rd circle is at C.

Let M be the foot of the perpendicular from C to C_1C_2.

Let radius of $S = 0$ be x and that of $S_1 = 0$ and $S_2 = 0$ be r

$$CM = x, CC_2 = r - x \Rightarrow C_2M = \sqrt{(r-x)^2 - x^2}$$

$$C_1C_2 = r, CC_1 = r + x, MC_1 = r + \sqrt{(r-x)^2 - x^2}$$

$\Rightarrow \quad \text{In } \Delta^{le}\ CMC_1, CC_1^2 = CM^2 + MC_1^2$

$$(r+x)^2 = x^2 + \left(r + \sqrt{(r-x)^2 - x}\right)^2$$

$$r^2 + 2rx = r^2 + (r-x)^2 - x^2 + 2r\sqrt{(r-x)^2 - x^2}$$

$$4rx - r^2 = 2r\sqrt{(r-x)^2 - x^2} \text{ (Squaring on both sides)}$$

$\Rightarrow \quad 16x^2 = 3r^2 \Rightarrow x = \frac{\sqrt{3}r}{4}$

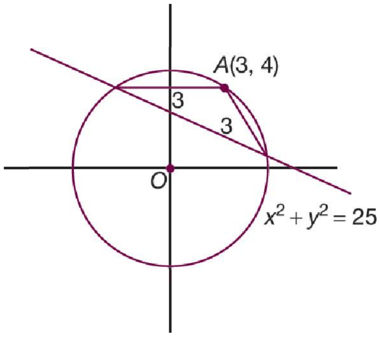
Hence, the correct answer is (B).

55. Clearly, point will lie on the director circle.

$\therefore$ Maximum value is $\sqrt{2}$.

Hence, the correct answer is (D).

56.



Equation of circle, whose centre is at (3, 4) and radius equal to 3 is

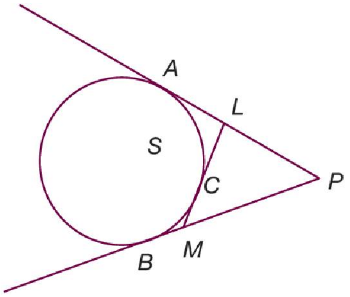
$$(x-3)^2 + (y-4)^2 = 9 \text{ and } x^2 + y^2 = 25$$

Now, the equation of straight line is the common chord of the two circles.

$\Rightarrow \quad 6x + 8y = 41$

Hence, the correct answer is (A).

57.



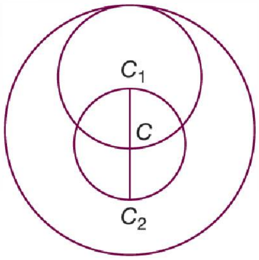
Perimeter of $\Delta PLM = PL + LC + CM + MP$

$= PL + LA + BM + MP$

$= PA + PB$

Hence, the correct answer is (B).

58.



C_1C_2 = Diameter of the given circle

$= 2\sqrt{1+4+4} = 2\times 3 = 6$

Hence, the correct answer is (C).

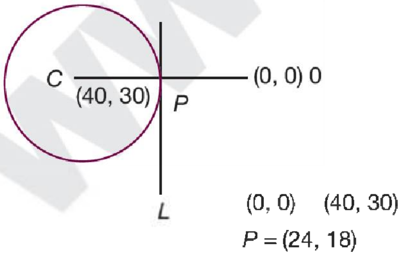
59. $P = (x, y) = (7 + 8\cos\theta, 3 + 8\sin\theta)$

$3x + 4y = 33 + 24\cos\theta + 32\sin\theta$

$\Rightarrow \quad \text{Max} = C + \sqrt{a^2 + b^2} = 73$

Hence, the correct answer is (B).

60.



Slope of $CO = \frac{3}{4} \Rightarrow$ Slope of tangent $= \frac{-4}{3}$

Point of contact = (24, 18)

$$y - 18 = \frac{-4}{3}(x - 24)$$

Hence, the correct answer is (A).

61.

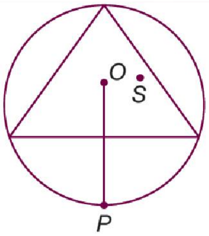
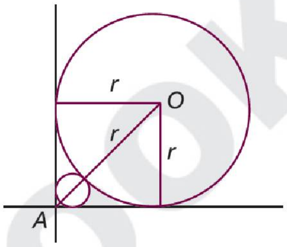


Image of (1, 2) w.r.t $2x - y - 3 = 0$ lies on circumcenter

$\Rightarrow sp$ = circumradius

Hence, the correct answer is (B).

62.

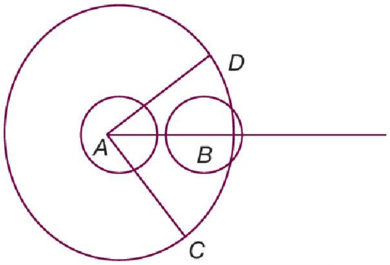


$OA = \sqrt{2} + 1 + r$

$$\therefore \quad \frac{r}{\sqrt{2}+1+r} = \sin 45^\circ \Rightarrow r = \frac{\sqrt{2}+1}{\sqrt{2}-1} = 3 + 2\sqrt{2}$$

Hence, the correct answer is (B).

63. $x^2 + y^2 - 4x - 3y = 0 \qquad \ldots(1)$



Given pair of lines is $x + 3 = 0$ and $x + 2y = 0$.

$x + 3 = 0$ and $x + 2y = 0$ are two normals to the required circle. By solving them, the centre of the circle is $A\left(-3, \frac{3}{2}\right)$.

Centre of circle (1) is $B\left(2, \frac{3}{2}\right)$ and radius is $\frac{5}{2}$.

$AB = 5 > r_1$

$\therefore$ A lies outside (1).

Radius of required circle is $AB + r_1 = \frac{15}{2}$

$$(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$$

Hence, the correct answer is (B).

64. When d is maximum, A and B coincide to each other and coincide with Q (5, 3) the other end of the diameter through $P(1, 1)$.

So secant $\overline{AB}$ coincides with tangent at $Q(5, 3)$.

Its equation is $5x + 3y - 3(x + 5) - 2(y + 3) + 8 = 0$, i.e., $2x + y - 13 = 0$.

Hence, the correct answer is (C).

65. Angle bisectors $x \pm y = 0$ and $r = 4\sqrt{2}, C_1 = (4,4)$

$$C_2 = \text{centre} = (\alpha,\alpha),\ \ \text{radius} = R \text{ and}$$

$$R.R = \left|\frac{17\alpha^2 - 18\alpha^2 + 7\alpha^2}{\sqrt{(18)^2}}\right| = \frac{4\alpha^2}{18} \Rightarrow R = \frac{\sqrt{2}\alpha}{3}$$

$$C_1C_2 = |r - R| \Rightarrow \alpha = 6, R = \frac{\sqrt{2}}{3} \times 6 = 2\sqrt{2}$$

$\therefore$ Equation of the circle is $x^2 + y^2 - 12x - 12y + 64 = 0$.

Hence, the correct answer is (B).

66. $(x-2)^2 + y^2 = 4$

Centre is (2, 0) and radius is 2.

Distance between (2, 0) and (5, 6) is $\sqrt{9+36} = 3\sqrt{5}$

$$\therefore \quad r_1r_2 = \frac{3\sqrt{5}-2}{2}, \frac{3\sqrt{5}+2}{2} = \frac{41}{4}$$

Hence, the correct answer is (B).

67. $AQ = 3 + 2\sqrt{2}$

$PQ = 3\sqrt{2} + 4$

Let 'r' be required radius $3\sqrt{2} + 4 = 3 + 2\sqrt{2} + r + \sqrt{2}$

$$\sqrt{2} + 1 = r\left(1 + \sqrt{2}\right) \Rightarrow r = 1$$

Hence, the correct answer is (D).

68. Let C be the centre of the circle $S = 0$.

Then circumcircle of the ΔRPQ passes through C.

$\therefore$ (2, 3) is the midpoint of RC.

$\therefore$ Co-ordinates of C are (– 1, – 2).

$\therefore$ Equation of the circle is $x^2 + y^2 + 2x + 4y - 20 = 0$.

Hence, the correct answer is (A).

69. Radical centre of the circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore \quad OD = 2$

$$HD = -BD \tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

$\therefore$ Co-ordinates of H are $\left(2, -\frac{1}{\sqrt{3}}\right)$

Hence, the correct answer is (C).

70. $(\alpha, 3-\alpha)$ be any point on $x + y = 3$

$\therefore$ Equation of chord of contact is $\alpha x + (3-\alpha)y = 9$

i.e., $\alpha(x-y) + 3y - 9 = 0$

$\therefore$ The chord passes through the point (3, 3) for all values of α.

Hence, the correct answer is (B).

71. Slope of $OQ = \frac{4}{3}$

Slope of $OR = -\frac{3}{4}$

$\therefore \quad \angle ROQ = 90^\circ \qquad \therefore \angle QPR = \frac{\pi}{4}$

Hence, the correct answer is (C).

72. Let, $y = mx$ be a chord.

The points of intersection are given by $x^2(1+m^2) - x(3+4m) - 4 = 0$.

$$\therefore \quad x_1 + x_2 = \frac{3+4m}{1+m^2} \text{ and } x_1x_2 = \frac{-4}{1+m^2}$$

($\because$ (0, 0) divides the point of (x_1, y_1) and (x_2, y_2) in the ratio 1:4)

$\therefore \quad x_2 = -4x_1$

$$\text{Then, } -3x_1 = \frac{3+4m}{1+m^2} \text{ and } -4x_1^2 = -\frac{-4}{1+m^2}$$

$\therefore \quad 9 + 9m^2 = 9 + 16m^2 + 24m$

i.e., $m = 0, -\frac{24}{7}$

$\therefore$ The lines are $y = 0$ and $7y + 24x = 0$.

Hence, the correct answer is (B).

73. $(x-1)^2 + (y+2)^2 = 16$

$(x-1)^2 + (y+2)^2 = 32$

$\Rightarrow \quad OS = 4\sqrt{2}$

$\therefore$ Required distance $TS = OT - SO = 12 - 4\sqrt{2}$

Hence, the correct answer is (D).

74. $\because \quad x^2 + y^2 - 2x - 2y - 6 = 0$

$\because \quad \text{Radius } = \sqrt{8}$

$\because \quad PT = \sqrt{2a^2 - 4a - 6}$

$\because \quad \tan\theta = \dfrac{2\sqrt{2}}{\sqrt{2}\sqrt{a^2 - 2a - 3}}$

$\therefore \quad \dfrac{\pi}{3} < 2\theta < \pi \Rightarrow \dfrac{\pi}{6} < \theta < \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} < \tan\theta < \infty \Rightarrow \dfrac{1}{\sqrt{3}} < \dfrac{2}{\sqrt{a^2 - 2a - 3}} < \infty$

$\Rightarrow \quad \dfrac{1}{3} < \dfrac{4}{a^2 - 2a - 3} < \infty$

$\Rightarrow \quad \dfrac{4}{a^2 - 2a - 3} - \dfrac{1}{3} > 0$

$\Rightarrow \quad \dfrac{(a-5)(a+3)}{(a-3)(a+1)} \Rightarrow a \in (-3,-1) \cup (3,5)$

Hence, the correct answer is (D).

75. $\because$ Reflection of (5, 8) in BC will lie on circumcircle.

$\because$ (8, 5) will lie on circumcircle

$\because$ Equation of circumcircle is

$$(x-2)^2 + (y-3)^2 = (8-2)^2 + (3-5)^2.$$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y - 27 = 0$

Hence, the correct answer is (B).

76. $\because$ Centre lies on line $y = x$

$\therefore$ Let centre (h, n)

$\therefore \quad \dfrac{\left|h - h\sqrt{3}\right|}{2} = 1$

$\Rightarrow \quad h = \left(\sqrt{3} + 1\right)$

$\therefore$ Equation of required circle is

$$x^2 + y^2 - 2x\left(\sqrt{3}+1\right) - 2y\left(\sqrt{3}+1\right) + 7 + 4\sqrt{3} = 0.$$

Hence, the correct answer is (C).

77. $\underset{x\to 0}{Lt}\left(\dfrac{1+Cx}{1-Cx}\right)^{\frac{1}{x}} = 4 \Rightarrow e^{\underset{x\to 0}{Lt}\frac{2Cx}{1-Cx}\frac{1}{x}} = 4 \Rightarrow e^{2C} = 4$

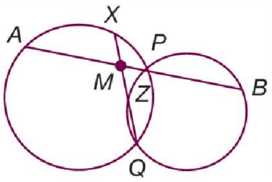
$x_1 = e^{4C} = \left(e^{2C}\right)^2 = 16, r = e^{C} = \sqrt{4} = 2$

Centres of the circles are $(16, \pm 2)$

$\therefore \quad x^2 + y^2 - 32x \pm 4y + 256 = 0$

Hence, the correct answer is (C).

78.



$MA \cdot MP = MX \cdot MQ$ ...(1)

$MP \cdot MB = MZ \cdot MQ$ ...(2)

$MX = MZ$

$\therefore \quad M$ divides XZ in the ratio 1:1.

Hence, the correct answer is (A).

79. $\displaystyle\sum_{n=0}^{\infty} \text{An} = \sum_{n=0}^{\infty} \pi\left[\left(\frac{4}{9}\right)^{2n} - \left(\frac{4}{9}\right)^{2n+1}\right]$

$$= \pi\left[\left(1 - \frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 - \left(\frac{4}{9}\right)^3 + \ldots\ldots\ldots\infty\right]$$

$$= \frac{9\pi}{13}$$

Hence, the correct answer is (C).

80. Chords are of lengths: $l = 1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1$

(no. of chords = 15)

$L.C = 2\sqrt{r^2 - d^2}$

$\therefore \quad 4(\Sigma r^2 - \Sigma d^2) = 2(1^2 + 2^2 + \ldots.. + 7^2) + 8^2$

$\Rightarrow \quad 4(\Sigma r^2 - \Sigma d^2) = \dfrac{2.(7)(8)(15)}{6} + 8^2$

$\Rightarrow \quad \Sigma d^2 = \Sigma r^2 - \dfrac{344}{4}$

$\Rightarrow \quad \Sigma d^2 = 15(16) - 86$

$\Rightarrow \quad \Sigma d^2 = 154$

Hence, the correct answer is (A).

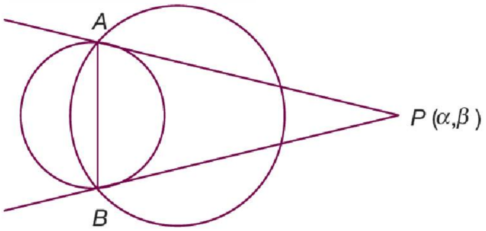
81. Equation of line AE is $x + y = 3$. Point C lies on AE such that the midpoint of AC lies on BD. Let C is (α, β).

$\Rightarrow \quad \alpha + \beta = 3$ and $4\left(\dfrac{\alpha+1}{2}\right) + 8\left(\dfrac{\beta+2}{2}\right) - 7 = 0$

$\Rightarrow \quad \alpha = \dfrac{15}{2}, \beta = -\dfrac{9}{2}$

Hence, the correct answer is (A).

82.



Given circles are S_1: $x^2 + y^2 - 12 = 0$ ...(1)

S_2: $x^2 + y^2 - 5x + 3y - 2 = 0$ ...(2)

Now, equation of common chord of circles (1) and (2) is $S_1 - S_2 = 0$

i.e., $5x - 3y - 10 = 0$ ...(3)

Let this line meet circle (1) (or (2)) at A and B

Let the tangents to circle (1) at A and B meet at $P(a, (B)$, then AB will be the chord of contact of the tangents to the circle (1) from P, therefore equation of AB will be

$$x\alpha + y\beta - 12 = 0 \quad \text{...(4)}$$

Now lines (3) and (4) are same.

$\therefore$ Equations (3) and (4) are identical.

$$\Rightarrow \frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10} \Rightarrow \alpha = 6, \beta = -\frac{18}{5}$$

Hence, $P = \left(6, -\frac{18}{5}\right)$

Hence, the correct answer is (B).

83. Required line is the common chord of the circles (1) and (2).

$\therefore$ $16x - 64 + 12y - 36 - 100 + 16 = 0$

$\Rightarrow$ $16x + 12y - 184 = 0$

$\Rightarrow$ $4x + 3y - 46 = 0$

Hence, the correct answer is (C).

84. $\therefore$ Equation of any tangent of $x^2 + y^2 - 4 = 0$ is

$$y = mx \pm 2\sqrt{1+m^2}$$

($\because$ It passes through $(-2, -4)$)

$$(2m-4)^2 = 4\left(1+m^2\right)$$

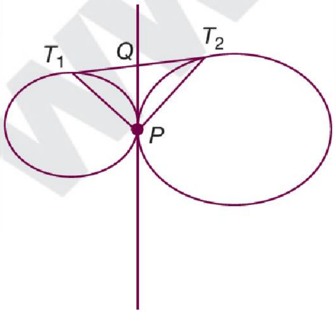
$$4m^2 - 16m + 16 = 4 + 4m^2$$

$\Rightarrow$ One tangent is vertical and slope of other tangent is $\frac{3}{4}$

$\Rightarrow$ $3x + 4y + 22 = 0$

Hence, the correct answer is (A).

85.



A common chord of two circles divides the direct common tangent into two equal parts.

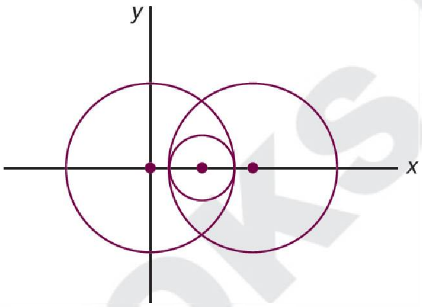
Length of direct common tangent $= 2\sqrt{r_1 r_2}$

$$PQ = QT_2 = \frac{1}{2}T_1T_2$$

$$= \frac{1}{2}.2\sqrt{4.9} = 6$$

Hence, the correct answer is (B).

86.



For the point P to enclose greatest area, the arbitrarily chosen point should be $\left(\frac{3}{2}, 0\right)$ and P should move in a circle of radius $\frac{1}{2}$. Locus of P is a circle of radius $\frac{1}{2}$.

$$\left(x - \frac{3}{2}\right)^2 + (y-0)^2 = \frac{1}{4} \Rightarrow x^2 + y^2 - 3x + 2 = 0$$

Hence, the correct answer is (D).

87. For $x^2 + y^2 = 9$, centre is $(0, 0)$ and the radius is $r_1 = 3$

For $x^2 + y^2 - 8x - 6y + n^2 = 0$, centre is $(4, 3)$ and radius is $r_2 = \sqrt{25 - n^2}$

$\Rightarrow$ $25 - n^2 > 0$

$\Rightarrow$ $-5 < n < 5$

Circles should cut, to have exactly two common tangents.

$\Rightarrow$ $r_1 + r_2 >$ Distance between centres

$\Rightarrow$ $3 + \sqrt{25 - n^2} > \sqrt{4^2 + 3^2}$

$\Rightarrow$ $-\sqrt{21} < n < +\sqrt{21}$

$\therefore$ Common values of n should be as $n = -4, -3, \ldots, 4$.

Hence, the correct answer is (C).

88. Consider normal to the circle, perpendicular to given line as $x - 2y + 12 = 0$ by solving it with the given line $2x + y + 4 = 0$, the nearest point will be $(-4, 4)$.

Hence, the required area

$$= 2 \times \frac{1}{2} \times 1 \times \sqrt{(-4)^2 + (4)^2 - 12 \times 4 + 35} = \sqrt{19}$$

Hence, the correct answer is (C).

89. Equation of AB is $x - y + 2 = 0$. Let $H(x, y)$ be the orthocentre of ΔABC

Let D be the reflection of H w.r.t. AB

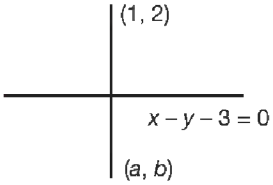
$$\therefore \quad \frac{h-x}{1}=\frac{k-y}{-1}=-2\frac{(x-y+2)}{2} \Rightarrow h=y-2,\ k=x+2$$

However, D lies on $(x-3)^2+(y+2)^2=25$

Equation to the locus of H is $(y-5)^2+(x+4)^2=25 \Rightarrow x^2+y^2+8x-10y+16=0$.

Hence, the correct answer is (B).

90.



Circle of the centre is (1, 2) and $r = 1$

Image of (1, 2) in $x - y - 3 = 0$

$$\frac{b-2}{a-1}=-1 \Rightarrow a+b=3 \qquad \ldots(1)$$

Also, $\frac{a+1}{2}, \frac{b+2}{2}$ lies on $x - y = 3$

$$(a+1)-(b+2)=6 \Rightarrow a-b=7 \qquad \ldots(2)$$

From (1) and (2), $a = 5$ and $b = -2$

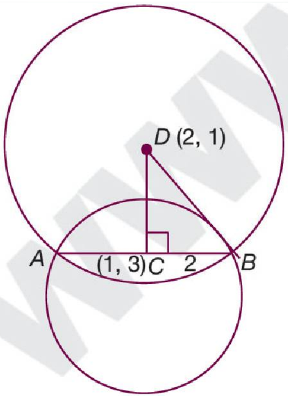
Hence, the required circle has the centre (5, – 2) and $r = 1$.

$$(x-5)^2+(y+2)^2=1$$

$$\Rightarrow \quad x^2+y^2-10x+4y+28=0$$

Hence, the correct answer is (A).

91.

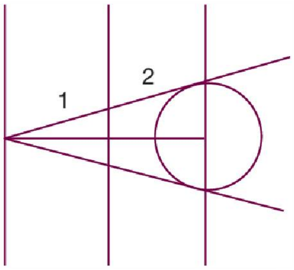


$$BD^2=CB^2+CD^2$$

$$=4+(2-1)^2+(1-3)^2=9 \qquad r=BD=3$$

Hence, the correct answer is (A).

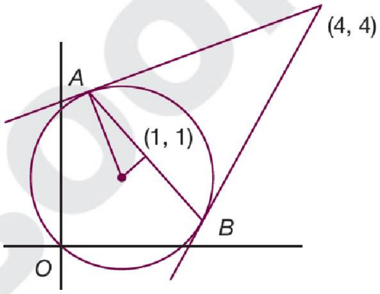
92.



Equation of AB is $3x+2y-28=0$ Equation of the line passing through P and parallel to AB is $3x+2y-25=0$. Line which divides PA and PB in the ratio 1:2 is $3x+2y-26=0$.

Hence, the correct answer is (C).

93.



Length of C.O.C. $= \dfrac{2RL}{\sqrt{R^2+L^2}}$

Equation of circle $= x^2+y^2-2x-2y-7=0$

Equation of C.O.C $= 4x+4y-(x+4)-(y+4)-7=0$
$= 3x+3y-15=0$

Radius = 3 perpendicular from (1, 1) $= \left|\dfrac{6-15}{\sqrt{18}}\right|$

$$=\frac{9}{\sqrt{18}}=\frac{9}{3\sqrt{2}}=\frac{3}{\sqrt{2}}$$

$$\therefore \quad \text{Length } \frac{AB}{2}=\sqrt{r^2-\left(\frac{3}{\sqrt{2}}\right)^2}=\frac{3}{\sqrt{2}}$$

$$\therefore \quad AB=3\sqrt{2}$$

Hence, the correct answer is (B).

94. $x-|y|-\alpha=0$

Centre (5, 0) and radius = 2

Perpendicular distance from centre to the line > Radius

$$\left|\frac{5-|0|-\alpha}{\sqrt{1+1}}\right|>2$$

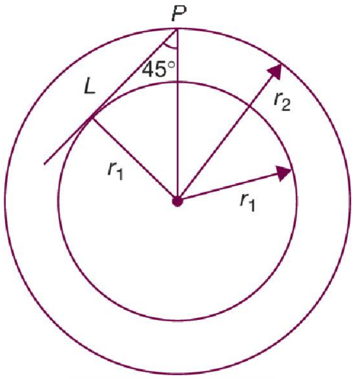
$$|5-\alpha|>2\sqrt{2}$$

$$\Rightarrow \quad 5-\alpha > 2\sqrt{2} \text{ or } 5-\alpha < -2\sqrt{2}$$

$$\alpha < 5-2\sqrt{2} \text{ or } \alpha > 5+2\sqrt{2}$$

Hence, the correct answer is (A).

95.



$$\pi r_1^2 = \pi r_2^2 - \pi r_1^2$$

$$\Rightarrow \quad 2\left(r_1^2\right) = r_2^2 \Rightarrow r_2 = \sqrt{2}\ r_1$$

Note: P lies on the director circle of radius r_1.

$\Rightarrow \quad L = r_1 = 2$ cm

Hence, the correct answer is (C).

96. $\dfrac{2r_1r_2}{\sqrt{r_1^2+r_2^2}} = \dfrac{16}{\sqrt{5}}$

Hence, the correct answer is (A).

97. Equation of tangent $2x - y + 5 = 0$ ...(1)

Point of contact = Foot of perpendicular of (–8, –6) on line (1) = (–6, –7)

Hence, the correct answer is (D).

98. $\sqrt{d^2-(r_1+r_2)^2} = 7,\ \sqrt{d^2-(r_1-r_2)^2} = 17$

$\Rightarrow \quad r_1r_2 = 18$

Hence, the correct answer is (C).

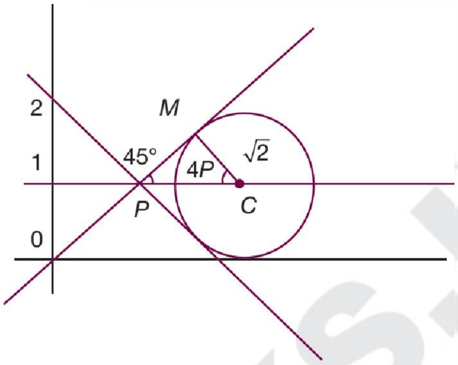
99. Equation of midpoint of chord is $xx_1 + yy_1 = x_1^2 + y_1^2$

By homogenising, $x^2 + y^2 = 16\left(\dfrac{xx_1+yy_1}{x_1^2+y_1^2}\right)^2$

Coefficient of x^2 + coefficient of $y^2 = 0 \Rightarrow x^2 + y^2 = 8$

Hence, the correct answer is (A).

100.



$$PC = \sqrt{2+2} = 2$$

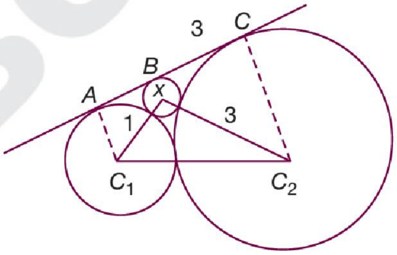
$$C = (1+2, 1) = (3, 1)$$

Required equation $(x-3)^2 + (y-1)^2 = \left(\sqrt{2}\right)^2$

$$x^2 + y^2 - 6x - 2y + 8 = 0$$

Hence, the correct answer is (C).

101.



The given circles are touching each other externally.

$$AB + BC = AC$$

$$\sqrt{4(x)} + \sqrt{4(3x)} = \sqrt{4(3)}$$

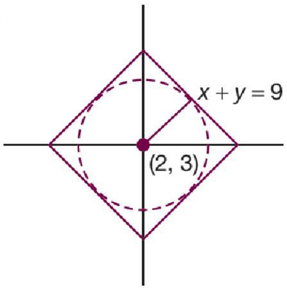
$$\sqrt{x} + \sqrt{3x} = \sqrt{3}$$

$$\sqrt{x} = \frac{\sqrt{3}}{1+\sqrt{3}}$$

$$x = \frac{3}{(1+\sqrt{3})^2} = \frac{3}{2}\left(2-\sqrt{3}\right)$$

Hence, the correct answer is (D).

102.



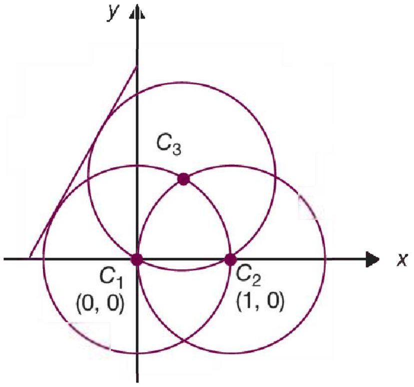
Perpendicular distance from centre to tangent = Radius

$$r = \frac{|2+3-9|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$\therefore$ Equation of circle is $(x-2)^2 + (y-3)^2 = 8$.

Hence, the correct answer is (D).

103.



Equation of any circle through (0, 0) and (1, 0) is

$$(x-0)(x-1) + (y-0)(y-0) + \lambda \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$

If it represents C_3, its radius =1

$$\Rightarrow \quad 1 = \left(\frac{1}{4}\right) + \left(\frac{\lambda^2}{4}\right) \Rightarrow \lambda = \pm\sqrt{3}$$

As the centre of C_3, lies above the x-axis,

we take $\lambda = -\sqrt{3}$ and thus an equation of C_3 is $x^2 + y^2 - x + \sqrt{3}y = 0$. $\because$ C_1 and C_2 intersect and are of unit radius, their common tangents are parallel to the joining their centres (0, 0) and (1/2, $\sqrt{3}$ /2).

So, let the equation of a common tangents be $\sqrt{3}x - y + k = 0$

It will touch C_1, if $\frac{k}{\sqrt{3+1}} = 1 \Rightarrow k = \pm 2$

From the figure, we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation to $\sqrt{3}x - y + 2 = 0$.

Hence, the correct answer is (B).

104. $c = (5, 5)$ and $(5 + 10\pi, 5)$

$(x - 5 - 10\pi)^2 + (y-5)^2 = 5^2$

Hence, the correct answer is (A).

105. $T = S_1$

i.e., $xx_1 + yy_1 - (y + y_1) = x_1^2 + y_1^2 - 2y_1$

Passes through (0, 0)

$\therefore \quad x^2 + y^2 - y = 0$

Hence, the correct answer is (A).

106. $c = \left(\frac{9}{2}, k\right)$

$$\left(x - \frac{9}{2}\right)^2 + (y-k)^2 = \frac{25}{4} + k^2$$

$\Rightarrow \quad x^2 + y^2 - 9x - 2ky + 14 = 0$

Hence, the correct answer is (C).

107. m = Distance from $A(a, o)$ to tangent = $\frac{a^2}{\sqrt{a^2+b^2}}$

n = Distance from (o, b) to tangent = $\frac{b^2}{\sqrt{a^2+b^2}}$

$$d = \sqrt{a^2+b^2} = m + n$$

Hence, the correct answer is (B).

108. $2x^2 + 6y^2 - x + y - 7xy - 1 = 0$

$x - 2y - 1 = 0 \quad \ldots(1)$

$2x - 3y + 1 = 0 \quad \ldots(2)$

Centre (– 5, – 3)

$\therefore \quad x^2 + y^2 + 10x + 6y - 19 = 0$

Hence, the correct answer is (A).

109. $\sin\theta = \frac{b}{a} \Rightarrow \theta = \sin^{-1}\frac{b}{a}$

Angle between the $2\theta = 2\sin^{-1}\frac{b}{a}$

Hence, the correct answer is (C).

110. Vertex $A(0, -\sqrt{12})$ and centroid $G\left(0, \frac{-2}{\sqrt{3}}\right)$

$$\text{Circumradius} = \sqrt{4 + \frac{4}{3}} = \frac{4}{\sqrt{3}}$$

$\therefore \quad \sqrt{3}(x^2 + y^2) + 4y - 4\sqrt{3} = 0$

Hence, the correct answer is (D).

Multiple Correct Choice Type Questions

1. Note that the triangle is right angled at (0, 6). Centre of the circle is (2, 3). Slope of the line joining the origin to the centre is $\frac{3}{2}$. Take parametric equation of a line through (2, 3) with $\tan\theta = -\frac{2}{3}$ as $\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = \pm r$, where $r = \sqrt{13}$ and get the co-ordinates on the circle.

Hence, (B) and (D) are correct.

2. $x^2 + y^2 - 4y + 3 + \lambda x = 0$

$\Rightarrow\ x = 0$ and $y^2 - 4y + 3 = 0$

$\Rightarrow\ y = 1, 3$

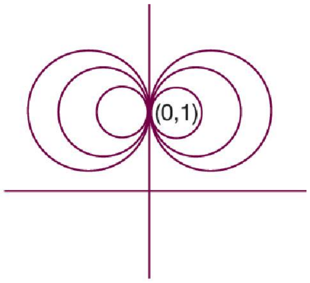
So, the common points are (0, 1) and (0, 3).

Hence, the radius of director circle is $\sqrt{2}$.

$\therefore$ Area of $S = 0$ is π sq. units.

Hence, (A), (B) and (D) are correct.

3. Given, $x^2 + y^2 + 2\lambda x - 2y + 1 = 0$



$$\Rightarrow\quad \left(x^2 + y^2 - 2y + 1\right) + 2\lambda x = 0$$

$$\Rightarrow\quad S + 2\lambda L = 0$$

Clearly, S and L touch.

$\Rightarrow$ There is only one limiting point, i.e., (0, 1).

$x = 0$ is radical axis each number touches y-axis.

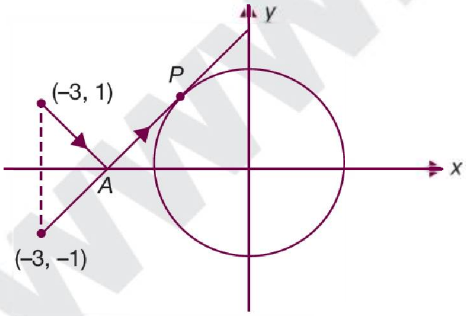
There are two circles touching x-axis.

Touch $y = x \Rightarrow \dfrac{|-\lambda - 1|}{\sqrt{2}} = |\lambda| \Rightarrow \lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda = 1 \pm \sqrt{2}$

Maximum radius $= 1 + \sqrt{2}$

Hence, (A), (B), (C) and (D) are correct.

4.



$\therefore\quad y = mx + \sqrt{1 + m^2}$

$(-3, -1)$

$\Rightarrow\quad (3m - 1)^2 = \sqrt{1 + m^2}$

$\Rightarrow\quad 8m^2 - 6m = 0$

$\Rightarrow\quad m = \dfrac{3}{4}$

Reflected ray is $3x - 4y - (-9 + 4) = 0 \Rightarrow 3x - 4y + 5 = 0$

Clearly, $A = \left(-\dfrac{5}{3}, 0\right)$

$$P = \left(\frac{-r^2 l}{n}, \frac{-r^2 m}{n}\right) = \left(\frac{-3}{5}, \frac{4}{5}\right)$$

Area of $\Delta OAP = \dfrac{1}{2}\left|\left(\dfrac{-4}{3}\right) - 0\right| = \dfrac{2}{3}$

Hence, (A), (C) and (D) are correct.

5. Interchanging x and y in one equation yields the other equation.

$\therefore$ (A) is true.

Since (A) is true, so is (B).

Since (B) is true, so is (C).

The centres are (a, b) and (b, a) and the radii are both $= \sqrt{a^2 + b^2}$

The distance between the centres $C_1C_2 = |a - b|\sqrt{2}$

The sum of the squares of the radii for orthogonal $C_1C_2^2 = r_1^2 + r_2^2$

Now, $2\left(a^2 + b^2\right) \neq 2\left(a^2 - b^2\right)$, for all a and b

$\therefore$ (D) is false.

Hence, (A), (B) and (C) are correct.

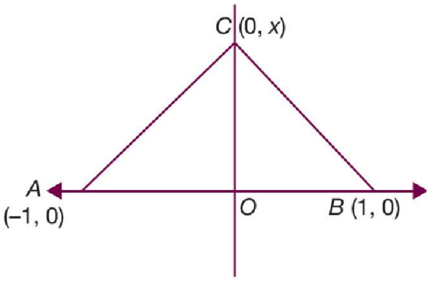
6. $x^2 + y^2 - 4\sin\theta\, x + 6\cos\theta\, y + 5\cos^2\theta + 1 = 0,\ \theta \in R$

Centre $C(2\sin\theta, -3\cos\theta)$, $r = \sqrt{3}$

$\therefore$ Locus of centre is ellipse.

Hence, (A), (B) and (D) are correct.

7.



Since, the two of the vertices $A(-1, 0)$ and $B(1, 0)$ lie on x-axis, the 3rd vertex C lies on y-axis.

Let the co-ordinate of C be $(0, x)$.

$$(CA)^2 = (AB)^2 = (BC)^2 = 4$$

$$1 + x^2 = 4 \Rightarrow x = \pm\sqrt{3}$$

$$\Rightarrow\quad OC = \pm\sqrt{3}$$

$\because$ The triangle is equilateral, the circumcentre of the triangle coincides with centroid, i.e., with $\left(O \pm \frac{1}{\sqrt{3}}\right)$.

Circumradius $= \frac{2}{\sqrt{3}}$

Possible equation of the circumcircle is

$$x^2 + \left(y \pm \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$$

$$\Rightarrow \quad x^2 + y^2 \pm \frac{2}{\sqrt{3}} y - 1 = 0$$

$$\Rightarrow \quad \sqrt{3}x^2 + \sqrt{3}y^2 \pm 2y - \sqrt{3} = 0$$

Hence, (A) and (C) are correct.

8. Radius of director circle $2\sqrt{2}$

$|x| + |y| = 4$ touches the director circle at 4 points ($r = d$).

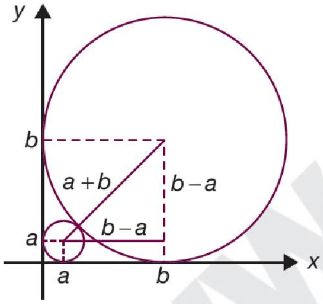
$|x| + |y| = 2\sqrt{2}$ intersects the circle, where it meets the co-ordinate axes $\rightarrow$ 4 ($r > d$ and points are repeated).

$|x| + |y| = 6$ does not intersect and touch the circle ($r < d$).

$|x| + |y| = 2$ intersects the circle at 8 distinct points ($r > d$).

Hence, (A), (C) and (D) are correct.

9.



Circle C_1 $(x - a)^2 + (y - a)^2 = a^2$

C_2 $(x - b)^2 + (y - b)^2 = b^2$

(A) $2(b - a)^2 = (a + b)^2$

$$\Rightarrow \quad \frac{b}{a} = 3 + 2\sqrt{2}$$

(B) $d^2 = r_1^2 + r_2^2 \Rightarrow 2(a - b)^2 = a^2 + b^2$

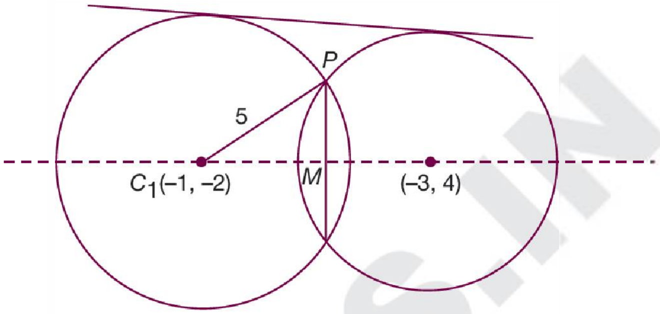
$$\Rightarrow \quad \frac{b}{a} = 2 + \sqrt{3}$$

(C) Centre of C_1 lies on common chord ($C_1 - C_2 = 0$)

(D) (a, a) satisfies the equation of C_2

Hence, (A), (B), (C) and (D) are correct.

10.



Circles are intersecting.

The number of common tangents = 2

Using $2g, g_2 + 2f_1f_2 = C_1 + C_2$

$\Rightarrow$ Circles are orthogonal.

Length of common tangent $= \sqrt{40 - (5 - \sqrt{15})^2}$

$$= 5\left(\frac{12}{5}\right)^{\frac{1}{4}}$$

Length of common chord $= 2\sqrt{25 - C_1M^2}$

Hence, (A), (B), (C) and (D) are correct.

11. $O(5, 3)$, $r = 2$

Equation of tangent at A is $x = 7$ and at B is $y = 1$.

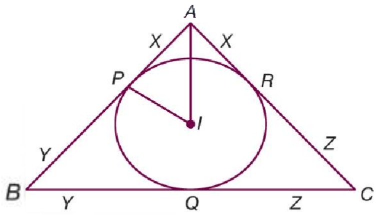
$\therefore$ $C(7, 1)$; area of $OACB = 4$

Equation of AB = radical axis

Equation of the smallest circle is $(x - 7)(x - 5) + (y - 3)(y - 1) = 0$.

Hence, (A), (C) and (D) are correct.

12.



Let, $AP = AR = x$

$BP = BQ = y$

$CQ = CR = z$

Then, $x + y + z = \frac{a + b + c}{2} = 21$

$y + z = 13, z + x = 14, x + y = 15$

$\Rightarrow \quad x = 8, y = 7, z = 6$

Also, area of the ΔABC is

$$\Delta = \sqrt{S(s - a)(s - b)(s - c)} = \sqrt{21 \times 8 \times 7 \times 6} = 84$$

The inradius is $r = \frac{\Delta}{s} = \frac{84}{21} = 4$

Area of the ΔAPR is $\dfrac{rL^3}{L^2+r^2} = \dfrac{4.(8)^3}{64+16} = \dfrac{128}{5}$ sq. units

Area of the quadrilateral $APIR = rL = 4\times 8 = 32$ sq. units

$$AI = \sqrt{AP^2+PI^2} = \sqrt{64+16} = 4\sqrt{5}$$

Hence, (A), (B), (C) and (D) are correct.

13. The given lines being parallel tangents to a circle, the diameter of the circle is equal to the distance between these lies, so that the required radius is $\dfrac{1}{2}\times\dfrac{\left|4+\dfrac{7}{2}\right|}{\sqrt{9+16}}$

$$= \frac{1}{2}\times\frac{15}{2}\times\frac{1}{5} = \frac{3}{4}$$

The centre of the circle lies on the line parallel to the given lines at a distance of $\dfrac{3}{4}$ from each of them.

So, let the equation be $3x - 4y + k = 0$ ...(1)

Then, $\left|\dfrac{k-4}{\sqrt{9+16}}\right| = \dfrac{3}{4}$

$$\Rightarrow \quad k = 4 \pm \left(\frac{15}{4}\right)$$

$$\Rightarrow \quad k = \frac{1}{4}$$

$$\Rightarrow \quad \frac{31}{4}$$

For $k = \dfrac{1}{4}$, distance of (1) from the other line is $\dfrac{3}{4}$.

Thus, the centre lies on the line $12x - 16y + 1 = 0$.

Hence, (A) and (C) are correct.

14. Let the tangents intersect at (α, β), then the line joining the two points at which tangents are drawn is:

(i) Common chord of C_1 and C_2 is $C_1 - C_2 = 0$ $\Rightarrow 2x + 4y - 18 = 0$.

(ii) Chord of contact when tangents are drawn from (α, β) to C_1.

$\Rightarrow$ Equations $2x + 4y - 18 = 0$ and $\alpha x + \beta y - 25 = 0$ must be identical on comparing,

we get $\dfrac{\alpha}{2} = \dfrac{\beta}{2} = \dfrac{25}{18}$

Hence, (A) and (C) are correct.

15. $A(\alpha) = (a\cos\alpha, a\sin\alpha)$

$B(\beta) = (a\cos\beta, a\sin\beta)$

$C(\gamma) = (a\cos\gamma, a\sin\gamma)$

ΔABC is equilateral $S = G$

$$(0,0) = \left(\frac{a(\cos\alpha+\cos\beta+\cos\gamma)}{3}\right),$$

$$b\left(\frac{\sin\alpha+\sin\gamma+\sin\beta}{3}\right)$$

$$\Rightarrow \quad \cos\alpha+\cos\beta+\cos\gamma = 0$$

$$\sin\alpha+\sin\beta+\sin\gamma = 0$$

Hence, (A) and (B) are correct.

16. Circle centre (r, r) radius $= r$

$$(x-r)^2 + (y-r)^2 = r^2$$

(2, 3) lies on it

$$(2-r)^2 + (3-r)^2 = r^2$$

$$r^2 - 10r + 13 = 0$$

Radii of S_1 and S_2 are the roots of the equation,

$$r_1 + r_2 = 10, \; r_1 r_2 = 13$$

Equation of S is $(x-2)^2 + (y-3)^2 = 13$

$$x^2 + y^2 - 4x - 6y = 0$$

$$\frac{r_1+r_2}{2} = 5$$

Also, it cuts $y = x$

Hence, (B), (C) and (D) are correct.

17. Clearly, $x^2+y^2+3x+y-6=0$ is circumcircle of $\Delta I_1I_2I_3$ and incentre of ΔABC is orthocentre of $\Delta I_1I_2I_3$ let (h, k).

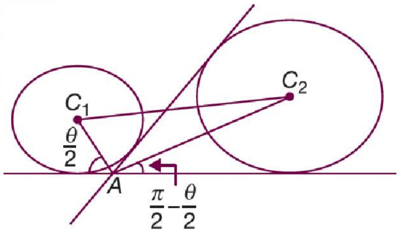
As mirror image of orthocentre about any side of triangle lies on the circumcircle. Hence, the required locus is $x^2+y^2+x-y-5=0$.

Hence, (A), (C) and (D) are correct.

18. Radical axis of $S_1 = 0$, $S_2 = 0$ is the line $L_1 = 0$, the point of concurrency of family of lines $ax + by + c = 0$ if a, b and c are in A.P. is $(1, -2)$.

Hence, (A) and (B) are correct.

19.



From the figure it is clear that, $\angle C_1AC_2 = 90^\circ$

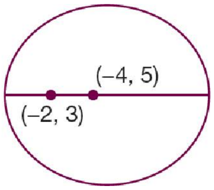
Similarly, $\angle C_1BC_2 = \angle C_1CC_2 = \angle C_1DC_2 = 90^\circ$

Thus, $ABCD$ is a cyclic quadrilateral with C_1C_2 as diameter.

$\therefore$ $ABCD$ is clearly not a square.

Hence, (A) and (B) are correct.

20.



$x^2+y^2+8x-10y-40=0$ and $(-2,3)$

Substitute in $x^2+y^2+8x-10y-40<0$

$C(-4,5)$ and $r=9$

Distance of $(-4,5)$ from the point $(-2,3)$ is $2\sqrt{2}$

$a+b=8$

$a=2\sqrt{2}+9$

$a-b=4\sqrt{2}$

$b=-2\sqrt{2}+9$

$ab=73$

Hence, (A), (C) and (D) are correct.

21. The point of intersection of the lines is (2, 0).

$\Rightarrow$ The centre of the required circle lies on $y = 0$.

Let it be $(x_1,0)$, $r=\dfrac{1x_1-21}{\sqrt{2}}$

$\Rightarrow \quad x_1=d=r+1$ (external touch)

$$\Rightarrow \quad \Rightarrow (x_1-1)^2=\frac{(x_1-2)^2}{(\sqrt{2})^2}\Rightarrow x_1=\pm\sqrt{2}$$

$$r_c=\frac{\left|\pm\sqrt{2}-2\right|}{\sqrt{2}}=\sqrt{2}+1 \text{ or } \sqrt{2}-1$$

$x_1=d=r-1$(internal touch)

$$(x_1+1)^2=\frac{(x_1-2)^2}{2}\Rightarrow x_1=-4\pm3\sqrt{2}$$

$$\Rightarrow \quad r=\frac{-6\pm3\sqrt{2}}{\sqrt{2}}=3\left(\sqrt{2}\pm1\right)$$

Hence, (A), (B), (C) and (D) are correct.

22. $S_1+\lambda S_2=0$

$$\Rightarrow \quad x^2\left(\sin^2\theta+\lambda\cos^2\theta\right)+2(h\tan\theta-\lambda h'\cot\theta)xy$$
$$+\left(\cos^2\theta+\lambda\sin^2\theta\right)y^2+(32+16\lambda)x$$
$$+(16+32\lambda)y+19(1+\lambda)=0$$

The above equation represents a circle.

$\Rightarrow \quad \sin^2\theta+\lambda\cos^2\theta=\cos^2\theta+\sin^2\theta\lambda$

$\Rightarrow \quad (\lambda-1)\sin^2\theta=(\lambda-1)\cos^2\theta$

$\Rightarrow \quad (\lambda-1)\left(\sin^2\theta-\cos^2\theta\right)=0$

$\lambda\neq1\Rightarrow\theta=\pi/4 \;\; h\tan\theta-h'\cot\theta=0\Rightarrow h-h'=0$

Hence, (B) and (C) are correct.

23. The angular bisectors of pair of tangents are $x + y - 3 = 0$ and $x - y + 1 = 0$.

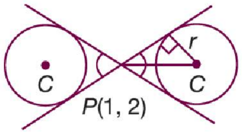
$x + y - 3 = 0$ is the acute angular bisector.

θ is angle between tangents $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$

$\Rightarrow \quad \tan\theta=\dfrac{4}{3}$

$$\cos\theta=\frac{3}{5}\Rightarrow\sin\frac{\theta}{2}=\sqrt{\frac{1-3/5}{2}}=\frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad CP=r\operatorname{cosec}\frac{\theta}{2}=\sqrt{10}\sqrt{5}=5\sqrt{2}$$



For this bisector, C lies at a distance of $5\sqrt{2}$ units from P.

$\Rightarrow \quad C=(1,2)\pm(5,-5)=(6,-3)$ or $(-4,7)$

For $x-y+1=0$, C lies at a distance of $\dfrac{5}{\sqrt{2}}$ units from P.

$$\Rightarrow \quad C=(1,2)\pm\left(\frac{5}{2},\frac{5}{2}\right)=\left(\frac{7}{2},\frac{9}{2}\right)\text{ or }\left(\frac{-3}{2},\frac{-1}{2}\right)$$

Hence, (A) and (C) are correct.

24. Centre of the circle $C_1 = (1, 2)$

Centre of 2nd circle = (–1, 0) and radius = $\sqrt{2}$

The distance between centres = $2\sqrt{2}$, which must be either r_1+r_2 or r_1-r_2

$\Rightarrow \quad r=\sqrt{2}$ or $3\sqrt{2}$

Hence, (B) and (C) are correct.

25. $16\ell^2+9m^2=24\ell m+6\ell+8m+1$

$$\Rightarrow \quad 25(\ell^2+m^2)=9\ell^2+16m^2+24\ell m+6\ell$$
$$+8m+1=(3\ell+4m+1)^2$$

$$\Rightarrow \quad \left\{\frac{\ell(3)+m(4)+1}{\sqrt{\ell^2+m^2}}\right\}^2=5 \text{ Centre}=(3,4),\text{ radius}=5$$

Hence, (A) and (B) are correct.

26. Given circle is $x^2+y^2=a^2$ ...(1)

Clearly, (0, 0) will belong to the interior of the circle (1). Also, other points interior of the circle (1) will have the co-ordinates of the form

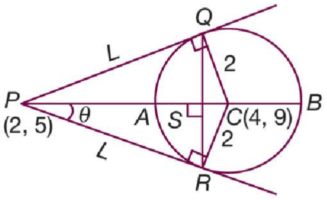
$(\pm\alpha,0),(0,\pm\alpha)$, where $\alpha^2<a^2$ and $(\pm\alpha,\pm\beta)$ and $(\pm\beta,\pm\alpha)$, where $\alpha^2+\beta^2<a^2$ and $\alpha, \beta \in I$

$\therefore$ Number of lattice points in the interior of the circle will be of the form $1 + 4k + 84$,
where $k, r = 0, 1, 2, \ldots\ldots$

$\therefore$ Number of such points must be of the form $4m + I$,
where $m = 0, 1, 2, \ldots\ldots$

Hence, (A), (B) and (C) are correct.

27.



$$\text{Radius} = \sqrt{16+81-93}=2$$

$$L=4$$

$$CP=\sqrt{20};\ AP=\sqrt{20}-2;\ BP=\sqrt{20}+2$$

$$\Rightarrow \quad CP=\frac{AP+BP}{2}\Rightarrow \text{A is correct}$$

Now, let

$$L=PR=\sqrt{(PC)^2-r^2}=\sqrt{20-4}=4=PQ;\ \tan\theta=\frac{2}{4}=\frac{1}{2}$$

Also, $\cos\theta=\dfrac{PS}{PR}\Rightarrow PS=PR\cos\theta=4\cdot\left(\dfrac{2}{\sqrt{5}}\right)=\dfrac{8}{\sqrt{5}}$

H.M. between PA and PB

$$=\frac{2\left(\sqrt{20}-2\right)\left(\sqrt{20}+2\right)}{2\sqrt{20}}=\frac{16}{2\sqrt{5}}=\frac{8}{\sqrt{5}}=PS$$

$\Rightarrow$ C is correct.

Hence, $(PS)(PC)=\left(\dfrac{8}{\sqrt{5}}\right)\left(\sqrt{20}\right)=16=(PR)^2$

$\Rightarrow$ PR is the G.M. of PS and PC

$\Rightarrow$ B is correct

Now, angle between the two tangents is 2θ, then $\left(\because \tan\theta=\dfrac{1}{2}\right)$

$$2\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{2.\frac{1}{2}}{1-\frac{1}{4}}\right)=\tan^{-1}\left(\frac{4}{3}\right)\Rightarrow \text{D is incorrect}$$

Hence, (A), (B) and (C) are correct.

28. Let $Q=(3\cos\theta,3\sin\theta),N=(3\cos\theta,o)$

Points of trisection are $(3\cos\theta,\sin\theta),(3\cos\theta,2\sin\theta)$

Locus is $\dfrac{x^2}{9}+y^2=1;\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

Hence, (A) and (D) are correct.

29. $\dfrac{x_1-1}{\cos\theta}=\dfrac{y_1-2\sqrt{2}}{\sin\theta}=2$

$x_1^2+y_1^2=9$

On solving, $\cos\theta=\dfrac{7}{9} or -1$

Then, get the point $\left(-1,2\sqrt{2}\right),\left(\dfrac{23}{9},\dfrac{10\sqrt{2}}{9}\right)$.

Hence, (A) and (C) are correct.

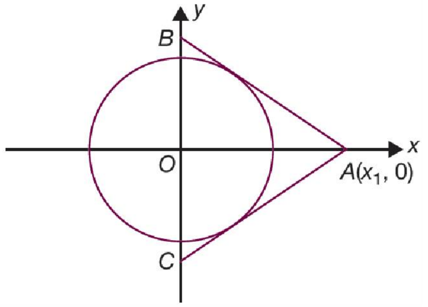
30. Tangent $\Rightarrow$ $\lambda=15,-35$

$\lambda=15\Rightarrow(a,b)=(5,0)$

$\lambda=-35\Rightarrow(a,b)=(-1,8)$

Hence, (A) and (D) are correct.

31.



Let the equation of the tangent $y-0$ be $m(x-x_1)$.

Perpendicular distance from $(0, 0) = 4\sqrt{2}$

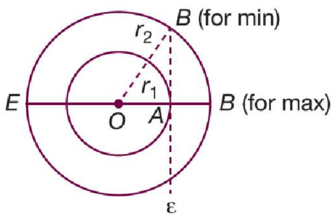
$$\left|\frac{mx_1}{\sqrt{1+m_1^2}}\right| = 4\sqrt{2} \Rightarrow m=\frac{4\sqrt{2}}{x_1^2-32}$$

$$\text{Area of }\Delta=2\times\frac{1}{2}\times x_1\times\frac{\sqrt{32}x_1}{\sqrt{x_1^2-32}}$$

$$\Delta'=0 \text{ at } x=\pm 8$$

Hence, (B) and (C) are correct.

32.



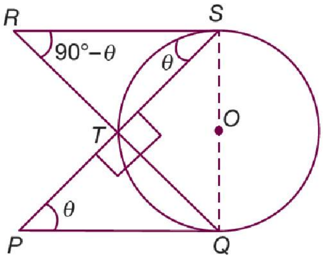
$(BE)_{\max} =$ Diameter of circle $C_2 = 2r_2$

$(BE)_{\min} = 2\sqrt{r_2^2 - r_1^2}$

$\left(OA^2 + OB^2 + BE^2\right)_{\min}$ is $r_1^2 + r_2^2 + 4r_2^2 - 4r_1^2 = 5r_2^2 - 3r_1^2$

$\left(OA^2 + OB^2 + BE^2\right)_{\max}$ is $r_1^2 + r_2^2 + 4r_2^2 = r_1^2 + 5r_2^2$

Hence, (A), (B), (C) and (D) are correct.

33. 

$\angle QTS = \angle PTQ = \frac{\pi}{2}$ (∵ QS is diameter)

$\frac{QT}{PQ} = \sin\theta, \frac{ST}{RS} = \cos\theta, \frac{4}{PQ} = \tan\theta, \frac{4}{RS} = \cot\theta$

$\therefore \left(\frac{QT}{PQ}\right)^2 + \left(\frac{ST}{RS}\right)^2 + PQ.RS$

$= \sin^2\theta + \cos^2\theta + (4\cot\theta)(4\tan\theta)$

Hence, (A), (B) and (C) are correct.

34. ∵ $\angle A = a$ is a constant, if 'I' is the incentre of ΔABC

$BIC = 90° + \frac{\alpha}{2}$, which is also fixed chord.

Hence, 'I' lies on a fixed circle, of which BC is a fixed chord.

∵ $\angle A = a$

If 'H' is orthocentre $\angle BHC = 180°$, which is fixed.

Hence, 'H' lies on a circle of which BC is a fixed chord.

(∵ $\angle A = a$, $\angle HGK$ where H and K are points of trisection of base BC which are fixed.)

∴ The fixed line segment formed by H and K subtends a constant angle α at a variable point G.

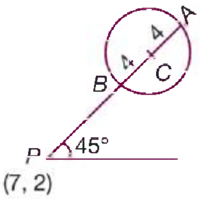
Hence, the locus of centroid also lies on circle.

Hence, (A), (B), (C) and (D) are correct.

35. Slope of $PC = \frac{-3-2}{2-7} = 1$

If $\tan\theta = 1$ (∵ $\theta = 45°$)

Equation of PA is $\frac{x-7}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = r$



$\therefore \left(7 + \frac{r}{\sqrt{2}}, 2 + \frac{r}{\sqrt{2}}\right)$ lies on circle.

Then,

$$\left(7+\frac{r}{\sqrt{2}}\right)^2 + \left(2+\frac{r}{\sqrt{2}}\right)^2 - 4\left(7+\frac{r}{\sqrt{2}}\right) + t\left(2+\frac{r}{\sqrt{2}}\right) - 3 = 0$$

$\Rightarrow \quad r^2 + 10\sqrt{2}r + 34 = 0$

$\therefore \quad r = -5\sqrt{2} \pm 4$

$\therefore \quad$ Point $\left(7 + \frac{-5\sqrt{2} \pm 4}{\sqrt{2}}, 2 + \frac{-5\sqrt{2} \pm 4}{\sqrt{2}}\right)$

$\Rightarrow \quad \left(2 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\right)$

Take + sign for A and – sign for B.

Hence, (A) and (C) are correct.

36. Radical axis of the given circles is $x + 5y - 6 = 0$, which passes through (1, 1).

Let the given circles intersect the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

orthogonally, then $2g(-3) + 2f(-6) = c + 1$

$2g(-2) + 2f(-1) = c - 11$

$\Rightarrow \quad 2g(-1) + 2f(-5) = 12 \Rightarrow -g - 5f - 6 = 0$

$\Rightarrow$ The radical axis passes through the centre $(-g, -f)$ of the 3rd circle.

Verify that it does not pass through the midpoint of the line joining the centres.

Hence, (A) and (C) are correct.

37. ∵ C_2 is the director circle of C_1

∴ Equation of C_2 is $x^2 + y^2 = 2(2)^2 = 8$

Again, C_3 is the director circle of C_2.

Hence, the equation of C_3 is $x^2 + y^2 = 2(8) = 16$.

Hence, (A) and (C) are correct.

38. $C_1 \equiv x^2 + y^2 - 2x - 4y - 4 = 0$ …(1)

$C_2 = x^2 + y^2 + 2x + 4y + 4 = 0$ …(2)

∴ Radical axis is $C_1 - C_2 = 0$

$\Rightarrow \quad -4x - 8y - 8 = 0$

$\Rightarrow \quad x + 2y + 2 = 0$, which is $L = 0$

∴ Option (A) is correct.

Centre and radius of $C_1 = 0$ are $(1, 2)$ and 3

$\because$ Length of perpendicular from $(1,2)$ on $L=0$ is $\frac{|1+4+2|}{\sqrt{1+4}}=\frac{7}{\sqrt{5}}\neq$ radius

$\therefore$ Option (B) is wrong.

L is also the common chord of C_1 and C_2

$\therefore$ Option (C) is correct.

L is perpendicular to the line joining the centres of C_1 and C_2.

$\therefore$ Option (D) is correct.

Hence, (A), (C) and (D) are correct.

39. Orthogonal tangents to a circle meet at the director circle.

$\therefore\quad S_2\equiv x^2+y^2=2.1$

$\Rightarrow\quad S_2\equiv x^2+y^2=2$

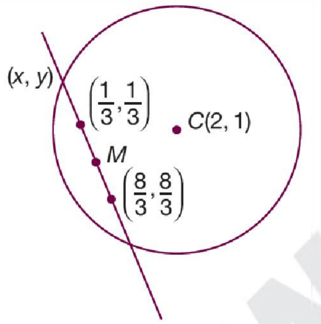
Also, $S_3\equiv x^2+y^2=4$

Ratio of radius of S_2 and $S_3=\sqrt{2}:2=1:\sqrt{2}$

Also, the three circles are concentric.

Hence, (A) and (C) are correct.

40.


(x, y)
$\left(\frac{1}{3},\frac{1}{3}\right)$
C(2, 1)
M
$\left(\frac{8}{3},\frac{8}{3}\right)$

$x+\frac{8}{3}=\frac{2}{3}=y+\frac{8}{3}$

$x=-2$ and $y=-2$

$(-2,-2)$ lies on circle

$C=20$

Midpoint of $\left(\frac{1}{3},\frac{1}{3}\right)$ and $\left(\frac{8}{3},\frac{8}{3}\right)$ is $\left(\frac{3}{2},\frac{3}{2}\right)$, which is the midpoint of the chord. So, using $T=S_1\Rightarrow y=x$.

radius $=\sqrt{4+1+20}=5$

$CM=\frac{1}{\sqrt{2}}$

$$\ell_{\text{chord}}=2\sqrt{25-\frac{1}{2}}=7\sqrt{2}$$

Hence, (A), (B) and (C) are correct.

41. $(x-1)^2+(y-1)^2+\lambda(x+y-2)=0$

$\Rightarrow\quad x^2+y^2+(\lambda-2)x+(\lambda-2)y+2-2\lambda=0\qquad\ldots(1)$

$x^2+y^2+4x+5y-6=0\qquad\ldots(2)$

Equation of common chord PQ is $s-s'=0$.

$\Rightarrow\quad(\lambda-6)x+(\lambda-7)y+8-2\lambda=0\qquad\ldots(3)$

(A) $PQ\parallel x+y-2=0$

$\Rightarrow\quad\frac{6-\lambda}{\lambda-7}=-1\Rightarrow 6=7$, which is impossible

(B) $PQ\perp x+y-2=0$

$\Rightarrow\quad\frac{6-\lambda}{\lambda-7}=1\Rightarrow\lambda=\frac{13}{2}$, which is possible

However, when $\lambda=\frac{13}{2}$, we can see that the circles (1) and (2) are not intersecting each other and their radical axis is perpendicular to the given line $x+y-2=0$.

Equation (3) can be written as $-6x-7y+8+\lambda(x+y-2=0)$, which is of the form $L_1+\lambda L_2=0$.

On solving L_1 and L_2, we get $(6,-4)$.

Hence, (A), (B) and (C) are correct.

42. $al^2+2dl+1=bm^2$

$bl^2+al^2+2dl+1=bm^2+bl^2$

$d^2l^2+2dl+1=b(l^2+m^2)$

$\Rightarrow\quad\left|\frac{dl+1}{\sqrt{l^2+m^2}}\right|=\sqrt{b}$

$\Rightarrow$ Fixed circle with centre $(d,0)$ and radius $\sqrt{b}$

$\therefore\quad(x-d)^2+y^2=b$

Hence, (A), (C) and (D) are correct.

43. $x^2+y^2-8x-16y+60=0\qquad\ldots(1)$

Equation of chord of contact from $(-2,\ 0)$ is $3x+4y-34=0\qquad\ldots(2)$

Intersecting point of (1) and (2) is

$x^2+\left(\frac{34-3x}{4}\right)^2-8x-16\left(\frac{34-3x}{4}\right)+60=0$

$\Rightarrow\quad 5x^2-28x-12=0$

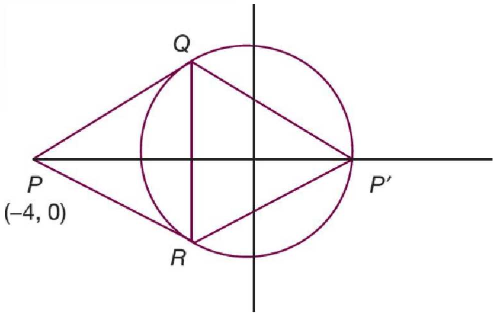
$\Rightarrow\quad x=6,\frac{-2}{5}$

$\therefore$ $(6,4)$ and $\left(\frac{-2}{5},\frac{44}{5}\right)$

Hence, (B) and (C) are correct.

Linked Comprehension Type Questions

1.



QR is a chord of contact $x + 1 = 0$

Circumcircle of PQR is $S + \lambda L = 0$

$\Rightarrow \quad x^2 + y^2 - 4 + \lambda(x+1) = 0$

Passes through $P(-4, 0)$

$\therefore \quad \lambda = 4$

Circumcentre $= (-2, 0)$ $P' = (2, 0)$ such that

Hence, the correct answer is (A).

2. $PQP'R$ is a rhombus

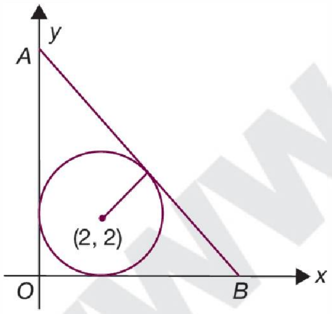
$\Delta PQP' : \Delta P'QR = 1:1$

Hence, the correct answer is (D).

3. Area of rhombus $PQP'R = 6\sqrt{3}$

Hence, the correct answer is (A).

4.



$x^2 + y^2 - 4x - 4y + d = 0 \Rightarrow (x - 2)^2 + (y - 2)^2 = 2^2$

AB is perpendicular to $y = x$ and distance from $(0, 0)$ is greater than 4.

So, equation of AB is $x + y = 2(2 + \sqrt{2})$.

Hence, the correct answer is (A).

5. Orthocentre of ΔOAB is $(0, 0)$.

Hence, the correct answer is (C).

6. Let circumcentre is (h, k) and $B(a, 0)$, $A(0, b)$

Co-ordinates of incentre are $(2, 2)$

$$\left(\frac{(b)(a) + (a)(b) + \left(\sqrt{a^2+b^2}\right)(0)}{a + b + \sqrt{a^2+b^2}}, \frac{(b)(0) + (a)(b) + (0)\left(\sqrt{a^2+b^2}\right)}{a + b + \sqrt{a^2+b^2}} \right) = (2, 2)$$

$$\Rightarrow \quad \frac{2ab}{a + b + \sqrt{a^2+b^2}} = 2,$$

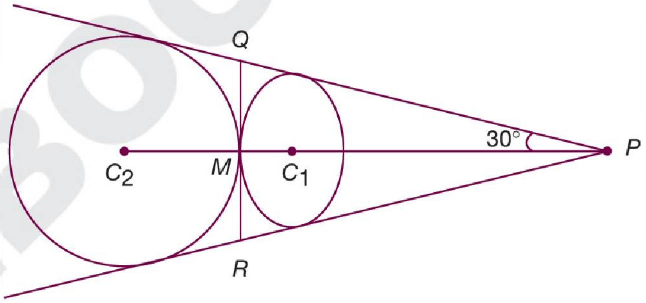
Now, put $a = 2b$, $b = 2k$

Locus is $x + y - xy + \sqrt{x^2 + y^2} = 0, k = 1$

Hence, the correct answer is (C).

7. Angle between the pair of lines is 60°.

The triangle formed by the common tangents is equilateral.



The given circle is the incircle of the $\Delta PQR \Rightarrow r = \sqrt{2}$

Radius of the other circle is $r_1 = 3\sqrt{2}$

Length of the direct common tangents $= 2\sqrt{rr_1} = 2\sqrt{6}$

Hence, the correct answer is (C).

8. $\sin 30° = \dfrac{r}{C_1P}$ or $C_1P = 2r$, M is the midpoint of QR

Area of the triangle is

$$\frac{1}{2} QR \cdot PM = \frac{1}{2}(2QM) \cdot (3r) = \frac{3r}{\sqrt{3}} \cdot (3r) = 6\sqrt{3}$$

Hence, the correct answer is (B).

9. If x is the radius of the required circle, then

$$2\sqrt{xr} + 2\sqrt{xr_1} = 2\sqrt{rr_1} \Rightarrow \sqrt{x} = \frac{\sqrt{rr_1}}{\sqrt{r} + \sqrt{r_1}}$$

$$\Rightarrow \quad x = \frac{rr_1}{r + r_1 + 2\sqrt{rr_1}} = \frac{6}{\sqrt{2} + 3\sqrt{2} + 2\sqrt{6}}$$

$$= \frac{3}{2\sqrt{2} + \sqrt{6}} = \frac{3\sqrt{2}}{4 + 2\sqrt{3}}$$

Hence, the correct answer is (A).

10. Centre and radius of the circle $x^2 + y^2 + 2ax + c = 0$ are $(-a, 0)$ and $\sqrt{a^2 - c}$

Centre and radius of the circle $x^2 + y^2 + 2by + c = 0$ are $(0, -b)$ and $\sqrt{b^2 - c}$

The two circles touch each other,

$$\text{if } \sqrt{a^2+b^2} = \left|\sqrt{a^2-c} \pm \sqrt{b^2-c}\right|$$

If $a^2 + b^2 = a^2 - c + b^2 - c \pm 2\sqrt{(a^2-c)(b^2-c)}$

$$c = \sqrt{(a^2-c)(b^2-c)}$$

$$c^2 = a^2b^2 - a^2c - b^2c + c^2$$

$$a^2b^2 = (a^2 + b^2)c$$

$$2c = \frac{2a^2b^2}{a^2+b^2}$$

$\Rightarrow$ $2c$ is H.M. between a^2, b^2

$\Rightarrow$ $a^2, 2c, b^2$ are in H.P.

Hence, the correct answer is (C).

11. Equation of common chord is $2ax - 2by = 0$

$$\Rightarrow \quad ax - by = 0 \Rightarrow \frac{a}{b} = \frac{3}{2}$$

Hence, the correct answer is (B).

12. $x^2 + y^2 + 2ax + c$ is a family of circles with $x = 0$, i.e., y-axis as its radical axis.

Hence, the correct answer is (B).

13-15. The correct answer is 13(C), 14 (C) and 15(B).

Combined solution to 13, 14, 15

Let $Q(x_1, y_1)$ be a point on C_2.

Let (m, n) be the point which divides PQ in the ratio $p{:}q$, then

$$m = \frac{hq + x_1p}{p+q} \quad \text{and} \quad n = \frac{kq + y_1p}{p+q}$$

$$x_1 = \frac{m(p+q) - hq}{p} \quad \text{and} \quad y_1 = \frac{n(p+q) - kq}{p}$$

Since, $Q(x_1, y_1)$ lies on $x^2 + y^2 = a^2$

We have, $\left[m(p+q) - hq\right]^2 + \left[n(p+q) - kq\right]^2 = a^2p^2$

$$\left[m - \frac{hq}{p+q}\right]^2 + \left[n - \frac{kq}{p+q}\right]^2 = \frac{a^2p^2}{(p+q)^2}$$

Locus of (m, n) is $\left[x - \frac{hq}{p+q}\right]^2 + \left[y - \frac{kq}{p+q}\right]^2 = \frac{a^2p^2}{(p+q)^2}$

Which is circle C_3, if (α, β) denote the centre of C_3,

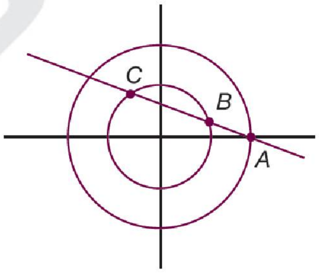
then $\alpha = \frac{hq}{p+q}$ and $\beta = \frac{kq}{p+q}$

Since $p(h, k)$ lies on C_1, $q^2\left[\frac{p+q}{q}\right]^2(\alpha^2 + \beta^2) = p^2a^2$

Locus of (α, β) is $x^2 + y^2 = \frac{p^2a^2}{(p+q)^2}$

$$\frac{\text{Radius of } C_1}{\text{Radius of } C_4} = \sqrt{\frac{a^2p^2}{q^2} \times \frac{(p+q)^2}{a^2+p^2}} = \frac{p+q}{q} = 1 + \frac{p}{q} = \frac{3}{2}$$ as (p, q) lies on $y = 2x$.

16.



$\therefore$ BC^2 (maximum) $= 4r_2^2$, i.e., $\overline{BC}$ becomes diameter of C_2.

Hence, the correct answer is (B).

17. Let B be the point (h, k), then midpoint of AB is given by

$$\left(\frac{h + r_1}{2}, \frac{K}{2}\right) = (x, y) \Rightarrow h = 2x - r_1, K = 2y$$

$$(2x - r_1)^2 + 4y_2 = r_2^2 \Rightarrow \left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}.$$

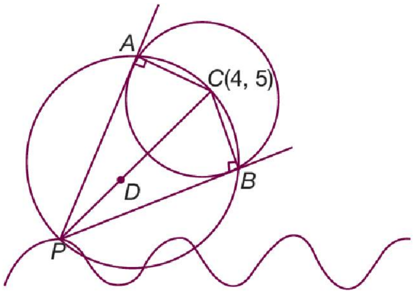
Hence, the correct answer is (A).

18. When BC is maximum locus of midpoint of AB is a fixed point only.

Hence, the correct answer is (D).

19-21. The correct answer is 19(D), 20(C) and 21(C).

Combined solution to 19, 20, 21



Centre of the given circle is $C(4, 5)$. Points P, A, C and B are concyclic such that PC is the diameter of the circle. Hence, centre D of the circumcircle of ΔABC is mid-point of PC, then we have:

$$h = \frac{t+4}{2} \text{ and } k = \frac{\sin t + 5}{2}$$

Eliminating t, we have

$$k = \frac{\sin(2h-4)+5}{2}$$

Or $y = \dfrac{\sin(2x-4)+5}{2}$

$$\Rightarrow \quad f^{-1}(x) = \frac{\sin^{-1}(2x-5)+4}{2}$$

Thus, range of $y = \dfrac{\sin(2x-4)+5}{2}$ is $(2, 3)$ and period is π.

Also, $f(x) = 4 \Rightarrow \sin(2x - 4) = 3$, which has no real solutions.

However, range of $y = \dfrac{\sin^{-1}(2x-5)+4}{2}$ is $\left[-\dfrac{\pi}{4}+2, \dfrac{\pi}{4}+2\right]$.

22. Equation of common chord is $2x - 2y = 0$

Equation of family of circle is $x^2 + y^2 + 2x + \lambda(2x - 2y) = 0$

Centre of circle is $(-\lambda - 1, +\lambda)$

Centre lies on $y = x$

$$\lambda = -\lambda - 1$$

$$2\lambda = -1$$

$$\lambda = -\frac{1}{2}$$

Equation is $x^2 + y^2 + x + y = 0$.

Hence, the correct answer is (D).

23. Let P be $(1 + \sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$ and c is $(1, 0)$.

Circumcentre of ΔABC is the midpoint of PC.

$$2h = 1 + \sqrt{2}\cos\theta + 1$$

$$2k = \sqrt{2}\sin\theta$$

$$(2(h-1))^2 + (2k)^2 = 2$$

Locus of $P(h, k)$ is $2x^2 + 2y^2 - 4x + 1 = 0$

Hence, the correct answer is (A).

24. Let the 2nd circle $x^2 + y^2 + 2gx + 2fy = 0$

Hence, $x^2 + y^2 + 2gx + 2fy = 0$ lies equal roots $f + g = 0$.

Equation of common chord is

$$2(g-3)x + 2(-g-4)y + 7 = 0$$

$$(-6x - 8y + 7) + g(2x - 2y) = 0$$

Passes through the intersection point of

$$-6x - 8y + 7 = 0 \text{ and } 2x - 2y = 0$$

$$\Rightarrow \quad \left(\frac{1}{2}, \frac{1}{2}\right)$$

Hence, the correct answer is (C).

25. Lines $x + 2y + a = 0$ will intersect the circle $x^2 + y^2 = 4$ if

$$\left|\frac{0+0+a}{\sqrt{1+4}}\right| < 2$$

$$\Rightarrow \quad -2\sqrt{5} < a < 2\sqrt{5}$$

Hence, the correct answer is (A).

26. Let lines $x + 2y + a = 0$ and $12x + 6y - 41 = 0$ intersect at p, then $PA \cdot PB = PT^2$ and $PC.PD = PT'^2$ $PC \cdot PD = PT'^2$, where T and T' are the points on the respective circles. A, B, C and D are concyclic.

$$PA \cdot PB = PC \cdot PD \Rightarrow PT^2 = PT'^2$$

Hence, point P will lie on the radical axis of both the circles. Now, equation of radical axis is $4x + 2y - 5 = 0$. Since, radical axis and the lines $x + 2y + a = 0$ and $12x - 6y - 41 = 0$ are concurrent at P, we have

$$\begin{vmatrix} 4 & 2 & -5 \\ 1 & 2 & a \\ 12 & -6 & -41 \end{vmatrix} = 0 \Rightarrow a = 2$$

Hence, the correct answer is (D).

27. Equation of the circle passing through point of intersection of circle $x^2 + y^2 - 4 = 0$ and $x + 2y + 2 = 0$ is

$$x^2 + y^2 - 4 + \lambda(x + 2y + 2) = 0 \qquad \ldots(1)$$

Common chord of circle represented by equation (1) and circle is

$$x^2 + y^2 - 4x - 2y + 1 = 0 \text{ is } (\lambda + 4)x + 2(\lambda + 1)y + 2\lambda - 5 = 0 \qquad \ldots(2)$$

Since, equation (2) and $12x - 6y - 41 = 0$ and represents the same line, we get

$$5x^2 + 5y^2 - 8x - 16y - 36 = 0.$$

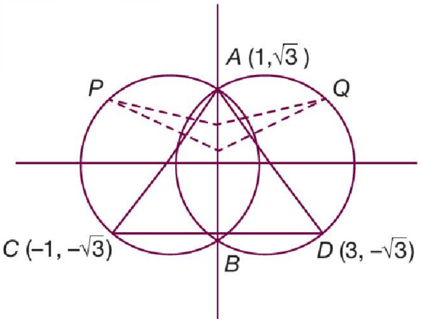
Hence, the correct answer is (A).

28-30. The correct answer is 28(A), 29(B) and 30(A).

Combined solution to 28, 29, 30

ΔACD is an equilateral triangle.

$$\Rightarrow \quad \text{Area} = \frac{\sqrt{3}}{4}\cdot 4^2 = 4\sqrt{3} \text{ sq. units}$$



The circles are images of each other w.r.t. line $x = 1$. Hence, locus is a straight line when particles moves in opposite direction and is a fixed point A, when moving in the same direction.

31. There will exist exactly one circle if the line passing through $A(2, -3)$ and $B(\lambda, 2\lambda - 1)$ is parallel to the given line $16x - 2y + 27 = 0$.

Also, if the point $B(\lambda, 2\lambda - 1)$ lies on the line $16x - 2y + 27 = 0$, then we will have exactly one circle.

Thus, two values of λ are possible.

Hence, the correct answer is (C).

32. The line joining $(3, -5)$ and $(5, -3)$ has slope 1 and thus it is perpendicular to $2x + 2y + 13 = 0$. Hence, the two circles will have same radii.

Hence, the correct answer is (A).

33. The equation to circle with $A(4, -3)$ and $B(-3, 4)$ as diameter is $(x - 4)(x + 3) + (y + 3)(y - 4) = 0$.

$$\Rightarrow \quad x^2 + y^2 - x - y - 24 = 0$$

The equation to line AB is $x + y - 1 = 0$

The system of circle is $x^2 + y^2 - x - y - 24 + \lambda(x + y - 1) = 0$, i.e., $x^2 + y^2 - (1 + \lambda)x - (1 + \lambda)y - 24 + \lambda = 0$.

By applying the condition that the circle touches the line $x + y - 5\sqrt{2} = 0$ to determine λ, we get $\lambda = -1$.

Then the circle is $x^2 + y^2 = 25$.

Hence, the correct answer is (A).

34-36. The correct answer is 34(B), 35(B) and 36(B).

Combined solution to 34, 35, 36

Equation of any circle through the given points is $(x - a)(x - 4a) + (y - 5a)(y - a) + \lambda(4x + 3y - 19a) = 0$, for some $\lambda \in R$.

As it touches the y-axis, $\left(-3a + \frac{3\lambda}{2}\right)^2 = 9a^2 - 19\lambda a$

$$(\because f^2 = c)$$

Solving $\lambda = 0, \frac{-40a}{9}$

The required circles are

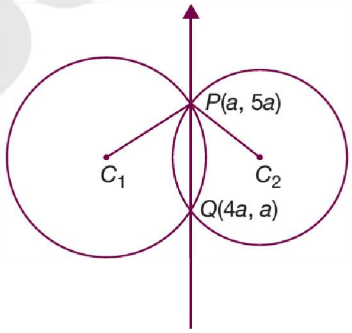
$$x^2 + y^2 - 5ax - 6ay + 9a^2 = 0 \text{ and}$$

$$x^2 + y^2 - 5ax - 6ay + 9a^2 - \frac{40a}{9}(4x + 3y - 19a) = 0$$

Hence, the centres are $\left(\frac{5a}{2}, 3a\right)$ and $\left(\frac{205a}{18}, \frac{29a}{3}\right)$.

The centres of the given circles are

$$C_1\left(\frac{205a}{18}, \frac{29a}{3}\right) \text{ and } C_2\left(\frac{5a}{2}, 3a\right)$$



Now, the angle of intersection θ of these two circles is the angle between the radius vectors at the common point P to the two circles, i.e., $\angle C_1PC_2 = \theta$.

$$\text{Slope of } C_1P = \frac{\frac{29}{3}a - 5a}{\frac{205}{18}a - a} = \frac{84}{187}$$

$$\text{Slope of } C_2P = \frac{5a - 3a}{a - \frac{5a}{2}} = -\frac{4}{3}$$

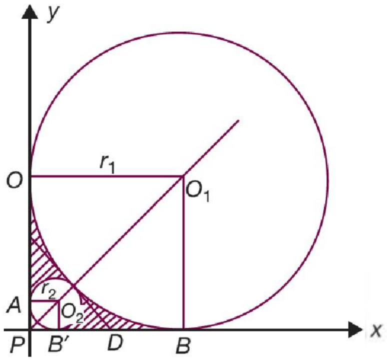
$$\text{So that, } \tan\theta = \frac{\frac{84}{187} + \frac{4}{3}}{1 - \frac{84}{187} + \frac{4}{3}} = \frac{252 + 748}{561 - 336} = \frac{40}{9}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{40}{9}\right)$$

$$\text{Area of } \Delta OC_1C_2 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{205}{18} & \frac{29}{3} & 1 \\ \frac{5}{2} & 3a & 1 \end{vmatrix} a^2 = 5a^2$$

37-39. The correct answer is 37(B), 38(B) and 39(C).

Combined solution to 37, 38, 39



In $\Delta PA'O_2$ and ΔPAO_1

$$\frac{AA'}{A'P} = \frac{O_1O_2}{O_2P}$$

$$\Rightarrow \quad \frac{r_1 - r_2}{r_2} = \frac{r_1 + r_2}{\sqrt{2}r_2}$$

$$\Rightarrow \quad \frac{r_1}{r_2} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = 3 + 2\sqrt{2}$$

Area of shaded region $= r_1^2 - \frac{\pi}{4} r_1^2 - \pi r_2^2 = A_1$

Area of circle $C_1 = A_2 \pi r_2^2$

$$\Rightarrow \quad \text{Let, } A = A_1 - A_2 = r_1^2 - \frac{\pi}{4} r_1^2 - 2\pi r_2^2$$

$$= r_2^2\left(\left(\frac{r_1}{r_2}\right)^2\left(1 - \frac{\pi}{4}\right) - 2\pi\right) > 0$$

Hence, $A_1 > A_2$ always.

Let, CD be the common tangent.

$\Rightarrow \quad PE = \left(\sqrt{2}+1\right)r_2 = CE \Rightarrow 2\left(\sqrt{2}+1\right)r_2$, where $r_2 = 1$

$\therefore \quad$ Length of $CD = 2\left(\sqrt{2}+1\right)$

40. Circumcentre of ΔPAB is midpoint of OP, where O is centre of $x^2 + y^2 = 1$.

$P = (\alpha, 4 - 2\alpha)$.

$S\left(\frac{\alpha}{2}, 2 - \alpha\right)$

Locus of S $2x + y = 2$

Hence, the correct answer is (B).

41. Let, $P = (a, 4 - 2\alpha)$.

Chord of contact of P w.r.t the circle

$x^2 + y^2 = 1$ is $(4y - 1) + \alpha(x - 2y) = 0$ ...(1)

Fixed point $Q\left(\frac{1}{2}, \frac{1}{4}\right)$

Hence, the correct answer is (B).

42. Let (x_1, y_1) be the midpoint of chord AB. So, equation of AB is

$T = S_1$

$xx_1 + yy_1 = {x_1}^2 + {y_1}^2$ ...(2)

and (1) and (2) represent the same line.

$\because \forall$ eliminate 'α'.

Hence, the correct answer is (C).

43. A line passing through $\left(\sqrt{2}, \sqrt{3}\right)$ is $y - \sqrt{3} = m\left(x - \sqrt{2}\right)$

$y - mx = \sqrt{3} - \sqrt{2}m$ ...(1)

Reflection of any point with respect to the diameter lies on the circle.

Thus, two rational points can't exist on the circle as their midpoint should satisfy equation (1).

Hence, the correct answer is (A).

44. $y - mx = -\sqrt{2}m - \sqrt{2}$

$y - mx = -\sqrt{2}(m + 1)$

For the above equation to have rational coefficients $m = -1$

$\Rightarrow \quad y + x = 0$

Hence, reflection of (1, 2) with respect to the above line lies on the circle.

Any other reflection point of (1, 2) will yield an irrational point.

Hence, the correct answer is (B).

45. (1, –1) lies on $y + x = 0$.

Hence, there is only one rational point.

Hence, the correct answer is (A).

46. For $x^2 + y^2 = 16$, a point (x, y) is internal if $-4 < x < 4$, $-4 < y < 4$ and $x^2 + y^2 - 16 < 0$.

$x = 0 \Rightarrow y = -3, -2, -1, 0, 1, 2, 3 \rightarrow 7$

$x = \pm 1 \Rightarrow y = -3, -2, -1, 0, 1, 2, 3 \rightarrow 14$

$x = \pm 2 \Rightarrow y = -3, -2, -1, 0, 1, 2, 3 \rightarrow 14$

$x = \pm 3 \Rightarrow y = -2, -1, 0, 1, 2, \rightarrow 10$

Total = 45

Hence, the correct answer is (B).

47. $x = \dfrac{m^2 - n^2}{m^2 + n^2}, y = \dfrac{2mn}{m^2 + n^2} \Rightarrow x^2 + y^2 = 1.$

As $m, n \in Q$, $\left(\dfrac{m^2 - n^2}{m^2 + n^2}, \dfrac{2mn}{m^2 + n^2}\right)$ is a rational point and others are not.

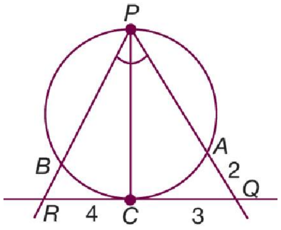
Hence, the correct answer is (B).

48. Maximum 2 points.

Hence, the correct answer is (B).

49-51. The correct answer is 49(A), 50(B) and 51(D).

Combined solution to 49, 50, 51



$$QA.QP = QC.QC$$

$$2.QP = 3^2 \Rightarrow PQ = \frac{9}{2}$$

$$\because \quad PQ : PR = QC : RC$$

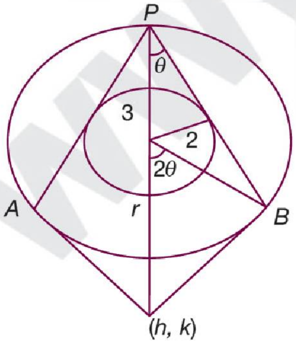
$$PQ : PR = 3:4$$

$$\frac{RC}{QC} = \frac{RP}{PQ} \Rightarrow \frac{4}{3} = \frac{RP}{\frac{9}{2}} \Rightarrow RP = 6$$

$$(RB)(RP) = (RC)(RC)$$

$$(RB)6 = (4)(4) \Rightarrow RB = \frac{8}{3}$$

52.



$$\sin\theta = \frac{2}{3}; \tan\theta = \frac{2}{\sqrt{5}}$$

$$\cos 2\theta = \frac{3}{r}$$

$$\frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{3}{r}$$

$$\frac{1 - \frac{4}{5}}{1 + \frac{4}{5}} = \frac{3}{r}$$

Then, $C \to x^2 + y^2 = 27^2$

Hence, the correct answer is (A).

53. Let $\sum_{i=1}^{6} x_i = \alpha$ and $\sum_{i=1}^{6} y_i = \beta$

Let O be orthocentre of triangle made by $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$\Rightarrow \quad O$ is $(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \Rightarrow (\alpha_1, \beta_1)$

Similarly, G be the centroid of triangle made by other three points.

$\Rightarrow \quad G$ is $\left(\dfrac{x_4 + x_5 + x_6}{3}, \dfrac{y_4 + y_5 + y_6}{3}\right)$

$\Rightarrow \quad G$ is $\left(\dfrac{\alpha - \alpha_1}{3}, \dfrac{\beta - \beta_1}{3}\right)$

Point dividing OG is ratio 3:1 is $\left(\dfrac{\alpha}{4}, \dfrac{\beta}{4}\right)$

$\Rightarrow \quad h + k = 3$

Hence, the correct answer is (B).

54. $T = 0$ and $x + y = 3$ should represent same equation, $xx_1 + yy_1 = 27$

$\therefore \quad x + y = 3$

Hence, the correct answer is (D).

55. Let $PQ = x$, $PT = \sqrt{x^2 - 1}$

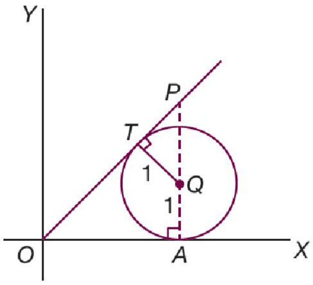
ΔTQP and ΔAPO are similar triangles, then

$$OT = OA = \frac{x + 1}{\sqrt{x^2 - 1}}$$

$$1 + x + \frac{2(x+1)}{\sqrt{x^2 - 1}} + \sqrt{x^2 - 1} = 8 \Rightarrow x = \frac{5}{3}$$

Hence, the correct answer is (C).

56.



$AP = \frac{8}{3}$, $OP = \frac{16}{3}$ Let $\angle AOP = 2\theta$

$\sin 2\theta = \frac{1}{2}$

From ΔOAQ, $\tan\theta = \frac{1}{OA}$

$OA = \frac{1}{\tan\theta} \Rightarrow \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{1}{2}$

$\tan\theta = 2 - \sqrt{3}$

Hence, $OA = 2 + \sqrt{3}$

Hence, the correct answer is (A).

57. $\frac{x-0}{\cos 2\theta} = \frac{y-0}{\sin 2\theta} \Rightarrow x - \sqrt{3}y = 0$

Hence, the correct answer is (A).

58. $9\ell^2 + 6\ell + 1 = 5\ell^2 + 5m^2$

$$\left(\frac{3\ell+1}{\sqrt{\ell^2+m^2}}\right) = \sqrt{5}$$

Hence, the centre is (3, 0) and radius = $\sqrt{5}$.

$\therefore$ Equation of circle $x^2 + y^2 - 6x + 4 = 0$

Hence, the correct answer is (C).

59. Let $(t, 1-t)$ be a point on the line $x + y - 1 = 0$.

Equation of chord of contact is $xt + y(1-t) - 3(x+t) + 4 = 0$ and passes through $\left(\frac{1}{2}, -\frac{5}{2}\right)$.

Hence, the correct answer is (A).

60-62. The correct answer is 60(C), 61(D) and 62(A).

Combined solution to 60, 61, 62

Point P is (3, 0)

ΔPQR is equilateral

$\Rightarrow$ orthocenter = incentre = (1, 0)

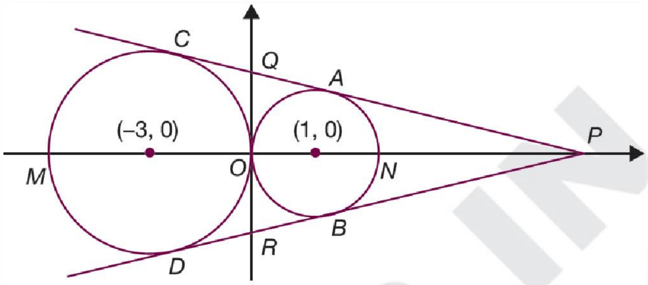
ΔPCD is equilateral.

$\Rightarrow$ incentre = circumcentre = (0, 0)

Centre of circle touching S_1 and S_2 internally is mid-point of M and N (–2, 0)

Now, $PQ = \sqrt{3}$, $PC = 3\sqrt{3}$

So, equation of $ABCD$ are $x - \frac{3}{2} = 0$ and $x + \frac{3}{2} = 0$, respectively.



Circles through CD and AB are

$$x^2 + y^2 + 6x + \lambda\left(x + \frac{3}{2}\right) = 0$$

$$x^2 + y^2 - 2x + \mu\left(x - \frac{3}{2}\right) = 0, \text{ respectively.}$$

$\Rightarrow \quad 6x + \lambda = \mu - 2$ and $\frac{3}{2}\lambda = -\frac{3}{2}\mu$

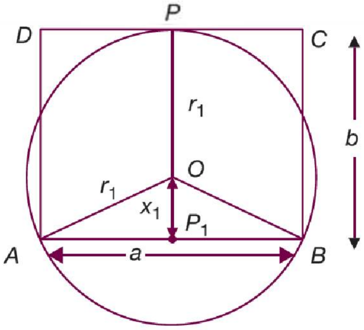
$\Rightarrow \quad \lambda = -\mu$ and $\mu = 4$

$\therefore$ Equation of circle through A, B, C and D is $x^2 + y^2 + 2x - 6 = 0$.

$\therefore$ Centre of circle is (–1, 0).

63-65. The correct answer is 63(B), 64(A) and 65(B).

Combined solution to 63, 64, 65



Let $r_1 = b - x_1 = OP = OA$

$\therefore \quad AP_1 = \frac{a}{2}$

$$r_1{}^2 = x_1{}^2 + \left(\frac{a}{2}\right)^2 = (b - x_1)^2$$

$\Rightarrow \quad x_1{}^2 + \frac{a^2}{4} = b^2 + x_1{}^2 - 2bx_1 \Rightarrow x_1 = \frac{4b^2 - a^2}{8b}$

$\therefore \quad r_1 = b - x_1 = \frac{4b^2 + a^2}{8b}$

Similarly, for the circle passing through B and C and touching side AD, $r_2 = \frac{4a^2+b^2}{8a}$

$$\text{Now, } r_1 + r_2 = \frac{4b^2+a^2}{8b} + \frac{4a^2+b^2}{8a}$$

$$= \frac{a^3+b^3+4ab(a+b)}{8ab}$$

$$= \frac{(a+b)(a^2+3ab+b^2)}{8ab} = \frac{(a+b)}{8}\cdot\frac{(a^2+2ab+b^2+5ab)}{ab}$$

$$= \frac{(a+b)}{8}\cdot\frac{[(a+b)^2+5ab]}{ab}$$

$$= \frac{(a+b)}{8}\cdot\frac{[(a-b)^2+5ab]}{ab}$$

However, $(a-b)^2 \ge 0$

$$\therefore \quad r_1 + r_2 \ge \frac{(a+b)}{8}\cdot\frac{5ab}{ab}$$

$$\Rightarrow \quad r_1 + r_2 \ge \frac{5(a+b)}{8}$$

66-68. The correct answer is 66(D), 67(B) and 68(C).

Combined solution to 66, 67, 68

The point of contact is at a distance of r units from $(-r, 0)$ on a line of inclination 60°.

∴ Point of contact $= (-r + r\cos 60°, 0 + r\sin 60°)$

$$= \left(-\frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$$

C_1 centre $= (-r + 2r\cos 60°, 0 + 2r\sin 60°) = \left(0, r\sqrt{3}\right)$

Direct common tangents are parallel to line of centres. Equation of direct common tangents is of the form $x\sqrt{3} - y + K = 0$.

To find K, use $CP = r$.

Transverse common tangent is perpendicular to the line of centres and passes through the point of contact.

Matrix Match/Column Match Type Questions

1. A → (p); B → (s); C → (q); D → (r)

$$x = 2+\cos\theta, y = 2+\sin\theta$$

$$\therefore \quad x+y = 4+\cos\theta+\sin\theta$$

$$\Rightarrow \quad \max(x+y) = 4+\sqrt{2}$$

$$x-y = \cos\theta-\sin\theta \Rightarrow \max(x-y) = \sqrt{2}$$

$$xy = 4+2(\sin\theta+\cos\theta)+\sin\theta\cos\theta$$

$$\max(xy) = \frac{9+4\sqrt{2}}{2} \text{ at } \theta = 45°$$

$$\frac{x}{y} = \frac{2+\cos\theta}{2+\sin\theta} \Rightarrow \max\left(\frac{x}{y}\right) = \frac{4+\sqrt{7}}{3} \text{ (using } AOD\text{)}$$

2. A → (r); B → (s); C → (q); D → (p)

(A) $C_1 : (x-2)^2 + (y+3)^2 = 5$

$C_2 : (x-5)^2 + (y-3)^2 = 20$

Distance between the centres

$$= \sqrt{(2-5)^2+(3+3)^2} = 3\sqrt{5}$$

$$r_1 + r_2 = \sqrt{5} + 2\sqrt{5} = 3\sqrt{5}$$

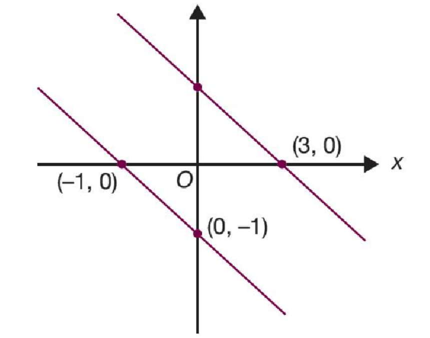
∴ Circle touch externally ⇒ 3 common tangents

(B) If C_2 bisects circumference of C_1, then their common chord must be the diameter of C_1.

Common chord: $10x + 4y = b + a$

Centre of C_1 lies on common chord

$$\Rightarrow \quad a+b = -40-16 = -56 \Rightarrow \frac{|a+b|}{14} = 4$$



(C) $f(x, y) = x^2 + y^2 + 2gx + 2fy + c$

$f(0, \lambda) = \lambda^2 + 2f\lambda + c = 0 \Rightarrow f^2 = c$

$f(\lambda, 0) = \lambda^2 + 2g\lambda + c = 0 \Rightarrow g^2 = c$

$f(1, 1) = -2 \Rightarrow 2g\lambda + 2f + c = -4$

$r = \sqrt{g^2+f^2-C} = \sqrt{C} \Rightarrow \sqrt{C} = 2$

(D) $\left|\frac{3}{\lambda}\times(-1)\right| = |(-1)\times 3|$

$\lambda = 1$

3. A → (p); B → (s); C → (q); D → (r)

Let θ be the angle of inclination of the line ABC.

Any point on this line is $(-1+r\cos\theta,\ 1+r\sin\theta)$

Substituting in the circle,

$$r^2 - 2r(\cos\theta - \sin\theta) + 1 = 0$$

AB, AP and AC are in A.P.

$$AP = \frac{1}{2}(AB + AC) = \frac{1}{2}2(\cos\theta - \sin\theta)$$

$$\tan\theta = \frac{y_1 - 1}{x_1 + 1} \Rightarrow \cos\theta = \frac{x_1 + 1}{AP}$$

$$\sin\theta = \frac{y_1 - 1}{AP}$$

$$\Rightarrow \quad AP^2 = x_1 + 1 - y_1 + 1$$

$$(x_1 + 1)^2 + (y_1 - 1)^2 = x_1 - y_1 + 2$$

Locus in $x^2 + y^2 + x - y = 0$

The distance from origin is zero.

(B) AB, AP and AC are in G.P.

$$AP^2 = AB \cdot AC$$

$$(x + 1)^2 + (y - 1)^2 = 1$$

Shortest distance from origin to the circle = $CP - r$ = $\sqrt{2} - 1$

(C) AB, AP and AC are in H.P.

$$\Rightarrow \quad AP = \frac{2AB \cdot AC}{AB + AC} \Rightarrow AP\frac{(AB + AC)}{2} = AB \cdot AC$$

$$\Rightarrow \quad AP(\cos\theta - \sin\theta) = 1 \Rightarrow x_1 + 1 - y_1 + 1 = 1$$

Locus is $x - y + 1 = 0$ and the distance from (0, 0) is $\frac{1}{\sqrt{2}}$

(D) $\frac{AP}{2} = \frac{AB + AC}{2} = 2(\cos\theta - \sin\theta)$

$$\Rightarrow \quad AP^2 = 2(x_1 + 1) - 2(y_1 - 1)$$

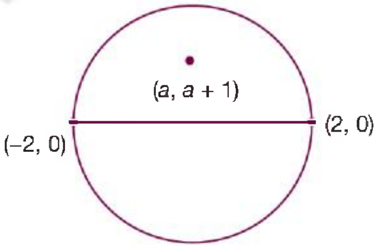
Locus $x^2 + y^2 = 2$. The distance from (0, 0) to locus = radius = $\sqrt{2}$

4. A → (s); B → (q); C → (p); D → (r)

(A) Let $a = r_1$, $b = r_2$, $c = r_3$, $d = r$

$$\frac{d}{a} + \frac{d}{b} + \frac{d}{c} = r\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = r\left(\frac{1}{r}\right) = 1$$

(B)



The point $(a, a + 1)$ lies inside the circle $x^2 + y^2 = 4$

For obtuse angle at $(a, a + 1)$

$$S_1 = a^2 + (a + 1)^2 - 4 < 0$$

$$2a^2 + 2a - 3 < 0$$

$$a \in \left(\frac{-1 - \sqrt{7}}{2}, \frac{-1 + \sqrt{7}}{2}\right)$$

(C) If ℓ_1 and ℓ_2 are the lengths of tangents

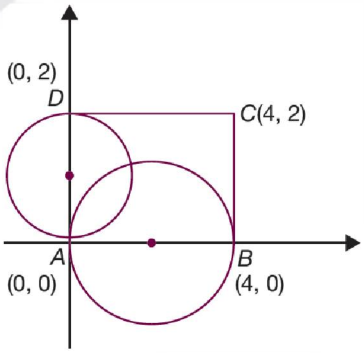
$$AB^2 = l_1^2 + l_2^2$$

$$AB^2 = 9 + 16$$

$$AB = 5$$

(D) Common chord = $\frac{2r_1r_2}{c_1c_2} = \frac{2.2.1}{\sqrt{5}}$

$$d = \frac{4}{\sqrt{5}}$$



5. A → (p); B → (r); C → (s); D → (q)

If the centre of circles lies on common chord, then $2x + 2fy + c + 4 = 0 \Rightarrow c = -4$.

Now, second circle with centre (3, 4) lies on common chord of variable circle and second circle $2gx + 2fy + c + 6x + 8y - 10 = 0$.

$$\Rightarrow \quad 3g + 4f + 18 = 0$$

Now, third circle with centre (–1, 2) lies on common chord of variable circle and 3rd circle lies on $2gx + 2fy - 2x + 4y + C + 2 = 0$.

$$\Rightarrow \quad -g + 2f + 4 = 0$$

$$g = -2 \text{ and } f = -3$$

6. A → (s); B → (r); C → (q); D → (p, q, s)

(A) $P(\lambda, \lambda)$ lies inside C_1 but outside C_2

$$\Rightarrow \quad \lambda^2 + \lambda^2 - 2\lambda < 0 \text{ and } \lambda^2 + \lambda^2 + 2\lambda > 0$$

$\Rightarrow \quad \lambda^2 - \lambda < 0$ and $\lambda^2 + \lambda > 0$

$\Rightarrow \quad \lambda(\lambda - 1) < 0$ and $\lambda(\lambda + 1) > 0$

$\Rightarrow \quad \lambda \in (0,1)$ and $\lambda \in (-\infty, -1) \cup (0, \infty)$

$\Rightarrow \quad \lambda \in (0,1)$ only

Similarly, check for other parts also.

7. A → (r); B → (p); C → (q); D → (s)

(A) Distance between the centres (–1, 0) and (0, –1) $= \sqrt{1+1} = \sqrt{2}$ sum of the radii $= \sqrt{1-c} + \sqrt{1-c}$

The two circles touch each other, if $2\sqrt{1-c} = \sqrt{2} \Rightarrow c = \frac{1}{2}$.

(B) The circles intersect orthogonally, if $2(1)\left(\frac{-1}{2}\right) + 2\left(\frac{3}{2}\right) = c + c \Rightarrow c = 1$.

(C) Centre of $x^2 + y^2 = 9$ is $(0, 0)$ and radius is $3 = OA$, centre of $x^2 + y^2 - 2x + 1 - c^2 = 0$ is $(1, 0)$ and radius is c. Since (1, 0) lies inside the circle $x^2 + y^2 = 9$, the circle with $O'(1,0)$ the circle $x^2 + y^2 = 9$, then circle with $O'(1,0)$ lies inside the circle with centre 0, if the radius $c < O'A = OA - OO' = 2$.

(D) Centre of $x^2 + y^2 = 9$ is $O(0, 0)$ and radius $= 3 = OA$, centre of $x^2 + y^2 - 6x - 8y + 25 - c^2 = 0$ are $O'(3,4)$ and C. Since, the point $O'(3,4)$ lies outside circle with centre O. The circle with centre O will be contained in the circle with centre $O'(3,4)$, if the radius $c > OA + OO' = 8$.

8. A → (r); B → (p); C → (p); D → (s)

(A) $C_1 = (2, 3)$ $r_1 = \sqrt{4+9+3} = 4$

$C_2 = (-1, -1)$ $r_2 = \sqrt{1+1-1} = 1$;

$C_1C_2 = 5$

$r_1 + r_2 = 5$

$C_1C_2 = r_1 + r_2$

∴ Number of common tangent is 3.

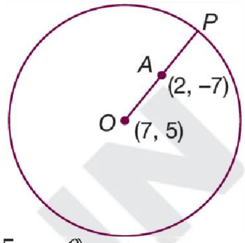
(B) Common chord is $S_1 = S_2$

$x + y = 4$

$x + y = 4$ at origin subtend an angle of $\frac{\pi}{2}$

(C) $C = (7, 5)$

$r = \sqrt{49+25+151} = 15$



$AP = OP - OA$

$OA = \sqrt{(7-2)^5 + (5+7)^2}$

$AP = 15 - 13 = 2$

(D) Let, $x + 5 = 14\cos\theta$

$y - 12 = 14\sin\theta$

$x^2 + y^2 = 365 + 28\,(12\sin\theta - 5\cos\theta)$

$\left(\sqrt{x^2+y^2}\right)_{\min} = \sqrt{365 - 28 \times 13} = 1$

9. A → (q); B → (q); C → (s); D → (p, r)

(A) Foot of perpendicular from centre to tangent or solve both equations.

(B) Circumcentre is the midpoint of $O(0, 0)$ and $(2, 4)$, i.e., $(1, 2)$.

(C) Point of intersection divides the distance between centres in the ratio of radii.

(D) Required S: $x^2 - x + y^2 + ly = 0$ passing through given two points,

$C_1\left(\frac{1}{2}, -\lambda\right)$ $r_1 = \sqrt{\frac{1}{4} + \lambda^2}$; $C_2(0, 0)$ $r_2 = 3$

$C_1C_2 = |r_1 - r_2|$

$\Rightarrow \quad \sqrt{\frac{1}{4} + \lambda^2} = \left|\sqrt{\frac{1}{4} + \lambda^2} - 3\right|$

$\Rightarrow \quad \frac{1}{4} + \lambda^2 = \frac{9}{4} \Rightarrow \lambda^2 = 2$

$\Rightarrow \quad \lambda = \pm\sqrt{2}$ we get centres $C\left(\frac{1}{2}, \mp\sqrt{2}\right)$

10. A → (q); B → (r); C → (q, s); D → (p)

Let the equations of the circles are $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

(A) Equation of the radical axis is $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

Slope of the radical axis $= -\dfrac{g_1 - g_2}{f_1 - f_2}$

Slope of the line joining the centres $= \dfrac{f_1 - f_2}{g_1 - g_2}$

So, the radical axis is perpendicular to the line joining the centre.

(B) Common tangent to the intersecting circles of equal radii is at the same distance from the centres of the two circles and hence is parallel to the line joining the centres.

(C) Since, the line joining the centres of the circles to the midpoint of the common chord is perpendicular to the chord, it bisects the chord.

(D) The line joining the centre of one to a point of intersection is tangent to the other circle. By definition of orthogonality, they are perpendicular.

11. A → (r); B → (q); C → (p, t); D → (q, s)

(A) Origin is the circumcentre ⇒ Circle is $x^2+y^2=1$

$$\Rightarrow \theta = \frac{\pi}{4}$$

(B) A tangent to $x^2+y^2=1$ is $y = mx \pm \sqrt{1+m^2}$. It touches $(x-2)^2 y^2 = 4$, if $\left|\frac{2m \pm \sqrt{1+m^2}}{\sqrt{1+m^2}}\right| = 2$

$$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$$

The common tangents are

$y = \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$ and $y = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$, which intersect at $(-2, 0)$

(C) Common chord of the given circles is

$$\left(x^2+y^2-8\right)-\left[\left(x-a\right)^2+y^2-8\right]=0 \Rightarrow 2x-a=0$$

$$\Rightarrow \frac{2x}{a} = 1$$

Homogenising $x^2+y^2-8=0 \Rightarrow x^2+y^2-8\left(\frac{2x}{a}\right)$

$= 0$ (It represents perpendicular lines)

$$\Rightarrow \quad 1-\frac{32}{a^2}+1=0 \Rightarrow a^2=16 \Rightarrow a=\pm 4$$

(D) $(4, k)$ must lie on the director circle of the given circle, which is $x^2+y^2=20$. Thus,

$$16+k^2=20 \Rightarrow k=\pm 2.$$

12. A → (r); B → (s); C → (p); D → (q)

C_1: Centre (0, 0), radius = 4

C_2: Centre (λ, 0), radius = 2

Distance between centres = $|\lambda|$

$r_1+r_2=6,\ |r_1-r_2|=2$

(A) $|r_1-r_2| < |\lambda| < r_1+r_2 \Rightarrow$ two common tangents

(B) $|\lambda| = |r_1-r_2| \Rightarrow$ one common tangent

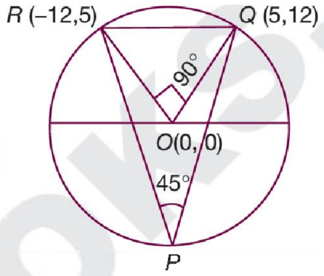
(C) $|\lambda| = r_1+r_2 \Rightarrow$ three common tangents

(D) $|\lambda| > r_1+r_2 \Rightarrow$ four common tangents

13. A → (p); B → (q); C → (r); D → (q)

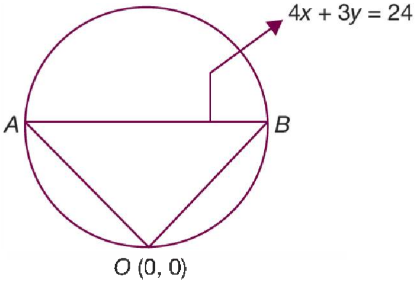
(A) Clearly, slope of OR × slope of $OQ = -1$

$$\therefore \angle ROQ = \frac{\pi}{2} \Rightarrow \angle QPR = \frac{\pi}{4}$$



(B) Clearly, $4x + 3y = 24$ passing through the centre and origin lies on the circle.

$$\therefore \angle AOB = \frac{\pi}{2}$$

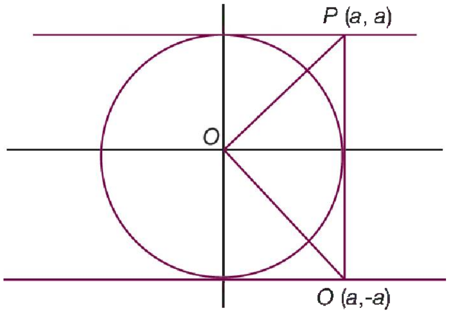


(C) With loss of generality, circle will be $x^2+y^2=a^2$

Clearly, slope of OP × slope of $OQ = -1$

$$\therefore \angle POQ = \frac{\pi}{2}$$

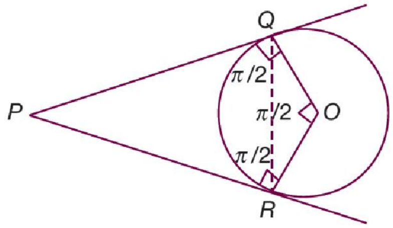
Double angle = π



(D) From figure,

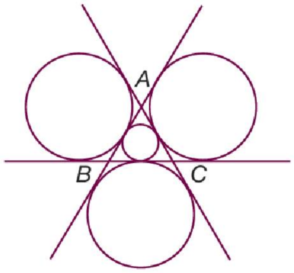
PQ and PR are tangents and QR is a chord.

$$\therefore \angle RPQ = \frac{\pi}{2}$$

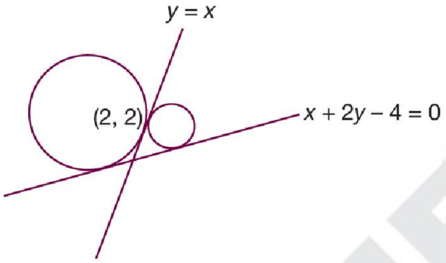


14. A → (r); B → (q); C → (q); D → (p)

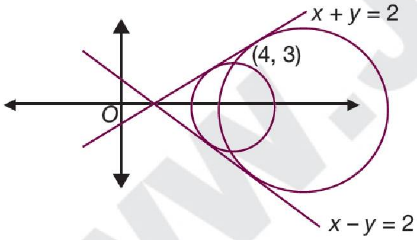
(A)



(B)



(C)



(D) One circle whose centre is radical centre and radius as length of tangent to any given circle

15. A → (r); B → (s); C → (q); D → (p)

(A) Since (2, 3) lies inside the circle, such chord is bisected at (2, 3), which has equation $y-3=-(x-2)$ or $x+y-5=0 \Rightarrow a=b=1$

(B) Let P be the point (α, β), then $\alpha^2+\beta^2+2\alpha+2\beta=0$

$$\text{Midpoint of } OP \text{ is } \left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

$$\therefore \quad \text{Locus of } \left(\frac{\alpha}{2}, \frac{\beta}{2}\right) \text{ is}$$

$$4x^2+4y^2+4x+4y=0 \text{, i.e., } x^2+y^2+x+y=0$$

$$\therefore \quad 2g=1, 2f=1$$

$$\therefore \quad g+f=1$$

(C) Centres of the circle are (1, 2) and $(5, -6)$

Equation of radical axis is $8x-16y-56=0$, i.e., $x-2y-7=0$

Points of intersection are $(3, -2)$

(D) Radical axis is $-8x+16y+58=0$, then $2p+q=0$

Integer/Numerical Answer Type Questions

1. The given line is a diameter.

$\therefore$ Required area = Area of semicircle

$$\Delta = \frac{1}{2}\pi \times 1 = \frac{\pi}{2}$$

$$[\Delta] = 1$$

Hence, the correct answer is (1).

2. Centre of circle $S_1 = (2, 4)$

Centre of circle $S_2 = (4, 2)$

Radius of circle S_1 = Radius of circle $S_2 = 4$

$\therefore$ Equation of circle S_2 is $(x-4)^2+(y-2)^2=16$

$$\Rightarrow \quad x^2+y^2-8x-4y+4=0 \qquad \ldots(1)$$

Equation of circle touching $y = x$ at (1, 1) can be taken as $(x-1)^2+(y-1)^2+\lambda(x-y)=0$

$$\Rightarrow \quad x^2+y^2+x(\lambda-2)+y(-\lambda-2)+2=0 \qquad \ldots(2)$$

As this is orthogonal to S_2,

$$\Rightarrow \quad 2\left(\frac{\lambda-2}{2}\right)(-4)+2\left(\frac{-\lambda-2}{2}\right)(-2)=4+2$$

$$\Rightarrow \quad -4\lambda+8+2\lambda+4=6$$

$\therefore$ Required equation of circle is $x^2+y^2+x-5y+2=0$

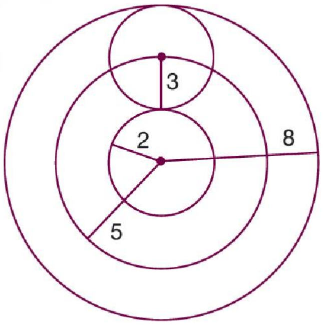
$$\text{Radius} = \sqrt{\frac{1}{4}+\frac{25}{4}-2} = \sqrt{\frac{26-8}{4}} = \sqrt{\frac{18}{4}} = \frac{3}{\sqrt{2}}$$

$$\lambda = 2$$

$$\lambda^2+2=6$$

Hence, the correct answer is (6).

3.



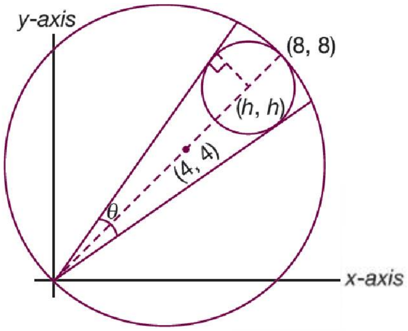
From figure, it is clear that the point will lie between two concentric circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 64$.

$\therefore$ Required locus is $4 \le x^2 + y^2 \le 64$

Required area $= \pi(64 - 4) = 60\pi$

Hence, the correct answer is (6).

4.



$y = x$ is angle bisector of given tangents

$\therefore$ Centre lies on it

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \frac{4\sqrt{2}}{7}$$

$$\tan\frac{\theta}{2} = \frac{1}{2\sqrt{2}}$$

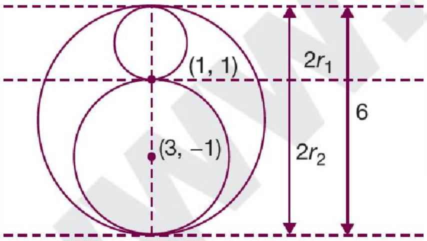
On solving,

$$\sin\frac{\theta}{2} = \frac{1}{3} = \frac{\sqrt{2}(8-h)}{\sqrt{2}h}$$

Hence, the equation of circle is $(x-6)^2 + (y-6)^2 = 8$

Hence, the correct answer is (8).

5.



$2r_1 + 2r_2 = 6$

$\Rightarrow \quad r_1 + r_2 = 3 \qquad \ldots(1)$

Also, $2r_2 - 2\sqrt{3} = 3$

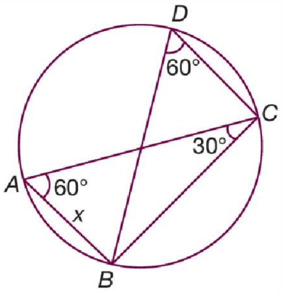
$$\Rightarrow \quad r_2 = \frac{3 + 2\sqrt{2}}{2}$$

$$\therefore \quad r_1 = \frac{3 - 2\sqrt{2}}{2}$$

$$\text{G.E} = 4r_1r_2 = 4 \times \frac{1}{4} = 1$$

Hence, the correct answer is (1).

6.



$\angle A = 60° = \angle D$

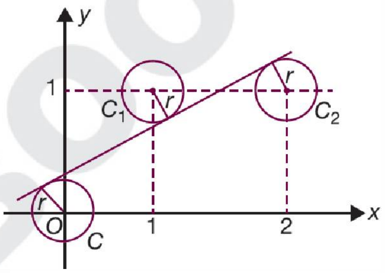
Given, $AC = 2$

$\angle ABC = 90°$

$\Rightarrow \quad x = 1$

Hence, the correct answer is (1).

7.



Equation of the line joining the origin and the centre of circle $C_2(2, 1)$ is $y = \dfrac{x}{2}$

$\Rightarrow \quad x - 2y = 0$

Let the equation of the common tangent is

$x - 2y + c = 0 \qquad \ldots(1)$

$\therefore$ Perpendicular distance from (0, 0) on this line

= Perpendicular distance from (1, 1)

$$\Rightarrow \quad \left|\frac{c}{\sqrt{5}}\right| = \left|\frac{c-1}{\sqrt{5}}\right|$$

$$\Rightarrow \quad c = 1 - c \Rightarrow c = \frac{1}{2}$$

$\therefore$ Equation of common tangent is $x - 2y + \dfrac{1}{2} = 0.$

$\Rightarrow \quad 2x - 4y + 1 = 0 \qquad \ldots(2)$

Perpendicular from (2, 1) on the line (2)

$$r = \left|\frac{4-4+1}{\sqrt{20}}\right| = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

Hence, the correct answer is (5).

8. The given two lines pass through the point (2, 3) and they are inclined at 45° and 135° to the x-axis. The other ends of chords can easily be calculated as

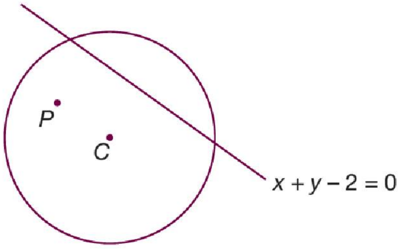
$$\left(2 + 3\sqrt{2}, 3 + 3\sqrt{2}\right) \text{ and } \left(2 - 3\sqrt{2}, 3 - 3\sqrt{2}\right).$$

There is a symmetry about the line $x = 2$ and therefore the centres of circles lie on $x = 2$.
As the chords subtend right angles at the centre, $2r^2 = 6^2$.

$\Rightarrow \quad r = 3\sqrt{2}$

Hence, the correct answer is (4).

9. Given, the circle $S(x, y) = x^2 + y^2 - x - y - 6 = 0 \quad \ldots(1)$



having centre at $C = \left(\frac{1}{2}, \frac{1}{2}\right)$

According to the required conditions, the given point $P(\alpha - 1, \alpha + 1)$ must lie inside the given circle,

i.e., $S(\alpha - 1, \alpha + 1) < 0$

$\Rightarrow \quad (\alpha - 1)^2 + (\alpha + 1)^2 - (\alpha - 1) - (\alpha + 1) - 6 < 0$

$\Rightarrow \quad \alpha^2 - \alpha - 2 < 0$, i.e., $(\alpha - 2)(\alpha + 1) < 0$

$\Rightarrow \quad -1 < \alpha < 2 \quad \ldots(2)$

Also, P and C must lie on the same side of the line (see figure)

$L(x, y) = x + y - 2 = 0, \quad \ldots(3)$

i.e., $L\left(\frac{1}{2}, \frac{1}{2}\right)$ and $L(\alpha - 1, \alpha + 1)$ must have the same sign.

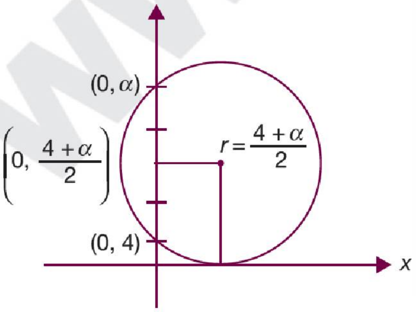
$\because \quad L\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} - 2 < 0$

$L(\alpha - 1, \alpha + 1) = (\alpha - 1) + (\alpha + 1) - 2 < 0$, i.e., $\alpha < 1 \ldots(4)$

Inequalities (2) and (4) together give the permissible values of α, as $-1 < \alpha < 1$.

Hence, the correct answer is (1).

10.



$$r = \frac{4 + \alpha}{2}, \alpha \geq 0$$

When $\alpha = 0$, smallest radius $= 2$

Hence, the correct answer is (2).

11. The centre of a circle passing through the points M and N lies on the perpendicular bisector $y = 3 - x$ of MN. By denoting the centre as $C(a, 3 - a)$, the equation of the circle is $(x - a)^2 + (y - 3 + a)^2 = 2(1 + a^2)$.

Since, for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle MPN$ reaches its maximum value, the circle through the three points M, N and P will be tangent to the x-axis at P, which means $2(1 + a^2) = (a - 3)^2$
$\Rightarrow a = 1$ or $a = -7$.

Thus, the points of contact are $P(1, 0)$ or $P'(-7, 0)$, respectively.

However, the radius of circle through the points M, N and P' is larger than that of the circle through the points M, N and P.

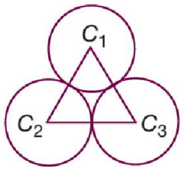
$\therefore \quad \angle MPN > \angle MP'N$.

Thus, $P = (1, 0)$

$\therefore \quad x$-co-ordinate of $P = 1$.

Hence, the correct answer is (1).

12. All the three circles touch each other externally.



$C_1C_2 = 7$

$C_2C_3 = 9$

$C_3C_1 = 8$

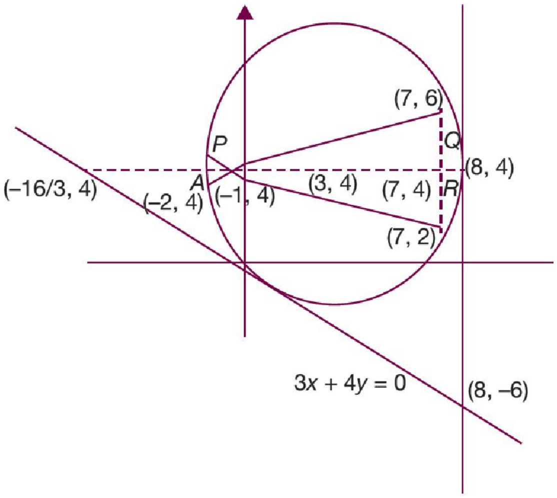
$$s = \frac{7 + 8 + 9}{2} = 12$$

$$\Delta = \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{12 \times 5 \times 3 \times 4}$$

$$r = \frac{\Delta}{s} = \sqrt{5}$$

Hence, the correct answer is (5).

13.



$$(x - 3)^2 + (y - 4)^2 - 25 < 0$$

Point at least distance from (–2, 4) is $P(a, b)$

$$\equiv (-1, 4)$$

Points which are at the greatest distance from (–2, 4) are $Q(c, d)$ and $R(e, f) = Q(7, 6)$ and $R(7, 2)$.

ΔPQR is an isosceles triangle and internal bisector of $\angle P$ is $y = 4$.

Equation of tangent at origin is $3x + 4y = 0$.

Equation of tangent at $\left(\frac{c+e}{2}+1, d\right) \equiv (8, 4)$ is $x = 8$.

Area of the triangle formed by the above three lines

$$\Delta = \frac{1}{2}\times 10\times\frac{40}{3} = \frac{200}{3} \Rightarrow \frac{3\Delta}{50} = 4$$

Hence, the correct answer is (4).

14. Let point of intersection of tangents is (x_1, y_1)

Equation of chord of contact $T = 0$

$$\left.\begin{array}{ll} xx_1 + yy_1 - 1 = 0 & \ldots(1) \\ S - S^1 = 0 \text{ (radical axis)} & \\ \Rightarrow (\lambda+6)x + (2\lambda-8)y + 2 = 0 & \ldots(2) \end{array}\right\} \text{represent same line}$$

$\therefore \quad -2x_1 = \lambda + 6$ and $-2y_1 = 2(\lambda - 8)$

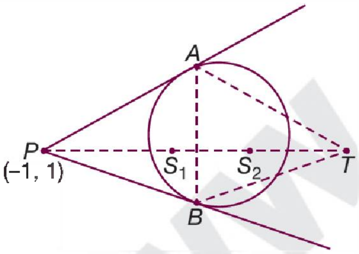
$\therefore \quad 5 - 2x_1 - 6 = -y_1 + 4$

$\Rightarrow \quad 2x - y + 10 = 0$

$p = 2, q = -1, p + q = 1$

Hence, the correct answer is (1).

15.



$PATB$ is clearly a parallelogram (ΔPAB, ΔTAB congruent),

Chord of contact AB

$\Rightarrow \quad x - 2y - 1 = 0$

S_1 = Circumcentre of ΔPAB = Midpoint of $P(-1, 1)$ and centre $(0, -1)$ of given circle $= \left(\frac{-1}{2}, 0\right)$

d = Required distance

$$= 2\left(\text{for distance from } \left(\frac{-1}{2}, 0\right) \text{ to } x - 2y - 1 = 0\right)$$

$$= \frac{2\left|-\frac{1}{2} - 0 - 1\right|}{\sqrt{5}} = \frac{3}{\sqrt{5}} = 1.3$$

Hence, the correct answer is (1).

16. $x^2 + y^2 - 4x - 2y + 1 + \lambda(12x - 6y - 41) = 0 \qquad \ldots(1)$

$x^2 + y^2 - 4 + \mu(x + 2y + t) = 0 \qquad \ldots(2)$

both represent same circle

$\therefore \quad -4 + 12\lambda = \mu \,|$ and $-2 - 6\lambda = 2\mu \,|$ and $1 - 41\lambda = \mu t - 4$

On solving

$$\left.\begin{array}{l} \lambda = 1/5 \\ \mu = -8/5 \end{array}\right\} \Rightarrow t = 2$$

Hence, the correct answer is (2).

17. Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

$\because$ The point $\left(m_i, \frac{1}{m_i}\right)$ lies on this circle

$$\therefore \quad m_i^2 + \frac{1}{m_i^2} + 2gm_i + \frac{2f}{m_i} + c = 0$$

$$\Rightarrow \quad m_i^4 + 2gm_i^3 + cm_i^2 + 2fm_i + 1 = 0$$

Clearly its roots are m_1, m_2, m_3 and m_4

$$\therefore \quad m_1m_2m_3m_4 = \frac{1}{1}$$

(Product of roots of biquadratic equation = 1)

Hence, the correct answer is (1).

18. $|x_1 - y_1| = 2$

Number of solutions such that points $(1,3), (2,4), (3,5)$

Number of such pairs = Three of circles

Hence, the correct answer is (3).

19. $d = \frac{|6 + 12 - 8|}{\sqrt{3^2 + 4^2}} = 2$

$$2\theta = \frac{1}{3}(360^\circ) = 120^\circ$$

$$\therefore \quad \cos 60^\circ = \frac{d}{r} \Rightarrow r = 4$$

Hence, the correct answer is (4).

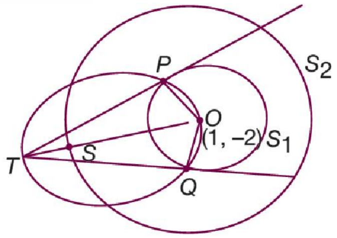
20. $(x - 2)^2 + y^2 = 4, C(2,0)$ and $r = 2$

Distance between $(2,0)$ and $(5,6)$ is $\sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$

$$r_1r_2 = \frac{3\sqrt{5} - 2}{2} \times \frac{3\sqrt{5} + 2}{2} = \frac{41}{4}$$

Hence, the correct answer is (4).

21.



$S_2 (x-1)^2 + (y-2)^2 = 32$ (director circle)

$S_1 (x-1)^2 + (y-2)^2 = 16$

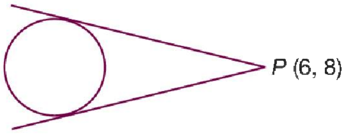
$OS = 4\sqrt{2}$

Require distance

$TS = OT - SO = 12 - 4\sqrt{2}$

Hence, the correct answer is (2).

22.



Area of the triangle $= \dfrac{r.(x_1^2 + y_1^2 - r^2)^{3/2}}{x_1^2 + y_1^2}$

$$A = \frac{r.(100-r^2)^{3/2}}{100}$$

$$\frac{dA}{dr} = \frac{r.\frac{3}{2}(100-r^2)^{\frac{1}{2}}(-2r) + (100-r^2)^{\frac{3}{2}}}{100}$$

$$= \frac{(100-r^2)^{\frac{1}{2}}}{100}(-3r^2 + 100 - r^2) = 0 \Rightarrow r = 5$$

$\therefore$ Area is maximum, when $r = 5$

Hence, the correct answer is (5).

23. Let $(r\cos\theta, r\sin\theta)$ be any point on the curve $2x^2 + 10y^2 + 6xy = 1$, then find

$$2r^2\cos^2\theta + 10r^2\sin^2\theta + 6r^2\sin\theta\cos\theta = 1$$

$$r^2 = \frac{1}{2 + 8\sin^2\theta + 3\sin 2\theta} = \frac{1}{6 + 3\sin 2\theta - 4\cos 2\theta}$$

$\Rightarrow \quad r_{\max} = 1$

Minimum distance between curves = 2

Hence, the correct answer is (2).

24. Let $PA = PB = l$

So, $\dfrac{1}{r^2} + \dfrac{1}{l^2} = \dfrac{1}{16}$

$$\Rightarrow \quad \frac{r^2 l^2}{r^2 + l^2} = 16$$

$$\Rightarrow \quad \frac{lr}{\sqrt{r^2 + l^2}} = 4$$

Now, length of chord of contact $= \dfrac{2lr}{\sqrt{r^2 + l^2}} = 8$

Hence, the correct answer is (8).

25. Both circles cut each other orthogonally. C and D will be centres of two circles. Also, CD will be the diameter of circumcircle of the quadrilateral $ACBD$.

Diameter $= \sqrt{3^2 + 4^2} = 5$

Hence, the correct answer is (5).

26. Equation of common chord of the circle is $6x + 14y + b + a = 0$ ...(1)

$\therefore$ Circumference of 1st circle is bisected by the 2nd circle.

$\Rightarrow$ Centre of 1st circle, i.e., $(1, -4)$ must lie on $6x + 14y + a + b = 0$

$\Rightarrow \quad a + b = 50$

Hence, the correct answer is (5).

27. $S_1 : C_1(-5, 12)$; $\quad S_2 : C_2(5, 12)$; $\quad r_1 = 16, r_2 = 4$

$CC_2 = r + 4$ (Where C is the centre of the circle touching C_2 externally and C_1 internally)

$CC_2 = r + 4$ (Not $r - 16$, $\because S_2$ is contained by S_1)

$\Rightarrow \quad CC_1 + CC_2 = 20$

$\therefore$ Locus of C is an ellipse with foci at C_1 and C_2 and length of major axis = 20

$\therefore$ Locus of C is $\dfrac{x^2}{100} + \dfrac{(y-12)^2}{75} = 1$ ...(1)

According to question, $y = ax$ is a tangent to this ellipse from (0, 0).

Equation of tangent to (1) is $y - 12 = mx + \sqrt{100m^2 + 75}$

However, this is passing through (0, 0)

$\Rightarrow \quad -12 = \sqrt{100m^2 + 75}$

$\Rightarrow \quad m^2 = \dfrac{69}{100}$

$\Rightarrow \quad p + q = 169$

$\Rightarrow \quad k = 2$

Hence, the correct answer is (2).

28. Radius $= \sqrt{51}$, perpendicular distance of the chord $7x + y - 4 = 0$ from centre $= \dfrac{|7\times 8 + 5 - 4|}{\sqrt{50}} = \sqrt{50}$

So, length of the chord $= 2 \times \sqrt{51 - 50} = 2$

Hence, the correct answer is (2).

29. Orthogonal tangents to a circle meet at the director circle.

$\therefore \quad S_2 \equiv x^2 + y^2 = 2.1$

$\Rightarrow \quad S_2 \equiv x^2 + y^2 = 2$

Also, $S_3 = x^2 + y^2 = 4$

Ratio of radius of S_2 and $S_3 = \sqrt{2} : 2 = 1 : \sqrt{2}$

Also, the three circles are concentric.

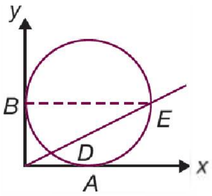
Hence, the correct answer is (3).

30. The equation of the circle is $(x-1)^2+(y-1)^2=1$

$\Rightarrow \quad x^2+y^2-2x-2y+1=0 \qquad \ldots(1)$

Let the equation of the variable straight line be $y = mx \qquad \ldots(2)$

Solving (1) and (2), we get $(1+m^2)x^2-2x(1+m)+1=0$



$\therefore \quad$ Length of $DE=\sqrt{\dfrac{8m}{1+m^2}}$

Area of ΔDEB,

$A=\dfrac{1}{2}DE\times$ Distance of B from DE

$$\therefore \quad A^2=\frac{1}{4}\cdot\frac{8m}{1+m^2}\cdot\frac{1}{1+m^2}=\frac{2m}{(1+m^2)^2}\Rightarrow A=\frac{\sqrt{2m}}{1+m^2}$$

$$\frac{dA}{dm}=\frac{1-3m^2}{\sqrt{m}(1+m^2)^2}=0\Rightarrow m=\pm\frac{1}{\sqrt{3}}$$

$$\frac{d^2A}{d^2m}<0,\text{ if } m=\frac{1}{\sqrt{3}}$$

$\therefore$ Area is maximum for $m=\dfrac{1}{\sqrt{3}}$

Hence, the correct answer is (3).

31. Area of $\Delta PQB=\dfrac{\sqrt{2m}}{(1+m^2)}$ is maximum, where $m=\dfrac{1}{\sqrt{3}}$

$$\Rightarrow\left[m+\frac{1}{2}\right]=1$$

Hence, the correct answer is (1).

32. Reflection of H about BC lies on circumcircle.

Let, $P\equiv(h,k)$ we have

$$\frac{(h-1)\sqrt{5}}{-2}=\frac{(k-2)\sqrt{5}}{1}=\frac{-6}{\sqrt{5}}\Rightarrow P\equiv\left[\frac{17}{5},\frac{4}{5}\right]$$

$$\Rightarrow \quad CP=\sqrt{\left(\frac{17}{5}-2\right)^2+\left(\frac{4}{5}-4\right)^2}=\sqrt{\frac{305}{25}}=\sqrt{\frac{61}{5}}\text{ units}$$

Hence, the correct answer is (1).

33. $x^2+y^2-2x-6y+6+k\,(4x)=0$

$\Rightarrow \quad x^2+y^2+(4k-2)x-6y+6=0 \qquad \ldots(1)$

$x^2+y^2+4x+6y+4=0 \qquad \ldots(2)$

Orthogonally intersections

$\Rightarrow \quad 2\,(2k-1)\,2+2\,(-3)\,(3)=10 \quad \Rightarrow \quad 8k-4-18=10$

$\Rightarrow \quad 8k=32 \quad \Rightarrow \quad k=4$

The required circle is $x^2+y^2+14x-6y+6=0$ and the centre is $(-7, 3)$.

Hence, the correct answer is (4).

34. Length of chord $AB=2\sqrt{x^2-d^2}$

$$=2\sqrt{4-\left(\frac{n}{\sqrt{2}}\right)^2}=2\sqrt{\frac{8-n^2}{2}}$$

$$AB^2=\frac{4(8-n^2)}{2}=2(8-n^2) \quad (\text{if } n=1, 2)$$

Req. $=2[(8-1)+(8-4)] \;\therefore k=14+8=22$

$$\therefore \quad \frac{4k}{11}=\frac{4\times 22}{11}=8$$

Hence, the correct answer is (8).

35. Consider $x^2+y^2=a^2$. Let, $P=(x_1y_1)$ and $Q=(x_2y_2)$

Chord of contact of P is $xx_1+yy_1=a^2$

Passes through $Q\Rightarrow x_1x_2+y_1y_2=a^2$

Also given $\Rightarrow x_2^2+y_1^2-a^2=5^2,\ x_2^2+y_2^2-a^2=(2\sqrt{6})^2$

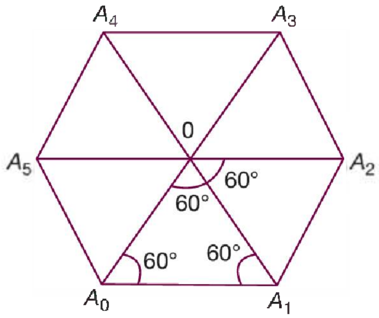
$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{x_1^2+y_1^2+x_2^2+y_2^2-2(x_1x_2+y_1y_2)}$$

$$=\sqrt{25+a^2+24+a^2-2a^2}=7$$

Hence, the correct answer is (7).

36.



$A_0A_1=1$

Also, $A_0A_2=A_0A_4=\sqrt{3}$

(Using cosine rule for ΔOA_0A_2)

$\therefore \quad A_0A_1\;A_0A_2\;A_0A_4=3$

Hence, the correct answer is (3).

37. $\sqrt{\sin^2\alpha+\cos^2\alpha-\sin^2\alpha}\le\dfrac{1}{2}\Rightarrow-\dfrac{1}{2}\le\cos\alpha\le\dfrac{1}{2}$

$$\frac{\pi}{3}\le\alpha\le\frac{2\pi}{3}$$

Hence, the correct answer is (3).

38. Let $y=K$ be a tangent

$\Rightarrow \quad r=d$

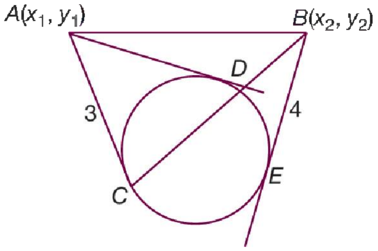
$\Rightarrow \quad 5 = |2 - K|$

$\Rightarrow \quad K = 7 \text{ or } -3$

$\therefore \quad \lambda = 4$

Hence, the correct answer is (4).

39. Chord of contact CD $xx_1 + yy_1 = a^2$



Lengths of tangents $AC = 3 = \sqrt{x_1^2 + y_1^2 - a^2}$

Lengths of tangents $BE = 4 = \sqrt{x_2^2 + y_2^2 - a^2}$

then length of AB is

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2}$$

$$= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2)}$$

$$= \sqrt{(3^2 + a^2) + (4^2 + a^2) - 2a^2} = 5$$

Hence, the correct answer is (5).

40. $x^2 + 2ax - 16 = 0$ is the quadratic equation whose roots are abscissae of A and B.

$y^2 + 24y - 9 = 0$ is the quadratic equation, whose roots are ordinates of A, B.

Equation of circle with AB as diameter is $(x^2 + 2ax - 16) + (y^2 + 24y - 9) = 0$

Radius $= \sqrt{a^2 + 12^2 + 16 + 9}$

$\Rightarrow R^2 = a^2 + 13^2 \Rightarrow (R - a)(R + a) = 169.1 \Rightarrow R - a = 1$

Hence, the correct answer is (1).

41. Let the radius of new circle $= r$

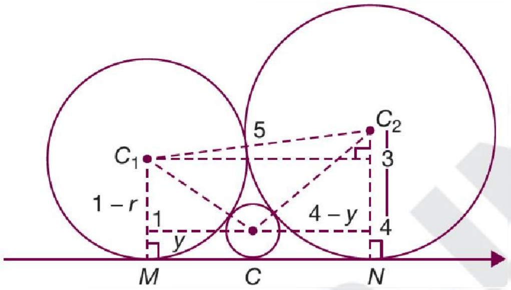
From symmetry, centre of the required circle is (0, 0).

$$\Rightarrow \quad C_1O = \sqrt{2(\sqrt{2} - 1)^2} = 2 - \sqrt{2} = r - (\sqrt{2} - 1) \Rightarrow r = 1$$

C_1O is the distance from origin to the centre of any of four circles.

Hence, the correct answer is (1).

42.



Let r be the radius of the new circle.

Let y be the perpendicular distance from C to C_1M from the figure

$$(1 - r)^2 + y^2 = (1 + r)^2$$

$\Rightarrow \quad y^2 = 4r$

$$(4 - r)^2 + (4 - y)^2 = (4 + r)^2$$

$\Rightarrow \quad (4 - y)^2 = 16r$

$\Rightarrow \quad (4 - y)^2 = 4y^2$

$\Rightarrow \quad y = \dfrac{4}{3}$

$\Rightarrow \quad r = \dfrac{4}{9}$

Hence, the correct answer is (2).

43. $d + r_1 + r_2 = 3\lambda$

$d =$ Distance between the centres $= 5$

$\Rightarrow \quad d + r_1 + r_2 = 5 + 1 + 3 = 9$

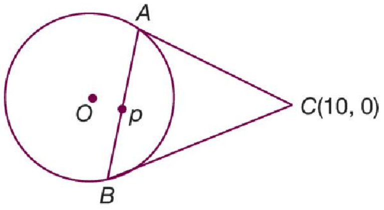
Hence, the correct answer is (3).

44. $OT^2 = OA \cdot OB,\ y = x^2 - \sqrt{5}\,x + 1 = 0$

$x_1x_2 = 1 \Rightarrow OT^2 = 1$

Hence, the correct answer is (1).

45. $PB = PC$



$\Rightarrow \quad \sqrt{OB^2 - OP^2} = PC$

Let (h, k) be the midpoint of AB.

Then locus $h^2 + k^2 - ch + \dfrac{c^2 - a^2}{2} = 0$, where $c = 10, a = 8$.

Hence, (h, k) is $x^2 + y^2 - 10x + 18 = 0$.

$\Rightarrow \quad \frac{k}{2} = 9$

Hence, the correct answer is (9).

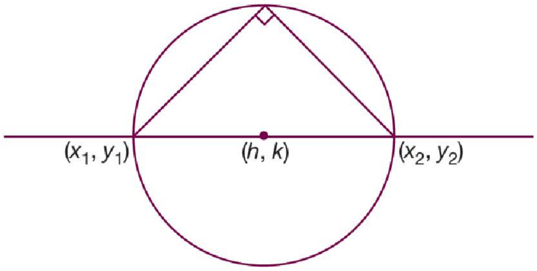
46. Maximum distance from origin $= \sqrt{x^2 + y^2}$

$O(0, 0), C(2, 0), \ r = \sqrt{4-3} = 1$

$\text{Maximum} = CO + r = \sqrt{4} + 1 = 3$

$\text{Minimum} = CO - r = \sqrt{4} - 1 = 1$

Difference $= 9 - 1 = 8$



Hence, the correct answer is (8).

47. Given pairs of lines are perpendicular.

Put $y = cx + 1$ in the given pair

$\therefore \quad c^2x^2 + (2c - c^2)x + 1 - c = 0$

Sum of the roots $= \frac{c^2 - 2c}{c^2}$

$\therefore \quad x_1 + x_2 = 1 - \frac{2}{c}$

However, $\frac{x_1 + x_2}{2} = h$

$c = \frac{2}{1 - 2h}$

Substitute in $k = ch + 1$

$\therefore \quad 2xy - y + 1 = 0 \qquad \therefore L^3 + M^3 + N^3 - 1 = 9$

Hence, the correct answer is (9).

ARCHIVE: JEE MAIN

1. Tangent to $x^2 + y^2 = 1$ at $P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is $x + y = \sqrt{2}$, slope is -1

$y = mx + c$ is tangent to $(x - 3)^2 + y^2 = 1$ the $m = 1$

Equation of required tangent is $x - y + c = 0$

$\left|\frac{c+3}{\sqrt{2}}\right| = 1 \Rightarrow c^2 + 6c + 7 = 0$

Hence, the correct answer is (C).

2. $R = \sqrt{16 + 4 - 16} = 2$ and length of tangent $L = \sqrt{S_1} = 4$

AB(Chord of contact) $= \frac{2LR}{\sqrt{L^2 + R^2}} = \frac{8}{\sqrt{5}}$

Now $(AB)^2 = \frac{64}{5}$

Hence, the correct answer is (D).

3. Equation of family of circle touching y-axis at $(0, 4)$ is given by $(x - 0)^2 + (y - 4)^2 + \lambda x = 0$.

$\because$ It passes through $(2, 0)$

we get $\lambda = -10$

$\Rightarrow$ Required circle is $(x - 0)^2 + (y - 4)^2 - 10x = 0$

$\Rightarrow$ $x^2 + y^2 - 10x - 8y + 16 = 0$

Centre of circle $\equiv (5, 4)$ and Radius $= 5$

distance of $4x + 3y - 8 = 0$ from $(5, 4)$

$= \left|\frac{24}{5}\right| \neq \text{radius}$

Hence, the correct answer is (B).

4. Common tangent is $S_1 - S_2 = 0 \Rightarrow -6x + 8y - 8 + k = 0$

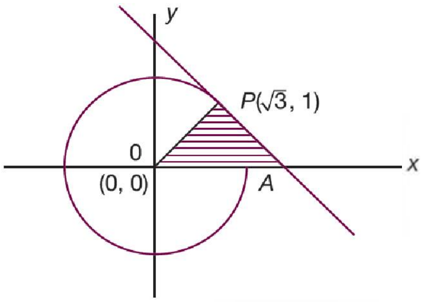
Use $p = r$ for I$^{\text{st}}$ circle centre $(3, 0)$, radius $= 1$

$\Rightarrow \quad \frac{|-18 - 8 + k|}{10} = 1$

$\Rightarrow \quad k = 36 \text{ or } 16 \Rightarrow k_{\max} = 36$

Hence, the correct answer is (36).

5.



Equation of tangent is $\sqrt{3}x + y = 4$

Point $A\left(\frac{4}{\sqrt{3}}, 0\right)$

Area of $\Delta OPA = \frac{1}{2} \times \frac{4}{\sqrt{3}} \times 1 = \frac{2}{\sqrt{3}}$

Hence, the correct answer is (D).

6. Circle $x^2 + y^2 = 4$

$\Rightarrow \quad C_1(0, 0)$ and $r_1 = 2$

and circle $x^2 + y^2 + 6x + 8y - 24 = 0$

$\Rightarrow \quad C_2(-3, -4)$ and $r_2 = 7$

$\Rightarrow \quad d = c_1c_2 = 5$

Also, $d = |r_1 - r_2|$

$\Rightarrow$ Circles touch externally.

Equation of common tangent $s_1 - s_2 = 0$

$\Rightarrow \quad 6x + 8x - 20 = 0$

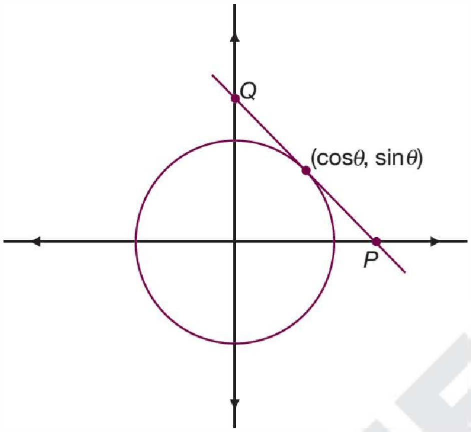
$\Rightarrow \quad 3x + 4y - 10 = 0$

$\therefore$ The point $(6, -2)$ satisfies.

Hence, the correct answer is (D).

7. Let the equation of tangent is $x \cos\theta + y \sin\theta = 1$.
Co-ordinates of P and Q are

$$P\left(\frac{1}{\cos\theta}, 0\right) \text{ and } Q\left(0, \frac{1}{\sin\theta}\right).$$



Let midpoint of P and Q is (h, k).

$$\text{So, } h = \frac{\frac{1}{\cos\theta} + 0}{2} \text{ and } k = \frac{0 + \frac{1}{\sin\theta}}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2h} \text{ and } \sin\theta = \frac{1}{2k}$$

Squaring and adding we get,

$$\frac{1}{4h^2} + \frac{1}{4k^2} = 1$$

$$\therefore \quad \text{Locus } \frac{1}{4x^2} + \frac{1}{4y^2} = 1$$

$\Rightarrow \quad x^2 + y^2 - 4x^2y^2 = 0$

Hence, the correct answer is (B).

8. Circles are $x^2 + y^2 - 16x - 20y + 164 = r^2 \Rightarrow C_1(8, 10)$
and $(x - 4)^2 + (y - 7)^2 = 36$
They intersect at two distinct points.

$$|r_1 - r_2| < C_2 < r_1 + r_2 \{C_1C_2 = \sqrt{16+9} = 5\}$$

Now, $|r - 6| < 5 < r + 6$

$$|r - 6| < 5 \qquad\qquad 5 < r + 6$$

$\Rightarrow \quad -5 < r - 6 < 5 \qquad -1 < r$ ...(1)

$\Rightarrow \quad 1 < r < 11$ ...(2)

From (1) and (2), $r \in (1, 11)$

Hence, the correct answer is (C).

9. Equation of common chord

$$4\,Kx + \frac{1}{2}\,y + K + \frac{1}{2} = 0 \qquad \text{...(1)}$$

and given line $4x + 5y - K = 0$ ...(2)

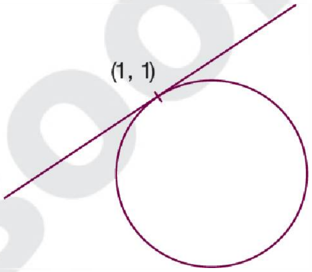
On comparing (1) and (2) we get,

$$K = \frac{1}{10} = \frac{K + 1/2}{-K}$$

$\Rightarrow$ No real value of K exist.

Hence, the correct answer is (B).

10.



Equation of circle is given as

$S + \lambda L = 0$

$(x - 1)^2 + (y - 1)^2 + \lambda(x - y) = 0$

Passes through $(1, -3)$

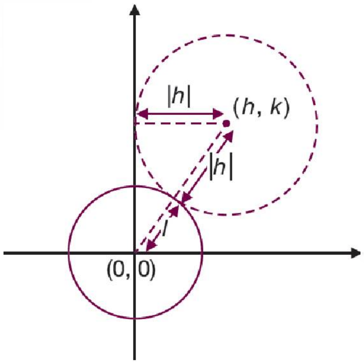
$16 + \lambda \times 4 = 0 \Rightarrow \lambda = -4$

$\therefore \quad (x - 1)^2 + (y - 1)^2 - 4(x - y) = 0$

$r = 2\sqrt{2}$

Hence, the correct answer is (C).

11.



$\sqrt{h^2 + k^2} = 1 + |h|$

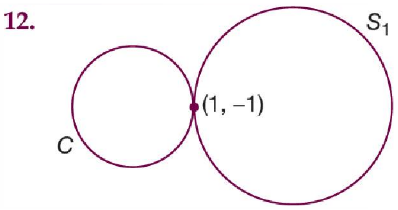
$h^2 + k^2 = 1 + h^2 + 2|h|$

$k^2 = 1 + 2|h|$

$y^2 = 1 + 2x$

Hence, the correct answer is (B).

12.



Circle C $\qquad\qquad$ $S_1 + \lambda S_2 = O$

$$x^2 + y^2 + 4x - 6y - 12 + \lambda((x-1)^2 + (y+1)^2) = 0$$

It passes through (4, 0)

$$16 + 0 + 16 - 0\ 12 + \lambda(9 + 1) = 0$$

$$\lambda = -2$$

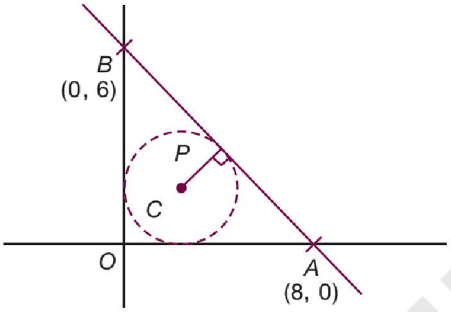
$$x^2 + y^2 + 4x - 6y - 12 - 2(x^2 + y^2 - 2x + 2y + 2) = 0$$

$$x^2 + y^2 - 8x + 10y + 16 = 0$$

$$r = \sqrt{16 + 25 - 16} = 5$$

Hence, the correct answer is (A).

13.



Circle $C(r, r)$, $\quad$ radius r, $\quad$ $3x + 4y - 24 = 0$ tangent

$$p = r$$

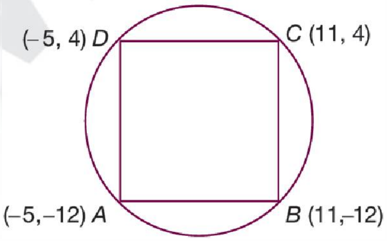
$$\left|\frac{3r + 4r - 24}{5}\right| = r$$

$$|7r - 24| = 5r$$

$7r - 24 = \pm 5\,r$ and, $r = 6$ (not possible) and $r = 2$ centre (2, 2)

Hence, the correct answer is (B).

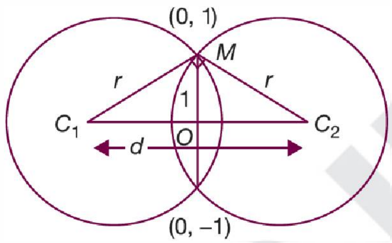
14. $R = \sqrt{9 + 16 + 103} = 8\sqrt{2}$



$OA = 13$ and $OB = \sqrt{265}$ and $OC = \sqrt{137}$ and $OD = \sqrt{41}$

Hence, the correct answer is (C).

15. $r^2 + r^2 = d^2$



In ΔC_1MO,

$$\left(\frac{\sqrt{2}r}{2}\right)^2 + 1^2 = r^2$$

$$r = \sqrt{2}$$

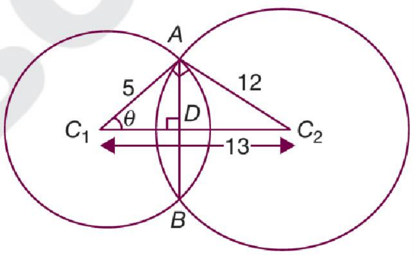
Distance between centres

$$d^2 = 2(2)$$

$$\Rightarrow d = 2$$

Hence, the correct answer is (C).

16.



ΔAC_1C_2

$$\tan\theta = \frac{12}{5} \quad \Rightarrow \quad \sin\theta = \frac{12}{13} \qquad \ldots(1)$$

ΔACD:

$$\sin\theta = \frac{AB/2}{5} \qquad \ldots(2)$$

From (1) and (2),

$$\Rightarrow \frac{AB}{2.5} = \frac{12}{13}$$

$$AB = \frac{120}{13}$$

Hence, the correct answer is (C).

17. Centre of circles are opposite side of line and $C_1(1, 1)$ and $C_2(9, 1)$

$$(3 + 4 - \lambda).(27 + 4 - \lambda) < 0$$

$$(\lambda - 7).(\lambda - 31) < 0$$

$$\lambda \in (7, 31)$$

Distance form S_1

$$\left|\frac{3 + 4 - \lambda}{5}\right| \geq 1 \qquad (P > r)$$

$$\lambda \in (-\infty, 2) \cup (12, \infty)$$

Distance form S_2

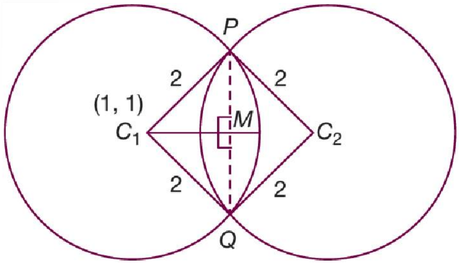
$$\left|\frac{27+4-\lambda}{5}\right| \ge 2$$

$$\lambda \in (-\infty, 21) \cup (41, \infty)$$

So, $\lambda \in (12, 21)$

Hence, the correct answer is (D).

18.



Equation of PQ and $S_1 - S_2 = 0$

$$(x-1)^2 - (x-3)^2 + (y-1)^2 - (y-3)^2 = 0 \Rightarrow x + y - 4 = 0$$

$$C_1M = \frac{|1+1-4|}{\sqrt{2}} = \frac{(-2)}{\sqrt{2}} = \sqrt{2}$$

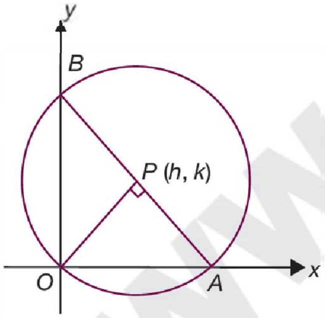
$$PM = \sqrt{4-2} = \sqrt{2}$$

$$PQ = 2\sqrt{2}$$

$\therefore$ Area of $PC_1QC_2 = 4$ sq. units

Hence, the correct answer is (A).

19.



Equation of AB is

$$y - k = -\frac{h}{k}(x - h)$$

$$\Rightarrow \quad hx + ky = h^2 + k^2$$

$$OA = \frac{h^2+k^2}{h}, \; OB = \frac{h^2+k^2}{k}$$

Now, $OA^2 + OB^2 = AB^2$

$$\frac{(h^2+k^2)^2}{h^2} + \frac{(h^2+k^2)^2}{k^2} = 4R^2$$

$$(x^2+y^2)^3 = 4R^2x^2y^2$$

Hence, the correct answer is (D).

20. Equation of required circle will be

$$(x-3)^2 + (y \pm r)^2 = r^2$$

$$x^2 + y^2 - 6x \pm 2ry + 9 = 0 \qquad \ldots(1)$$

Length of y-intercept $= 2\sqrt{f^2 - c}$ $\qquad f = \pm r$

$$\Rightarrow \quad 8 = 2\sqrt{r^2 - 9} \Rightarrow r = 5$$

So, equation of the required circle will be

$$x^2 + y^2 - 6x \pm 10y + 9 = 0$$

two circles

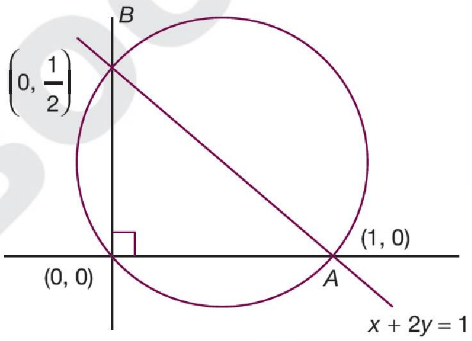
$$x^2 + y^2 - 6x + 10y + 9 = 0 \qquad \ldots(2)$$

$$x^2 + y^2 - 6x - 10y + 9 = 0 \qquad \ldots(3)$$

$\therefore$ Option D, i.e., (3, 10) satisfies equation (3).

Hence, the correct answer is (D).

21.



Equation of circle $x(x-1) + \left(y - \frac{1}{2}\right)y = 0$

$$x^2 + y^2 - x - \frac{y}{2} = 0$$

Equation of tangent at (0, 0)

$$x(0) + y(0) - \frac{x+0}{2} - \frac{y+0}{2\times 2} = 0$$

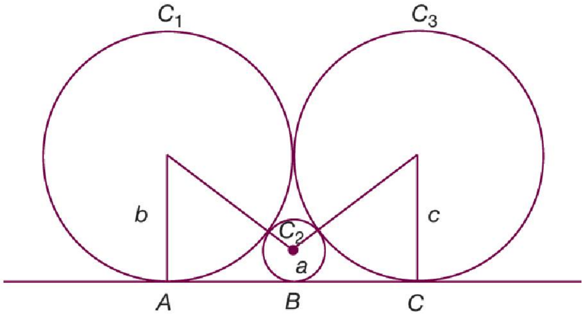
$$2x + y = 0 \qquad \ldots(1)$$

Sum of distance of A and B from line (1) is

$$\frac{2}{\sqrt{5}} + \frac{\frac{1}{2}}{\sqrt{5}} = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$$

Hence, the correct answer is (A).

22.



Length of direct common tangent for circle C_1 and C_2 is

$$AB = \sqrt{(a+b)^2 - (a-b)^2}$$

For C_2 and C_3

Length of direct common tangent is

$$BC = \sqrt{(a+c)^2 - (a-c)^2}$$

For C_1 and C_3,

length of direct common tangent is

$$AC = \sqrt{(b+c)^2 - (b-c)^2}$$

$$AB + BC = AC$$

$$\sqrt{(a+b)^2 - (a-b)^2} + \sqrt{(a+c)^2 - (a-c)^2}$$

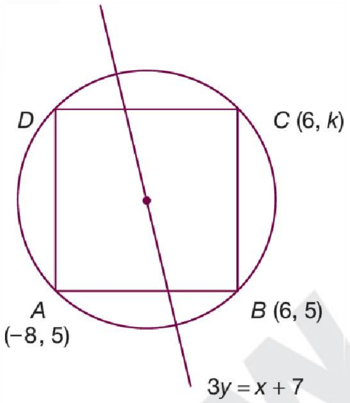
$$= \sqrt{(b+c)^2 - (b-c)^2}$$

$$\sqrt{ab} + \sqrt{ac} = \sqrt{bc}$$

$$\frac{1}{\sqrt{c}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}}$$

Hence, the correct answer is (A).

23.



Let vertex C is $(6, k)$, then centre of circle

$$\left(-1, \frac{5+k}{2}\right)$$

It lies on diameter $3y = x + 7$

$$\Rightarrow \quad 3\left(\frac{5+k}{2}\right) = -1 + 7$$

$$\Rightarrow \quad k = -1$$

So, $AB = 14$ and $BC = 6$

Area $= 14 \times 6 = 84$

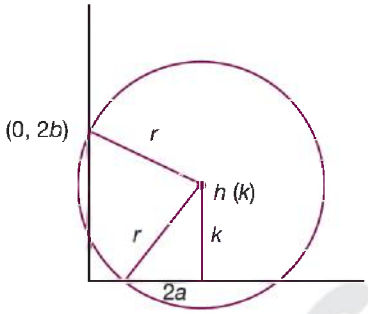
Hence, the correct answer is (B).

24. $r = \sqrt{25 + 36 - c} = \sqrt{36}$

$c = 25$

Hence, the correct answer is (D).

25.



$$k^2 + 4a^2 = r^2 \text{ and } (h-0)^2 + (k-2b)^2 = r^2$$

$$\Rightarrow \quad h^2 + (k-2b)^2 = k^2 + 4a^2$$

$$\Rightarrow \quad h^2 = 4bk - 4b^2 + 4a^2$$

Locus is $x^2 = 4\,(by - b^2 + a^2)$

$\therefore$ The locus of the centre of this circle is a parabola.

Hence, the correct answer is (D).

26. Equation of circle is $x^2 + y^2 = 16$.

Equation of chord is $x + y = n \quad (n \in N)$

$$\text{Length of chord} = 2\sqrt{16 - \frac{n^2}{2}}$$

$$(\text{Length of chord})^2 = 2(32 - n^2)$$

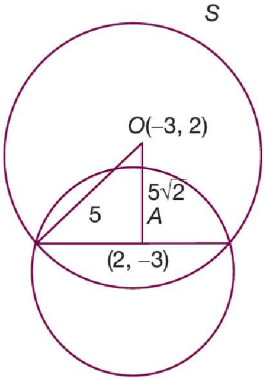
Possible value of $n = 1, 2, 3, 4, 5$

Sum of squares of the length of chords

$= 2(31) + 2(28) + 2(23) + 2(16) + 2(7) = 210$

Hence, the correct answer is (C).

27.



Centre of S: $O(-3, 2)$ centre of given circle $A(2, -3)$

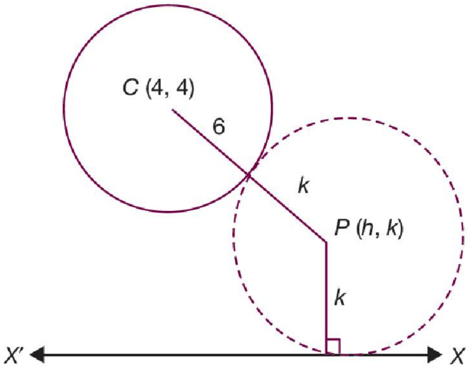
$$\Rightarrow \quad OA = 5\sqrt{2}$$

Also, $AB = 5$ ($\because$ $AB = r$ of the given circle)

$\Rightarrow$ Using Pythagoras theorem, ΔOAB $r = 5\sqrt{3}$

Hence, the correct answer is (C).

28.



For the given circle,

Centre: (4, 4)

Radius: 6

$$6 + k = \sqrt{(h-4^2)+(k-4)^2}$$

$$(h-4)^2 = 20k + 20$$

$\therefore$ Locus of (h, k) is $(x-4)^2 = 20(y+1)$, which is a parabola.

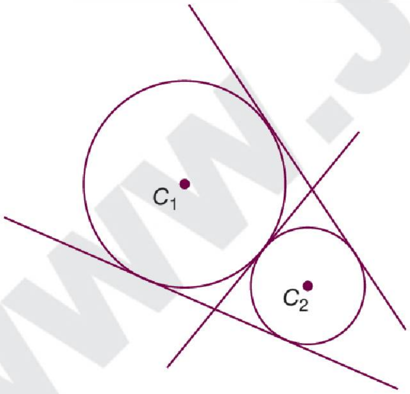
Hence, the correct answer is (A).

29. $x^2 + y^2 - 4x - 6y - 12 = 0$ …(1)

Centre, $c_1 = (2, 3)$ and radius, $r_1 = 5$ units

$$x^2 + y^2 + 6x + 18y + 26 = 0 \quad \text{…(2)}$$

Centre, $c_2 = (-3, -9)$ and radius, $r_2 = 8$ units



$$C_1C_2 = \sqrt{(2+3)^2+(3+9)^2} = 13 \text{ units}$$

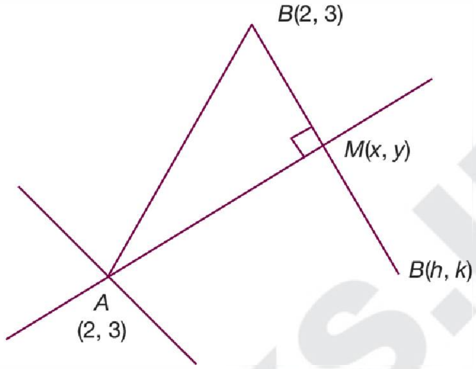
$$r_1 + r_2 = 5 + 8 = 13$$

$\therefore$ $C_1C_2 = r_1 + r_2$

$\therefore$ There are three common tangents.

Hence, the correct answer is (A).

30.



Let M is the midpoint of BB' and AM is perpendicular bisector of BB' (where A is point concurrency).

$$(x-2)(x-1) + (y-2)(y-3) = 0$$

$$\Rightarrow \left(\frac{h+2}{2}-2\right)\left(\frac{h+2}{2}-1\right)+\left(\frac{k+3}{2}-2\right)\left(\frac{k+3}{2}-3\right)=0$$

$$\Rightarrow h^2 + k^2 - 2h - 4k + 3 = 0$$

$\therefore$ It is circle with $C(1, 2)$ and $r = \sqrt{2}$.

Hence, the correct answer is (A).

31. Equation of circle C: $(x-1)^2 + (y-1)^2 = 1$

Radius of $T = |y|$

T touches C externally

$\therefore$ Distance between the centres = Sum of their radii

$$\Rightarrow \sqrt{(0-1)^2+(y+1)^2} = 1+|y|$$

$$\Rightarrow (0-1)^2+(y-1)^2 = (1+|y|)^2$$

$$\Rightarrow 1+y^2+1-2y = 1+y^2+2|y|$$

If $y > 0$, then $2y = 1 - 2y$

$$\Rightarrow y = \frac{1}{4}$$

If $y < 0$, then $1-2y = -2y$

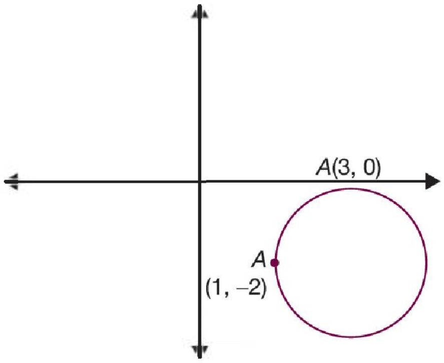
$\Rightarrow$ $1 = 0$ (not possible)

$$\therefore y = \frac{1}{4}$$

Hence, the correct answer is (D).

32. The circle touches x-axis at (3, 0).

$\therefore$ The equation of circle be $(x-3)^3 + (y-0)^2 + ly = 0$.



As it passes through (1, –2), put $x = 1, y = -2$.

$\Rightarrow$ $(1-3)^2 + (-2)^2 + \lambda(-2) = 0 \Rightarrow \lambda = 4$

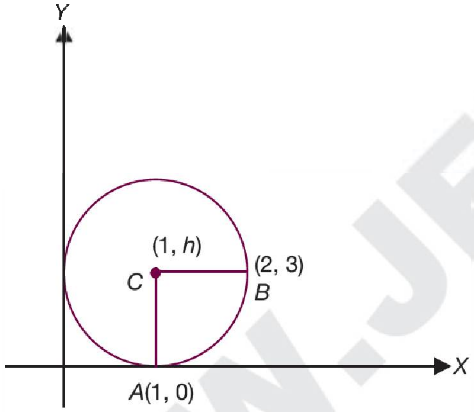
$\therefore$ Equation of circle is $(x-3)^2 + y^2 - 8 = 0$

Now, from the options (5, –2) satisfies the equation of circle.

Hence, the correct answer is (C).

33. Let the centre of the circle be $(1, h)$.

($\because$ Circle touches x-axis at (1, 0))



Let the circle passes through the point $B(2, 3)$.

$\therefore$ $CA = CB$ (Radius)

$\Rightarrow$ $CA^2 = CB^2$

$\Rightarrow$ $(1-1)^2 + (h-0)^2 = (1-2)^2 + (h-3)^2$

$\Rightarrow$ $h^2 = 1 + h^2 + 9 - 6h \Rightarrow h = \frac{10}{6} = \frac{5}{3}$

Thus, the diameter is $2h = \frac{10}{3}$

Hence, the correct answer is (B).

34. The centres of the circles are $C_1\left(\frac{a}{2}, 0\right)$ and $C_2(0,0)$

Also, $r_1 = \frac{|a|}{2}, r_2 = c$

$$C_1C_2 = r_1 - r_2 = \frac{|a|}{2} \Rightarrow c - \frac{|a|}{2} = \frac{|a|}{2} \Rightarrow c = |a|$$

If the two circles touch each other, then they must touch each other internally.

Hence, the correct answer is (D).

35. The given circle is $x^2 + y^2 - 4x - 8y - 5 = 0$.

If the circle is intersecting the line $3x - 4y = m$ at two distinct points.

$\Rightarrow$ Length of perpendicular from centre to the line < radius

$\Rightarrow$ $\frac{|6 - 16 - m|}{5} < 5 \Rightarrow |10 + m| < 25$

$\Rightarrow$ $-25 < m + 10 < 25$

$\Rightarrow$ $-35 < m < 15$

Hence, the correct answer is (B).

36. $P(1, 0)$ and $Q(-1, 0)$

Let $A = (x, y)$

$$\frac{AP}{AQ} = \frac{BP}{BQ} = \frac{CP}{CQ} = \frac{1}{3}$$

$\Rightarrow$ $3\,AP = AQ$

$\Rightarrow$ $9\,AP^2 = AQ^2$

$\Rightarrow$ $9(x-1)^2 + 9y^2 = (x+1)^2 + y^2$

$\Rightarrow$ $x^2 + y^2 - \left(\frac{5}{2}\right)x + 1 = 0$ ...(1)

$\therefore$ A lies on a circle.

Similarly, B and C also lie on the same circle.

Hence, the circumcentre of ABC is centre of circle (1) which is $\left(\frac{5}{4}, 0\right)$.

Hence, the correct answer is (D).

37. The given circles are

$S_1 = x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ ...(1)

$S_2 = x^2 + y^2 + 2x + 2y - p^2 = 0$ ...(2)

$\therefore$ Equation of common chord PQ is $S_1 - S_2 = 0$.

$\Rightarrow$ $L = x + 5y + p^2 + 2p - 5 = 0$

$\Rightarrow$ Equation of circle passing through P and Q is $S_1 + \lambda L = 0$.

$\Rightarrow$ $(x^2 + y^2 + 3x + 7y + 2p - 5) + \lambda(x + 5y + p^2 + 2p - 5) = 0$

As it passes through (1, 1)

$\therefore$ $(7 + 2p) + \lambda(2p + p^2 + 1) = 0$

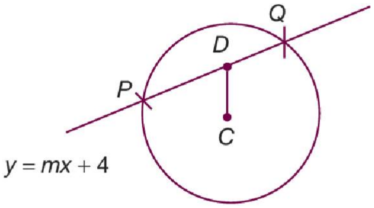
$\Rightarrow$ $\lambda = -\frac{2p + 7}{(p+1)^2}$, which does not exist for $p = -1$

Hence, the correct answer is (A).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1.



Let the line $y = mx + 1$ intersect the circle with centre $(3, -2)$ and radius 5 at points P and Q. The co-ordinate 'D' is the midpoint of PQ having x-co-ordinate $\frac{-3}{5}$.

Then, $D\left(\frac{-3}{5}, 1-\frac{3m}{5}\right)$

$$\text{Slope of } CD = \frac{-2-\left(1-\frac{3m}{5}\right)}{3-\left(\frac{-3}{5}\right)} = \frac{-1+\frac{m}{5}}{\frac{6}{5}}$$

$M_{PQ} \cdot M_{CD} = -1$ ($\because$ Both are perpendicular)

$$\Rightarrow \quad m\frac{\left(-1+\frac{m}{5}\right)}{\frac{6}{5}} = -1 \Rightarrow \frac{m(m-5)}{6} = -1$$

$\Rightarrow \quad m^2 - 5m + 6 = 0$

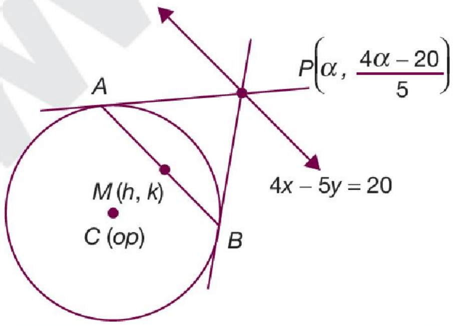
$m = 3$ and $m = 2$

$\therefore \quad 2 \le m < 4$

Hence, the correct answer is (D).

2. Any point P on line $4x - 5y = 20$ is $\left(\alpha, \frac{4\alpha-20}{5}\right)$

Equation of chord of contact AB to the circle $x^2 + y^2 = 9$.



Drawn from point $P\left(\alpha, \frac{4\alpha-20}{5}\right)$ is

$$x\alpha + y\left(\frac{4\alpha-20}{5}\right) = 9 \qquad \ldots(1)$$

Also, the equation of chord AB, whose midpoint is (h, k) is

$$hx + ky = h^2 + k^2 \qquad \ldots(2)$$

$\because$ Equation (1) and (2) represent the same line.

$$\therefore \quad \frac{h}{\alpha} = \frac{k}{\frac{4\alpha-20}{5}} = \frac{h^2+k^2}{9}$$

$\Rightarrow \quad 5k\alpha = 4h\alpha - 20h$ and $9h = \alpha(h^2 + k^2)$

$$\Rightarrow \quad \alpha = \frac{20h}{4h-5k} \text{ and } \alpha = \frac{9h}{h^2+k^2}$$

$$\Rightarrow \quad \frac{20h}{4h-5k} = \frac{9h}{h^2+k^2} \Rightarrow 20(h^2+k^2) = (4h-5k)$$

$\therefore$ Locus of (h, k) is $20(x^2 + y^2) - 36x + 45y = 0$.

Hence, the correct answer is (A).

3. Equation of circle touching the y-axis at $(0, 2)$ is

$$(x-0)^2 + (y-2)^2 + \lambda x = 0$$

It passes through $(-1, 0)$. $\therefore 1 + 4 - \lambda = 0$ or $\lambda = 5$

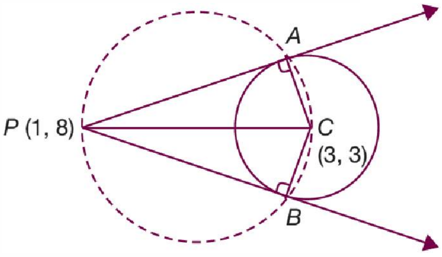
$\therefore$ Equation of circle is $x^2 + y^2 + 5x - 4y + 4 = 0$.

Putting $y = 0$, we get $x = -1, -4$.

$\therefore$ The circle passes through $(-4, 0)$.

Hence, the correct answer is (D).

4. Tangents PA and PB are drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ with centre $C(3, 2)$.



Clearly, the circumcircle of ΔPAB will pass through C and as $\angle A = 90°$, PC must be a diameter of the circle.

$\therefore$ Equation of required circle is

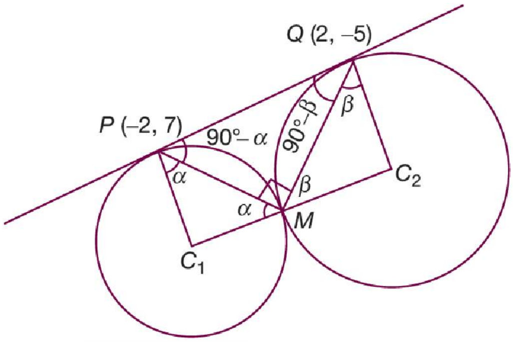
$$(x-1)(x-3) + (y-8)(y-2) = 0.$$

$\Rightarrow \quad x^2 + y^2 - 4x - 10y + 19 = 0$

Hence, the correct answer is (B).

Multiple Correct Choice Type Problems

1. According to the question, we have the following figure.



Clearly, $\angle PMQ = \frac{\pi}{2}$

Hence, locus of M is $(x + 2)(x - 2) + (y - 7)(y + 5) = 0$.

$\Rightarrow \quad E_1 = x^2 + y^2 - 2y - 39 = 0 \qquad \ldots(1)$

Clearly, E_2 is the locus of midpoints of chords of E_1 passing through (1, 1).

Let the midpoint of chord of E_1 be (h, k).

Its equation is $h + k - (1 + k) - 39 = h^2 + k^2 - 2k - 39$.

$\Rightarrow \quad h^2 + k^2 - 2k - h + 1 = 0$

Hence, $E_2 = x^2 + y^2 - x - 2y + 1 = 0 \qquad \ldots(2)$

Option (A) is incorrect, although, it satisfies equation (1) otherwise the line T would touch the second circle at two points.

Also, $\left(\frac{4}{5}, \frac{7}{5}\right)$ satisfies equation (2) but again in this case one end of the chord would be (–2, 7), which is not included in E_1.

$\therefore \left(\frac{4}{5}, \frac{7}{5}\right)$ does not lie in E_2.

$\left(\frac{1}{2}, 1\right)$ does not satisfy equation (2), therefore it does not lie in E_2.

$\left(0, \frac{3}{2}\right)$ does not satisfy equation (1), so it does not lie in E_1.

Hence, (B) and (D) are correct.

2. Circle: $x^2 + y^2 = 1$

Equation of tangent at $P(\cos\theta, \sin\theta)$

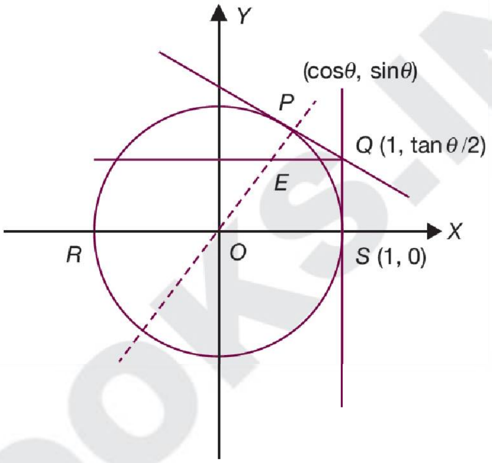
$x \cos\theta + y \sin\theta = 1 \qquad \ldots(1)$

Equation of normal at P

$y = x \tan\theta$

Equation of tangent at S is $x = 1$

$$\therefore \quad Q\left(1, \frac{1-\cos\theta}{\sin\theta}\right) = Q\left(1, \tan\frac{\theta}{2}\right)$$



$\therefore$ Equation of line through Q and parallel to RS is $y = \tan\frac{\theta}{2}$

$\therefore$ Intersection point E of normal and $y = \tan\frac{\theta}{2}$

$$\tan\frac{\theta}{2} = x\tan\theta \Rightarrow x = \frac{1-\tan^2\theta/2}{2}$$

$\therefore$ Locus of E: $x = \frac{1-y^2}{2}$ or $y^2 = 1 - 2x$

It is satisfied by the points $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{3}, \frac{-1}{\sqrt{3}}\right)$.

Hence, (A) and (C) are correct.

3. Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

It passes through (0, 1)

$\therefore \quad 1 + 2f + c = 0 \qquad \ldots(1)$

This circle is orthogonal to $(x - 1)^2 + y^2 = 16$

i.e., $x^2 + y^2 - 2x - 15 = 0$

and $x^2 + y^2 - 1 = 0$

$\therefore$ We should have

$2g(-1) + 2f(0) = c - 15$

$\Rightarrow \quad 2g + c - 15 = 0 \qquad \ldots(2)$

$2g(0) + 2f(0) = c - 1$

$\Rightarrow \quad c = 1 \qquad \ldots(3)$

Solving (1), (2) and (3), we get

$c = 1, g = 7, f = -1$

$\therefore$ The required circle is $x^2 + y^2 + 14x - 2y + 1 = 0$, with centre (–7, 1) and radius = 7.

$\therefore$ (B) and (C) are correct options.

Alternate

Given circles

$$S_1: x^2 + y^2 - 2x - 15 = 0$$

$$S_2: x^2 + y^2 - 1 = 0$$

Centre of the circle which intersects above two circles orthogonally lies on the radical axis of the circles which is $S_1 - S_2 = 0$.

$\Rightarrow \quad x + 7 = 0$

Let the centre of the required circle be $C(-7, k)$.

Circle passes through the point $A(0, 1)$.

$\therefore \quad$ Radius, $r = \sqrt{7^2 + (k-1)^2}$

Also, radius = length of the tangent from C to the any one of the given circles.

$\therefore \quad r = \sqrt{7^2 + k^2 - 1}$

On comparing, we get $7^2 + (k-1)^2 = 7^2 + k^2 - 1$

$\Rightarrow \quad -2k + 1 = -1$

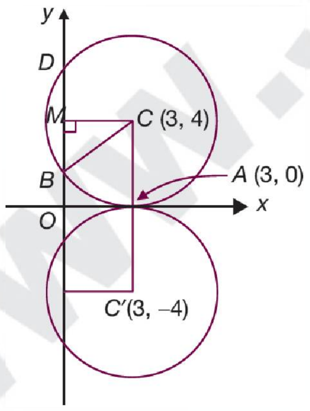
$\Rightarrow \quad k = 1$

$\therefore \quad r = 7$

Hence, (B) and (C) are correct.

4. The figure shows circles touching the x-axis at $A(3, 0)$ and having intercept $BD = 2\sqrt{7}$ on the y-axis.

From the figure, the abscissa of centre is 3.



Also, BC = Radius of the circle

$$= \sqrt{CM^2 + BM^2} = \sqrt{9+7} = 4$$

$\therefore \quad AC = 4$

So, the centres of the circles are

$C(3, 4)$ and $C'(3, -4)$.

Hence, the equations of circles are

$$(x-3)^2 + (y \pm 4)^2 = 16$$

$$\Rightarrow \quad x^2 + y^2 - 6x \pm 8y + 9 = 0$$

Hence, (A) and (C) are correct.

Reasoning Based Questions

1. The given circle is $x^2 + y^2 + 6x - 10y + 30 = 0$.

Centre $(-3, 5)$, radius $=2$

$L_1: 2x + 3y + (p-3) = 0$ and

$L_2: 2x + 3y + p + 3 = 0$

Clearly $L_1 \parallel L_2$

Distance between L_1 and L_2

$$= \left|\frac{p+3-p+3}{\sqrt{2^2+3^2}}\right| = \frac{6}{\sqrt{13}} < 2$$

$\Rightarrow$ If one line is a chord of the given circle, other line may or may not the diameter of the circle.

$\therefore$ Statement 1 is true and statement 2 is false.

Hence, the correct answer is (C).

2. Equation of director circle of the given circle $x^2 + y^2 = 169$ is $x^2 + y^2 = 2 \times 169 = 338$.

We know from every point on director circle, the tangents drawn to given circle are perpendicular to each other.

Here, $(17, 7)$ lies on director circle.

$\therefore$ The tangent from $(17, 7)$ to the given circle are mutually perpendicular.

Hence, the correct answer is (A).

Linked Comprehension Type Questions

1. Equation of $E_1 E_2$: $y = 1$

Equation of $F_1 F_2$: $x = 1$

Equation of $G_1 G_2$: $x + y = 2$

By symmetry, tangents at E_1 and E_2 will meet on y-axis and tangents at F_1 and F_2 will meet on x-axis.

$$E_1 \equiv \left(\sqrt{3}, 1\right) \text{ and } F_1 \equiv \left(1, \sqrt{3}\right)$$

Equation of tangent at E_1: $\sqrt{3}x + y = 4$

Equation of tangent at F_1: $x + \sqrt{3}y = 4$

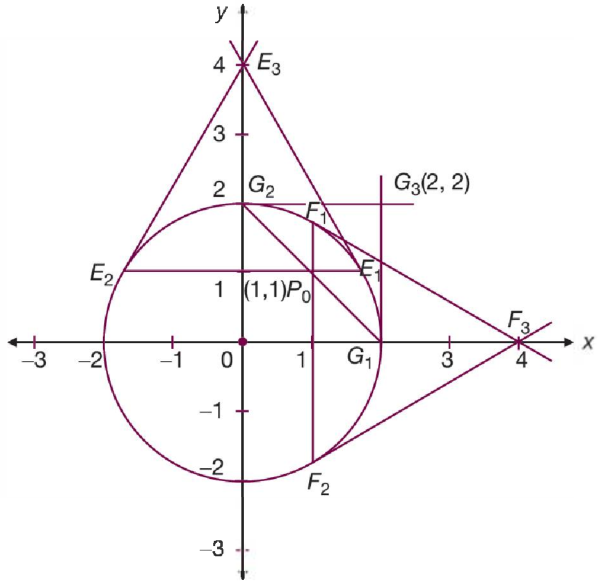
$\therefore$ Points $E_3\,(0, 4)$ and $F_3\,(4, 0)$

Tangents at G_1 and G_2 are $x = 2$ and $y = 2$ intersecting each other at $G_3(2, 2)$.

Clearly, E_3, F_3 and G_3 lie on the curve $x + y = 4$.

Hence, the correct answer is (A).

2.



Let point P be $(2\cos\theta, 2\sin\theta)$

Tangent at $x\cos\theta + y\sin\theta = 0$

$$\therefore \quad M\left(\frac{2}{\cos\theta}, 0\right) \text{ and } N\left(0, \frac{2}{\sin\theta}\right)$$

Midpoint of $MN = \left(\dfrac{1}{\cos\theta}, \dfrac{1}{\sin\theta}\right)$

For locus of midpoint (x, y) of MN,

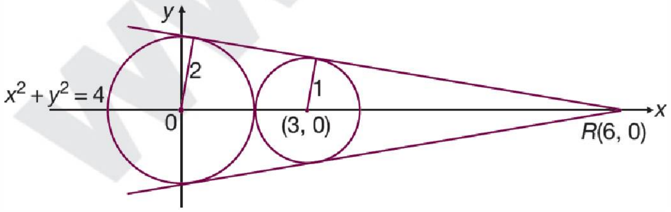
$$x = \frac{1}{\cos\theta}, y = \frac{1}{\sin\theta}$$

$$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} = 1$$

$$\Rightarrow \quad x^2 + y^2 = x^2y^2$$

Hence, the correct answer is (D).

3. The point of intersection of direct common tangents is (6, 0).



So, let the equation of common tangent be

$$y - 0 = m(x - 6)$$

As it touches $x^2 + y^2 = 4$, we have

$$\left|\frac{0 - 0 - 6m}{\sqrt{1 + m^2}}\right| = 2$$

$$\Rightarrow \quad 9m^2 = 1 + m^2$$

$$\Rightarrow \quad m = \pm\frac{1}{2\sqrt{2}}$$

So, the equations of common tangents are

$$y = \frac{1}{2\sqrt{2}}(x - 6), y = -\frac{1}{2\sqrt{2}}(x - 6) \text{ and also } x = 2$$

Hence, the correct answer is (D).

4. Equation of tangent PT to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$ *is* $x\sqrt{3} + y = 4$

Let the line L, perpendicular to tangent PT be $x - y\sqrt{3} + \lambda = 0$

As it is tangent to the circle $(x - 3)^2 + y^2 = 1$

∴ Length of perpendicular from centre of circle to the tangent = Radius of circle

$$\Rightarrow \quad \left|\frac{3 + \lambda}{2}\right| = 1 \Rightarrow \lambda = -1 \text{ or } -5$$

∴ Equation of L can be $x - \sqrt{3}y = 1$ or $x - \sqrt{3}y = 5$

Hence, the correct answer is (A).

5. Centre of C lies on the line through D perpendicular to PQ. Thus, centre of C lie on

$$y - \frac{3}{2} = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{x - 3\sqrt{3}}{2}\right) \Rightarrow x = \sqrt{3}y$$

Let the centre of the circle C be $\left(\sqrt{3}y_1, y_1\right)$, then

$$\left(\frac{3\sqrt{3}}{2} - \sqrt{3}y_1\right)^2 + \left(\frac{3}{2} - y_1\right)^2 = 1$$

$$\Rightarrow \quad 4\left(\frac{3}{2} - y_1\right)^2 = 1 \Rightarrow \frac{3}{2} - y_1 = \pm\frac{1}{2} \Rightarrow y_1 = 1, 2$$

Thus, centre of C can be $\left(\sqrt{3}, 1\right)$ or $\left(2\sqrt{3}, 2\right)$.

∵ Centre of the circle and origin lie on the same side of $\sqrt{3}x + y - 6 = 0$

and $\sqrt{3}(0) + 0 - 6 < 0$ and $\sqrt{3}\left(\sqrt{3}\right) + 1 - 6 < 0$, we get centre of the circle C to be $I\left(\sqrt{3}, 1\right)$.

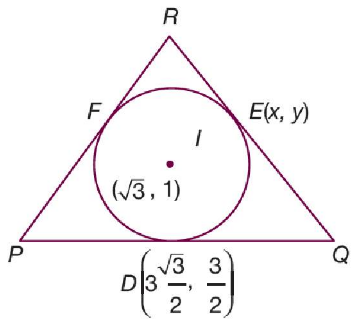
Hence, its equation is $\left(x-\sqrt{3}\right)^2+(y-1)^2=1$.

Hence, the correct answer is (D).

6. Next, if m is the slope of QR, then $\pm\tan\left(\frac{\pi}{3}\right)$ $=\frac{m-\left(-\sqrt{3}\right)}{1+m\left(-\sqrt{3}\right)}$

$\Rightarrow\quad \pm\sqrt{3}\left(1-m\sqrt{3}\right)=m+\sqrt{3}\Rightarrow m=0$ or $\sqrt{3}$

Let the slope of QR be 0 and the co-ordinate of E be (x, y), then since IE is perpendicular to



$\frac{x-\sqrt{3}}{y-1}=0\Rightarrow x=\sqrt{3}$ and as $IE=1$, $\left(x-\sqrt{3}\right)^2+(y-1)^2$ $=1\Rightarrow y=0$ or 2

So the co-ordinates of E are $\left(\sqrt{3},0\right)$, as $y=2$ is not given in the choices.

Similarly, let the slope of PR be $\sqrt{3}$. If the co-ordinates of F are (p, q),

$$\frac{q-1}{p-\sqrt{3}}\sqrt{3}=-1$$

$\Rightarrow\quad p-\sqrt{3}=-\sqrt{3}\,(q-1)$

Also, $\left(p-\sqrt{3}\right)^2+(q-1)^2=1\Rightarrow q-1=\pm\frac{1}{2}\Rightarrow q=\frac{3}{2}$ or $\frac{1}{2}$

If $q=\frac{3}{2}$, $p=\sqrt{3}-\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}$

If $q=\frac{1}{2}$, $p=\sqrt{3}+\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$

So, the co-ordinates of F are $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$.

Hence, the correct answer is (A).

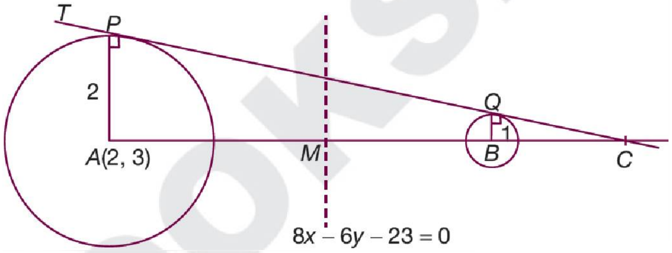
7. Now, equation of RP is $y-0=0\left(x-\sqrt{3}\right)\Rightarrow y=0$

Equation of QR is $y-\frac{3}{2}=\sqrt{3}\left(x-\frac{\sqrt{3}}{2}\right)\Rightarrow y=\sqrt{3}x$

Hence, the correct answer is (D).

Integer/Numerical Answer Type Questions

1.



$$AM=\left|\frac{8\times2-6\times3-23}{\sqrt{8^2+6^2}}\right|=\frac{5}{2}$$

$\Rightarrow\quad AB=2AM=5$

ΔAPC and ΔBQC are similar.

$$\frac{BC}{AC}=\frac{1}{2}\Rightarrow AC-AB=\frac{1}{2}AC$$

$\Rightarrow\quad AC=2\,AB=10$

Hence, the correct answer is (10).

2. Centre (–1, –2).

Geometrically, circle will have exactly 3 common points with axes in the cases.

(i) Passing through origin $\Rightarrow p=0$

(ii) Touching x-axis and intersecting y-axis at two points, i.e., $f^2>C$ and $g^2=C$, i.e., $4>-p$ and $1=-p$.

$\Rightarrow\quad p>-4$ and $p=-1$

$\Rightarrow\quad p=-1$

(iii) Touching y-axis and intersecting x-axis at two points, i.e., $f^2=c$ and $g^2>C$.

$\Rightarrow\quad 4=-p$ and $1>-p$

$\Rightarrow\quad p=-4$ and $p>-1$

It is not possible.

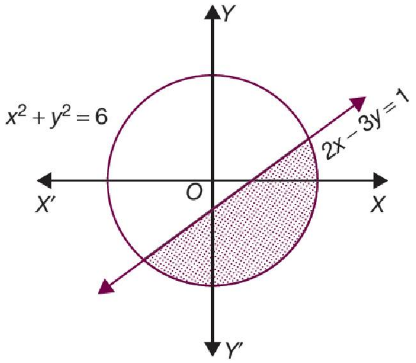
$\therefore$ Only two values of p are possible.

Hence, the correct answer is (2).

3. The smaller region of circle is the region given by

$$x^2+y^2\le 6 \qquad \ldots(1)$$

$$2x-3y\ge 1 \qquad \ldots(2)$$

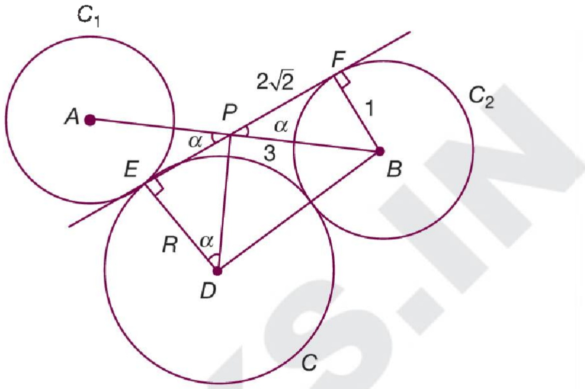


We observe that only two points $\left(2, \frac{3}{4}\right)$ and $\left(\frac{1}{4}, -\frac{1}{4}\right)$ satisfy both the inequalities (1) and (2).

$\therefore$ Two points lie inside the smaller part.

Hence, the correct answer is (2).

4. P is the midpoint of AB and DP is perpendicular to AB.



In ΔPFB, $\cos \alpha = \frac{2\sqrt{2}}{3}$ $\therefore \sin\alpha = \frac{1}{3}$

In ΔDEP,

$$\tan\alpha = \frac{2\sqrt{2}}{R}$$

$$\Rightarrow \quad R = \frac{2\sqrt{2}}{\tan\alpha} = 8 \text{ units}$$

Hence, the correct answer is (8).

CHAPTER 4: PARABOLA

Test Your Concepts-I

1. Compare the given equation with $ax^2+2hxy+by^2+2gx+2fy+c=0$

∴ $a=3, h=-9, b=27, g=1, f=7, c=-2$

$\Delta = abc+2fgh-af^2-bg^2-ch^2$

$=(3)(27)(-2)+2(7)(1)(-9)-3(7)^2-27(1)^2+2(-9)^2$

$=-162-126-147-27+162\neq 0$

Also, $h^2-ab=81-3.27=0$

So, we have, h^2-ab and $\Delta\neq 0$.

Hence, the given equation represents a parabola.

2. Given parabola $y^2=kx-8$

⇒ $y^2=k\left(x-\frac{8}{k}\right)$

Shifting the origin $Y^2=kX$, where $Y=0,\ X=\frac{x-8}{k}$

Directrix of standard parabola is $X=\frac{-k}{4}$

Directrix of original parabola is $x-\frac{8}{k}=\frac{k}{4}$

⇒ $x=\frac{8}{k}-\frac{k}{4}=1$

$32-k^2=4k$

⇒ $k^2+4k-32=0$

⇒ $(k-4)(k+8)=0$ or $k=-8, 4$

3. $x=t^2-2t+2,\ y=t^2+2t+2$

$x+y=2(t^2+2),\ y-x=4t$

⇒ $x+y=2\left\{\left(\frac{y-x}{4}\right)^2+2\right\}$

⇒ $x^2+y^2-2xy-8x-8y+32=0$

$h^2=ab, \Delta\neq 0$, which is a parabola

4. Length of latus rectum is $4\sqrt{2}=4a$ units

⇒ $a=\sqrt{2}$

Equation of directrix is $x+y=\lambda$

Whose distance from (4, 4) is $2\sqrt{2}$ units

∴ $\left|\frac{4+4-\lambda}{\sqrt{2}}\right|=2\sqrt{2}\Rightarrow 8-\lambda=\pm 4$

⇒ $\lambda=4$ or 12

∴ $\frac{\lambda_1+\lambda_2}{2}=8$

5. Let other end of diameter be (h, k), centre $\left(\frac{h+3}{2},\frac{k+4}{2}\right)$

Hence centre is $\sqrt{\left(\frac{3+h}{2}-3\right)^2+\left(\frac{k+4}{2}-4\right)^2}=\left|\frac{4+k}{2}\right|$

⇒ $\left(\frac{h-3}{2}\right)^2+16-4(k+4)=0$

⇒ $(h-3)^2=16k$

Locus (h, k) $(y-3)^2=16y$

which gives the equation of parabola.

6. The vertex is the midpoint of the perpendicular dropped from the focus N to the directrix and focus.

We get $N(0, 3)$ vertex $(-1, 2)$

7. $(x-1)^2+(y-3)^2=\left(\frac{5x-12y+17}{13}\right)^2$

Perpendicular distance from (1, 3) to the line $5x-12y+17=0$ is

$2a=\frac{|5\times1-12\times3+17|}{\sqrt{169}}=\frac{14}{13}$ units

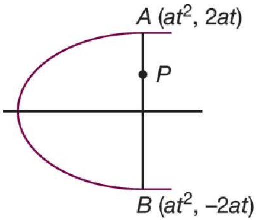
Length of latus rectum $=4a=\frac{28}{13}$

Test Your Concepts-II

1. Let AB be a double ordinate, where $A(at^2, 2at)$, $B(at^2, -2at)$. If $P(h, k)$ be its trisection point, which divides AB in the ratio 1:2, then $3h=2at^2+at^2$, $3k=4at-2at$

⇒ $t^2=\frac{h}{a}$, $t=\frac{3k}{2a}$

Thus, locus is $\frac{9k^2}{4a^2}=\frac{h}{a}$, i.e., $9y^2=4ax$.



2. $\frac{x-0}{\cos\theta}=\frac{y-2}{\sin\theta}=r$; $\tan\theta=\frac{3}{4}$, $\cos\theta=\frac{4}{5}$, $\sin\theta=\frac{3}{5}$

Now, any point on line

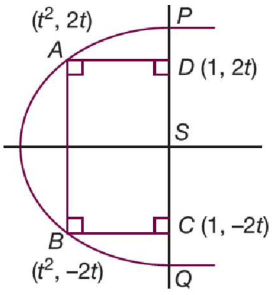
$$P(x_1, y_1)=(r\cos\theta, 2+r\sin\theta)=\left(\frac{4r}{5}, 2+\frac{3r}{5}\right)$$

If 'P' lies on parabola, then $16r^2 + 15r - 50 = 0$

Let, $PA = r_1$ and $PB = r_2$

$$\therefore \quad r_1 r_2 = PA \cdot PB = \frac{25}{8}$$

3. Take $A = (t^2, 2t)$



Perimeter of $ABCD = 2\,(1 - t^2 + 4t) = \dfrac{11}{2}$

$$\Rightarrow \quad t = \frac{1}{2} \text{ units}$$

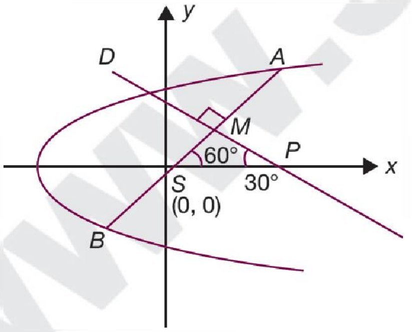
Now, area of $ABCD = 4t\,(1 - t^2) = \dfrac{3}{2}$ units

4. Let E be the midpoint of AB.

$$3x^2 - 8x - 16 = 0$$

$$x_1 + x_2 = \frac{8}{3}$$

$$\frac{x_1 + x_2}{2} = \frac{4}{3}$$



$\therefore$ x-co-ordinate of M is $\dfrac{4}{3}$; $M\left(\dfrac{4}{3}, \dfrac{4\sqrt{3}}{3}\right)$

So, $SM = \dfrac{8}{3}$, $SP \cos 60° = SM$

$$SP = \frac{16}{3}$$

5. Circle $(x-6)^2 + y^2 = 2$ and tangent is $y = m(x - 6) + \sqrt{2m^2 + 2}$

It is passing through (4, 0) focus of parabola

$$0 = -2m + \sqrt{2m^2 + 2} \Rightarrow 2m^2 + 2 = 4m^2$$

$$m^2 = 1 \Rightarrow m = \pm 1$$

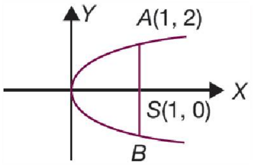
6. $FA = 4$, $FB = 5$

We know that $\dfrac{1}{a} = \dfrac{1}{AF} + \dfrac{1}{FB}$

$$\Rightarrow \quad a = \frac{20}{9} \Rightarrow 4a = \frac{80}{9}$$

7. If x is a perfect square, then px will be a perfect square, only if p is a perfect square, which is not possible as p is a prime number. Hence, y cannot be a perfect square. So, number of such points will be only one, i.e., (0, 0).

8. The parabola $y^2 = 4x$.
Here, $a = 1$ and focus is (1, 0)
The focal chord is ASB. This is clearly the latus rectum of parabola, whose value is 4 units.



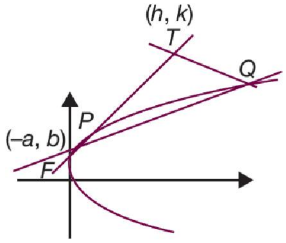
9. Since, the length of latus rectum is $\dfrac{8}{7}$, it is the smallest focal chord

Hence, focal chord of length $\dfrac{4}{7}$ does not exist.

Test Your Concepts-III

1. Chord of contact of (h, k) is $ky = 2a\,(x + h)$.

It passes through $(-a, b)$.



$\Rightarrow$ $bk = 2a\,(-a + h)$

$\therefore$ The required locus is $by = 2a\,(x - a)$.

2. Let the point (x_1, y_1) lie on the line $x + 2y = 4$.

$$\therefore \quad x_1 + 2y_1 = 4 \qquad \ldots(1)$$

Equation of chord of contact $T = 0$ or $yy_1 = 2(x + x_1)$

$\Rightarrow$ $yy_1 = 2x + 2(4 - 2y_1)$

$\Rightarrow$ $2(x + 4) - y_1(4 + y) = 0$

$\Rightarrow$ $L_1 + \lambda L_2 = 0$

For fixed point,

$L_1 = 0 \Rightarrow x + 4 = 0$ or $x = -4$

$L_2 = 0 \Rightarrow 4 - y = 0$ or $y = -4$

$\therefore$ Fixed point is $(-4, -4)$.

3. Any tangent of the parabola, $y = mx + \dfrac{a}{m}$

$$\Rightarrow \quad 4 = m + \frac{1}{m}$$

$$\Rightarrow \quad 4m = m^2 + 1$$

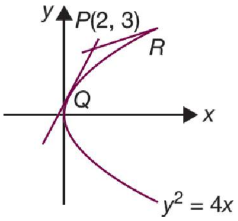
$$\Rightarrow \quad m^2 - 4m + 1 = 0$$

$$\Rightarrow \quad m_1 + m_2 = 4 \text{ and } m_1m_2 = 1$$

$$\therefore \quad |m_1 - m_2| = \sqrt{16-4} = 2\sqrt{3}$$

4. $t_1t_2 = 2, t_1 + t_2 = 3$

$$\Rightarrow \quad t_1 = 1 \text{ and } t_2 = 2$$



Hence, the points are $\left(t_1^2, 2t_1\right)$ and $\left(t_2^2, 2t_2\right)$, i.e., (1, 2) and (4, 4).

5. For $y^2 = 4(x + 2)$, the parametric co-ordinates are $(t^2 - 2, 2t)$. So, the equation of tangents is

$$y(2t) = 2(x + t^2 - 2) + 8$$

$$ty = x + t^2 + 2$$

$$A(-t^2 - 2, 0)$$

$$B\left(0, \frac{t^2+2}{t}\right)$$

$$\Rightarrow \quad h = -t^2 - 2, k = \frac{t^2+2}{t}$$

$$\Rightarrow \quad k = -\frac{h}{t} \Rightarrow t = -\frac{h}{k}$$

So, locus is $h = -2 - \left(\dfrac{h^2}{k^2}\right)$

$$h + 2 + \left(\frac{h^2}{k^2}\right) = 0$$

$$\Rightarrow \quad xy^2 + x^2 + 2y^2 = 0$$

$$\therefore \quad x^2 + y^2(x + 2) = 0$$

6. $d_1 = 4$ and $d_2 = 9$

Required distance $= \sqrt{d_1d_2} = \sqrt{4 \times 9} = 6$ units

7. $y = mx + \dfrac{1}{m}$ is the tangent to the parabola $y^2 = 4x$

$$\Rightarrow \quad y = m + \frac{1}{m} \Rightarrow m^2 - 4m + 1 = 0$$

$$m_1 + m_2 = 4,\ m_1m_2 = 1$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

8. Equation of tangent is $y = mx + \dfrac{a}{m}$

$$\Rightarrow \quad m^2x - my + a = 0$$

$$\Rightarrow \quad m_1 + m_2 = \frac{y}{x},\ m_1m_2 = \frac{a}{x}$$

$$\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)^2 - 4\left(\frac{a}{x}\right) = \left(1 + \frac{a}{x}\right)^2$$

$$\Rightarrow \quad x^2 - y^2 + 6ax + a^2 = 0$$

9. The tangents are drawn from $\left(0, -\dfrac{c}{b}\right)$ on y-axis, which is directrix of the given parabola $(x = 0)$.

$$\Rightarrow \quad \left(-\frac{a_1}{b}\right)\left(-\frac{a_2}{b}\right) = -1$$

$$\Rightarrow \quad a_1a_2 + b^2 = 0$$

10. The parabola be $(y - a)^2 = 4b(x - c)$

Equation of tangent is $(y - a) = -\dfrac{p}{q}(x - c) - \dfrac{bq}{p}$

Comparing with $px + qy = 1$, we get $cp^2 - bq^2 + apq - p = 0$

$$\therefore \quad \frac{c}{2} = \frac{b}{3} = \frac{a}{4} = 1$$

$\therefore$ The equation is $(y - 4)^2 = 12(x - 2)$

11. Solving tangent $y = 2x - 3$ and parabola $y^2 = 4a\left(x - \dfrac{1}{3}\right)$

$$\Rightarrow \quad (2x - 3)^2 = 4a\left(x - \frac{1}{3}\right)$$

$$\Rightarrow \quad 4x^2 + 9 - 12x = 4ax - \frac{4a}{3}$$

$$\Rightarrow \quad 4x^2 - 4(3+a)x + 9 + \frac{4a}{3} = 0$$

For tangency, $D = 0$.

$$\Rightarrow \quad 16(3+a)^2 - 16\left(9 + \frac{4a}{3}\right) = 0$$

On solving, $a = 0, \dfrac{-14}{3}$

However, a is non-zero.

So, $a = \dfrac{-14}{3}$ is acceptable.

12. Slope of tangent $= \dfrac{x_1}{2} = m$

$$\Rightarrow \quad x_1 = 2m \Rightarrow y_1 = m^2$$

So, equation of tangent is $y - m^2 = m(x - 2m)$ which passes through $(-1, -1)$

$$\Rightarrow \quad -1 - m^2 = m(-1 - 2m)$$

$$\Rightarrow \quad m^2 + m - 1 - 0 \Rightarrow m = \frac{-1 \pm \sqrt{5}}{2}$$

13. Any tangent is $y = mx + \dfrac{a}{m}$. Required difference is

$$\left[\frac{m(a+k) + a/m}{\sqrt{1+m^2}}\right]^2 - \left[\frac{m(a-k) + a/m}{\sqrt{1+m^2}}\right]^2$$

$$= \frac{1}{1+m^2} \times 4\left(ma + \frac{a}{m}\right)mk = 4ak$$

14. Image of $S(2, 3)$ w.r. to x-axis is $A(2, -3)$

Image of $S(2, 3)$ w.r. to y-axis is $B(-2, 3)$ directrix is AB $3x + 2y = 0$.

15. Common tangents to $y^2 = 4ax$ and $x^2 = 4ay$ is $xa^{\frac{1}{3}} + yb^{\frac{1}{3}} + a^{\frac{2}{3}}\, b^{\frac{2}{3}} = 0$

Here, $a = 8$, $b = 8$

$\therefore$ The required tangent is $x + y + 8 = 0$.

Test Your Concepts-IV

1. Parabola $y^2 = 4a(x - \ell_1)$ and $x^2 = 4a(y - \ell_2)$

Let point of contact is (x_1, y_1)

Tangent T_1, $yy_1 = 2a(x - \ell_1)$; $m_1 = \dfrac{2a}{y_1}$

Tangent T_2, $xx_1 = 2a(y - \ell_2)$; $m_2 = \dfrac{x_1}{2a}$

$$m_1 = m_2 \Rightarrow \frac{2a}{y_1} = \frac{x_1}{2a}$$

$$\Rightarrow \quad x_1 y_1 = 4a^2$$

So, the required locus of point of contact is $xy = 4a^2$.

2. $P = (t^2, 2t) \quad C(-3, 2)$

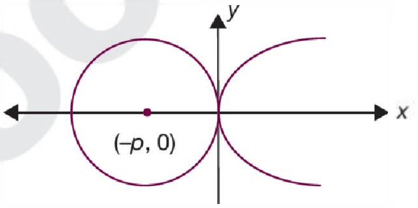
Circumcentre of ΔPAB = Midpoint of CP

$$\Rightarrow \quad \left(\frac{t^2 - 3}{2}, t + 1\right) = (x, y)$$

$$(y-1)^2 - 3 = 2x\,,\ (y-1)^2 = 2x + 3$$

Director circle of $(x+3)^2 + (y-2)^2 = 1$ is $(x+3)^2 + (y-2)^2 = 2$, which has no solutions with $x + 1 = 0$.

3. Centre of the circle is $(-p, 0)$. If it touches the parabola, then according to figure, only one case is possible.



Hence, $p > 0$

4. Let Q be $(at^2, 2at)$, (for $a = 2$)

$\therefore \quad Q$ be $(2t^2, 4t)$

Then, the locus of midpoint of PQ

$$x = \frac{1 + 2t^2}{2}, \ y = \frac{0 + 4t}{2}$$

$$\therefore \quad y^2 - 4x + 2 = 0$$

5. Let $P(at_1^2, 2at_1)\, Q(at_2^2, 2at_2)$ and $T(at_1t_2, a(t_1 + t_2))$

$$ST = \sqrt{(a(1 - t_1t_2))^2 + a^2(t_1 + t_2)^2}$$

$$= a\sqrt{1 + t_1^2 + t_2^2 + t_1^2t_2^2}$$

$$2\sqrt{10} = a\sqrt{(1 + t_1^2)(1 + t_2^2)}$$

Tangent at P $yt_1 = x + at_1^2$ and at Q $yt_2 = x + at_2^2$

$$P_1P_2 = \left|\frac{a + at_1^2}{\sqrt{1 + t_1^2}}\right|\left|\frac{a + at_2^2}{\sqrt{1 + t_2^2}}\right|$$

$$= a^2\sqrt{1 + t_1^2}\sqrt{1 + t_2^2} = 8\sqrt{10}$$

$$\Rightarrow \quad a = 4$$

6. Chord of contact of $(t_1t_2, t_1 + t_2)$ with respect to $y^2 = 4x$ is $(t_1 + t_1)y = 2(x + t_1t_2)$

$$\Rightarrow \quad \frac{(t_1 + t_2)y - 2x}{2t_1t_2} = 1$$

Now homogenization of parabola with chord of contact

$$y^2 = 4(x)(1)$$

$$\Rightarrow \quad y^2 = 4x\left(\frac{(t_1+t_2)y - 2x}{2t_1t_2}\right)$$

$\theta = 90°$, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \quad t_1t_2 + 4 = 0$

$\Rightarrow \quad t_1t_2 = -4$

$x = -4$

$\therefore \quad x + 4 = 0$

Test Your Concepts-V

1. Equation of normal $y = mx - 2am - am^3$, $a = 2$, $m = 2$

$\lambda = -2am - am^3$

$\therefore \quad \lambda = -24$

2. Given, parabola $y^2 = 4Kx$ and $y^2 = 4(x - 1)$

Normal to $y^2 = 4Kx$ is $y = mx - 2Km - Km^3$

Normal to $y^2 = 4(x - 1)$ is $y = m(x - 1) - 2m - m^3$

For common normal, $y = mx - 3m - m^3$

$\Rightarrow \quad -2Km - Km^3 = -3m - m^3$

$m \neq 0 \qquad -2K - Km^2 = -3 - m^2$

$m^2(K - 1) = 3 - 2K$

$m^2 = \dfrac{3-2K}{K-1} > 0 \Rightarrow \dfrac{2K-3}{K-1} < 0 \;\; K \in \left(1, \dfrac{3}{2}\right)$ for having common normal other than x-axis.

3. $P(at^2,\ 2at); Q\left(a\left(-t-\dfrac{2}{t}\right)^2, 2a\left(-t-\dfrac{2}{t}\right)\right)$

$$PQ = \sqrt{\left(\frac{4a}{t^2}+4a\right)^2 + \left(4at+\frac{4a}{t}\right)^2}$$

$$= 4a\sqrt{\frac{(1+t^2)^2}{t^4} + \frac{(1+t^2)^2}{t^2}}$$

$$PQ = \frac{4a(1+t^2)^{\frac{3}{2}}}{t^2}$$

$$\frac{dPQ}{dt} = 4a\left[\frac{t^2\frac{3}{2}(1+t^2)^{\frac{1}{2}}2t - (1+t)^{\frac{3}{2}}2t}{t^4}\right] = 0$$

$$\Rightarrow \quad t(1+t^2)^{\frac{1}{2}}[3t^2 - 2 - 2t^2] = 0$$

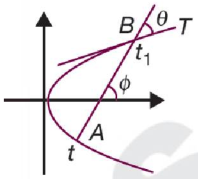
$$t^2 = 2$$

$PQ = 6a\sqrt{3}$, $OP = 2a\sqrt{3}$, $OQ = 4a\sqrt{6}$

4. Let P be $(t^2, 2t)$, then the normal at P is $y + tx = 2t + t^3$, which meets x-axis at $G(2 + t^2, 0)$.

As N is $(t^2, 0)$, $NG = 2$

5. Normal at t: $y + tx = 2at + at^2$



m_N at $A = -t = \tan\phi$

$t = -\tan\phi = m_1$

Now, tangent at B is $(t_1)y = xt + a\ t_1^2$

m_T at $A = \dfrac{1}{t_1} = m_2$

Also, $t_1 = -t - \dfrac{2}{t}$

$$\tan\theta = \left|\frac{\frac{1}{t_1}+t}{1-\frac{t}{t_1}}\right| = \left|\frac{1+tt_1}{t-t_1}\right| = \left|\frac{1+t^2}{2\left(t+\frac{1}{t}\right)}\right| = \frac{t}{2}$$

(As $t\,t_1 = -t^2 - 2$)

Hence, $\tan\theta = \dfrac{\tan\phi}{2}$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{\tan\phi}{2}\right)$$

6. $x_1 = at_1t_2$, $y_1 = a(t_1 + t_2)$; $x_2 = a(t_1^2 + t_2^2 + t_1t_2 + 2)$, $y_2 = -at_1t_2(t_1 + t_2)$ with $t_1t_2 = -1$

$x_1 = -a$, $y_1 = a(t_1 + t_2)$; $x_2 = a(t_1^2 + t_2^2 + 1)$; $y_2 = a(t_1 + t_2)$

Clearly, $y_1 = y_2$

7. $t_1t_2 = -1$

N: $y + tx = 2at + at^3$, which passes through (h, k).

Hence, $at^3 + (2a - h)t + k = 0$; $t_1\,t_2\,t_3 = -\dfrac{k}{a}$; $t_1\,t_2 = -1$

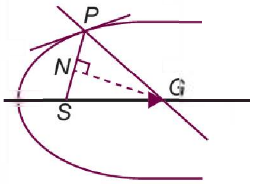
Chord joining t_1 and t_2 is $2x - (t_1 + t_2)y + 2at_1\,t_2 = 0$

$(2x - 2a) - (t_1 + t_2)y = 0$

$\Rightarrow \quad x = a$ and $y = 0$

$\therefore \quad$ The fixed point is $(a, 0)$.

8.



Normal to parabola $y^2 = 4ax$

$$y + tx = 2at + at^3$$

meets x-axis at $G(2a + at^2, 0)$

$$P(at^2, 2at) \quad S(a, 0)$$

$$SP \qquad y - 0 = \frac{2t}{t^2 - 1}(x - a)$$

$$2tx - (t^2 - 1)y - 2at = 0$$

$$GN = \frac{2t(2a + at^2) - 0 - 2at}{\sqrt{4t^2 + (t^2 - 1)^2}}$$

$$= \frac{2at + 2at^3}{t^2 + 1} = 2at$$

$$NP = \sqrt{PG^2 - GN^2} = \sqrt{4a^2 + 4a^2t^2 - 4a^2t^2}$$

Given parabola

$$\left(\frac{12x + 5y + 3}{13}\right)^2 = 4\left(\frac{5x - 12y - 1}{13}\right)$$

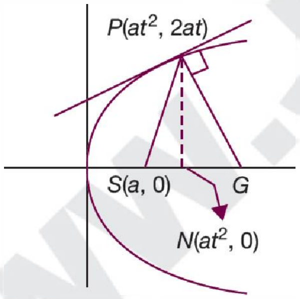
$$NP = 2a = \frac{1}{2}\ell_{LR}$$

$$\ell_{LR} = 4$$

NP is equal to semi–latus rectum = 2

9. The circle passes through focus (2, 0)
We get $c = 4$ and radius as 3.

10. $y^2 = 4ax$ is the parabola



Given, N divides SG internally in the ratio 1:3.

$$S = (a, 0), N = (at^2, 0)$$

Equation of the normal PG is $Y = -t + 2at + at^3$.
It meets x-axis at $y = 0$

$$tx = 2at + at^3$$

$$x = 2a + at^2$$

$$G = (at + at^2, 0)$$

N divides SG internally in the ratio 1:3

$$at^2 = \frac{2a + at^2 + 3a}{4}$$

Or $\quad 3at^2 = 5a$

$$\Rightarrow \quad t = \pm\sqrt{\frac{5}{3}}$$

11. Let t_1, t_2 and t_3 be the co-normal points drawn from (x_1, y_1) to $y^2 = 4ax$.

Equation of the normal at point 't' to $y^2 = 4ax$ is $y + tx = 2at + at^3$.

This passes through (x_1, y_1)

$$\Rightarrow \quad y_1 + tx_1 = 2at + at^3$$

$$\Rightarrow \quad at^3 + (2a - x_1)t - y_1 = 0$$

t_1, t_2 and t_3 are the roots of the equation.

$$\therefore \quad t_1 + t_2 + t_3 = 0$$

$\therefore$ The ordinate of the centroid of the triangle formed by the points t_1, t_2 and t_3 is $\frac{2a}{3}(t_1 + t_2 + t_3) = 0$.

12. $y + tx = 2t + t^3$, passing through (6, 0)

$$6t = 2t + t^3$$

$$\Rightarrow \quad t^2 + 2 - 6 = 0$$

$$t = \pm 2$$

$\therefore A(4, 4)$

$\therefore$ Minimum distance $= \sqrt{4 + 16} - \sqrt{5} = \sqrt{20} - \sqrt{5}$

$= \sqrt{5}$ unit

Single Correct Choice Type Questions

1. $x = t^2 + 2t + 3 = (t + 1)^2 + 2 = y^2 + 2$

Hence, the correct answer is (C).

2. $(y - 1)^2 = a\left(x + \frac{3}{a}\right)$

Equation of directrix is $x + \frac{3}{a} + \frac{a}{4} = 0$ and the given equation is $x + a = 0$, then $\frac{3}{a} + \frac{a}{4} = a$

$$4a^2 = 12 + a^2$$

$$\therefore \quad a = 2$$

Hence, the correct answer is (A).

3.

$$\left.\begin{aligned} x = t^2 + t + 1 &\Rightarrow t^2 + t + 1 - x = 0 \\ y = t^2 + 2t + 3 &\Rightarrow t^2 + 2t + 3 - y = 0 \end{aligned}\right\} \text{eliminate } t$$

$$\frac{t^2}{3 - y - 2 + 2x} = \frac{t}{1 - x - 3 + y} = \frac{1}{1}$$

$$\left.\begin{aligned} t &= -x + y - 2 \\ t &= \frac{1 - y + 2x}{-x + y - 2} \end{aligned}\right\} (x - y + 2)^2 = (2x - y + 1)$$

$$(x - y)^2 + 4(x - y) + 4 = (2x - y + 1)$$

$$(x - y)^2 = -2x + 3y - 3$$

$$\therefore \quad (x - y + \lambda)^2 = -2x + 3y - 3y - 3 + 2\lambda(x - y) + \lambda^2$$

$$(x - y + \lambda)^2 = x(2\lambda - 2) + y(-2\lambda + 3) + \lambda^2 - 3$$

$$\therefore \left.\begin{array}{l}\text{Slope of } x-y+1=0 \text{ is } 1\\ \text{Slope line on RHS is } \dfrac{2-2\lambda}{3-2\lambda}\end{array}\right\} \dfrac{2-2\lambda}{3-2\lambda}=-1$$

$$2-2\lambda=-3+2\lambda$$

$$4\lambda=5\Rightarrow\lambda=\frac{5}{4}$$

Equation of parabola is $\left(x-y+\frac{5}{4}\right)^2=\frac{x}{2}+\frac{y}{2}+\frac{25}{16}-3$

$$\left(x-y+\frac{5}{4}\right)^2=\frac{1}{2}\left(x+y-\frac{23}{16}\right)$$

$$\left(\frac{x-y+\frac{5}{4}}{\frac{1}{\sqrt{2}}}\right)^2=\frac{1}{2\sqrt{2}}\left(\frac{x+y-\frac{23}{16}}{\sqrt{2}}\right) \quad \text{LR}=\frac{1}{2\sqrt{2}}$$

Hence, the correct answer is (C).

4. $25\left(x-\frac{1}{5}\right)^2+25\left(y-\frac{2}{5}\right)^2=25\lambda\left(\frac{3x-4y-1}{5}\right)^2$

$$\Rightarrow \left(x-\frac{1}{5}\right)^2+\left(y-\frac{2}{5}\right)^2=\lambda\left(\frac{3x-4y-1}{5}\right)^2$$

$$\Rightarrow \lambda=1$$

Hence, the correct answer is (A).

5. $y=\frac{a^3x^2}{3}+\frac{a^2x}{2}-2a$

$$y=\frac{2a^3}{6}\left(x^2+\frac{3}{2a}x-\frac{12a}{2a^3}\right)$$

$$y=\frac{2a^3}{6}\left(x^2+2.\frac{3}{4a}x+\frac{9}{16a^2}-\frac{9}{16a^2}-\frac{12a}{2a^3}\right)$$

$$y=\frac{2a^3}{6}\left(\left(x+\frac{3}{4a}\right)^2-\frac{105}{16a^2}\right)$$

$$\left(y+\frac{105a}{48}\right)=\frac{2a^3}{6}\left(x+\frac{3}{4a}\right)^2$$

Vertex of parabola is $x=\frac{-3}{4a}$, $y=\frac{-105a}{48}$

$$xy=\frac{105a}{48}\times\frac{3}{4a}=\frac{105}{64}$$

Hence, the correct answer is (A).

6. Equation of tangent to $y^2=36x$ at (x_1, y_1) is $yy_1=18(x+x_1)$, which meets axis at $(-x_1,0)$, $\left(0,\frac{18x_1}{y_1}\right)$ and centroid $=(x,y)=\left(\frac{-x_1}{3},\frac{6x_1}{y_1}\right)$

$$x_1=-3x \text{ and } y_1=\frac{6x_1}{y}$$

$$y_1{}^2=\frac{36x_1{}^2}{y^2}$$

$$36x_1=\frac{36x_1{}^2}{y^2}$$

$$y^2=x_1\Rightarrow y^2=-3x$$

Hence, the correct answer is (B).

7. $y^2=4ax$, $x^2=4ay$ intersect at $(0, 0)$, $(4a, 4a)$.

So, common chord equation is $x-y=0$.

Given, $2bx+3cy+4d=0$ is a common chord.

$$\Rightarrow d=0,\ 2b+3c=0$$

$$\therefore d^2+(2b+3c)^2=0$$

Hence, the correct answer is (A).

8. Equidistant points from focus are symmetric about axis of parabola and perpendicular to axis.

$\therefore$ Tangent at vertex is parallel to the line joining the two points.

$\therefore$ Its slope is $p=\frac{3-2}{2-3}=-1$

Hence, the correct answer is (B).

9. Given parabolas are symmetric about the line $y=x$, so they have a common normal with slope 1, which meets the parabolas at $\left(\frac{-1}{2},\frac{13}{4}\right)$, $\left(\frac{13}{4},\frac{-1}{2}\right)$. Hence, the required circles is $x^2+y^2-\frac{11}{4}x-\frac{11}{4}y-\frac{13}{4}=0$.

Hence, the correct answer is (B).

10. If $A=\left(at_1^2, 2at_1\right)$, $B\left(at_2^2, 2at_2\right)$ be the extremities of a focal chord for the parabola $y^2=4ax$, then

$$t_1t_2=-1.$$

Let, $C(\alpha,\beta)$ be the centre of the circle A and B as the extremities of the diameter.

$$\alpha=\frac{a}{2}\left(t_1^2+t_2^2\right) \text{ and } \beta=a\left(t_1+t_2\right)$$

Eliminating t_1, t_2, we get $\beta^2=2a(\alpha-a)$

Locus of $C(\alpha,\beta)$ is $y^2=2a(x-a)$

Hence, the correct answer is (C).

11. $x=aT^2$, $y=2aT$ represents a parabola.

In (A), $a=1$, $T=t+1$

In (B), $a=1$, $T=(t-1)$

In (C), $a = 3$, $T = \sin t$

However, in (D), if $2aT = 2a \cos t$

$\Rightarrow T = \cos t$, which does not satisfy $x = aT^2$.

Hence, the correct answer is (D).

12. Let the vertices be $O(0, 0)$, $A(at^2, 2at)$, $B\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ then

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ at^2 & 2at & 1 \\ \frac{a}{t^2} & \frac{-2a}{t} & 1 \end{vmatrix} = \frac{5a^2}{2}$$

$\Rightarrow \quad 2t^2 - 5t + 2 = 0$

$\Rightarrow \quad t = 2 \text{ or } \frac{1}{2}$

So, the vertices of a focal chord are $(4a, 4a)$ and $\left(\frac{a}{4}, -a\right)$.

By taking $t = 2$, the length of this focal chord is $25\frac{a}{4}$.

Hence, the correct answer is (C).

13. One extremity of the focal chord be $(2at, at^2)$. Equation of the tangent is $tx = y + at^2$, which meets the tangent at the vertex, $y = 0$ at $x = at$.

So, $x_1 = at$ and $x_2 = a\left(-\frac{1}{t}\right)$

Thus, $x_1x_2 = -a^2$.

Hence, the correct answer is (D).

14. Focus of parabola is (1, 0)

$\Rightarrow$ Diagonals are focal chords

As $= 1 + t^2 = CE \qquad \frac{1}{C} + \frac{1}{\frac{25}{4} - c} = 1 \qquad \Rightarrow C = \frac{5}{4}, 5$

For $C = \frac{5}{4} \qquad t = \pm\frac{1}{2}$

For $C = 5 \qquad t = \pm 2$

$\Rightarrow \quad A = \left(\frac{1}{4}, 1\right) \quad B = (4, 4) \quad C = (4, -4) \quad D = \left(\frac{1}{4}, -1\right)$

$AD = 2$ and $BC = 8$

Thus, the distance between AD and $BC = \frac{15}{4}$ units

Area of trapezium $= \frac{75}{4}$ sq. units

Hence, the correct answer is (A).

15. Equation of normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$ and for $x^2 = 4by$ is $y = mx + 2b + \frac{b}{m^2}$

We get, $2b + \frac{b}{m^2} + 4m + am^3 = 0$

$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$

$\therefore$ Maximum number of normals is 5.

Hence, the correct answer is (C).

16. Roots of $ax^2 + bx + c = 0$ are α and β

$$\alpha + \beta = -\frac{b}{a}, \ \alpha\beta = \frac{c}{a}$$

Equation of circle through $(\alpha, 0)$ and $(\beta, 0)$ is $S = (x - \alpha)(x - \beta) + y^2 + \lambda y = 0$.

Length of tangent from origin $= \sqrt{\alpha\beta} = \sqrt{\frac{c}{a}}$

Hence, the correct answer is (D).

17. $y = ax^2 + bx + c$, vertex (4, 2)

$\Rightarrow \quad b = -8a$

$$\frac{4ac - b^2}{4a} = 2$$

$\Rightarrow \quad c = 2 + 16a$

$\therefore$ Product $abc = \Delta = -16(a^2 + 8a^3) \quad \frac{d\Delta}{da} < 0 \ \forall a \in [1,3]$

$\Rightarrow \quad \Delta_{\max} = -144$ and $\Delta_{\min} = -3600$

$\therefore \quad$ Difference is 3456

Hence, the correct answer is (C).

18. Give equation is $(x + y)^2 = 6x + 2y - 3$

$$(x + y + \lambda)^2 = 6x + 2y - 3 + 2\lambda x + 2\lambda y + \lambda^2$$

$\because \quad \left(\frac{2\lambda + 6}{2\lambda + 2}\right) = -1$ Then, $\lambda = -2$

$\therefore \quad (x + y - 2)^2 = 2x - 2y + 1$

Substitute $\frac{x + y - 2}{\sqrt{2}} = Y$ and $\frac{2x - 2y + 1}{\sqrt{8}} = X$

$$Y^2 = \sqrt{2}X, \qquad 4a = \sqrt{2}$$

Focus $X = a$, $Y = 0$

$$x - y + 1 = 1, \ x + y = 2$$

Solving will give focus as (1, 1).

Hence, the correct answer is (D).

19. For points $P(t_1)$ and $Q(t_2)$, $l(at^2) + m(2at) = 1$

Product of roots $= t_1t_2 = \frac{-1}{la} = 2$

So, $a = \frac{-1}{2l}$

Hence, the correct answer is (D).

20. Equation of circle is $x^2+y^2=\frac{9a^2}{4}$ and parabola is $y^2=4ax$

So, for point of intersection, on solving, we get,

$$x^2+4ax-\frac{9a^2}{4}=0$$

$\Rightarrow \quad (2x-a)(2x+9a)=0$

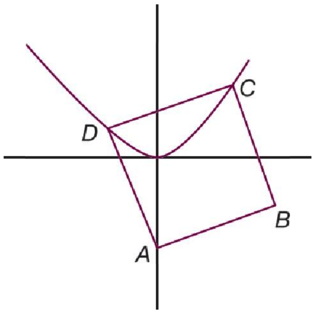
Or $x=\frac{a}{2},\frac{-9a}{2}$; $x=\frac{-9a}{2}$ rejected (quadrants I and IV only)

Hence, the common chord cuts the axis at $\left(\frac{a}{2},0\right)$.

So, the required ratio is 1:1.

Hence, the correct answer is (A).

21.



Let $\overline{AB}\equiv y=2x-17$ and $\overline{CD}=y=2x+c$, where

$C=(x_1,y_1)$ and $D=(x_2,y_2)$

$$CD=\frac{y_1-y_2}{x_1-x_2}=2 \qquad \ldots(1)$$

Let, $CD=a=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$\Rightarrow \quad a^2=5(x_1-x_2)^2$ (From (1))

$$=5\left[(x_1-x_2)^2-4x_1x_2\right] \qquad \ldots(2)$$

$\therefore \quad y=x^2, y=2x+c\Rightarrow x^2-2x-c=0$

$\therefore \quad x_1+x_2=2, x_1x_2=-c$

From (2) $\Rightarrow a_2=5(4+4c)$ $\qquad \ldots(3)$

Let $A=(k,2k-17)$ be any point on $y=2x-17$

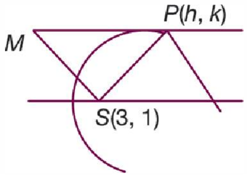
$$\therefore \quad AD=a=\frac{|2k-2k+17+c|}{\sqrt{5}}\Rightarrow a^2=\frac{(17+c)^2}{5} \qquad \ldots(4)$$

From (4) and (5), $a^2=1280$

Hence, the correct answer is (B).

22. Image of (3, 1) with respect to $x+2y-10=0$ is (5, 5).

The horizontal line through (5, 5) is $y=5$, which meets $x+2y-10=0$ at P.



$\Rightarrow \quad P=(0,5)\Rightarrow h+k=5$, $SP=PM=5$

$\Rightarrow$ Perpendicular distance from (0, 5) to $x+k=0$ is 5

$\Rightarrow \quad k=5$

Equation of directrix is $x+5=0$, $M=(-5,5)$

$$\cos(\angle SPM)=\frac{25+25-80}{2\times5\times5}=\frac{-3}{5}\Rightarrow \sin(\angle SPM)=\frac{4}{5}$$

Area of $\Delta SPM=\frac{1}{2}\times5\times5\times\frac{4}{5}=10$ sq. units

Hence, the correct answer is (A).

23. Required area is half of the area of triangle formed by the points (4, 6), (10, 8) and (2, 4)

$\therefore$ The required area is 2 sq. units.

Hence, the correct answer is (B).

24. Portion of tangent intercepted between parabola and directrix subtends a right angle at the focus.

Hence, the correct answer is (C).

25. $A(2t_1,t_1^2) \qquad B(2t_2,t_2^2)$

Again $P(-1,0)$, A and B are collinear.

$$\Rightarrow \quad \frac{t_2^2-t_1^2}{2(t_2-t_1)}=\frac{t_2^2}{2t_2+1}$$

$\Rightarrow \quad (t_1+t_2)=-2t_1t_2$ $\qquad \ldots(1)$

Let, the centroid is (h,k)

$$h=\frac{2(t_1+t_2)}{3} \qquad \ldots(2)$$

$$k=\frac{t_1^2+t_2^2}{3} \qquad \ldots(3)$$

From (1), (2) and (3), eliminate t_1 and t_2

$$3h^2+2h=4k$$

Hence, locus is $3x^2+2x=4y$.

Hence, the correct answer is (C).

26. Line PQ is $(t_1+t_2)y=2(x+at_1t_2)$ $\qquad \ldots(1)$

$\lambda y=-x+b$ $\qquad \ldots(2)$

Equations (1) and (2) are same.

$$\frac{t_1+t_2}{\lambda}=\frac{2}{-1}=\frac{2at_1t_2}{b}$$

$\therefore \quad b=-at_1t_2$ and $2a\le -at_1t_2\le 4a$

$\therefore \quad -4\le t_1t_2\le -2$

Hence, the correct answer is (A).

27. Let tangent at $P\left(at^2, 2at\right)$ make an angle θ with x-axis,

then $\tan\theta = \frac{1}{t}$

Projection of BC on tangent

$$= BC\sin\theta = \frac{4a}{\sqrt{1+t^2}} \geq 2a\sqrt{2} \qquad (\because - \leq t \leq 1)$$

Hence, the correct answer is (B).

28. Any point on $x + y = 1$ is $(t, 1 - t)$.

Equation of chord with this point is $y(1-t) - 2a(x+t) = (1-t)^2 - 4at$

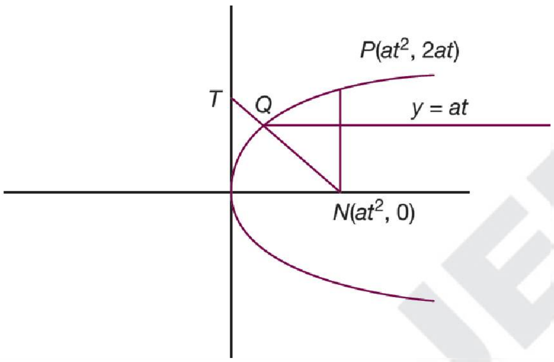
It passes through $(a, 2a) \Rightarrow t^2 - 2t + 2a^2 - 2a + 1 = 0$

This should have two district real roots

$\Rightarrow \quad a^2 - a < 0 \Rightarrow 0 < a < 1 \Rightarrow 0 < 4a < 4$

Hence, the correct answer is (B).

29.



Equation of the line parallel to the axis and bisecting the ordinate PN of the point $P\left(at^2, 2at\right)$ is $y = at$, which meet the parabola $y^2 = 4ax$ at the point $Q\left(\frac{1}{4}at^2, at\right)$

Co-ordinates of N are $\left(at^2, 0\right)$

Equation of NQ is

$y - 0 = \dfrac{0 - at}{at^2 - \frac{1}{4}at^2}\left(x - at^2\right)$, which meets the tangent at the vertex.

$x = 0$ at the point $y = \frac{4}{3}at$

Hence, the correct answer is (A).

30. Let the normals are drawn from $P(h, k)$ to the parabola $y^2 = 4x$.

Then, $k = mh - 2m - m^3$, i.e., $m_1 + m_2 + m_3 = 0$

The equation of the normal at $(1, 2)$ is $x + y = 3$ ...(1)

The equation of the normal at $(1, -2)$ is $x - y = 3$ ...(2)

From (1) and (2), we have,

$x = 3 \quad y = 0$

$\therefore \quad h = 3, \ k = 0$

The equation of the third normal is $y - 0 = 0(x - 3)$

$y = 0, \quad x = 0$

$\therefore$ The third foot is $(0, 0)$.

Hence, the correct answer is (B).

31. $A(0, 2)$, $B\left(t_1^2 - 4, t_1\right)$, $C(t^2 - 4, t)$

$$\frac{2 - t_1}{4 - t_1^2} \cdot \frac{t_1 - t}{t_1^2 - t^2} = -1$$

$$\Rightarrow \quad \frac{1}{2 + t_1} \cdot \frac{1}{t + t_1} = -1$$

$$\Rightarrow \quad t_1^2 + (2 + t)t_1 + (2t + 1) = 0$$

For real t_1, $(2 + t)^2 - 4(2t + 1) \geq 0$

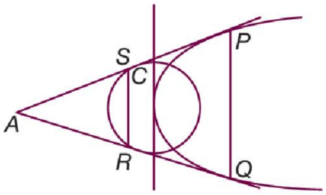
$\Rightarrow \quad t^2 - 4t \geq 0$

$\Rightarrow \quad t \in (-\infty, 0] \cup [4, \infty)$

Hence, the correct answer is (B).

32. Tangent at t on the parabola is $x - yt + at^2 = 0$. If it is also a tangent to the circle

$\Rightarrow \quad t = \pm 1$



$R\left(\frac{-a}{2}, \frac{-a}{2}\right) S\left(\frac{-a}{2}, \frac{a}{2}\right)$ $P(a, 2a)$, $Q(a, -2a)$.

Area $= \frac{1}{2}(a + 4a)\left(a + \frac{a}{2}\right) = \frac{15}{4}a^2$ sq. units

Hence, the correct answer is (C).

33. Let foot of perpendicular from Q to the directrix be N.
$\Rightarrow SQ + PQ = QN + PQ$ is minimum if P, Q and N are collinear.
So, minimum value of $SQ + PQ = PN = 1 - b$

Hence, the correct answer is (D).

34. Normal is $y = mx - 2m - m^3 \Rightarrow 0 = 2km - 2m - m^3$

Let m_1, m_2, m_3 be the roots, then $m_3 = 0$ and $m_1 m_2 = -1$

$\Rightarrow \quad k = \frac{3}{2}$

Hence, the correct answer is (A).

35. Any point on the parabola is $(x, x^2 + 7x + 2)$

Its distance from the line $y = 3x - 3$ is given by

$$P = \left|\frac{3x-(x^2+7x+2)-3}{\sqrt{9+1}}\right| = \left|\frac{x^2+4x+5}{\sqrt{10}}\right| = \frac{x^2+4x+5}{\sqrt{10}}$$

(as $x^2 + 4x + 5 > 0\ \forall\ x \in R$)

$$\frac{dp}{dx} = 0 \Rightarrow x = -2$$

$\therefore$ The required point is $(-2, -8)$.

Hence, the correct answer is (A).

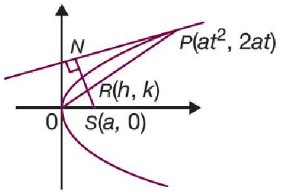
36. We know that any side of the triangle is more than the difference of the remaining two sides.

So, $|PR - PQ| < RQ$

The required point P will be the point of intersection of the line RQ with parabola which is $(a, 2a)$ as PQ is a tangent to the parabola.

Hence, the correct answer is (A).

37. T: $ty = x + at^2$ $\quad\ldots(1)$



Line perpendicular to (1) through $(a, 0)$ is

$$tx + y = ta \quad\ldots(2)$$

Equation of OP: $y - \frac{2}{t}x = 0$ $\quad\ldots(3)$

From Equations (2) and (3), by eliminating t, we get, locus as $x^2 + 2y^2 - ax = 0$.

Hence, the correct answer is (A).

38. It is a well-known property of a parabola that a tangent and normal to it from focus intersects at tangent at vertex.

Hence, the correct answer is (A).

39. By solving equation of parabola with x-axis $(y = 0)$, we get $(a - b)x^2 + (b - c)x + (c - a) = 0$, which should have two equal values of x, as x-axis touches the parabola.

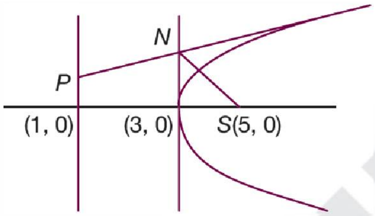
$\Rightarrow\quad (b - c)^2 - 4(a - b)(c - a) = 0$

$\Rightarrow\quad (b + c - 2a)^2 = 0 \Rightarrow 2a + b + c = 0 \Rightarrow ax + by + c = 0$ always passes through $(-2, 1)$

Hence, the correct answer is (A).

40. $\angle QNS = 90°$

x-co-ordinate of $N = 3$



Hence, the correct answer is (C).

41. The equation of the tangent at $(-3, 2)$ to the parabola $y^2 + 4x + 4y = 0$ is $2y + 2(x - 3) + 2(y + 2) = 0$

$\Rightarrow\quad x + 2y - 1 = 0$

The tangent at one end of the focal chord is parallel to the normal at the other end.

$\Rightarrow\quad$ Slope of normal at Q = Slope of tangent at $P = -\frac{1}{2}$

Hence, the correct answer is (A).

42. Let the focus be (h, k).

Feet of the perpendicular from (h, k) on to tangents are (a, k) (h, b).

$$\text{Slope of directrix} = \frac{b-k}{h-a}$$

$$\Rightarrow \quad \frac{b-k}{h-a} = m$$

The locus is $y + mx = am + b$

Hence, the correct answer is (A).

43. For $x = 1$, $y = a + b + c$

Tangent at $(1, a + b + c)$ is

$$\frac{1}{2}(y + a + b + c) = ax + \frac{b}{2}(x + 1) + c$$

Comparing with $y = 2x$, $c = a$, $b = 2(1 - a)$

Which are true for only option (A).

Hence, the correct answer is (A).

44. The circle, whose diameter ends are $(at^2, 2at)$ $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - at^2)\left(y + \frac{2a}{t}\right) = 0 \quad\ldots(1)$$

Let t_1, t_2, t_3, t_4 be the points of intersection of (1) and parabola $y^2 = 4ax$, where t_1, t_2 are the ends of diameter, then $t_1t_2t_3t_4 = \frac{-3a^2}{a^2}$

$\therefore\quad t_3t_4 = 3$

Hence, the correct answer is (B).

45. Let, A = vertex, AP = chord of $x^2 = 4y$ such that slope of AP is $\cot\alpha$

Let, $P = (2t, t^2)$

Slope of $AP = \frac{1}{2}$

$\Rightarrow \quad \cot a = \frac{1}{2}$

$\Rightarrow \quad t = 2 \cot \alpha$

Now, $AP = \sqrt{4t^2 + t^4} = t\sqrt{4 + t^2}$

$= 4 \cot \alpha \operatorname{cosec} \alpha$

$= 4 \cos \alpha \operatorname{cosec}^2 \alpha$

Hence, the correct answer is (A).

46. The family of curves passing through the points of intersection of two parabolas is

$$y^2 - 3ax + \lambda(x^2 + ax + 5 - 2y) = 0 \qquad \ldots(1)$$

Since, Equation (1) is circle, a belongs to $(-\infty, -2) \cup (2, \infty)$.

Hence, the correct answer is (D).

47. Line of shortest distance is normal for both parabola and circle.

Centre of circle is (2, 1).

Equation of normal to circle is $y - 1 = m(x - 2)$

$\Rightarrow \quad y = mx + (1 - 2m) \qquad \ldots(1)$

Equation of normal for a parabola is $y = mx - 2am - am^3 \qquad \ldots(2)$

Comparing (1) and (2),

$am^3 = -1 \Rightarrow m^3 = 1 \Rightarrow m = -1 \qquad (a = 1)$

Equation is $y - 1 = -x + 2 \Rightarrow x + y = 3$

Hence, the correct answer is (A).

48. $(x + 5)^2 = 16y$ comparing it with $x^2 = 4ay$,

We get extremities of LR (3, 4), (–13, 4)

Hence, the correct answer is (C).

49. $y = mx + \frac{a}{m}$

$\Rightarrow \quad m^2x - 3y + a = 0,\ m_1 \cdot m_2 = -1$

Hence, the correct answer is (C).

50. The parabola is $(x - 1)^2 = 4(y - 1)$

Equation of circle $(x - 1)^2 + (y - r - 2)^2 = r^2$

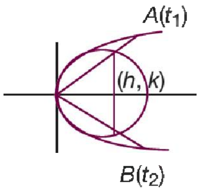
Solving with one $y^2 + \{4 - 2(r + 2)\}y + 4r = 0$

It has equal roots $D = 0$

$\Rightarrow \quad r = 4$

Hence, the correct answer is (D).

51. $t_1t_2 = -4 \qquad A(at_1^2, 2at_1)\ B(at_2^2, 2at_2)$



$P\left(\frac{at_1^2}{2}, at_1\right) \qquad Q\left(\frac{at_2^2}{2}, at_2\right)$

C (h, k)

$h = \frac{a}{4}\left(t_1^2 + t_2^2\right), k = \frac{a}{2}(t_1 + t_2)$

$k^2 = \frac{a^2}{4}\left(t_1^2 + t_2^2 + 2t_1t_2\right) = a \cdot \frac{a}{4}\left(t_1^2 + t_2^2\right) - 2a^2$

$k^2 + 2a^2 = a \cdot h$

$\Rightarrow \quad y^2 = a(x - 2a)$

Hence, the correct answer is (B).

52. $y + 4 = x^2$

$x^2 = 4 \cdot \frac{1}{4}(y + 4)$

Focal distance $= \frac{25}{4}$ units

Distance from directrix $y = \frac{-15}{4}$ units

Ordinate of points on the parabola, whose focal distance is $\frac{25}{4}$ units

$= \frac{-17}{4} + \frac{25}{4} = 2$

$\therefore$ The points are $\left(\pm\sqrt{6}, 2\right)$

$\Rightarrow \quad a + b = 8$

Hence, the correct answer is (A).

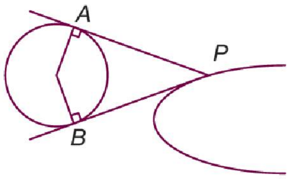
53. Length of side $= 8\sqrt{3}a = 8\sqrt{3}\frac{1}{2} = 4\sqrt{3}$ units

Hence, the correct answer is (B).

54. $P(t^2, 2t), C(-3, 2)$

$APBC$ is a cyclic quadrilateral.

Circumcentre of ΔPAB is the midpoint of CP.



$h = \frac{t^2 - 3}{2}$

$\Rightarrow \quad t^2 = 2h + 3; \qquad k = \frac{2t + 2}{2}$

$\Rightarrow \quad t = k - 1$

$\therefore$ Locus $(y - 1)^2$ is $2x + 3$.

Hence, the correct answer is (D).

55. Length of double ordinate = $8\sqrt{2}$ units

Hence, the correct answer is (B).

56. Equation of pair of tangent is $SS_1 = T^2$.

$\Rightarrow \quad (y^2 - 4x)(8) = 4(y - x + 1)^2$

$\Rightarrow \quad y^2 - 2y(1 - x) - (x + 6x + 1) = 0$

Put $x = 2$, we get,

$\Rightarrow \quad y^2 + 2y - 17 = 0$

$\Rightarrow \quad |y_1 - y_2| = 6\sqrt{2}$

Hence, the correct answer is (B).

57. Let $P(t_1)$ and $Q(t_2)$ be the points on $y^2 = 8x$. Here, $4a = 8$ or $a = 2$

$$\text{Required distance} = z = at_2^2 = a\left(t_1^2 + \frac{4}{t_1^2} + 4\right)\left(\because t_2 = -\frac{2}{t_1}\right)$$

Z is least, if $\dfrac{dz}{dt_1} = 0$ or $t_1^2 = 2$

$\therefore$ Least value of Z is 16.

Hence, the correct answer is (A).

58. QR is a focal chord

$\Rightarrow \quad R(at^2, 2at)$ and $Q\left(\dfrac{a}{t^2}, \dfrac{2a}{t}\right)$

$$\Rightarrow \quad d = \left|2at + \frac{2a}{t}\right| = 2a\left|t + \frac{1}{t}\right|$$

$$\text{Now, } A = \frac{1}{2}\begin{vmatrix} at^2 & 2at & 1 \\ \frac{a}{t^2} & -\frac{2a}{t} & 1 \\ 0 & 0 & 1 \end{vmatrix} = a^2\left|t + \frac{1}{t}\right|$$

$$\Rightarrow \quad 2a\left|t + \frac{1}{t}\right| = \frac{2A}{a}$$

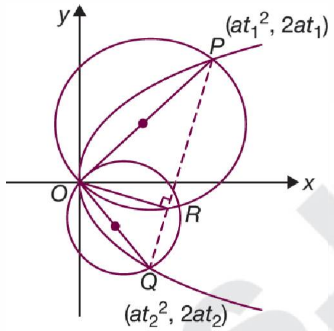
Hence, the correct answer is (C).

59. Let, $P(t_1)$ and $Q(t_2)$.

$\Rightarrow \quad$ Slope of tangent at $P\left(\dfrac{1}{t_1}\right)$ and at $Q\left(\dfrac{1}{t_2}\right)$

$\Rightarrow \quad \cot\theta_1 = t_1$ and $\cot\theta_2 = t_2$

$$\text{Slope of } PQ = \frac{2}{t_1 + t_2} = \tan\phi$$



$\Rightarrow \quad \tan\phi = -\dfrac{1}{2}(\cot\theta_1 + \cot\theta_2)$

$\Rightarrow \quad \cot\theta_1 + \cot\theta_2 = -2\tan\phi$

Hence, the correct answer is (A).

60. Let any tangent is tangent at vertex $x = 0$

Let $B(t_1)$ and $C(t_2)$

$\Rightarrow \quad A(at_1t_2, a(t_1 + t_2))$

$\Rightarrow \quad p_1 = at_1^2,\ p_2 = at_2^2$ and $p_3 = at_1t_2$

$\Rightarrow \quad p_1, p_2$ and p are in G.P.

Hence, the correct answer is (B).

61. Let the midpoint of AB be $M(h, k)$.

Then equation of AB is $hx - 2b(y + k) = h^2 - 4bk$

Let $T(2at, -at^2)$

$\Rightarrow$ Equation of tangent $AB = x(2at) = -2a(y - at^2)$

By comparing these two equations, we get $\dfrac{h}{2at} = \dfrac{-2b}{2a}$ $= \dfrac{h^2 - 2bk}{2a^2t^2}$

By eliminating t and Locus (h, k), we get $(a + 2b)x^2 = 4b^2y$

Hence, the correct answer is (D).

62. Let $S = y^2 - 4x + 6y$. $S(-1, 4) = 4^2 - 4(-1) + 6(4) = 16 + 4 + 24 = 44 > 0$

$\therefore$ The point $(-1, 4)$ lies outside the parabola and hence one normal can be drawn from $(-1, 4)$ to the parabola.

Hence, the correct answer is (D).

63. Let $A(t_1)$, $B(t_2)$ be the extremities of a focal chord $y^2 = 4ax$.

$\therefore \quad t_1t_2 = -1$

$(x_1, y_1) = [at_1t_2, a(t_1 + t_2)]$ and

$(x_2, y_2) = [a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2)]$

$y_2 = -at_1t_2(t_1 + t_2) = -a(-1)(t_1 + t_2) = a(t_1 + t_2) = y_1$

Hence, the correct answer is (C).

64. Let $P(t_1)$, $Q(t_2)$ and $R(t_3)$

Equation of a normal to $y^2 = 4ax$ is $y + tx = 2at + at^3$

This passes through (h, k)

$\Rightarrow \quad k + th = 2at + at^3$

$\Rightarrow \quad at^3 + (2a - h)t - k = 0$

Where, t_1, t_2 and t_3 are the roots of this equation $t_1 + t_2 + t_3 = 0$.

Centroid of ΔPQR is $G\left[\frac{a}{3}\left(t_1^2 + t_2^2 + t_3^2\right), \frac{2a}{3}(t_1 + t_2 + t_3)\right]$

$t_1 + t_2 + t_3 = 0$

$\Rightarrow \quad \frac{2a}{3}(t_1 + t_2 + t_3) = 0$

$\therefore$ Centroid lies on $y = 0$

Hence, the correct answer is (B).

65. Let the given parabola be $y^2 = 4ax$ ...((1)

If the vertex of moving parabola (a, b), then its equation is

$(y - b)^2 = -4a(x - a)$ ...((2)

Solving (1) and (2), we get,

$2y^2 - 2by + b^2 - 4aa = 0$

Since curve touch each other discriminant = 0

$\Rightarrow \quad b^2 = 8ax$

Locus is $y^2 = 8ax$

$\therefore$ Latus rectum $= 8a$

Hence, the correct answer is (C).

66. Let, $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

Since the normals at P and Q meet on the parabola, $t_1t_2 = 2$.

Point of intersection of the normal

$$(x_1, y_1) = \left(a\left[t_1^2 + t_2^2 + t_1t_2 + 2\right], -at_1t_2[t_1 + t_2]\right)$$

$\Rightarrow \quad x_1 = a(t_1^2 + t_2^2 + t_1t_2 + 2) = a(t_1^2 + t_2^2 + 4)$

$\Rightarrow \quad a(t_1^2 + t_2^2) = x_1 - 4a$

$$PQ^2 = (at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2$$
$$= a^2(t_1 - t_2)^2[(t_1 + t_2)^2 + 4]$$
$$= a(t_1^2 + t_2^2 - 4)\, a(t_1^2 + t_2^2 + 8) = (x_1 - 8a)(x_1 + 4a)$$

Hence, the correct answer is (B).

67. Let, $a(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$

The normal at A cuts the curve again at B.

$\therefore \quad t_1 + t_2 = -\frac{2}{t_1}$ ...(1)

Again, AB subtends a right angle at the vertex $O(0, 0)$ of the parabola.

Slope $OA = \frac{2at_1}{at_1} = \frac{2}{t_1}$ and slope of $OB = \frac{2}{t_2}$

$OA \perp OB$

$\Rightarrow \quad \frac{2}{t_1}, \frac{2}{t_2} = -1, t_1t_2 = -4$ ...(2)

Slope of AB is $\frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2} = -t_1$ (By (1))

Again from (1) and (2), on putting for t_2, we get, $t_1 = -\frac{4}{t_1} = -\frac{2}{t_1}$

$\therefore \quad t_1^2 = 2$ or $t_1 = \pm\sqrt{2}$

$t_2 = \frac{-4}{t_1} = \frac{-4}{(\pm\sqrt{2})} = \pm 2\sqrt{2}$

$(\therefore A = (2a, \pm 2a\sqrt{2}), B = (8a, 4\sqrt{a}))$

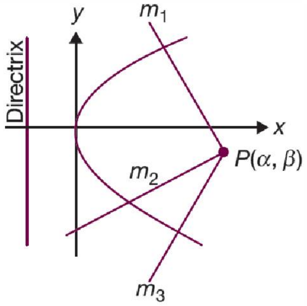
$$AB = \sqrt{(2a - 8a)^2 + (2a\sqrt{2} + 4\sqrt{2}a)^2} = \sqrt{36a^2 + 72a^2}$$
$$= \sqrt{108a^2} = 6\sqrt{3}a$$

Hence, the correct answer is (C).

68. Equation of normal to $y^2 = 4x$ is $y = mx - 2m - m^3$ ...(1)

It passes through (α, β) $\beta = m\alpha - 2m - m^3$

$\Rightarrow \quad m^3 + (2 - \alpha)m + \beta = 0$ ...(2)



(Let m_1, m_2 and m_3 are roots)

$\therefore \quad m_1, m_2, m_3 = -\beta$ (as $m_1m_2 = a$)

$\Rightarrow \quad m_3 = -\frac{\beta}{a}$

Now, $-\frac{\beta^3}{a^3} - (2 - a) \times \frac{\beta}{a} + \beta = 0$

$\Rightarrow \quad b^2 + (2 - \alpha)a^2\beta - ba^3 = 0$

$\Rightarrow$ Locus of P is $y^3 + (2 - x)ya^2 - ya^3 = 0$

As P is not the axis of parabola

$\Rightarrow \quad y^2 = a^2x - 2a^2 + a^3$ as it is the part of $y^2 = 4x$

$\therefore \quad a^2 = 4$ or $-2a^2 + a^3 = 0$, $a = \pm 2$ or $a^2(a - 2) = 0$

$a = \pm 2$ or $a = 0, a = 2$

$\therefore$ $a = 2$ is the required value of a.

Hence, the correct answer is (A).

69. One end of the latus rectum $= (a, 2a)$

Equation of the normal at $(a, 2a)$ is $2a(x - a) + 2a(y - 2a) = 0 \Rightarrow x + y - 3a = 0$

Solving $y^2 = 4ax$, $x + y - 3a = 0$, we get the ends of normal chord as $(a, 2a)$, $(9a, -6a)$.

$$\text{Length of the chord} = \sqrt{(9a-a)^2 + (-6a-2a)^2}$$

$$= \sqrt{64a^2 + 64a^2} = 8\sqrt{2}a \text{ units}$$

Hence, the correct answer is (B).

70. Let, $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$.

The normal at A cuts the curve again at B.

$$\therefore \quad t_1 + t_2 = \frac{-2}{t_1} \qquad \ldots(1)$$

Again, AB subtends a right angle at the vertex $O(0, 0)$ of the parabola.

$$\text{Slope of } OA = \frac{2at_1}{at_1^2} = \frac{2}{t_1}$$

$$\text{Slope of } OB = \frac{2}{t_2}$$

$$OA \perp OB \Rightarrow \frac{2}{t_1}, \frac{2}{t_2} = -1$$

$\Rightarrow \quad -1$

$$\Rightarrow \quad t_1 t_2 = -4 \qquad \ldots(2)$$

$$\text{Slope of } AB \text{ is } \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2} = -t_1 \qquad \text{(By (1))}$$

Again from (1) and (2), on putting for t_2, we get,

$$t_1 - \frac{4}{t_1} = \frac{2}{t_1}.$$

$$\therefore \quad t_1^2 = 2 \Rightarrow t_1 = \pm\sqrt{2}$$

$$\therefore \quad \text{Slope} = \pm\sqrt{2}$$

Hence, the correct answer is (A).

71. Let $P(at^2, 2at)$

Equation of the normal at $P(t)$ is $y + tx = 2at + at^3$

Equation to y-axis is $x = 0$.

Solving $G(2a + at^2, 0)$, we get Focus $s(a, 0)$

ΔSPG is an equilateral triangle.

$\Rightarrow \quad PG = GS$

$$\Rightarrow \quad \sqrt{4a^2 + 4a^2t^2} = \sqrt{a^2(1 + t^2)^2}$$

$\Rightarrow \quad 4a^2(1 + t^2) = a^2(1 + t^2)^2$

$\Rightarrow \quad 4 = 1 + t^2$

$\Rightarrow \quad t^2 = 3$

$\therefore$ Length of the side $= SG = a(1 + t^2) = a(1 + 3) = 4a$ units

Hence, the correct answer is (D).

72. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

Normals at P and Q on the parabola intersect on the parabola.

$\Rightarrow \quad t_1 t_2 = 2$

$$at_1^2 \times at_2^2 = a^2(t_1t_2)^2 = a^2(2)^2 = 4a^2$$

Hence, the correct answer is (A).

73. Let the normals at $P(t_1)$ and $Q(t_2)$ intersect on the parabola at $R(t_3)$.

Equation of any normal is $tx + y = 2at + at^3$

Since it passes through Q, we get, $t \cdot at_3^2 + 2at_3 = 2at + at^3$

$\Rightarrow at^3 + (2a - at_3^2)t - 2at_3 = 0$, which is a cubic equation in t.

Hence, its roots are t_1, t_2, t_3.

$$\text{Product of the roots} = t_1t_2t_3 = \frac{-(-2at_3)}{a} = 2t_3$$

$\Rightarrow \quad t_1t_2 = 2$

Product of the abscissae of P and $Q = at_1^2 \cdot at_2^2 = a^2(t_1t_2)^2 = a^2(2)^2 = 4a^2$

Product of the ordinates of P and $Q = 2at_1 \cdot 2at_2 4a^2 \cdot t_1t_2 = 4a^2(2) = 8a^2$

Hence, the correct answer is (B).

74. Let $P(x_1, y_1)$ be the point of intersection of the two perpendicular normals at $A(t_1)$, $B(t_2)$ on the parabola $y^2 = 4ax$.

Let t_3 be the foot of the third normal through P.

Equation of a normal at t to the parabola is $y + xt = 2at + at^3$

If this normal passes through P, then $y_1 + x_1t = 2at + at^3$

$$\Rightarrow \quad at^3 + (2a - x_1)t - y_1 = 0 \qquad \ldots(1)$$

Now t_1, t_2 and t_3 are the roots of (1).

$$\therefore \quad t_1t_2t_3 = \frac{y_1}{a}$$

Slope of the normal at t_1 is $-t_1$

Slope of the normal at t_2 is $-t_2$

Normals at t_1 and t_2 are perpendicular.

$\Rightarrow \quad (-t_1)(-t_2) = -1$

$\Rightarrow \quad t_1t_2 = -1$

$\Rightarrow \quad t_1t_2t_3 = -t_3$

$$\Rightarrow \quad \frac{y_1}{a} = -t_3$$

$$\Rightarrow \quad t_3 = -\frac{y_1}{a}$$

t_3 is a root of (1) $\Rightarrow a\left(-\frac{y_1}{a}\right)^3 + (2a - x_1)\left(-\frac{y_1}{a}\right) - y_1 = 0$

$$\Rightarrow \quad -\frac{y_1^3}{a^2} - \frac{(2a - x_1)y_1}{a} - y_1 = 0$$

$$\Rightarrow \quad -y_1^2 + a(2a - x_1) + a^2 = y_1^2 = a(x_1 - 3a)$$

$\therefore$ The locus of P is $y^2 = a(x - 3a)$

Hence, the correct answer is (A).

75. A, B and C are respectively $\left(t_1^2, 2t_1\right), \left(t_2^2, 2t_2\right)$ and $\left(t_3^2, 2t_3\right)$

Since, AB lie on $x - y = 1$, $t_1^2 - 2t_1 = 1$, $t_2^2 - 2t = 1$

Subtracting, $t_1 + t_2 = 2$

Now, $t_1 + t_2 + t_3 = 0$

$\Rightarrow \quad t_3 = -2$

So, $D(4, -4)$

Hence, the correct answer is (B).

76. Parabola is $y^2 = 4ax$ …(1)

Polar of a pole $(x_1, y_1) = yy_1 - 2ax = 2ax_1$ …(2)

Making Equation (1) as homogeneous with respect to Equation (2), we get,

$$y^2 - 4ax\left(\frac{yy_1 - 2ax}{2ax_1}\right) = 0$$

$$x_1y^2 - 2xyy_1 + 4ax^2 = 0$$

Angle between these pair of lines is 45°

$$\therefore \quad \tan 45° = \frac{2\sqrt{y_1^2 - 4ax_1}}{(x_1 + 4a)}$$

Locus of (x_1, y_1) is $(x + 4a)^2 = 4(y^2 - 4ax)$.

$\therefore \quad \lambda = 4$

Hence, the correct answer is (D).

77. Putting (λ, λ) in $y^2 = 4ax$, gives $\lambda = 4a$

Slope of normal at $(4a, 4a)$ is -2

Equation of normal at $(4a, 4a)$ is $y - 4a = -2(x - 4a)$

$\Rightarrow \quad y + 2x - 12a = 0$

The co-ordinates of intersection points of the above normal $= y^2 + 2ay - 24a^2 = 0$

$y = 4a, -6a$ and $x = 4a, 9a$

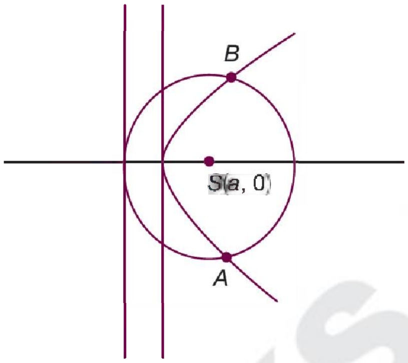
Then, slope of SA, $m_1 = \frac{4}{3}$

Slope of SB, $m_2 = \frac{6a}{8a} = \frac{-3}{4}$,

$m_1m_2 = -1$

Hence, the correct answer is (C).

78.



Centre of circle is $(a, 0)$ and radius is $2a$.

Equation of circle is $(x - a)^2 + y^2 = 4a^2$

$x^2 + y^2 - 2ax - 3a^2 = 0$ and $y^2 = 4ax$

Solving $x^2 + 4ax - 2ax - 3a^2 = 0$, we get,

$x^2 + 2ax - 3a^2 = 0$

$x = -3a, a$ and $y = \pm 2a$

$\therefore$ Length of $AB = 4a$ units.

Hence, the correct answer is (A).

79. Normal at $(at^2, 2at)$ is $y + tx = 2at + at^3 \left(a = \frac{1}{4}\right)$

If this passes through $(c, 0)$, we have $ct = 2\,at + at^3 = \frac{t}{2} + \frac{t^3}{4}$

$\Rightarrow \quad t = 0$ or $t^2 = 4c - 2$

If $t = 0$, the point at which the normal is drawn is $(0, 0)$. If $t \neq 0$, then the two values of t represents slope of normals through $(c, 0)$.

If these normals are perpendicular, then $(-t_1)(-t_2) = -1$

$$\Rightarrow \quad \left(\sqrt{4c - 2}\right)\left(-\sqrt{4c - 2}\right) = -1$$

$$\therefore \quad c = \frac{3}{4}$$

Hence, the correct answer is (A).

80. Tangent at extremities of focal chord intersects at right angle at directrix.

Let $P(2t^2, 4t) = (2, 4)$

$\Rightarrow \quad t = 1$

PQ is focal chord $t_1t_2 = -1$

$\Rightarrow \quad t_1 = -1$

$\Rightarrow \quad Q(2, -4)$

Equation of tangent at 'P' $ty = x + at^2$

$\Rightarrow \quad y = x + 2$

Co-ordinate of R by putting $x = -2$ and $y = 0 = (-2, 0)$

R is the midpoint of Q and Q' is the mirror image of Q

$\therefore \quad Q' = (-6, 4)$

Hence, the correct answer is (B).

81. Let the midpoint of chord of circle $x^2 + y^2 = 9$ is (h, k)

Equation of chord of circle is $hx + ky = h^2 + k^2$.

Equation of pair of lines joining the point of intersection of chord and the parabola with the origin is

$$y^2 - 4(x + y)\,\frac{(hx + ky)}{(h^2 + k^2)} = 0$$

Since the angle between these lines is 90°, then the required locus is $x^2 + y^2 = 4(x + y)$.

Hence, the correct answer is (A).

82. $y^2 = 16(x - k)$ $\qquad$ $x^2 = 16(y - \lambda)$

$$2y\frac{dy}{dx} = 16 \qquad 2x = 16\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8}{y} = m_1 \qquad \frac{dy}{dx} = \frac{x}{8} = m_2$$

Since two circle touch each other, $m_1 = m_2$

$$\Rightarrow \quad \frac{8}{y} = \frac{x}{8}$$

$$\therefore \quad xy = 64$$

Hence, the correct answer is (D).

83. $T = (at_1t_2, a(t_1 + t_2))$

$P' = (at_1t_3\ a(t_1 + t_3))$

$Q' = (at_2t_3\ a(t_2 + t_3))$

$TP' : TP = \lambda : 1$

$$\lambda = \frac{at_1t_3 - at_1t_2}{at_1t_2 - at_1^2}$$

$$= \frac{t_3 - t_2}{t_2 - t_1}$$

$$\therefore \quad \frac{TP'}{TP} = \frac{t_3 - t_2}{t_2 - t_1}$$

Similarly, let $TQ' : TQ = \mu : 1$

$$\frac{TQ'}{TQ} = \frac{at_2t_3 - at_1t_2}{at_1t_2 - at_2^2} = \frac{t_3 - t_1}{t_1 - t_2}$$

$$\therefore \quad \frac{TP'}{TP} + \frac{TQ'}{TQ} = \frac{t_3 - t_2}{t_2 - t_1} + \frac{t_3 - t_1}{t_1 - t_2} = \frac{t_1 - t_2}{t_2 - t_1} = -1$$

Hence, the correct answer is (A).

84. The locus is the directrix of the parabola.

Hence, the correct answer is (D).

85. Let $y^2 = 4ax$ be the parabola

$y = mx + \dfrac{a}{m}$ and $y = \left(-\dfrac{1}{m}\right)x - am$ are perpendicular tangents

$$S = (a, 0),\ A = \left(-\frac{a}{m^2}, 0\right),\ B = (-am^2, 0)$$

$$|SA| = a\left(1 + \frac{1}{m^2}\right) = \frac{a(1 + m^2)}{m^2}$$

$$|SB| = a(1 + m^2)$$

Hence, the correct answer is (C).

86. Lengths are invariant under change of axes. Consider $y^2 = 8x$ and focal chord at $(2t^2, 4t)$

Focus = (2, 0).

Equation of focal chord at t is $y = \dfrac{2t}{t^2 - 1}(9x - 2)$

$$\Rightarrow \quad 2tx + (1 - t^2)y - 4t = 0$$

$$\frac{4|t|^2}{\sqrt{4t^2 + (1 - t^2)^2}} = 2 \Rightarrow (|t| - 1)^2 = 0$$

$\therefore$ Length of focal chord at 't' $= 2\left(t + \dfrac{1}{t}\right)^2 = \dfrac{2(t^2 + 1)^2}{t^2}$

= 8 units

Hence, the correct answer is (A).

87. $x^2 = 4(y + 2)$ is the given parabola.

Any normal is $x = m(y + 2) - 2m - m^3$.

If $(10, -1)$ lies on this line, then $10 = +\,m - 2 - m^3$

$$\Rightarrow \quad m^3 + m + 10 = 0$$

$$\Rightarrow \quad m = -2$$

Slope of normal $= \dfrac{1}{m}$.

Hence, the correct answer is (C).

88. The point $(9, -6)$ lies on $y = mx - 2m - m^3$.

$$\therefore \quad -6 = 9m - 2m - m^3$$

$$\Rightarrow \quad m^3 - 7m - 6 = 0$$

Roots are $-1, -2$ and 3

$$\therefore \quad |A| = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = (-2)(-4) - (3 - 1) + 2 = 8$$

Hence, the correct answer is (D).

89. Let (α, β) be the vertex of the moving parabola and its equation is

$$(y, \beta)^2 = -4a(x - \alpha) \qquad \ldots(1)$$

Let the equation of fixed parabola be $y^2 = 4ax$ $\qquad \ldots(2)$

(Here, $4a = u$)

From (1) and (2), we get, $(y - \beta)^2 = -4a\left(\dfrac{y^2}{4a} - \alpha\right)$

$$\Rightarrow \quad 2y^2 - 2\beta y + \beta^2 - 4a\alpha = 0$$

The above is a quadratic equation in y having same roots.

$\Rightarrow \quad \Delta = 0$

$\Rightarrow \quad \beta^2 = 8a\alpha$

Hence, locus is $y^2 = 8ax$, i.e., $y^2 = 2ux$

Hence, the correct answer is (A).

90. The equation of the axis of the parabola is $y - 2 = 0$, which is parallel to the x-axis.
So, a ray parallel to the x-axis of parabola.
Any ray parallel to the axis of a parabola passes through this focus after reflection.
Here, (0, 2) is the focus.

Hence, the correct answer is (A).

91. The normal at $P(at_1^2, 2at_1)$ is $y + xt_1 = 2at_1 + at_1^3$ with slope.

Say, $\tan\theta = -t_1 = \frac{1}{\sqrt{3}}$

If it meets curve at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1} = \frac{7}{\sqrt{3}}$

Then, angle θ between parabola (tangent at Q) and normal at P is given by $\tan\theta = \dfrac{-t_1 - \frac{1}{t_2}}{1 - \frac{t_1}{t_2}}$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$$

Hence, the correct answer is (D).

92. $y = ax^2 + 2a^2x + 1$

$$\Rightarrow \quad \frac{y - (1 - a^3)}{a} = (x + a)^2$$

$\therefore \quad$ Vertex $= (\alpha, \beta) = (-a, 1 - a^3)$

$\Rightarrow \quad \beta = 1 + a^3$

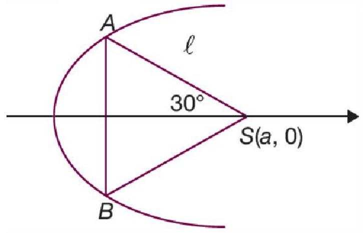
$\Rightarrow \quad$ Curve is $y = 1 + x^3$

Hence, the correct answer is (C).

93. Circle will be touching both parabolas and the centre of the circle will be on the common normal.

Hence, the correct answer is (B).

94.



$A(a - \ell\cos 30°, \ell\sin 30°)$

Point 'A' lies on $y^2 = 4ax \Rightarrow \left(\frac{\ell}{2}\right)^2 = 4a\left(a - \frac{\ell\sqrt{3}}{2}\right)$

$\Rightarrow \quad$ It is a quadratic in 'ℓ', get ℓ

Hence, the correct answer is (B).

95. Let, $Q = \left(at_2^2, 2at_2\right)$

$R = \left(at_2^2, 2at_2\right)$

Normals at Q and R meet on parabola

Also, $T = (at_1t_2, a(t_1 + t_2))$

Let (α, β) be the centroid of ΔQRT

Then, $3\alpha = a\left(t_1^2 + t_2^2 + t_1t_2\right)$ and $\beta = a(t_1 + t_2)$

Eliminate $(t_1 + t_2)$

Hence, the correct answer is (C).

96. $A = (a, 0)$, $H = (0, 2at + at^3)$, $G = (2at + at^2, 0)$, $Q = (h, k)$
$(h, k) = (2a + at^2, 2at + at^3)$
By eliminating 't', we get, $x^3 = 2ax^2 + ay^2$

Hence, the correct answer is (C).

97. Origin and focus line on off side of $x + y = 1$

$\Rightarrow \quad \alpha + \beta > 0$

Origin and focus line on same side of $x + y = 3$

$\Rightarrow \quad \alpha + \beta < 2$

Hence, the correct answer is (C).

98. Let POC be (h, k). Then, tangent at (h, k) to both parabolas represents same line.

$yk = 2a(x + h) - 4a\alpha$

$xh = 2a(y + k) - 4a\beta$

Both are same line

$$\frac{2a}{k} = \frac{h}{2a} \Rightarrow hk = 4a^2$$

Locus $xy = 4a^2$ Hyperbola

Hence, the correct answer is (D).

99. $3y^2 + 4y = 6x - 8$

$$\Rightarrow \quad 3\left(y^2 + \frac{4}{3}y\right) = 6x - 8$$

$$\Rightarrow \quad 3\left(y + \frac{2}{3}\right)^2 = 2x - \frac{8}{3} + \frac{4}{9}$$

$$\Rightarrow \quad \left(y + \frac{2}{3}\right)^2 = 2\left(x - \frac{10}{9}\right)$$

Let any point on the axis be $\left(a, -\frac{2}{3}\right)$

$$y + \frac{2}{3} = m\left(x - \frac{10}{9}\right) - m - \frac{1}{2}m^3$$

$$\Rightarrow \quad 0 = m\left[a - \frac{10}{9} - 1 - \frac{1}{2}m^2\right]$$

$$\Rightarrow \quad a - \frac{19}{9} = \frac{1}{2}m^2$$

$$\Rightarrow \quad m^2 = 2\left(a - \frac{19}{9}\right)$$

$$\therefore \quad a > \frac{19}{9}$$

Hence, the correct answer is (B).

100. Let $P(h, k)$

$$y = mx + \frac{a}{m}$$

$$k = mh + \frac{a}{m}$$

$$\Rightarrow \quad m^2h + a - mk = 0$$

$$\Rightarrow \quad m_{PA} + m_{PB} = \frac{k}{h}$$

$$m_{PA} \cdot m_{PB} = \frac{a}{h}$$

$$\frac{k^2}{h^2} - \frac{2a}{h} = 4$$

$$\Rightarrow \quad k^2 - 2ah = 4h^2$$

$$\therefore \quad y^2 = 2ax + 4x^2 = 2x(2x + a)$$

Hence, the correct answer is (A).

101. Equation of a tangent to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$. If it touches the circle $x^2 + y^2 = \frac{a^2}{2}$ then,

$$\frac{a}{m} = \left(\frac{a}{\sqrt{2}}\right)\sqrt{1 + m^2}$$

$$\Rightarrow \quad 2 = m^2(1 + m^2)$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 - 1)(m^2 + 2) = 0$$

$$\Rightarrow \quad m^2 = 1$$

$$\Rightarrow \quad m = \pm 1$$

Hence, the common tangents are $y = x + a$ and $y = -x - a$, which intersect at the point $(-a, 0)$, which is the focus of the parabola $y^2 = -4ax$.

Hence, the correct answer is (C).

102. Let $P(h^2, 2h)$

$$\tan\theta = \frac{1}{h}$$

$$\text{And } \Delta PTM = \frac{1}{2} \times 2h \times 2h \cot\theta = 2h^3$$

$$a^2 \le h^2 \le 4a^2$$

$$\therefore \quad \text{Maximum area} = 2(2a)^3 = 16a^3 \text{ sq. units}$$

Hence, the correct answer is (C).

103. Since two parabolas are symmetrical about $y = x$, on solving $y = x$ and $y^2 - 4x - 8y + 40 = 0$, we get, $x^2 - 12x + 40 = 0$, which has no real solution.

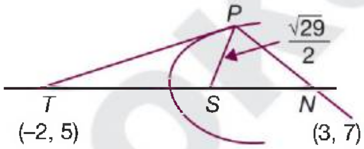
$\therefore$ They don't intersect.

Point on $(x - 4)^2 = 4(y - 6)$ is $(6, 7)$ and the corresponding point on $(y - 4)^2 = 4(x - 6)$ is $(7, 6)$.

$\therefore$ Minimum distance is $\sqrt{2}$ units.

Hence, the correct answer is (D).

104.



$TN = \sqrt{29}$

We know that, $SP = ST = SN$

So, focal distance $SP = \frac{1}{2}\, TN = \frac{\sqrt{29}}{2}$

Hence, the correct answer is (A).

105. Length of focal chord is $a\left(t + \frac{1}{t}\right)^2$ units, if $(at^2, 2at)$ is one extremity of the parabola $y^2 = 4ax$.

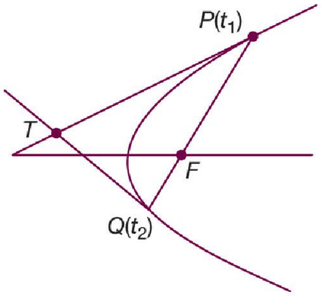
$$\therefore \quad t + \frac{1}{t} \ge 2 \ (AM \ge GM)$$

$$\Rightarrow \quad a\left(t + \frac{1}{t}\right)^2 \ge 4a$$

Here, $4a = 32$

Hence, the correct answer is (C).

106.



Equation of tangent at $P \to ty = x + at^2$ ...(1)

Equation of tangent at $Q \to \frac{-1}{t}y = x + \frac{a}{t^2}$ ...(2)

$$\Rightarrow \quad x = -a$$

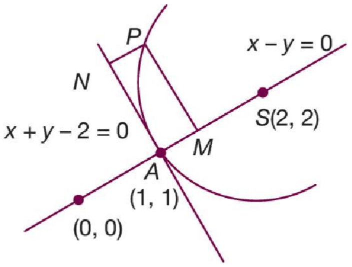
$\therefore$ t lies on the directrix and thus ΔPQT is right angled triangle.

Thus, the circle passing through P, Q and T must have P and Q as end points of diameter.

Thus, area of required circle is $\frac{\pi(PQ)^2}{4}$ sq. units.

Hence, the correct answer is (A).

107.



$$PM^2 = 4a(PN)$$

$$\Rightarrow \quad \frac{(x-y)^2}{2} = 4\sqrt{2}\ \frac{(x-y-2)}{\sqrt{2}}$$

$$\Rightarrow \quad (x-y)^2 = 8\,(x+y-2)$$

Hence, the correct answer is (A).

108. Any line through (α, β) is $y - \beta = m(x - a)$ ...(1)

Solving Equation (1) with equation of the parabola, we get, $2at - \beta = m(at^2 - \alpha)$.

$$\Rightarrow \quad 2amt^2 - 2at + \beta - m\alpha = 0$$

$$\Rightarrow \quad t_1t_2 = \frac{\beta - m\alpha}{am} = -4$$

$$\Rightarrow \quad m = \left(\frac{\beta}{\alpha - 4a}\right)$$

Hence, the required equation is $y - \beta = \frac{\beta}{\alpha - 4a}(x - \alpha)$.

$$\Rightarrow \quad y(\alpha - 4a) - \alpha\beta + 4a\beta = \beta x - \alpha\beta$$

$$\Rightarrow \quad \beta x + (4a - \alpha)y - 4a\beta = 0$$

Hence, the correct answer is (A).

109. Family of lines passes through focus.

Hence, latus rectum will makes the shortest intercept.

Hence, the correct answer is (C).

110. If the tangents at P and Q intersect at T, then axis of parabola is parallel to TR, where R is the midpoint of P and Q. So, slope of the axis is 1.

$\therefore$ Slope of the directrix = –1.

Hence, the correct answer is (C).

111. Let $P(t_1, 4t_1^2)$, $Q(t_2, 4t_2^2)$

Slope of $OP \times$ slope of $OQ = -1$

$$\Rightarrow \quad 4t_1 \cdot 4t_2 = -1$$

Equation of tangent at $\left(t_1, 4t_1^2\right)$ is $y - 4t_1^2 = 8t_1(x - t_1)$

$$\Rightarrow \quad y + 4t_1^2 = 8t_1x \qquad ...(1)$$

Equation of tangent at $\left(t_2, 4t_2^2\right)$ is $y + 4t_2^2 = 8t_2x$ ...(2)

Let (x_1, y_1) is the point of intersection.

Equation (1) – Equation (2) $\Rightarrow x_1 = \frac{t_1 + t_2}{2}$

$$y_1 = 8t_1\left(\frac{t_1+t_2}{2}\right) - 4t_1^2 = 4t_1t_2 = \frac{-1}{4}$$

$$\Rightarrow \quad 4y_1 + 1 = 0$$

Hence, the correct answer is (D).

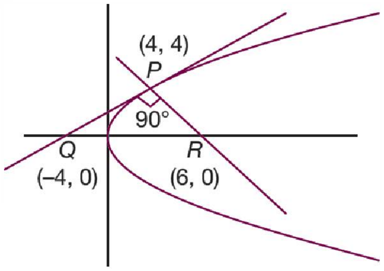
112. Let P be $(t^2, 2t)$

$$\text{Area of } \Delta PAB = \frac{1}{2}\begin{vmatrix} t^2 & 2t & 1 \\ 9 & 6 & 1 \\ 4 & -4 & 1 \end{vmatrix} = |\,5t^2 - 5t - 30\,|$$

$\therefore$ It is maximum at $t = \frac{1}{2}$

Hence, the correct answer is (D).

113. Equation of tangent is $2y = x + 4$.



$\therefore \quad Q(-4, 0)$

Equation of normal is $y - 4 = -2\,(x - 4)$

$$\Rightarrow \quad y + 2x = 12$$

Clearly, QR is the diameter of the required circle.

$$\Rightarrow \quad (x+4)(x-6) + y^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 24 = 0$$

$\therefore$ Centre is (1, 0).

Hence, the correct answer is (C).

114. Any point on the given parabola is $(t^2, 2t)$. The equation of the tangent at (1, 2) is $x - y + 1 = 0$. The image (h, k) of the point $(t^2, 2t)$ in $x - y + 1 = 0$ is given by

$$\frac{h - t^2}{1} = \frac{k - 2t}{-1} = \frac{-2(t^2 - 2t + 1)}{1+1}$$

$\therefore h = t^2 - t^2 + 2t - 1 = 2t - 1$ and $k = 2t + t^2 - 2t + 1 = t^2 + 1$

Eliminating t from $h = 2t - 1$ and $k = t^2 + 1$, we get, $(h+1)^2 = 4(k-1)$.

$\therefore$ The required equation of reflection is $(x+1)^2 = 4(y-1)$.

Hence, the correct answer is (C).

115. Foot of perpendicular from focus upon tangent is say (α, β).

So, $\frac{\alpha+1}{3} = \frac{\beta+1}{-1} = \frac{-(-3+1-8)}{3^2+(-1)^2} = 1$

⇒ $(\alpha, \beta) = (2, -2)$

Images of (7, 13) and (– 1, – 1) with respect ot (2, – 2) will lie on respectively the axis and the directrix of the parabola. The two points are respectively (– 3, – 17) and (5, –3). Slope of axis is $\frac{-1+17}{-1+3} = 8$.

So, equation of directrix is $y + 3 = -\frac{1}{8}(x-5)$

⇒ $x + 8y + 19 = 0$

Hence, the correct answer is (D).

116. Parabola is $y^2 - 4y = 4x \Rightarrow (y-2)^2 = 4(x+1)$

Focus = (0, 2)

Let, radius of circle = r, then centre = $(r, 2)$

Circle is $(x-r)^2 + (y-2)^2 = r^2$

⇒ $(x-r)^2 + 4(x+1) = r^2$ has equal roots $\Delta = 0$

⇒ $r = 4$

Hence, the correct answer is (A).

117. The focus of the parabola, $y^2 = 4ax$ is $S(a, 0)$. Let $P(at^2, 2at)$ be any point on the parabola, then co-ordinates of the midpoint of SP are given by

$$x = \frac{a(t^2+1)}{2} \text{ and } y = \frac{2at+0}{2}$$

Eliminating 't', we get the locus of the midpoint.

As $y^2 = 2ax - a^2$ or $y^2 = 2a\left(x - \frac{a}{2}\right)$ ...(1)

Equation (1) is a parabola of the form $Y^2 = 4AX$ ...(2)

Where, $Y = y$, $X = x - \frac{a}{2}$ and $A = \frac{a}{2}$

Equation of the directrix of Equation (2) is $X = -A$

So, equation the directrix of Equation (1) is $x - \frac{a}{2} = \frac{-a}{2}$

⇒ $x = 0$

Hence, the correct answer is (C).

118. Equation of the tangent at $P(x_1, y_1)$ to the parabola $y^2 = 4ax$

$$yy_1 = 2a(x + x_1) \text{ or } 2ax - y_1y + 2ax_1 = 0 \quad ...(1)$$

If $M(h, k)$ is the midpoint of QR, then equation of QR a chord of the parabola $y^2 = 4a(x + b)$ in terms of its midpoint is $ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$.

(using $T = S'$) or $2ax - ky + k^2 - 2ah = 0$ ...(2)

Since (1) and (2) represent the same line, we have,

$\frac{2a}{2a} = \frac{y_1}{k} = \frac{2ax_1}{k^2 - 2ah}$

⇒ $k = y_1$ and $k^2 - 2ah = 2ax_1$

⇒ $y_1^2 - 2ah = 2ax_1$

⇒ $4ax_1 - 2ax_1 = 2ah$

(as $P(x_1, y_1)$ lies on the parabola $y^2 = 4ax$)

⇒ $h = x_1$ so that $h = x_1$ $k = y_1$ and the midpoint of QR is (x_1, y_1).

Hence, the correct answer is (B).

119. Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$, which will touch the circle $(x - 3) + y^2 = 9$, whose centre is (3, 0) and radius is 3, if

$\left|\frac{0 + m(3) + (1/m)}{\sqrt{1+m^2}}\right| = 3$

Or if $\left(3m + \frac{1}{m}\right)^2 = 9(1 + m^2)$

Or if $9m^2 + 6 + \frac{1}{m^2} = 9 + 9m^2$

Or if $m^2 = \frac{1}{3}$, i.e., $m = \pm 1$

As the tangent is above the x-axis, we take $m = \frac{1}{\sqrt{3}}$ and thus the required equation is $\sqrt{3}y = x + 3$.

Hence, the correct answer is (C).

120. Equation of the normal at 't' to the parabola $y^2 = 4ax$ is $y = -tx + 2at + at^3$ ...(1)

The joint equation of the lines joining the vertex (origin) to the points of intersection of the parabola and the line (1) is $y^2 = 4ax\left[\frac{y + tx}{2at + at^3}\right]$

⇒ $(2t + t^3)y^2 = 4x(y + tx)$

⇒ $4tx^2 - (2t + t^3)y^2 + 4xy = 0$

Since these lines are at right angles, coefficient of x^2 + coefficient of $y^2 = 0$.

⇒ $4t - 2t - t^3 = 0$

⇒ $t^2 = 2$

For $t = 0$, the normal line is $y = 0$, i.e., axis of the parabola which passes through the vertex (0, 0).

Hence, the correct answer is (C).

121. H.M. of 3 and 2 = $\frac{12}{5}$ = Half of the length of latus rectum

Hence, the length of the latus rectum = $\frac{24}{5}$ units

Alternate

Let $y^2 = 4ax$ be the equation of the parabola, then the focus is $S(a, 0)$. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be vertices of a focal chord of the parabola, then $t_1t_2 = -1$.

Let, $SP = 3$ and $SQ = 2$

$$SP = \sqrt{a^2\left(1-t_1^2\right)+4a^2t_1^2} = a\left(1+t_1^2\right) = 3 \qquad \ldots(1)$$

$$SQ = a\left(1+\frac{1}{t_1^2}\right) = 2 \qquad \ldots(2)$$

From (1) and (2), we get, $t_1^2 = \frac{3}{2}\ a = \frac{6}{5}$

Hence, the length of the latus rectum is $\frac{24}{5}$ units.

Hence, the correct answer is (D).

122. Focus (2, 0)

$y - 0 = m(x-2)$

$\Rightarrow \frac{y}{m} + 2 = x$

$\Rightarrow y^2 - \frac{4y}{m} - 1 = 0$

$B^2 - 4AC > 0$

$\frac{1+m^2}{m^2} > 0$

$\Rightarrow m \in R - \{0\}$

Hence, the correct answer is (D).

Multiple Correct Choice Type Questions

1. $\sqrt{(x-2)^2+(y+5)^2} = \left|\frac{3x+4y-1}{5}\right|$

$S(2,-5)$, Directrix $3x+4y-1=0$

$\Rightarrow$ Tangent at vertex $3x+4y+K=0$ at a distance of a units from focus $(2,-5)$

$\therefore$ Axis is $4(x-2)-3(y+5)=0$

Hence, (A), (B), (C) and (D) are correct.

2. $y^2 = 8x, \quad a = 2, \quad x_1 + a = 4$

$\therefore \quad (x_1, y_1) = (2, \pm 4)$

Hence, (A) and (C) are correct.

3. $R = \left(\frac{\lambda+1}{\lambda+1}, \frac{3\lambda+1}{\lambda+1}\right) = \left(1, 1+\frac{2\lambda}{\lambda+1}\right)$

$\Rightarrow \quad y_1^2 - 4x_1 < 0$

$\Rightarrow \quad \left(1+\frac{2\lambda}{\lambda+1}\right)^2 - 4 < 0$

$\Rightarrow \quad \left(1+\frac{2\lambda}{\lambda+1}+2\right)\left(1+\frac{2\lambda}{\lambda+1}-2\right) < 0$

$\Rightarrow \quad \left(\frac{2\lambda}{\lambda+1}+3\right)\left(\frac{2\lambda}{\lambda+1}-1\right) < 0$

$\Rightarrow \quad \frac{2\lambda}{\lambda+1} > -3$ and $\frac{2\lambda}{\lambda+1} < 1$

$\Rightarrow \quad 5\lambda > -3$ and $\lambda < 3$

$\Rightarrow \quad \lambda > -\frac{3}{5}$ and $\lambda < 1$

$\Rightarrow \quad \lambda \in \left(-\frac{3}{5}, 1\right)$

Hence, (A), (B), (C) and (D) are correct.

4. Equation of normal chords at $p\left(2t_1^2, 4t_1\right)$ and $R\left(2t_2^2, 4t_2\right)$ are $y + t_1x - 4t_1 - 2t_1^3 = 0$ and $y + t_2x - 4t_2 - 2t_2^3 = 0$

Equation of curve through P, Q, R and S is $\left(y+t_1x-4t_1-2t_1^3\right)\left(y+t_2x-4t_2-2t_2^3\right)+\lambda\left(y^2-8x\right)=0$

P, Q, R, S are concyclic, $t_1 + t_2 = 0$ and $t_1t_2 = 1 + \lambda$

Points of intersection of tangents $\left(at_1t_2, a\left(t_1+t_2\right)\right)$ lies on x-axis, slope of $PR = \frac{z}{t_1+t_2}$

$\therefore$ PR is parallel to y-axis.

Hence, (A) and (C) are correct.

5. Point 'p' lies on the directrix of $y^2 = 8x$ and slopes of PA and PB are 1 and –1, respectively.

Equation of PA: $y = x + 2$

Equation of PB: $y = -x - 2$

Equation of AB: $x = 2$

Let $(h, 0)$ be the centre and the radius be 'r'.

$\Rightarrow \quad \frac{|h+2|}{\sqrt{2}} = \frac{|h-2|}{1} = r$

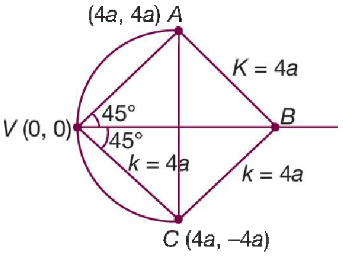
$\Rightarrow \quad h^2 - 12h + 4 = 0$

$\Rightarrow \quad h = 6 \pm 4\sqrt{2}$

$r = |h-2| = 4\left(\sqrt{2}-1\right), 4\left(\sqrt{2}+1\right)$

Hence, (A) and (B) are correct.

6.



Let K be the length of the side, such that $A(4a, 4a)$, $B(4a, -4a)$; $V(0, 0)$, $B(8a, 0)$.

Hence, (A), (B), (C) and (D) are correct.

7. Slope of $AS = \dfrac{2at_1}{t_1^2 - a} = \dfrac{2t_1}{t_1^2 - 1}$

Slope of $BC = \dfrac{2a(t_3 - t_2)}{a(t_3^2 - t_2^2)} = \dfrac{2}{t_3 + t_2}$

$$\therefore \quad \frac{2t_1}{t_1^2 - 1} \times \frac{2}{t_3 + t_2} = -1$$

$$4t_1 = t_3 t_1^2 + t_1^2 t_2 - t_3 - t_2$$

$$\therefore \quad (yt)\cdot(t_1) = 0$$

$$\Rightarrow \quad t_3 + t_2 = 0$$

$\therefore$ (C) is correct.

$$t_1^2(t_2 + t_3) + 4t_1 = t_3 + t_2$$

Similarly, $t_2^2(t_1 + t_3) + 4t_2 = t_1 + t_3$

$$(-)\ t_1^2 t_2 + t_1^2 t_3 - t_2^2 t_1 - t_2^2 t_3 + 4(t_1 - t_2) = t_2 - t_1$$

$$t_1 t_2(t_1 - t_2) + t_3(t_1^2 - t_2^2) = +5(t_2 - t_1)$$

$$t_1 t_2 + t_3(t_1 + t_2) = -5$$

$$\therefore \quad \sum t_1 t_2 = -5$$

$\therefore$ (A) is true.

Now, $t_1^2(t_2 + t_3) + 4t_1 = t_3 + t_2$

$$t_1(t_1 t_2 + t_3 t_1) + 4t_1 = t_2 + t_3$$

$$t_1(-5 - t_2 t_3) + 4t_1 = t_2 + t_3$$

$$-t_1 t_2 t_3 - t_1 = t_2 + t_3$$

$$\therefore \quad \frac{1}{t_1 t_2} + \frac{1}{t_2 t_3} + \frac{1}{t_3 t_1} = -1$$

$\therefore$ (B) is correct.

Hence, (A), (B) and (C) are correct.

8. Normals at $P(t_1)$ and $Q(t_2)$ meet the parabola at t_3.

$$t_3 = -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2} \Rightarrow (t_1 - t_2)\left(1 - \frac{2}{t_1 t_2}\right) = 0$$

We get, $t_1 t_2 = 2$

From first relation, we get, $t_3 = -t_1 - \dfrac{2}{t_1} = -t_1 - t_2$

$$\left|t_1 + \frac{2}{t_1}\right| \geq 2\sqrt{t_1 . \frac{2}{t_1}} = 2\sqrt{2} \Rightarrow |t_3| \geq 2\sqrt{2}$$

Hence, (B), (C) and (D) are correct.

9. $x = t^2 - 2t + 2;\ y = t^2 + 2t + 2$

$$x + y = 2(t^2 + 2) \text{ and } y - x = 4t$$

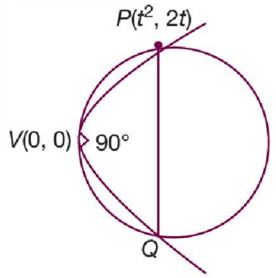
$$\frac{x+y}{2} = \frac{(y-x)^2}{16} + 2 \Rightarrow (y-x)^2 = 8(x + y - 4)$$

$$\left(\frac{y-x}{\sqrt{2}}\right)^2 = 4\sqrt{2}\left(\frac{x+y-h}{\sqrt{2}}\right)$$

This is the parabola, for which $y = x$ is axis, $x + y = 4$ is tangent at vertex and length of latus rectum is $4\sqrt{2}$ units.

Hence, (A), (B) and (D) are correct.

10. Slope of $PV = \dfrac{2t - 0}{t^2 - 0} = \dfrac{2}{t}$



Equation of QV is $y = -\dfrac{t}{2}(x)$

On solving with $y^2 = 4x$, $Q = \left(\dfrac{16}{t^2}, \dfrac{-8}{t}\right)$

Area of ΔPVQ is $\dfrac{1}{2}.PV.VQ = 20$

$$\Rightarrow PV \cdot VQ = 40$$

By solving the above equation, we get, $t = \pm 4, \pm 1$

Hence, (A) and (B) are correct.

11. $-Cx = Ay^2 + By + D$

$$-Cx = A\left(y^2 + \frac{B}{A}y + \frac{D}{A}\right)$$

$$x = -\frac{A}{C}\left(\left(y + \frac{B}{2A}\right)^2 + \frac{D}{A} - \frac{B^2}{4A^2}\right)$$

$$= -\frac{A}{C}\left[\left(y + \frac{B}{2A}\right)^2\right] - \frac{A}{C}\left[\frac{4AD - B^2}{4A^2}\right]$$

$$\Rightarrow \quad \left(x+\frac{4AD-B^2}{4AC}\right)=-\frac{A}{C}\left(y+\frac{B}{2A}\right)^2$$

$$\Rightarrow \quad \left(y+\frac{B}{2A}\right)^2=-\frac{C}{A}\left(x+\frac{4AD-B^2}{4AC}\right)$$

Hence, (A), (C) and (D) are correct.

12. The parabola can be written as $\left(x+\frac{2}{3}\right)^2=2(y-\frac{10}{9})$, i.e., $X^2=2Y\left(X=x+\frac{2}{3}, Y=y-\frac{10}{9}\right)$.

A point on axis is $\left(\frac{-2}{3}, Y\right)$ from which three normals can be drawn, if $Y>1$.

$$\Rightarrow \quad y>\frac{19}{9}$$

Hence, (B) and (C) are correct.

13. Curve is $\left(\frac{3x-4y}{5}\right)^2=\left(\frac{20x+15+60}{25}\right)$

Now get parameters

Hence, (A), (B) and (C) are correct.

14. $(x-3)^2=-4y-k+9=-4\left(y+\frac{k-9}{4}\right)$

Equation of the directrix is $y+\frac{k-9}{4}=1$

$$\Rightarrow \quad y=\frac{13-k}{4}$$

$$\frac{13-k}{4}=-1$$

$$\Rightarrow \quad k=17$$

Vertex is $\left(3,\frac{-k+9}{4}\right)=(3,-2)$

Focus $(3,-3)$

Hence, (A) and (C) are correct.

15. $t\left(-t-\frac{2}{t}\right)=-4$

$t=\pm\sqrt{2}$

Slope $=-t=\mp\sqrt{2}$

Hence, (A) and (B) are correct.

16. Let the equation of any tangent to $y=x^2$ be $y=mx-\frac{m^2}{4}$

$$mx-\frac{m^2}{4}=-x^2+4x-4$$

$$x^2+x(m-4)-\frac{m^2}{4}+4=0$$

Put $D=0$, we get, $m=0$ and 4

The equation is $y=4x-4$ and $y=0$.

Hence, (A) and (B) are correct.

17. Focus of the parabola is $\left(\frac{p}{2},0\right)$.

Since, the circle touches the directrix $x=-\frac{p}{2}$ of the parabola, the radius of the circle $=\frac{p}{2}+\frac{p}{2}=p$

$\Rightarrow$ Equation of the circle is $\left(x-\frac{p}{2}\right)^2+y^2=p^2$

$$\Rightarrow \quad x^2+y^2-px-\frac{3p^2}{4}=0$$

This circle meets the parabola $y^2=2px$ at points, whose abscissae are given by

$$x^2+2px-px-\frac{3p^2}{4}=0$$

$$\Rightarrow \quad x=\frac{p}{2}, x=-\frac{3p}{4}$$

However, $x=-\frac{3p}{4}$ is not possible on parabola, $y^2=2px$.

$$\therefore \quad x=\frac{p}{2}$$

Hence, (A) and (B) are correct.

18. The equation of the parabola can be written as $\left(y-\frac{5}{2}\right)^2=-3\left(x-\frac{25-4k}{12}\right)$

The length of the latus rectum is 3 units and the horizontal normal is $y=2x+1$, which is a tangent. The quadratic $(2x+1)^2-5(2x+1)+3x+k=0$ must have equal roots, which is $4x^2-3x+k-4=0$. Now, the roots are equal, if $b^2=4ac$.

$\Rightarrow$ $9=16(k-4)$, etc.

Since, $y=\frac{5}{2}$ is normal at the vertex.

Hence, (B), (C) and (D) are correct.

19. Given parabola is $y^2=8x$ ...(1)

Here, $a=2$.

Let $P(2t^2, 4t)$ be a point on parabola.

Equation (1) and S be the focus

Given, $SP=4$

$\therefore \quad a(1+t^2) = 4$

$\quad\; 2(1+t^2) = 4$

$\Rightarrow \quad t = \pm 1$

$\therefore \quad P(2, 4)$ or $(2, -4)$

Given, $f(x+y) = f(x)\,f(y)$, for all x and y

Given, $f(1) = 2 \qquad \ldots(2)$

From Equation (2), $f(2) = f(1+1) = f(1)\cdot f(1) = 2^2 = 4$

Similarly, $f(n) = 2^n$

$\because \quad a_r = f(r)$

$\therefore \quad a_1 = f(1) = 2,$

$\quad\; a_2 = f(2) = 4$

Hence, $P(a_1, a_2)$ or $(a_1, -a_2)$

Hence, (A) and (B) are correct.

20. For parabola $y^2 = 4ax$, equation of tangent at $A\,(at^2, 2at)$ is $ty = x + at^2$

$$\therefore \quad C(0, at),\ B\left(-a, at - \frac{a}{t}\right)$$

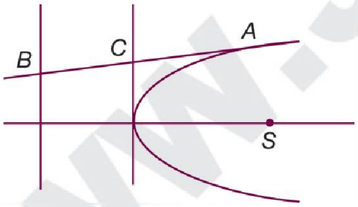
$$AC = at\sqrt{1+t^2}\ ;\ BC = \frac{a}{t}\ \sqrt{1+t^2}\ ;\ CS = a^2(1+t^2)$$

$$\Rightarrow \quad AC\cdot BC = (CS)^2$$

Slope of $CS \times$ Slope of $AB = \dfrac{0-at}{a-0} \times \dfrac{1}{t} = -1$

Hence, CS is perpendicular to AB

$$\therefore \quad CS = \frac{|2+2|}{\sqrt{2}} = 2\sqrt{2}$$



Hence, (A), (B) and (C) are correct.

21. $y = tx + 2at + at^3$

(α, β) lies on normal

$$\Rightarrow \quad at^3 + t(2a-\alpha) - \beta = 0$$

If two normals are coincident, then roots of above equation will be t_1, t_1, t_2.

$$2t_1 + t_2 = 0 \qquad \ldots(1)$$

$$t_1{}^2 t_2 = \frac{\beta}{a} \qquad \ldots(2)$$

$$t_1{}^2 + 2t_1t_2 = 2 - \frac{\alpha}{a} \qquad \ldots(3)$$

From (1) and (2), we get, $-2t_1^3 = \dfrac{\beta}{a}$

From (1) and (3), we get, $-3t_1^3 = 2 - \dfrac{\alpha}{a}$

Eliminate t_1 to get $27\,a\beta^2 = 4(\alpha - 2a)^3$

Let $y = mx - 2am - am^3$ be the equation of normal (α, β) that lies on normal.

$$\beta = mx - 2am - am^3 \qquad \ldots(4)$$

To get distance of point of intersection of normal and axis from origin, put $y = 0$.

$$\Rightarrow \quad x = \frac{2am + am^3}{m} = 2a + am^2$$

There are three normals with slope m_1, m_2 and m_3, which are the roots of Equation (4).

The distances are $2a + am_1{}^2$, $2a + am_2^2$, $2a + am_3^2$

Now, these distances are in A.P.

$$m_1^2 + m_3^2 = 2m_2^2 \qquad \ldots(5)$$

Of course, m_1, m_2 and m_3 are roots of Equation (4) and thus $am^3 + m(2\alpha - \beta) + \beta = 0$

$$\Rightarrow \quad m_1 + m_2 + m_3 = 0$$

$$m_1\,m_2\,m_3 = \frac{\beta}{a} \qquad \ldots(6)$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-\alpha}{a}$$

From Equations (2) and (6), $27a\beta^2 = 2(\alpha - 2a)^3$

Hence, (A) and (C) are correct.

22. Parabola touches x-axis at $(2, 0)$ and y-axis at $(0, 3)$.

Tangent at vertex is the line joining $(2, 0)$ and $(0, 3)$, which is $3x + 2y = 6$.

The point $(0, 0)$ lies on directrix.

Hence, directrix is $3x + 2y = 0$.

Hence, (B), (C) and (D) are correct.

23. $(y+b)^2 = -2a\left(x - \dfrac{b^2}{2a} + \dfrac{c}{2a}\right)$, which is a parabola.

Latus rectum $= 2a$.

Hence, (B) and (D) are correct.

24. The equation of the locus is $y = 2(x^2+1)$

$L(t, 2)$, $M(0, y)$

$$\frac{y_1 - 2}{0 - t} = -t$$

$y_1 = 2 + t^2$

Let the midpoint of LM is (h, k)

$$\frac{t+0}{2} = h, \ \frac{4+t^2}{2} = k$$

$y = 2(x^2 + 1)$

Hence, (A), (C) and (D) are correct.

25. Let the parabola is $y = ax^2 + bx + c$

$\Rightarrow \quad 1 - x = ax^2 + bx + c$

$\Rightarrow \quad ax^2 + (b + 1)\,x + c - 1 = 0$

For tangency, $D = 0$

$(b + 1)^2 - 4(a)\,(c - 1) = 0$

Now, check with options $(b + 1)^2 + 4a(1- c) = 0$

Hence, (A), (B) and (C) are correct.

26. Let $A = \left(t_1^2, \ 2t_1\right)$ and $B = \left(t_2^2, \ 2t_2\right)$

Slope of the chord $AB = \dfrac{2}{t_1 + t_2} = 2$

Put $t_2 = 1 - t_1$ in $y = \dfrac{2}{3}\,(t_2 + 2t_1)$

$\therefore \quad 3y = 2(1 - t_1 + 2t_1) = 2(1 + t_1)$

$\therefore \quad t_1 = \dfrac{3y-2}{2}$ and $t_2 = 1 - t_1 = 1 - \dfrac{3y-2}{2} = \dfrac{4-3y}{2}$

$\therefore \quad 3x = t_2^2 + 2\,t_1^2 = \left(\dfrac{4-3y}{2}\right)^2 + 2\left(\dfrac{3y-2}{2}\right)^2$

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$$

Hence, (A) and (D) are correct.

27. $y = 2x - 1$ and $y^2 = 4x$

Solving $y^2 - 2y - 2 = 0$

However, $y_1 + y_2 + y_1 = 0$

$\Rightarrow \quad y_3 = -2$

$\therefore \quad x_3 = 1 \ \ C(1, -2)$

$x_3 = 1$ and $4x^2 - 8x + 1 = 0 = 1 \qquad x_1 + x_2 = 2$

$$\text{Centroid}\left(\frac{2+1}{3}, 0\right) = (1, 0)$$

Equation of normal at C is $x - y - 3 = 0$.

Hence, (A), (B), (C) and (D) are correct.

28. The chord with midpoint $(\alpha, 1-\alpha)$ is $y(1-\alpha)$ $- 2a(x+\alpha) = (1-\alpha)^2 - 4a\alpha$

It passes through $(a, 2a)$

$\therefore \quad (1-\alpha)^2 = 2a(1-a) > 0$

$\Rightarrow \quad a \in (0, 1)$

$\Rightarrow \quad 4a \in (0, 4)$

Hence, (A), (B) and (C) are correct.

29. Solving equation of parabola with x-axis, i.e., $y = 0$, we get, $(a - b)\,x^2 + (b - c)$ $x + (c - a) = 0$, which should have two equal values of x, as x-axis touches the parabola.

$\Rightarrow \quad (b - c)^2 - 4(a - b)\,(c - a) = 0$

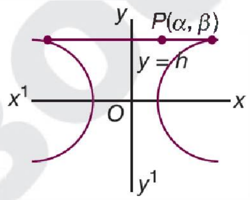
$\Rightarrow \quad (b + c - 2a)^2 = 0$

$\Rightarrow \quad b + c = 2a$

That is, $ax + by + c = 0$, which always pass through $(-2, 1)$.

Hence, (A) and (D) are correct.

30.



Given

$y^2 = 4a(x-k), y^2 = 4b(x+k)$

$$A = \left(\frac{h^2}{4a} + k, h\right), B\left(-k - \frac{h^2}{4b}, h\right)$$

Let $P(\alpha, \beta)$ then $\alpha = \dfrac{1}{2}\left(\dfrac{h^2}{4a} + k - k - \dfrac{h^2}{4b}\right), \beta = h$

$$\therefore \quad 2\alpha = \frac{h^2}{4}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Locus of P is $2x = \dfrac{y^2}{2}\dfrac{(b-a)}{ab}$

If $a = b \Rightarrow x = 0$

If $a \neq b \Rightarrow y^2 = \dfrac{hab}{(b-a)}\,x$, which is a parabola.

Hence, (A) and (B) are correct.

31. Let $y = mx + c$ be the common tangent

$mx + c = x^2$ has equal roots

$\Rightarrow \quad m^2 + 4c = 0 \qquad \ldots(1)$

$(mx + c) = -(x - 2)^2$

$\Rightarrow \quad (m - 4)^2 - 4(4 + c) = 0 \qquad \ldots(2)$

From (1) and (2), $m = c = 0$, $m = 4$ and $c = -4$

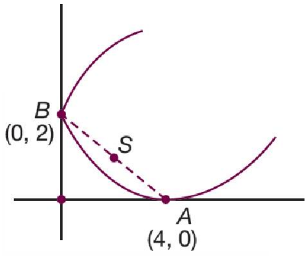
Hence, (A) and (B) are correct.

32. The diameter circle $x^2 + y^2 - 4x - 2y = 0$

Directrix is tangent at (0, 0)

$$4x + 2y = 0$$

$$\Rightarrow \quad 2x + y = 0$$



$$\text{BS:SA} = \frac{2}{\sqrt{5}} : \frac{8}{\sqrt{5}} = 1:4$$

$$\Rightarrow \quad S = \left(\frac{4}{5}, \frac{8}{5}\right)$$

The required equation is $x - 2y = \frac{-12}{5}$, the x-intercept is $\frac{-12}{5}$ and the latus rectum is $\frac{32}{5\sqrt{5}}$

Hence, (A), (B) and (C) are correct.

33. The given equation is $(x + y + 1)^2 = 24(x - y - 1)$

So, the vertex is $(0, -1)$

Length of the latus rectum is $12\sqrt{2}$ units

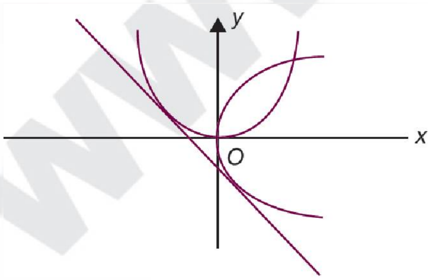
Equation of the axis is $x + y + 1 = 0$

The standard form is $\left(\frac{x+y+1}{\sqrt{2}}\right)^2 - 12\sqrt{2}\left(\frac{x-y-1}{\sqrt{2}}\right) = 0$

Here, $S_{11} > 0$ for the point $P(1, 2)$,

Hence, (A), (C) and (D) are correct.

34.



Equation of the common tangent is $a^{\frac{1}{3}}x + b^{\frac{1}{3}}y + (ab)^{\frac{2}{3}} = 0$

$$\Rightarrow \quad x + y + 1 = 0$$

The points P and Q are $P = (1, -2)$ and $Q = (-2, 1)$

$$\therefore \quad PQ = 3\sqrt{2}$$

Area of the region $POQ = -2$

$$\Delta QMO = \int_{-2}^{0} \frac{x^2}{4} dx - \frac{1}{2} \times 1 \times 1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\therefore \quad \text{Required area} = \frac{1}{6} + \frac{1}{2} \times 1 \times 1 + \frac{1}{6} = \frac{5}{6} \text{ sq. units}$$

$$P = (1, -2) = \left(t_1^2, 2t_1\right)$$

$$\Rightarrow \quad t_1 = -1$$

Now, $t_2 = -t_1 - \frac{2}{t_1} = 1 + 2 = 3$

$$\therefore \quad R = \left(t_2^2, 2t_2\right) = (9, 6)$$

Hence, (A), (B), (C) and (D) are correct.

35. Equation of parabola is $y \ (y + 2)^2 = -3\left(x - \frac{4}{3}\right)$

Equation of directrix is $x - \frac{4}{3} = \frac{3}{4}$

$$12x = 25$$

$$\Rightarrow \quad k = 25$$

Equation of latus rectum is $x - \frac{4}{3} = -\frac{3}{4}$

$$12x - 7 = 0$$

Equation of tangent at vertex is $3x - 4 = 0$

Also, $\frac{1}{PS} + \frac{1}{SQ} = \frac{1}{a} = \frac{1}{3/4} = \frac{4}{3}$.

Hence, (A), (B), (C) and (D) are correct.

36. Let P and Q be points $-m_1$ and $-m_2$

$$\Rightarrow \quad \frac{-2}{m_1 + m_2} = 1$$

$$\Rightarrow \quad m_1 + m_2 = -2$$

$\Rightarrow \quad m_3 = 2$, as $x - y = 1$ is a focal chord

$$\Rightarrow \quad m_1 m_2 = -1$$

$\Rightarrow \quad m_1 = \sqrt{2} - 1$ and $m_2 = -\left(\sqrt{2} + 1\right)$

Hence, (A), (B), (C) and (D) are correct.

37. As mirror image of focus about any tangent lies on directrix.

Hence, slope of directrix $= \frac{2-1}{1-2} = -1$

Hence, equation of directrix and axis are $x + y - 4 = 0$ and $y - x = 0$.

Hence, (A), (B) and (D) are correct.

38. The required condition can be obtained if circle $x^2+(y-a)^2=1$ is inside the parabola $y=x^2\ y^2$ $+(1-2a)y+(a-1)\le 0$

$$\Rightarrow \quad a\ge \frac{5}{4}$$

Hence, (A) and (B) are correct.

39. Equation of tangent at (I, m) is $my = 2a\,(x + I)$ $\quad\ldots(1)$

Any point on (1) is $\left(h, \dfrac{2a(h+I)}{m}\right)$

Equation of chord of contact to $x^2+y^2=a^2$ is $hx+y\left(\dfrac{2a(h+I)}{m}\right)=a^2$

$$\Rightarrow \quad mhx+2ahy+2aly-ma^2=0$$

$$\Rightarrow \quad h(mx+2ay)+(2aly-ma^2)=0$$

The fixed point $\left(-\dfrac{a^2}{I},\dfrac{ma}{2I}\right)$

Now, $\dfrac{I}{\alpha}=\dfrac{I}{-a^2/I}=-\dfrac{I^2}{a^2}$

$$\frac{m}{\beta}=\frac{m}{ma/2I}=\frac{2I}{a}$$

Clearly, $\left[\dfrac{1}{2}\left(\dfrac{m}{\beta}\right)\right]^2+\dfrac{1}{\alpha}=0$

$$\Rightarrow \quad 4\left(\frac{1}{\alpha}\right)+\left(\frac{m}{\beta}\right)^2=0$$

Hence, (B) and (D) are correct.

40. The points of intersection P, Q, R, S lie on $2x^2+12x-8y+26=0$, i.e., $(x+3)^2=4(y-1)$

So, vertex is $(-3, 1)$ and focus is $(-3, 2)$.

Points of intersection also lie on $y^2-20y+59=0$

Sum of two of the ordinates = 20

$$AP+AQ+AR+AS=40$$

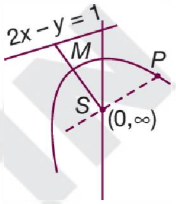
Hence, (B) and (C) are correct.

41. Let the focus be $(0,\alpha)$

$$SM=SP$$

$$SM=\frac{\alpha+1}{\sqrt{5}}$$

$\therefore$ P lies on the line $2x-y+\alpha=0$ at a distance of $\dfrac{\alpha+1}{\sqrt{5}}$ units from S.



$$\frac{x-0}{\frac{1}{\sqrt{5}}}=\frac{y-\alpha}{\frac{2}{\sqrt{5}}}=\pm\frac{(\alpha+1)}{\sqrt{5}}$$

$$(x,y)=\left(\frac{\alpha+1}{5},\alpha+\frac{2(\alpha+1)}{5}\right) \text{ or}$$

$$(x,y)=\left(-\frac{(\alpha+1)}{5},\alpha-\frac{2(\alpha+1)}{5}\right)$$

$$x=\frac{\alpha+1}{5},y=\frac{7a+2}{5} \text{ or } x=\frac{-\alpha-1}{5},y=\frac{3\alpha-2}{5}$$

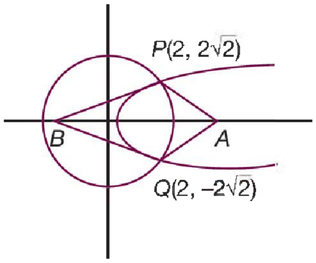
$$\alpha=5x-1=\frac{5y-2}{7} \text{ or } \alpha=-5x-1=\frac{5y+2}{3}$$

$$35x-7=5y-2 \text{ or } 3x+y+1=0$$

$$7x-y-1=0$$

Hence, (A) and (D) are correct.

42. $y^2=4x$ $\quad\ldots(1)$



$$x^2+y^2=12 \quad\ldots(2)$$

On solving, we get,

$$P=(2,2\sqrt{2}),\ Q=(2,-2\sqrt{2})$$

$$\left.\begin{aligned} m_1&=\frac{1}{\sqrt{2}}\\ m_2&=-\frac{1}{\sqrt{2}}\end{aligned}\right\} m_1+m_2=0$$

$$\angle PAB=\angle PBA$$

$\therefore$ ΔPAB isosceles.

$\therefore$ $PABQ$ is a rhombus (not concyclic)

$\overline{PB}$ is $x-\sqrt{2}y+2=0\Rightarrow B=(-2,0)$

$\overline{PA}$ is $x+\sqrt{2}y+2=0\Rightarrow A=(6,0)$

Area of $PABQ=2\left(\dfrac{1}{2}AB\times\text{Height}\right)=(8)(2\sqrt{2})$

$$=16\sqrt{2} \text{ sq. units}$$

Hence, (A), (B) and (C) are correct.

43. $h = \frac{a+at^2}{2}$; $k = at$

$$h = \frac{a}{2}\left(1+\left(\frac{k}{a}\right)^2\right)$$

$y^2 = 2a\left(x-\frac{a}{2}\right)$ Now get parameters

Hence, (A), (B), (C) and (D) are correct.

44. Circle: $(x+a)^2 + (y-2at)\left(y+\frac{2a}{t}\right) = 0$

From $y = 0$, $x^2 + 2ax - 3a^2 = 0$

$\Rightarrow$ $x = a$ or $-3a$

Hence, (A) and (D) are correct.

45. Slope of line joining the given points, $m = \frac{1-0}{4-3} = 1$

Equation of tangent is $y = x + c$

Parabola $y = (x-3)^2$

On solving these two, we get, $x^2 - 7x + 9 - c = 0$

For tangency $D = 0$; $49 - 4(9-c) = 0$ or $c = -\frac{13}{4}$

$\therefore$ The required equation is $4y = 4x - 13$.

Hence, (C) and (D) are correct.

46. $(t^2, 2t)\left(\frac{1}{t^2}, \frac{-2}{t}\right)$ are the extremities of a focal chord.

Centre $\left(\frac{t^2+\frac{1}{t^2}}{2}, \frac{2\left(t-\frac{1}{t}\right)}{2}\right) = (h,k)$

$t^2 + \frac{1}{t^2} = 2h$ $\quad t - \frac{1}{t} = k$

Eliminating 't', we get, $k^2 + 2 = 2h$

Locus of (h, k) is $y^2 = 2(x-1)$

$\therefore$ $y = m(x-1) + \frac{1}{2m}$ is always a tangent to $y^2 = 2(x-1)$.

Hence, (A), (B), (C) and (D) are correct.

47. Foot of perpendicular from focus onto any tangent lies on tangent at vertex

$\therefore (3,0)$ and $(0,4)$ are on tangent at vertex

Equation of tangent at vertex is $\frac{x}{3} + \frac{y}{4} = 1$

$\Rightarrow$ $4x + 3y = 12$

Latus rectum = 4 (Distance from focus to tangent at vertex)

$$= 4\left(\frac{12+12-12}{\sqrt{4^2+3^2}}\right) = \frac{48}{5}$$

Directrix is $4x + 3y = 0$

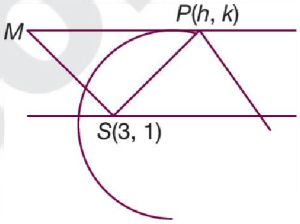
Axis is $3x - 4y = -7$

Hence, (A) and (C) are correct.

48. Image of (3, 1) with respect to $x + 2y - 10 = 0$ is (5, 5).

The horizontal line through (5, 5) is $y = 5$, which meets $x + 2y - 10 = 0$ at $P \Rightarrow P = (0,5) \Rightarrow h + k = 5$

$SP = PM = 5$



$\Rightarrow$ Perpendicular distance from (0, 5) to $x + k = 0$ is 5

$\Rightarrow$ $k = 5$

$\Rightarrow$ Equation of directrix is $x + 5 = 0$, $M = (-5, 5)$

$$\cos\angle SPM = \frac{25+25-80}{2\times5\times5} = \frac{-3}{5}$$

$\Rightarrow$ $\sin\angle SPM = \frac{4}{5}$

Area of $\Delta SPM = \frac{1}{2}\times5\times5\times\frac{4}{5} = 10$ sq. units

$2a = |3+5| = 5 \Rightarrow 4a = 16.$

Vertex = Midpoint of (–5, 1) and (3, 1) = (–1, 1)

Hence, (A), (B), (C) and (D) are correct.

Linked Comprehension Type Questions

1. The correct answer is 1(B), 2(A) and 3(A).

Combined Solution to 1, 2 and 3

Point of intersection is $P(0, 1)$.
Tangent at P is $y = ax + 1$

Touches circle $\Rightarrow r^2 = \frac{1}{1+a^2}$

Point does not lie below x-axis

$\Rightarrow$ $y \geq 0 \Rightarrow x^2 + ax + 1 \geq 0 \ \forall x$

$\Rightarrow$ $a^2 - 4 \leq 0$

$$\Rightarrow \quad a \in [-2,2]$$

When a is maximum, $r = \dfrac{1}{\sqrt{1+4}} = \dfrac{1}{\sqrt{5}}$

Radius is maximum, when $a = 0$

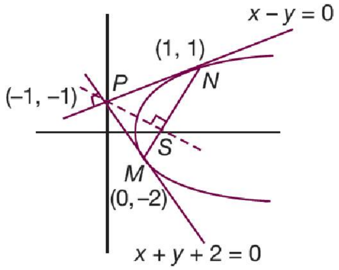
Area of triangle $= \dfrac{1}{2}\left|\left(\dfrac{-1}{a}\right)(1)\right| = \dfrac{1}{2a}$ sq. units

Area is minimum, where a is maximum.

4. **The correct answer is 4(D), 5(A) and 6(D).**

Combined Solution to 4, 5 and 6

The equations of the bisectors are given by $x - y = 0$ and $x + y + 2 = 0$.



These bisectors intersect at the point (–1,–1).
Focus S, is the foot of perpendicular from P to MN, i.e., point of intersection of lines.

$$MN(3x - y - 2 = 0) \text{ and } PS(x + 3y + 4 = 0)$$

$$\therefore \quad \text{Focus is } \left(\frac{1}{5}, -\frac{7}{5}\right)$$

$$MS = \sqrt{\frac{2}{5}}, \; NS = 4\sqrt{\frac{2}{5}}$$

$$\text{Length of latus rectum} = 2\left[\frac{2\left(\sqrt{\frac{2}{5}}\right)\left(4\sqrt{\frac{2}{5}}\right)}{\sqrt{\frac{2}{5}} + 4\sqrt{\frac{2}{5}}}\right]$$

($\because$ H.M. of segments of focal chord is semi-latus rectum)

$$= \frac{16 \times \frac{2}{5}}{5\sqrt{\frac{2}{5}}} = \frac{16}{5} \times \sqrt{\frac{2}{5}} = \frac{16\sqrt{2}}{(5)^{3/2}}$$

Circle passing through PMN is

$$(x-0)(x-1) + (y-1)(y+2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x + y - 2 = 0$$

$\therefore$ Equation of tangent at (–1, –1), which is also equation of directrix is $3x + y + 4 = 0$.

7. $$\tan\left(\frac{\pi}{4}\right) = \left|\frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{2}{t_1} - \left(\frac{2}{t_2}\right)}\right| = 2\left|\frac{t_2 - t_1}{t_1 t_2 + 4}\right|,$$ where P and Q are t_1, t_2

$$\Rightarrow \quad (t_1 t_2 + 4)^2 = 4(t_2 - t_1)^2$$

$$\text{Area of } \Delta APQ = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \cdot 2a^2 |t_1 - t_2| t_1 t_2 \text{ sq. units}$$

$$= \left|a^2 t_1 t_2 \cdot \frac{1}{2}(t_1 t_2 + 4)\right| \qquad \text{(From (1))}$$

$$= \frac{a^2}{2}|-1(-1+4)| = \frac{3a^2}{2}$$

Hence, the correct answer is (B).

8. Equation of PQ is $(t_1 + t_2)y = 2x + 2at_1 t_2$ …(2)

From (1),

$$(t_1 + t_2)^2 = (t_2 - t_1)^2 + 4t_2 t_1 = \frac{(t_1 t_2 + 4)^2}{4} + 4t_3 t_1$$

$$= \frac{(t_1 t_2 + 12)^2}{4} - 32$$

Let $t_1 + t_2 = m$, then $(t_1 t_2 + 12)^2 = 4(m^2 + 32)$

So the Equation (2) is

$$my = 2x + 2a\{-12 \pm 2\sqrt{m^2 + 32}\}$$

$$= 2(x - 12a) \pm 4a\sqrt{m^2 + 32}$$

$\therefore \; y = \dfrac{2}{m}(x - 12a) \pm 4a\sqrt{1 + \dfrac{32}{m^2}}$, which touches the ellipse $\dfrac{(x-12a)^2}{128a^2} + \dfrac{y^2}{16a^2} = 1$

$$\Rightarrow \quad (x - 12a)^2 + 8y^2 = 128a^2$$

Hence, the correct answer is (D).

9. Latus rectum of $S_2 = \dfrac{2 \times 16a^2}{8\sqrt{2}a} = 2\sqrt{2}a = \dfrac{1}{\sqrt{2}}(4a) = \dfrac{1}{\sqrt{2}} l_1$

Hence, the correct answer is (C).

10. Point at intersection of $P_1, L = 0$ is $(4a, 4a)$

$$S_1 = 0 \Rightarrow x - 2y + 4a = 0$$

Hence, the correct answer is (A).

11. Ends of latus rectum of $y^2 = 4ax$ are $(a, 2a)\ (a, -2a)$

Ends of latus rectum of $y^2 = -4ax$ are $(-a, 2a)\ (-a, -2a)$

Area of rectangle $= 4a \times 2a = 8a^2$ sq. units

Hence, the correct answer is (C).

12. The correct answer is 12(D), 13(B) and 14(B).

Combined Solution to 12, 13 and 14

$$(3x-4y)^2 = 5(4x+3y+12)$$

$$\left(\frac{(3x-4y)}{5}\right)^2 = \frac{1}{5}(4x+3y+12)$$

Let $X = \dfrac{4x+3y+12}{5},\ Y = \dfrac{3x-4y}{5}$

$$Y^2 = 4AX \Rightarrow 4A = 1$$

Directrix, $X = -A \Rightarrow \dfrac{4x+3y+12}{5} = \dfrac{-1}{4}$

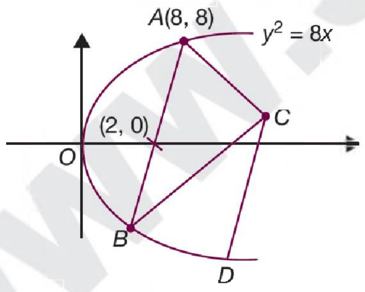
$$\Rightarrow \quad 4x+3y+\frac{53}{4} = 0$$

Equation of axis, $Y = 0 \Rightarrow 3x - 4y = 0$

Solving equation of latus rectum ($X = A$) and axis we get, focus $\left(\dfrac{-43}{25}, \dfrac{-129}{100}\right)$

15. The correct answer is 15(B), 16(D) and 17(C).

Combined Solution to 15, 16 and 17



Let point be $A(t)$.

$$at^2 = 8,\ 2at = 8 \text{ and } a = 2$$

So, $t = 2$

Hence, point $B\left(\dfrac{-1}{2}\right)$,

i.e., $\left(2\left(\dfrac{-1}{2}\right)^2, 2\times 2\times\left(\dfrac{-1}{2}\right)\right) = \left(\dfrac{1}{2}, -2\right)$

Slope of normal at 't' is $-t$.

$\Rightarrow$ Slope of normal at B is $\dfrac{1}{2}$

So, equation of normal is $y + 2 = \dfrac{1}{2}\left(x - \dfrac{1}{2}\right)$, i.e.,

$$x - 2y = \frac{9}{2} \qquad \ldots(1)$$

Now, $y_A + y_B + y_D = 0$

$\Rightarrow$ $8 - 2 + y_D = 0$ or $y_D = -6$

So, $x_D = \dfrac{36}{8} = \dfrac{9}{2}$

So, $D\left(\dfrac{9}{2}, -6\right)$

Equation of normal at point A is $y - 8 = -2(x - 8)$

i.e., $2x + y = 24$, $\qquad \ldots(2)$

Solving (1) and (2), we get, C is $\left(\dfrac{21}{2}, 3\right)$.

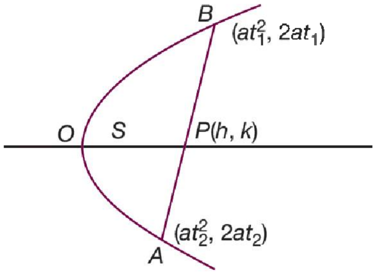
18. Let $P(h, k)$ be the midpoint, equation of chord through midpoint (h, k).

Equation of chord through midpoint is $ky - 2ax + 2ah - k^2 = 0$.

Combined equation of OA and OB will be

$$y^2 - 4ax\frac{(ky-2ax)}{k^2-2ah} = 0$$

$$\tan\alpha = \frac{4a\sqrt{4ah-k^2}}{k^2-2ah+8a^2}$$



$$(k^2 - 2h + 8a^2)^2 \tan^2\alpha = 16a^2(4ah - k^2)$$

$$(y^2 - 2ax + 8a^2)^2 \tan^2\alpha = 16a^2(4ah - y^2)$$

Hence, the correct answer is (C).

19. $t_1^2 + t_2^2 = \dfrac{2h}{a} \qquad \ldots(1)$

$$t_1 + t_2 = \frac{k}{a}$$

$$\frac{SB}{SA} = \frac{(t_1^2+1)}{t_2^2+1} = \frac{2}{1} \qquad \ldots(2)$$

Solving all the equations, we get,

$$9(k^2 - 2ah) = 4a^2(2h - a)(4h + a)$$

Hence, the correct answer is (A).

20. Normal passing through (α, β)

$$at^3 + (2a - \alpha)t - \beta = 0$$

$$\Rightarrow \quad t_1 + t_2 + t_3 = 0,\ t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-\alpha}{a}$$

$$t_1t_2t_3 = \frac{\beta}{a}$$

If P, Q, R are $\left(at_1^2, 2at_1\right)$, $\left(at_2^2, 2at_2\right)$ and $\left(at_3^2, 2at_3\right)$, then centroid of ΔPQR

$$= \left(\frac{at_1^2+at_2^2+at_3^2}{3}, \frac{2at_1+2at_2+2at_3}{3}\right)$$

$$= \left(\frac{2\alpha-4a}{3}, 0\right)$$

$\therefore$ (B) is correct.

Hence, the correct answer is (B).

21. The orthocenter of ΔPQR can be easily found as $\left(-4a - a(t_1t_2+t_2t_3+t_3t_1), \frac{-a}{2}t_1t_2t_3\right)$

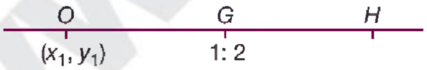
$$= \left(\alpha - 6a, \frac{-\beta}{2}\right)$$

$\therefore$(C) is correct

Hence, the correct answer is (C).

22. Using $\frac{OG}{GH} = \frac{1}{2}$ and last question

$$\frac{2\alpha-4a}{3} = \frac{1+(\alpha-6a)+2\times x_1}{3}$$



$$O = \frac{1\left(\frac{\beta}{2}\right)+2y_1}{3}$$

$\Rightarrow$ Circumcenter (x_1, y_1) is $\left(\frac{\alpha+2a}{2}, \frac{-\beta}{4}\right)$

$\therefore$ (A) is correct.

Hence, the correct answer is (A).

23. $|PA - PB|$ is max, when P, A and B are collinear and P divides AB externally.

Equation of AB is $-x + y = 1$, i.e., $y = x + 1$

$$(x + 1)^2 = 4x$$

$$\Rightarrow \quad x = 1$$

$\therefore$ AB intersect parabola at (1, 2).

Hence, the correct answer is (C).

24. Minimum value of $|PA - PB| = 0$, i.e., P lies on the perpendicular bisector of AB, which is $y = -x$.

This line meets the parabola at (0, 0), (4, –4).

Hence, the correct answer is (A).

25. $m_{SP} \cdot m_{SQ} = -1$

$$\Rightarrow \quad \frac{2t_1}{(t_1^2-1)} \cdot \frac{2t_2}{(t_2^2-1)} = -1$$

where $t_2 = -t_1 - \frac{2}{t_1}$ Now get PQ

Hence, the correct answer is (B).

26. Point of intersection of tangents at $A(t_1)$ and $B(t_2)$ is $(at_1t_2, a(t_1 + t_2)) = (-16a, k)$

$$t_1t_2 = -16;\ t_2 = \frac{-16}{t_1}$$

Equation of AB is $y(t_1 + t_2) = 2x + 2at_1t_2$

$$y\left(t_1 - \frac{16}{t_1}\right) = (2x - 32a)$$

$$(2x - 32a) - \left(t_1 - \frac{16}{t_1}\right)y = 0$$

Fixed point (16a, 0)

Hence, the correct answer is (B).

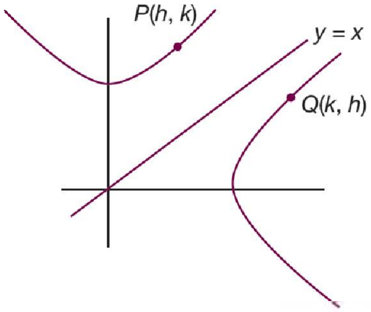
27. Equation of tangent of $y^2 = 4(x - 2)$

at $y^2 = 4(x - 2)$

with slope 1 is

$y = x - 1$

Point of intersection is (3, 2)



Centre of circle is midpoint of (3, 2) (2, 3) is $C\left(\frac{5}{2}, \frac{5}{2}\right)$

Both parabolas have parallel tangents $y = x - 1$ and $x = y - 1$

Distance between them is diameter = $\sqrt{2}$ units

$\therefore$ Required circle is $x^2 + y^2 - 5x - 5y + 12 = 0$

Hence, the correct answer is (B).

28. The correct answer is 28(B), 29(A) and 30(A).

Combined Solution to 28, 29 and 30

Any point on circle $x^2 + y^2 = 4$ is $(2\cos\alpha, 2\sin\alpha)$

$\therefore$ Equation of directrix is $x(\cos\alpha) + y(\sin\alpha) - 2 = 0$.

Let focus be (x_1, y_1). Then, as $A(1, 0)$, $B(-1, 0)$ lie on parabola we must have,

$$\left.\begin{aligned}(x_1-1)^2 + y_1^2 &= (\cos\alpha - 2)^2\\ (x_1-1)^2 + y_1^2 &= (\cos\alpha + 2)^2\end{aligned}\right\}$$

$\Rightarrow \quad x_1 = 2\cos\alpha$ and $y_1 = \pm\sqrt{3}\sin\alpha$

$\therefore$ Locus of focus is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ and focus is of the form $(2\cos\alpha, \pm\sqrt{3}\sin a)$

$\therefore$ Length of semi-latus rectum of parabola = Perpendicular distance from focus to directrix $= |2 \pm \sqrt{3}|\sin^2\alpha$

Hence, maximum possible length $= 2 + \sqrt{3}$ units

31. $A = (a, a^2 - 3)$

Equation of tangent L is $S_1 = 0$ is $2(a^2 - 3)x - ay - a^3 + 3a = 0$

Hence, the correct answer is (D).

32. The line L meets the parabola C: $y = x^2 - 3$ at the points, for which

$$x^2 - 3 = \frac{2(a^2-3)}{a}x - a^2 + 3$$

$\Rightarrow \quad (x - a)(ax + 6 - a^2) = 0$

However, $x = 1$ and $x \neq a$

$$x = \frac{a^2 - 6}{a} = 1$$

$\Rightarrow \quad a = -2, 3$

$$x = \frac{a^2 - 6}{a} = 1$$

$a = -2, 3$

Hence, the correct answer is (B).

33. If $a > 0$, then $a = 3$, $A(3, 6)$, $B(1, -2)$ equation of C is $y = x^2 - 3$ or $x^2 = y + 3$

Vertex 'O' of the parabola C is $(0, -3)$, slope $OA = 3$ and slope $OB = 1$

Hence, the correct answer is (B).

34. $am^3 + m(2a - x) + y = 0$, $m_1 + m_2 + m_3 = 0$,

$m_1m_2 + m_2m_3 + m_1m_3 = \frac{2a - x}{a}$; $m_1m_2m_3 = -\frac{y}{a}$;

given $m_1m_2 = 1$; $m_3 = \frac{-y}{a}$

We get curve $y^2 = a(x - a)$

Hence, the correct answer is (B).

35. $\frac{m_1 - m_2}{1 + m_1m_2} = \frac{m_2 - m_3}{1 + m_2m_3}$ compute m_2 and use $am_2^3 + (2a - x)m_2 + y = 0$ for locus

Hence, the correct answer is (A).

36. $y = mx - 2ax - ax^3$

$ax^3 + (2a - x)m + y = 0$

$$\sum m = 0, \quad \sum m_1m_2 = \frac{2a - x}{a}, \quad \sum m_1m_2m_3 = \frac{-y}{a}$$

$\theta_1 + \theta_2 + \theta_3 = k$

$\Rightarrow \quad \tan(\theta_1 + \theta_2 + \theta_3) = \frac{S_1 - S_3}{1 - S_2} = k$

$\Rightarrow \quad 0 - S_3 = k(1 - S_2)$

$\Rightarrow \quad y = a\left(\frac{x}{a} - 1\right)$

It is a straight line.

Hence, the correct answer is (A).

37. The correct answer is 37(B), 38(D) and 39(A).

Combined Solution to 37, 38 and 39

We have $(t_2 + t_3)(t_2 + t_1) = -4$

$$t_2^2 + t_2(t_1 + t_3) + t_1t_3 + 4 = 0 \qquad \text{...(1)}$$

$(t_1 + t_3)^2 - 4(t_1t_3 + 4) \geq 0$; $(t_1 - t_3)^2 \geq 16$

$t_2 + t_4 = -1(t_1 + t_3)$ and $t_2t_4 = t_1t_3 + 4$

As t_2 and t_4 are y-co-ordinates of intersection points of the tangents at (A, C) and (B, D), then $y_1 = t_1 + t_3$, $y_2 = t_2 + t_4 \Rightarrow y_1 + y_2 = 0$

40. $\tan\frac{\pi}{3} = \left|\frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1t_2}}\right|$

Or $\quad 3(4 + t_1t_2)^2 = 4[(t_1 + t_2)^2 - 4t_1t_2]$

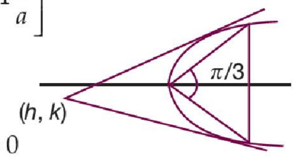
Or $\quad 3\left[4 + \frac{h}{a}\right]^2 = 4\left[\left(\frac{k}{a}\right)^2 - 4\frac{h}{a}\right]$

$h = at_1t_2$

$k = a(t_1 + t_2)$

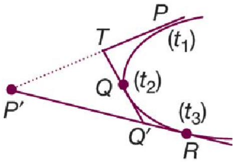
Or $\quad 3x^2 - 4y^2 + 40ax + 48a^2 = 0$

Hence, the correct answer is (B).



41. $TP = a(t_1 - t_2)\sqrt{1+t_1^2}$ $\dfrac{TP'}{TP} = \dfrac{t_2 - t_3}{t_1 - t_2}$

Similarly, $\dfrac{TQ'}{TQ} = \dfrac{t_1 - t_3}{t_2 - t_1}$



$$\frac{TP'}{TP} + \frac{TQ'}{TQ} = 1$$

$T(at_1\,t_2, a(t_1 + t_2))$

$P'(at_1\,t_3, a(t_1 + t_3))$

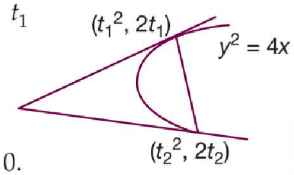
$Q'(at_2\,t_3, a(t_2 + t_3))$

Hence, the correct answer is (A).

42. $h = t_1t_2 = t_1\left(-t_1 - \dfrac{2}{t_1}\right) = -t_1^{\,2} - 2$

$k = t_1 + t_2 = t_1 + \left(-t_1 - \dfrac{2}{t_1}\right) = \dfrac{-2}{t_1}$

$\therefore \quad h = -\left(\dfrac{-2}{k}\right)^2 - 2$



Hence, locus is $\dfrac{4}{y^2} + x + 2 = 0.$

Hence, the correct answer is (C).

43. The equation of the normal to the parabola $y^2 = 4ax$ at P is $y + tx = 2at + at^3$. Suppose it meets the parabola at Q. If O is the vertex of the parabola, then the combined equation of OP and OQ is a homogenous equation of second degree, which is given by $y^2 = 4ax\left(\dfrac{y+tx}{2at+at^3}\right)$

$\Rightarrow \quad y^2(2at + at^3) = 4ax(y + tx)$

If OP and OQ are at right angles, then the co-efficient of x^2 + co-efficient of $y^2 = 0$

$\Rightarrow \quad 4at - 2at - at^3 = 0$

$\therefore \quad t^2 = 2$

Hence, the correct answer is (B).

44. The co-ordinates of the ends of the latus of the parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$ is

$$y - 2a = \left(\frac{-2a}{2a}\right)(x - a) \text{ or } x + y - 3a = 0$$

Hence, the correct answer is (B).

45. Let $P\,(at_1^{\,2}, 2at_1)$ and $Q\,(at_2^{\,2}, 2at_2)$ be two points on the parabola $y^2 = 4ax$.

It is given that the normal at P and Q intersect at R on the parabola.

$\therefore \quad t_1t_2 = 2$

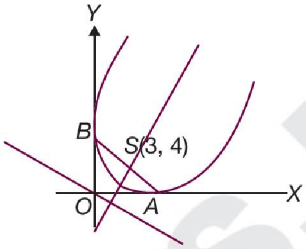
So, product of the ordinates at P and Q

$= (2at_1)(2at_2) = 4a^2 \times t_1t_2 = 8a^2$

Hence, the correct answer is (C).

46. The correct answer is 46(A), 47(B) and 48(A).

Combined Solution to 46, 47 and 48



x-axis and y-axis are perpendicular tangents to the parabola. Hence, their point of intersection (0, 0), lies on the directrix.

Hence equation of directrix is $y = mx$.

Let $A(h, 0)$, $B(0, k)$ are points of contact.

AB must be a focal chord.

$\angle OSA = 90°$ and $\angle OSB = 90°$

$\therefore \quad \dfrac{4}{3} \times \dfrac{4}{3-h} = -1$

$16 = -9 + 3h$

$h = \dfrac{25}{3}$

Similarly, we get, $k = \dfrac{25}{4}$

$\therefore \quad A\left(\dfrac{25}{3}, 0\right), B\left(0, \dfrac{25}{4}\right)$

Since B is a point on parabola, $SB^2 = BM^2$, i.e.,

$$3^2 + \left(4 - \frac{25}{4}\right)^2 = \left(\frac{\left|m.0 - \frac{25}{4}\right|}{\sqrt{m^2+1}}\right)^2$$

On solving, we get,

$m = \pm\dfrac{4}{3}$

However, as directrix make obtuse angle with x-axis, $m = -\dfrac{4}{3}$

Equation of axis is $y - 4 = \dfrac{3}{4}(x - 3)$

Equation of axis is $3x - 4y + 7 = 0$

$y = \dfrac{-4}{3}x$

$\Rightarrow \quad 4x + 3y = 0$

Distance between focus and directrix is $2a$

$\Rightarrow \quad 2a = \dfrac{24}{5}$

Latus rectum, $4a = \dfrac{48}{5}$

49. Point of intersection of $y = 2$ and $y^2 = 8x$ is $P\left(\frac{1}{2}, 2\right)$ and focus of the parabola is $S(2, 0)$

$\therefore$ Equation of the reflected ray is $y - 0 = \dfrac{2-0}{\frac{1}{2}-2}(x-2)$

$\Rightarrow \quad y = -\frac{4}{3}(x-2)$

$\Rightarrow \quad 4x + 3y = 8$

Hence, the correct answer is (C).

50. $\because \quad y^2 + 10y - 4x + 17 = 0$

$\Rightarrow \quad (y+5)^2 - 25 - 4x + 17 = 0$

$\Rightarrow \quad (y+5)^2 = 4x + 8$

$\Rightarrow \quad (y+5)^2 = 4(x+2)$

Let $y + 5 = Y$, $x + 2 = X$, then $Y^2 = 4X$

Focus is $X = 1$, $Y = 0$, i.e., $(-1, -5)$

After reflection, the ray must pass through focus $(-1, -5)$.

Hence, the correct answer is (B).

51. Solving $y = 1$ and $y^2 = x$

Then $A(1, 1)$

On solving, $y = -2$ and $y^2 = x$

Then, $B(4, -2)$

$\because$ After reflection, both reflected rays pass through focus of the parabola $y^2 = x$

i.e., $C\left(\frac{1}{4}, 0\right)$

$$\therefore \quad \text{Required area} = \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 4 & -2 \\ \frac{1}{4} & 0 \\ 1 & 1 \end{vmatrix} = \frac{21}{8} \text{ sq. units}$$

Hence, the correct answer is (A).

52. Here $P(1, 2) \Rightarrow$ Equation of tangent at P is $y = x + 1$

Now, to find image of the parabola, find image of a variable point on the parabola and then take its locus.

Let the variable point on $y^2 = 4x$ is $A(t^2, 2t)$

Now, image of A with respect to $y = x + 1$ is $(2t - 1, t^2 + 1)$

Hence, $h = 2t - 1$, $k = t^2 + 1$

Locus(h, k) is $(x+1)^2 = 4(y-1)$

Hence, the correct answer is (C).

53. By using family of circle, equation of the circle is

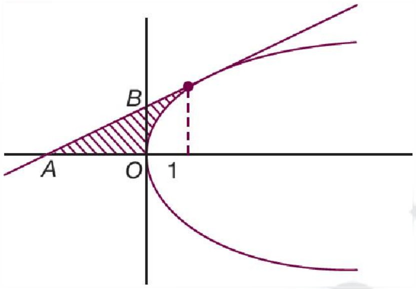
$$(x-1)^2 + (y-2)^2 + \lambda(x - y + 1) = 0$$

If this circle passes through $(1, 0)$, then $\lambda = -2$

Then equation of the circle is $(x-2)^2 + (y-1)^2 = 2$

Hence, the correct answer is (B).

54.



Area enclosed by parabola, x-axis and $y = x + 1$ is

(Area of $\Delta AOB = \frac{1}{2}$)

$$= \frac{1}{2} + \int_0^1 (x + 1 - \sqrt{4x})dx = \frac{2}{3}$$

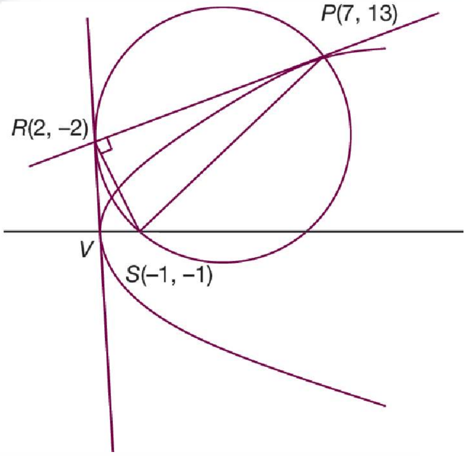
Hence, the correct answer is (A).

55. $\dfrac{\alpha+1}{3} = \dfrac{\beta+1}{-1} = -\dfrac{(-3+1-8)}{10}$

$\Rightarrow \quad (a, b) = (2, -2)$

Hence, the correct answer is (B).

56.



Normal at $(2, -2)$ will pass through the centre of circle.

Centre = Midpoint of $(7, 13)$ and $(-1, -1)$

$\Rightarrow \quad C = (3, 6)$

Slope of $RC = \dfrac{6+2}{3-2} = 8$

Hence, the correct answer is (B).

57. Tangent at vertex will be perpendicular to RC and it passes through R.

$$y + 2 = -\left(\frac{1}{8}\right)(x-2)$$

$\Rightarrow \quad x + 8y + 14 = 0$

Hence, the correct answer is (C).

Matrix Match/Column Match Type Questions

1. A → (r); B → (p); C → (q); D → (s)

(A) If the parameters t_1, t_2 and t_3 correspond to the points A, B and C respectively, then

$$t_3 = 3,\ t_1 + t_2 = -3,\ t_1t_2 = 2 \qquad \ldots(1)$$

$$AB^2 = (t_1 - t_2)^2\,((t_1 + t_2)^2 + 4) = 13 \text{ by (1)}$$

$$\Rightarrow \quad AB = \sqrt{13}$$

(B) Equation of AB is $2x - (t_1 + t_2)y + 2t_1t_2 = 0$

Then distance of C(9,6) from this line is $\frac{40}{\sqrt{13}}$ units

$$\therefore \quad \Delta ABC = \frac{\sqrt{13}}{2} \cdot \frac{40}{\sqrt{13}} = 20 \text{ sq. units}$$

(C) Distance of (0, 0) from the line $2x + 3y + 4 = 0$ is $\frac{4}{\sqrt{13}}$ units

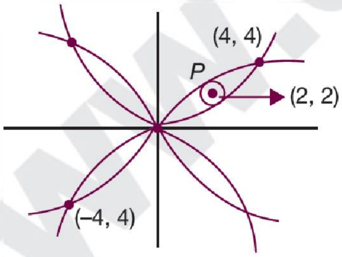
(D) The line $2x + 3y + 4 = 0$ makes intercepts of 2, $\frac{4}{3}$ on the axes.

$\therefore$ The area is $\frac{4}{3}$ sq. units

2. A → (r); B → (s); C → (q); D → (q)

(A) $\therefore (\pm 4, \pm 4)$ (0, 0) are the 5 points of intersection.

(B) Centre is (2, 2) move (2, 2) at right angles to $y = x$ by 'r' units to reach $y^2 = 4|x|$, $x^2 = 4|y|$.



$$\left(\frac{x-2}{\frac{-1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = r\right)$$

$\left(2 - \frac{r}{\sqrt{2}}, 2 + \frac{r}{\sqrt{2}}\right)$ lies on $y^2 = 4x$.

$$\left(2 + \frac{r}{\sqrt{2}}\right)^2 = 4\left(2 - \frac{r}{\sqrt{2}}\right)$$

$$\frac{r^2}{2} + \frac{4r}{\sqrt{2}} + 4 = 8 - \frac{4r}{\sqrt{2}}$$

$$r^2 + \frac{16}{\sqrt{2}}r - 8 = 0$$

$$r = \frac{-\frac{16}{\sqrt{2}} + \sqrt{128 + 32}}{2}$$

$$= -4\sqrt{2} + 2\sqrt{10}$$

(C) 8 points of intersection

(D) 8 points of intersection

3. A → (r); B → (s); C → (p); D → (q)

Focus (1, 2) and directrix is $12x - 5y + 3 = 0$

(A) Locus of point of intersection of perpendicular tangents is directrix = $12x - 5y + 3 = 0$.

(B) Locus of foot of perpendicular from focus upon any tangent is the line parallel to directrix and passing through vertex = $24x - 10y + 1 = 0$.

(C) Required line is the line parallel to directrix and passing through focus = $12x - 5y - 2 = 0$.

(D) Required line is the line perpendicular to directrix and passing through focus = $5x + 12y - 29 = 0$.

4. A → (q); B → (q); C → (s); D → (p)

(A) Equation of tangent at (3, 6): $y = x + 3 \Rightarrow T(-3, 0)$

Equation of normal at (3, 6): $y = -x + 9 \Rightarrow G(9, 0)$

Hence, midpoint is (3, 0).

(B) The point is obviously focus (3, 0)

(C) If variable chord is PQ, then

$P(t_1)$ and $Q(t_2)$

$\Rightarrow \quad t_1t_2 = -4$

$\Rightarrow$ Chords are concurrent at $(4a, 0)$

$\Rightarrow$ (12, 0)

(D) Let $A(t_1)$ and $B(t_2)$, $C(t_3)$ and $D(t_4)$

If AB and CD intersect at a point E on the axis, then by solving the equations of AB and CD, we get the relation $t_1t_2 = t_3t_4$

Now, equations of the circles with AB and CD as diameters are

$$\left(x - at_1^2\right)\left(x - at_2^2\right) + (y - 2at_1)(y - 2at_2) = 0$$

$$\left(x - at_3^2\right)\left(x - at_4^2\right) + (y - 2at_3)(y - 2at_4) = 0$$

If we solve these two circles, then equation of their radical axis is of the form, $y = mx$.

5. A → (r, s); B → (s); C → (p); D → (p, q)

(A) Any point is $(t, 1 - t)$

The chord with this as midpoint is $y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$.

$\Rightarrow \quad (1 - t)^2 = 2a(1 - a) \geq 0$

$\Rightarrow \quad 0 < a \leq 1$

$\therefore \quad LR \in (0, 4)$

(B) $P(1, 0), Q(4, 0)$

$(x - 1)(x - 4) + y^2 + \lambda y = 0$

The length of the tangent from $(0, 0)$ is $\sqrt{4} = 2$ units

(C) $m_1 m_2 = -1$

(D) $1 - |h| > 0$

$\Rightarrow \quad -1 < h < 1$

6. A → (s); B → (r); C → (s); D → (p, q)

(A) We know that, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ at the point $(am^2, -2am)$.

The given equation can be written as $y = -2x - 2a(-2) - (-2)^3 a$.

It represents a normal to the parabola corresponding to $m = -2$ at the point $(4a, 4a)$, whose distance from the focus $(a, 0)$ is $\sqrt{(4a - a)^2 + (4a)^2}$

$= \sqrt{(9a^2 + 16a^2)} = 5a = 5$ units

(B) Length of the latus rectum $= 4a$ units

$= 2$ (Distance from the focus of the directrix)

$$= 2 \times \frac{|3 \times 2 + 4 \times 0 + 4|}{\sqrt{(9 + 16)}} = \frac{2 \times 10}{5} = 4 \text{ units}$$

(C) We know that, $m_1 + m_2 + m_3 = 0$

$$\Rightarrow \quad m_3 = -m_1 - m_2 = \frac{2}{3} - 1 = -\frac{1}{3} = -\frac{k}{15}$$

$\Rightarrow \quad k = 5$

(D) Any point on the line $x + y = 1$ can be taken as $(t, 1 - t)$. Equation of the chord, with this as midpoint is $y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$, which passes through $(a, 2a)$.

So, $t^2 - 2t + 2a^2 - 2a + 1 = 0$, this should have two distinct real roots

Hence, $a^2 - a < 0, 0 < a < 1$

$\therefore$ The length of latus rectum < 4

7. A → (p, q, r, s, t); B → (s, t); C → (q, r, s, t); D → (s, t)

If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$.

(A) $\because \quad y^2 - 4x - 2y + 5 = 0$

$\Rightarrow \quad (y - 1)^2 = 4(x - 1)$

Let, $y - 1 = Y$ and $x - 1 = X$

$\therefore \quad Y^2 = 4X$

On comparing with $Y^2 = 4aX$

$\therefore \quad a = 1$

According to question, $X > 2a$

$\Rightarrow \quad x - 1 > 2$

Or $\quad x > 3$

$\therefore \quad x = 4, 5, 6, 7, 8$ (P, Q, R, S, T)

(B) $\because \quad 4y^2 - 32x + 4y + 65 = 0$

$\Rightarrow \quad 4(y^2 + y) = 32x - 65$

$$\Rightarrow \quad 4\left(\left(y + \frac{1}{2}\right)^2 - \frac{1}{4}\right) = 32x - 65$$

$$\Rightarrow \quad 4\left(y + \frac{1}{2}\right)^2 = 32x - 64$$

$$\text{Or} \quad 4\left(y + \frac{1}{2}\right)^2 = 8(x - 2)$$

Let, $y + \frac{1}{2} = Y$ and $x - 2 = Y$

$\therefore \quad Y^2 = 8X$

On comparing with $y^2 = 4ax$

$\therefore \quad a = 2$

According to question $X > 2a$

$\Rightarrow \quad x - 2 > 4$

$\therefore \quad x > 6$

$\therefore \quad x = 7, 8$ (S, T)

(C) $\because \quad 4y^2 - 16x - 4y + 41 = 0$

$\Rightarrow \quad 4(y^2 - y) = 16x - 41$

$$\Rightarrow \quad 4\left\{\left(y - \frac{1}{2}\right)^2 - \frac{1}{4}\right] = 16x - 41$$

$$\Rightarrow \quad 4\left(y - \frac{1}{2}\right) = 16x - 40$$

$$\text{Or} \quad \left(y - \frac{1}{2}\right) = 4\left(x - \frac{5}{2}\right)$$

Let $\quad y - \frac{1}{2} = Y$ and $x - \frac{5}{2} = X$

$\therefore \quad Y^2 = 4X$

On comparing with $Y^2 = 4aX$

∴ $a = 1$

According to question, $X > 2a$

⇒ $x - \frac{5}{2} > 2$ or $x > \frac{9}{2}$

∴ $x = 5, 6, 7, 8$ (Q, R, S, T)

8. A → (s, t); B → (p, q, r, s); C → (p, q, r); D → (q, r, s, t)

The other vertices of the triangle are $(5, 2\sqrt{5})$ and $(5, -2\sqrt{5})$

Therefore, the centroid is $\left(\frac{10}{3}, 0\right)$, the circumcenter is $\left(\frac{9}{2}, 0\right)$ and the incenter is (3, 0).

9. A → (s); B → (q); C → (r); D → (p)

(A) $(x - h)^2 + y^2 = (h - 1)^2$

$y^2 = 4x$

$(x - h)^2 + 4x = (h - 1)^2$

Apply, $D = 0$

(B) Find the common normal to

$y^2 = 4x, y^2 = 2x - 6$

$y = mx - 2m - m^3$

$y = m(x - 3) - m - \frac{1}{2}m^3$

Apply, $c = -2am - am^3$

$-4m - \frac{1}{2}m^3 = -2m - m^3$

$\frac{m^3}{2} - 2m = 0 \qquad m = 0, m = \pm 2$

(C) Length of the semi-latus rectum is the H.M. between segments of the focal chord.

(D) Tangents at the ends of focal chord are perpendicular.

10. A → (q); B → (s); C → (q); D → (p)

(A) Equation of normal is $y = -tx + 2at + at^3$ at $P(t)$

It intersect the curve again at point $Q(t_1)$ on the parabola such that $t_2 = -t - \frac{2}{t}$

Again, slope of OP is $\frac{2}{t} = M_{OP}$

Also, slope of OQ is $\frac{2}{t_1} = M_{OQ}$

Since $M_{OP}.M_{OQ} = -1 = \frac{4}{tt_1} \Rightarrow tt_1 = -4$

$t\left(-t - \frac{2}{t}\right) = -4 \Rightarrow t^2 = 2$

(B) $P(1, 2), Q(4, 4), R(16, 8)$

Now, $ar(\Delta PQR) = 6$ sq. units

(C) Equation of normal from any point $P(am^2, -2m)$ is $y = mx - 2am - am^3$

It passes through $\left(\frac{11}{4}, \frac{1}{4}\right)$

⇒ $4m^3 + 8m - 11m + 1 = 0$

⇒ $4m^3 - 3m + 1 = 0$

⇒ $(m + 1)(2m - 1)^2 = 0$

$m = -1, \frac{1}{2}$

∴ Only 2 normals are possible.

(D) Since, normal at $P(t_1)$ meets the curve again at (t_2), then $t_2 = -t_1 - \frac{2}{t_1}$

Such that, here normal at $P(1)$ meets the curve again at $Q(t)$.

⇒ $t = -3$

⇒ $|t - 1| = 4$

11. A → (q); B → (r); C → (s); D → (p, r, s)

(A) Point of intersection of perpendicular tangents lie on directrix

$x - 1 = +1 \Rightarrow x = 2$

(B) Equation of parabola $(y - 1)^2 = 4(x - 1)$

Equation of directrix $x - 1 = -1$

⇒ $x = 0$

(C) Equation of latus rectum $(x - 2) + 2(y - 1) = 0$

⇒ $x + 2y - 4 = 0$

Equation of tangent at vertex $\frac{x + 2y - 4 + x + 2y + 10}{2} = 0$

⇒ $x + 2y + 3 = 0$

(D) Equation of tangent will be of form $y = mx + \frac{3}{4m}$

$m = \frac{-1}{2}; x + 2y + 3 = 0$

$m = 1; 4x - 4y + 3 = 0$ and $x = 0$

12. A → (q); B → (s); C → (p); D → (r)

(A) $x^2 + (y-5)^2 = r^2,\ x^2 = 4y$

$\Rightarrow \quad 4y + (y-5)^2 - r^2 = 0$, which has equal roots

$\Rightarrow \quad r = 4$

(B) $S_1 = S_{11} \Rightarrow 2y = x$

Length of the chord $= (2)\ \sqrt{2^2+1^2}\ = 5\sqrt{2}$

(C) $3y^2 = kx \qquad x = 1 - 2y$

$3y^2 = k(1-2y)$ should have equal roots

$D = 0$

$\Rightarrow \quad k = 3$

(D) $a = \dfrac{l_1 l_2}{l_1 + l_2} = 2$

13. A → (s); B → (q); C → (r); D → (r)

(A) $y^2 = 4x \qquad \ldots(1)$

Clearly, if $P\left(at^2, 2at\right)$

Equation of tangent is $ty = x + at^2$ for t,

$y = 0, x_1 = -at^2$

and equation of normal is

$y = -tx + 2at + at^3$

For $R, y = 0, x_2 = \left(2at + at^2\right) \qquad \ldots(2)$

Here, $a = 1$

$\Rightarrow \quad x_1 = -t^2$

$x_2 = \left(2 + t^2\right) \Rightarrow 3 = 2 + t^2$

$\Rightarrow \quad t^2 = 1 \Rightarrow t = \pm 1$

Take $t = 1$, then $x_1 = -1$

$\therefore \quad PM = 2at = 2 \times 1 \times 1 = 2$

$RT = x_1 + x_2 = (1+3) = 4$

$\therefore$ Area of quadrilateral $PTQR = 2 \times \left(\dfrac{1}{2} \times 4 \times 2\right)$

$= 8$ sq.units

(B) Length of latus rectum + Tangent $PT = 4a + \sqrt{S_1}$

$= 4 \times 1 + \sqrt{0^2 - 4(-1)}$

$= 4 + 2 = 6$ sq. units

(C) Clearly, RT will be the diameter of circle.

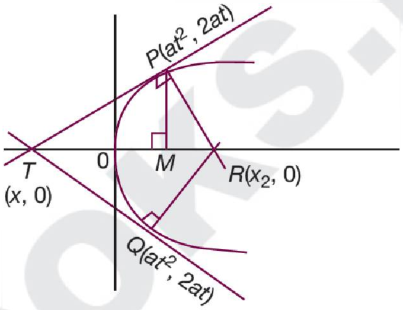
$\therefore$ Circumference $= (\pi \times$ diameter$)$

$= \pi RT = \pi \times 4$ units

(D) Since, in the part (1), we get

$x_2 = \left(2a + at^2\right) > 2a,\ \ if\,(t \neq 0)$

$\therefore \quad$ For three real normals, $x_2 > 2a = 2 \times 1 = 2$



14. A → (q); B → (r); C → (r); D → (r)

$P = (4a, 4a),\ Q = (9a, -6a)$

(A) $S = (a, 0)$

$m_1 =$ Slope of $SP = \dfrac{4}{3} \quad \Rightarrow m_1 m_2 = -1$

$m_2 =$ Slope of $SQ = \dfrac{-6}{8} = \dfrac{-3}{4}$

$\Rightarrow \quad t = 2$

(B) $t_1 t_2 = 2,\ t_1 + t_2 = -3 \Rightarrow\ t_1 = -1,\ t_2 = -2$

If $t_1 t_2 = 2$, the normals at t_1 and t_2 meet on the parabola at $t = 1 + \dfrac{2}{1} = 3 \Rightarrow x_1 = 9,\ \ y_1 = 6 \Rightarrow \dfrac{2x_1}{y_1} = 3$

(C) If $P = t, Q = \dfrac{-1}{t},\ P' = -t - \dfrac{2}{t},\ Q' = \dfrac{1}{t} + 2t$

$PQ = a\left(t + \dfrac{1}{t}\right)^2,$

$P'Q' = a\left|3\left(t + \dfrac{1}{t}\right)\right|\sqrt{\left(t - \dfrac{1}{t}\right)^2 + 4}$

$= 3a\left(t + \dfrac{1}{t}\right)^2 \Rightarrow \dfrac{P^1Q^1}{PQ} = 3$

(D) Let $P = (at^2, 2at)$

$\Rightarrow \quad t = \sqrt{2} \Rightarrow Q = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}$

$\Rightarrow \quad P(2a, 2\sqrt{2}a),\ Q(8a, -4\sqrt{2}a)$

$\Rightarrow \quad \dfrac{SQ}{SP} = \dfrac{a + 8a}{a + 2a} = 3$

Integer/Numerical Answer Type Questions

1. $x + y + 1 = 0$ is a common tangent.

$\therefore \quad l = m = n = 1$

$\therefore \quad l + 2m + 3n = 6$

Hence, the correct answer is (6).

2. Parabolas are equal, if their latus rectums are equal.

Latus rectum of $y^2 = \lambda x$ is λ rewriting second parabola $(x-3)^2 + (y-(-2))^2 = \left(\frac{3x-4y-2}{5}\right)^2$

$\therefore$ Focus $= (3, -2)$ and directrix is $3x - 4y - 2 = 0$

Distance between focus and directrix $\frac{\lambda}{2} = \frac{|9+8-2|}{\sqrt{9+16}}$

$\Rightarrow \quad \lambda = 6$ units

Hence, the correct answer is (6).

3. Let $P(t)$ and $Q\left(-\frac{1}{t}\right)$, as they are focal chords.

$$\Rightarrow \quad P'\left(-t-\frac{2}{t}\right) \text{and } Q'\left(\frac{1}{t}+2t\right)$$

Hence, $P'Q' = a\left|3\left(t+\frac{1}{t}\right)\right|\sqrt{\left(t-\frac{1}{t}\right)^2+4} = 3a\left(t+\frac{1}{t}\right)^2$

and $PQ = a\left(t+\frac{1}{t}\right)^2$

$$\Rightarrow \frac{P'Q'}{PQ} = 3$$

Hence, the correct answer is (3).

4. The curve is $(y + 1)^2 = 4(x - 1)$

Equation of the normal to the given curve is $y + 1 = m(x - 1) - 2m - m^3$, which passes through (h, k).

$m^3 + m(3 - h) + 1 + k = 0$

$m_1m_2m_3 = -(1 + k)$

$\Rightarrow \quad m_3 = (1 + k)$

$(\because m_1m_2 = -1)$

$(1+k)^3 + (1+k)(3-h) + (1+k) = 0$

$(1+k)^2 + 3 - h + 1 = 0$

$(y+1)^2 = (x-4)$

$\therefore \quad \frac{\mu}{2} = -1$

$\Rightarrow \quad \mu = -2, \lambda = 4$

$\therefore \quad \lambda - 2\mu = 8$

Hence, the correct answer is (8).

5. $x_1 + x_2 = 0$

$\Rightarrow \quad y = mx + a$ is a horizontal line $m = 0$

Hence, the correct answer is (0).

6. Equation of pair of tangents

$SS_1 = T^2$

Then angle between tangents is θ, we get

$$\tan\theta = \frac{\sqrt{{y_1}^2 - 12x_1}}{x_1 + a} = \frac{\sqrt{16-12}}{1+3} = \frac{1}{2}$$

$\therefore \quad 2k = 1$

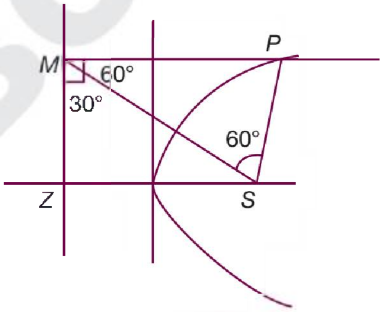
Hence, the correct answer is (1).

7. Tangents are perpendicular. Then $3x - 4y + c = 0$ is a focal chord, which passes through focus (0, 1)

$\therefore$ We get, $c = 4$.

Hence, the correct answer is (4).

8.



$SZ =$ Semi-latus rectum $= 4$

$$\sin 30° = \frac{SZ}{SM}$$

$\Rightarrow \quad SM = 8$

Hence, the correct answer is (8).

9. Since the normal at $(ap^2, 2ap)$ to $y^2 = 4ax$ meets the parabola at $(aq^2, 2aq)$

$$\therefore \quad q = -p - \frac{2}{p} \qquad \text{...(1)}$$

Since, $OP \perp OQ$,

$$\frac{2ap-0}{ap^2-0} \times \frac{2aq-0}{aq^2-0} = -1$$

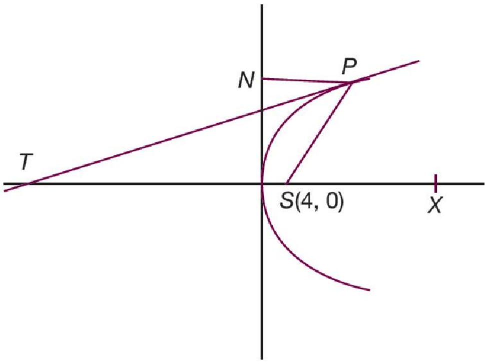
$\Rightarrow \quad pq = -4$

$$\Rightarrow \quad p\left(-p-\frac{2}{p}\right) = -4 \qquad \text{(Using (1))}$$

$\therefore \quad p^2 = 2$

Hence, the correct answer is (2).

10.



Since PT bisects $\angle NPS$

$\therefore \quad \angle NPS = \angle PSX = m_{PS} = \tan^{-1}\frac{16}{12} = \tan^{-1}\frac{4}{3} = \theta.$

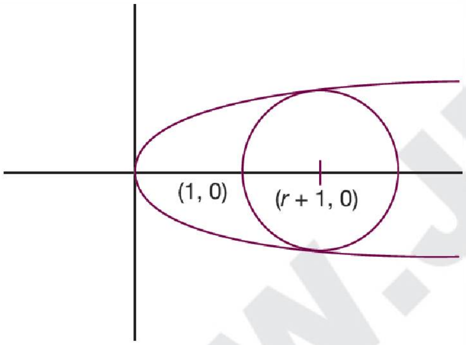
$$\tan\theta = \frac{4}{3} = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{k}{2}} \Rightarrow \tan\frac{\theta}{2} = \frac{1}{2}$$

$\Rightarrow \quad \frac{\theta}{2} = \tan^{-1}\frac{1}{2} = \cot^{-1}2$

$\Rightarrow \quad k = 2$

Hence, the correct answer is (2).

11.



Circle $(x-r-1)^2 + y^2 = r^2$

Solving with parabola

$$x^2 + (4-2(r+1))x + (2r+1) = 0$$

$$D = 0$$

$\Rightarrow \quad 4(1-r)^2 - 4(2r+1) = 0$

$\therefore \quad r = 4$ units

Hence, the correct answer is (4).

12. $B(at_1^2, 2at_1), C(at_2^2, 2at_2)$

Let (α, β) be the circumcentre of ΔABC, then

$$\alpha = \frac{at_1^2 + at_2^2}{2} \text{ and } \beta = \frac{2at_1^2 + 2at_2}{2}$$

$\therefore$ ABC is a right angled triangle.

$$t_1t_2 = 4$$

Hence, $y^2 = 2ax - 8a^2$

Hence, the correct answer is (8).

13. $m = \left(\frac{dy}{dx}\right)_{(1,1)} = b+2$

Equation of the tangent is $y - 1 = (b+2)(x-1)$

$\Rightarrow \quad (b+2)x - y - (b+1) = 0 \Rightarrow \left(\frac{b+2}{b+1}\right)x - \frac{y}{b+1} = 1$

Now, area of the $\Delta OAB = 2$ sq. units

$\Rightarrow \quad \frac{(b+1)^2}{2|b+2|} = 2$

$\Rightarrow \quad b^2 + 2b + 1 = \pm 4(b+2)$

$\Rightarrow \quad b^2 - 2b - 7 = 0$ or $b^2 - 6b + 9 = 0$

$\Rightarrow \quad b_1 = -3,\ b_2 = 1-2\sqrt{2},\ b_3 = 1+2\sqrt{2}$

$\Rightarrow \quad |b_1| + |b_2 + b_3| = 5$

Hence, the correct answer is (5).

14. Equation of tangent on parabola is $y = mx + \frac{4}{m}$

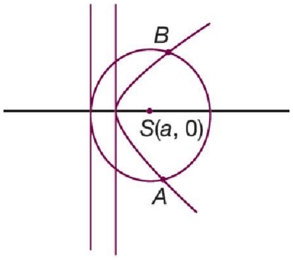
$$\left|\frac{0 - 0 - \frac{4}{m}}{\sqrt{1+m^2}}\right| = 2\sqrt{2}$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\therefore \quad m^2 = 1$

Hence, the correct answer is (1).

15. Centre of circle $(a, 0)$ and radius $2a$ units



Equation of circle $(x-a)^2 + y^2 = 4a^2$

$x^2 + y^2 - 2ax - 3a^2 = 0$ and $y^2 = 4ax$

On solving, we get, $x^2 + 4ax - 2ax - 3a^2 = 0$

$\Rightarrow \quad x^2 + 2ax - 3a^2 = 0$

$x = -3a, a$ and $y = \pm 2a$

$\therefore \quad$ Length of $AB = 4a$ units

Hence, the correct answer is (4).

16. The parabola touches x-axis at $A(a, 0)$ and y-axis at $B(0, b)$. Then, focus is the point of intersection of circles with diameter OA and OB.

Circle

$$\left.\begin{array}{l} x(x-a)+y^2=0 \\ \\ x^2+y(y-b)=0 \end{array}\right] \begin{array}{l} S\left(\frac{2}{5},\frac{4}{5}\right) \text{ lies on it} \\ a=2 \\ b=1 \end{array}$$

$$\text{area} = \frac{1}{2}ab = 1$$

Hence, the correct answer is (1).

17. Let $R = \left(t_1^2, 2t\right)$ be a point on the parabola.

$m_{PQ} = 2$

$PQ = 5\sqrt{5}$

Equation of PQ

$$y + 4 = 2(x - 4)$$

$$\Rightarrow \quad 2x - y - 12 = 0$$

Perpendicular distance of R to PQ is $\left|\frac{2t^2 - 2t - 12}{\sqrt{5}}\right|$ maximum for $t = \frac{1}{2}$

$$\text{Maximum area, } S = \frac{125}{4}$$

$$\Rightarrow \quad (4S)^{\frac{1}{3}} = 5$$

Hence, the correct answer is (5).

18. Let (x_1, y_1) be a point on $y_1{}^2 = -8(x_1 + 4)$ ...(1)

Equation of chord of contact is $2x - y_1 y + 2x_1 = 0$.

If $p(h, k)$ be its midpoint, then its equation will be $2x - ky + k^2 - 2h = 0$.

By comparing both, we get, $k = y_1$, $2x_1 = k^2 - 2h$ using (1)

So, $k^2 = -4(k^2 - 2h + 8)$

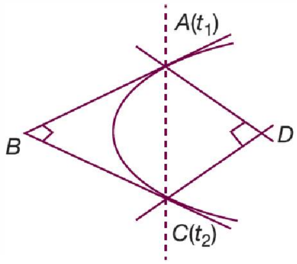
$$\Rightarrow \quad k^2 = \frac{8}{5}(h - 4)$$

$$\therefore \quad \lambda = \frac{8}{5}$$

Hence, the correct answer is (8).

19. $t_1 t_2 = -1$

Clearly, $ABCD$ is a rectangle.



Co-ordinate of $B(t_1 t_2, (t_1 + t_2))$

$$AB = (t_2 - t_1)|\sqrt{1 + t_1^2}$$

$$BC = (t_2 - t_1)\sqrt{1 + t_2^2}$$

$$\text{Area} = AB \times BC = (t_2 - t_1)^2 \sqrt{(1 + t_1^2)(1 + t_2^2)}$$

$$= \left(t_1 + \frac{1}{t_1}\right)^3$$

$\therefore$ Minimum area of $ABCD = 8$ sq. units.

Hence, the correct answer is (8).

20. Any point on the circle $x^2 + y^2 = 4$ is $(2 \cos \alpha, 2 \sin \alpha)$

$\therefore$ Equation of directrix is $x(\cos \alpha) + y(\sin \alpha) - 2 = 0$

Let focus be (x_1, y_1). Then, as $A(1, 0)$, $B(-1, 0)$ lie on parabola, we must have

$$\left.\begin{array}{l} (x_1 - 1)^2 + y_1^2 = (\cos\alpha - 2)^2 \\ (x_1 + 1)^2 + y_1^2 = (\cos\alpha + 2)^2 \end{array}\right|$$

$\Rightarrow \quad x_1 = 2 \cos \alpha, y_1 = \pm \sqrt{3} \sin \alpha$

$\therefore$ Length of semi-latus rectum of parabola = Perpendicular distance from focus to directrix

$$|2 \pm \sqrt{3}| \sin^2\alpha$$

Hence, maximum possible length is $2 + \sqrt{3}$ units.

Hence, the correct answer is (3).

21. Equation of the normal is $y = mx - 2m - m^3$

If it passes through $(21, 30)$, we have $30 = 21m - 2m - m^3$

$$\Rightarrow \quad m^3 - 19m + 30 = 0$$

Then, $m = -5, 2, 3$

However, if $m = 2$ or 3, then the point, where the normal meets the curve will be $\left(am^2, -2am\right)$. At this point, the curve does not exist.

$\therefore \quad m = -5 \qquad (\because m + 6 = 1)$

Hence, the correct answer is (1).

22. $\sqrt{x} = 1 - \sqrt{y}$

$x = 1 + y - 2\sqrt{y}$

$(x - y - 1)^2 = 4y$

$$(x - y)^2 = 2\left(x + y - \frac{1}{2}\right)$$

$$\left(\frac{x - y}{\sqrt{2}}\right)^2 = \left(\frac{x + y - \frac{1}{2}}{\sqrt{2}}\right) \times \sqrt{2}$$

$$\therefore \quad \ell_{LR} = \sqrt{2}\,;\, k = 2$$

Hence, the correct answer is (2).

23. $\frac{1}{AF}+\frac{1}{FB}=\frac{1}{a}$

$\Rightarrow \quad \frac{1}{a}=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$

$\Rightarrow \quad a=2$

$\Rightarrow \quad LR=8$

Hence, the correct answer is (8).

24. Sum of slopes of normal $=0$

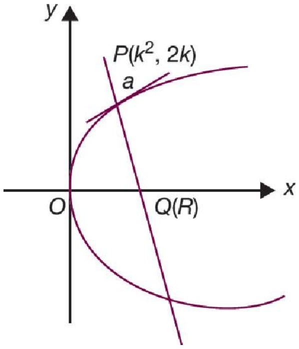
$\Rightarrow \quad \left(\frac{-2}{3}\right)+1+\left(\frac{-k}{15}\right)=0$

$\Rightarrow \quad \frac{k}{15}=\frac{1}{3}$

$\Rightarrow \quad k=5$

Hence, the correct answer is (5).

25.



Consider the parabola, $y^2=4ax$

We have to find the locus of $R(h, k)$, since Q has ordinate 'O', ordinate of P is $2k$.

Also, P is on the curve, then the abscissa of P is $\frac{k^2}{a}$

Now, PQ is normal to the curve.

Slope of tangent to curve at any point $\frac{dy}{dx}=\frac{2a}{y}$

Hence, slope of normal at point P is $-\frac{k}{a}$

Also, slope of normal joining P and $R(h, k)$ is $\frac{2k-k}{\frac{k^2}{a}-h}$

Hence, comparing slopes $\frac{2k-k}{\frac{k^2}{a}-h}=-\frac{k}{a}$

Or $\quad y^2=a(x-a)$

For $\quad y^2=16x,\ a=4.$

Hence, locus is $y^2=4(x-a)$.

Hence, the correct answer is (4).

26. Both the curves are symmetrical about the line $y=x$. Distance between any pair of points = 2(Distance of (t, t^2+1) from $y=x$)

$$2\left[\frac{t^2+1-t}{\sqrt{2}}\right]=\sqrt{2}\left[t^2-t+1\right]$$

$$\text{Min }(t^2-t+1)=-\left[\frac{1-4}{4}\right]=\frac{3}{4}$$

$\therefore$ The shortest distance is $\frac{3\sqrt{2}}{4}$ units.

Hence, the correct answer is (4).

27. $h=\frac{t_1^2+t_2^2}{2}$

$$k=\frac{2(t_1+t_2)}{2}=t_1+t_2$$

Also, $t_1t_2=-4$

$\Rightarrow \quad 2h=(t_1+t_2)^2-2t_1t_2$

$$2h=k^2+8$$

$$y^2=2(x-4)$$

$\therefore$ Latus rectum = 2

Hence, the correct answer is (2).

28. Length of latus rectum is unchanged, when origin is shifted to any point and it is

$$\left|\frac{\text{Coeffecient of } y}{\text{Coeffecient of } x^2}\right|=\frac{41}{7}$$

Hence, the correct answer is (5).

29. Equation of normal at (α, β) is $2(y-\beta)+\beta(x-\alpha)=0$ which is through (6, 2).

$\Rightarrow \quad \alpha\beta-4\beta=4$

Hence, the correct answer is (4).

30. Given, $P(t^2, 2t)$

Given parabola is $y^2=4x$ ...(1)

Here, $a=1$

Let $Q=(t_1^2, 2t_1)$

Since, the normal at P meets the curve again at Q.

$\therefore \quad t_1=-t-\frac{2}{t}=\frac{-(t^2+2)}{t}$ ...(2)

Now, $O(0,0)$, $P(t^2, 2t)$, $Q(t_1^2, 2t_1)$

Given, $\frac{k}{|t|}\left(1+t^2\right)\left(2+t^2\right)=$ Area of ΔPOQ

$$=\frac{1}{2}\left|t^2.2t_1-2t.t_1^2\right|$$

$$=\left|t^2t_1-tt_1^2\right|$$

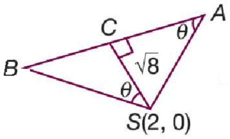
$$=\left|t^2\left(\frac{t^2+2}{t}\right)+t\frac{\left(t^2+2\right)^2}{t^2}\right|$$

$$=\left(t^2+2\right)\left|t+\frac{t^2+2}{t}\right|=\left(t^2+2\right)2\frac{\left(1+t^2\right)}{|t|}$$

$\therefore \quad k=2$

Hence, the correct answer is (2).

31. Circle on AB as diameter passes through the focus.



$AC=\sqrt{8}\cot\theta$

$BC=\sqrt{8}\tan\theta \quad AC\cdot BC=8$

Hence, the correct answer is (8).

32. Any normal will be $y+tx=3t+\frac{3}{2}t^2$, which passes through (2, 8). So, $3t^3+2t-16=0$

Let, $f(t)=3t^3+2t-16$

$\Rightarrow \quad f'(t)=9t^2+2<0$

So, only one normal exists.

Hence, the correct answer is (1).

33. Tangents B and K are perpendicular

Clearly, BK is a focal chord.

H.M. of $\alpha\beta=\frac{1}{2}LR$

$$\frac{2\alpha\beta}{\alpha+\beta}=2$$

$$\alpha+\beta=\alpha\beta\Rightarrow\alpha\beta=\frac{100}{9}$$

For real and distinct roots, $D>0$

$\Rightarrow \quad a^2-5a-14<0$

$\Rightarrow \quad -2<a<7$

$\therefore \quad$ The number of integral values of a are 8.

Hence, the correct answer is (8).

34. Parabola $P_1 \; x^2=k(y-4)$

Parabola $P_2 \; (x-6)^2=-ky$

Writing tangent and comparing, we get, $x_1=3, y_1=2$

Hence, the correct answer is (5).

35. Let $x\cos\theta+y\sin\theta=6$ is any tangent to the circle $x^2+y^2=36$, which is a directrix to the parabola.

Now, if $P(\alpha,\beta)$ is focus of parabola, then $\sqrt{(\alpha-5)^2+\beta^2}$

$$=\frac{|5\cos\theta-6|}{\sqrt{\cos^2\theta+\sin^2\theta}}$$

$$\Rightarrow \quad \sqrt{(\alpha+5)^2+\beta^2}=\frac{|-5\cos\theta-6|}{\sqrt{\cos^2\theta+\sin^2\theta}}.$$

$$\Rightarrow \quad (5-\alpha)^2+\beta^2=(5\cos\theta-6)^2 \qquad \ldots(1)$$

$$(5+\alpha)^2+\beta^2=(5\cos\theta+6)^2 \qquad \ldots(2)$$

$\Rightarrow \quad 20\alpha=120\cos\theta$

$\Rightarrow \quad \cos\theta=\frac{\alpha}{6}$

On subtracting $\cos\theta=\frac{\alpha}{6}$ in Equation (1), we get,

$$(5-\alpha)^2+\beta^2=\left(\frac{5\alpha}{6}-6\right)^2$$

$\Rightarrow \quad \frac{\alpha^2}{36}+\frac{\beta^2}{11}=1$

$\Rightarrow \quad k=6$

Hence, the correct answer is (6).

36. Let the other two vertices be on $2x-y+k=0$.

Length of the side $=\frac{|K+17|}{\sqrt{5}}=4.\sqrt{\frac{1}{4}(5)(1+k)}$ (Length of chord)

$\Rightarrow \quad (K+17)^2=100(1+K)$

$\Rightarrow \quad K^2-66K+189=0$

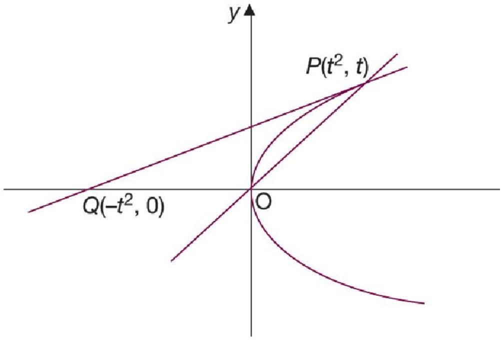
$\Rightarrow \quad (K-3)(K-63)=0$

$\Rightarrow \quad \frac{L}{l}=\frac{80}{20}=4$

Hence, the correct answer is (4).

ARCHIVE: JEE MAIN

1.



Tangent at $P(t^2, t)$ is $2yt = x + t^2$

Point $Q(-t^2, 0)$

Now area of $\Delta OPQ = 4$

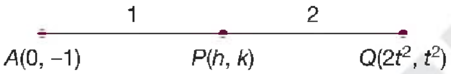
$$\frac{1}{2}\left\|\begin{matrix} 0 & 0 & 1 \\ t^2 & t & 1 \\ -t^2 & 0 & 1 \end{matrix}\right\| = 4$$

$t = 2 \quad (\because t > 0)$, $P(4, 2)$ lies on $y = mx$

$$\therefore \quad m = \frac{1}{2}$$

Hence, the correct answer is (0.50).

2.



$\Rightarrow \quad 3h = 2t$ and $3k = t^2 - 2$

$$\Rightarrow \quad 3y = \left(\frac{3x}{2}\right)^2 - 2 \Rightarrow 12y = 9x^2 - 8$$

Hence, the correct answer is (B).

3. $y = mx + 4$ is tangent to $y^2 = 4x$ then we get $m = \dfrac{1}{4}$

$y = \dfrac{1}{4}x + 4$ is also tangent to $x^2 = 2by$, then solving them we get $x^2 - \dfrac{b}{2}x - 8b = 0$

$D = 0$ for tangency, we get $b^2 + 128b = 0$

$\Rightarrow \quad b = -128, 0$

$b \neq 0$,we get $b = -128$

Hence, the correct answer is (C).

4. If one end of a focal chord AB of the parabola $y^2 = 8x$ is Parabola $y^2 = 8x$

$$4t_1 = -2 \Rightarrow t_1 = -\frac{1}{2}.$$

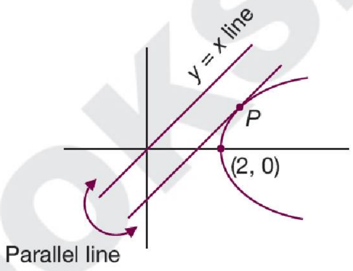
$$t_1 \cdot t_2 = -1 \Rightarrow t_2 = -\frac{1}{t_1} \Rightarrow t_2 = 2$$

So coordinate of B is (8, 8)

$\therefore$ Equation of tangent at B is $8y = 4(x + 8) \Rightarrow 2y = x + 8$

Hence, the correct answer is (D).

5. At point $P(x_1, y_1)$, draw a tangent parallel to given line.



$$yy_1 = \frac{1}{2}(x + x_1) - 2$$

$$\text{Slope of tangent} = \frac{1}{2y_1} = 1$$

$$y_1 = \frac{1}{2} \text{ and } x_1 = \frac{9}{4} \qquad \text{Point } P\left(\frac{9}{4}, \frac{1}{2}\right)$$

Shortest distance = Length of perpendicular from P on line $x - y = 0 = \dfrac{\left|\frac{9}{4} - \frac{1}{2}\right|}{\sqrt{2}} = \dfrac{7}{4\sqrt{2}}$

Hence, the correct answer is (A).

6.
$$y^2 = 4x \qquad \ldots(1)$$
$$x^2 + y^2 = 5 \qquad \ldots(2)$$

For point of intersection

$x^2 + 4x - 5 = 0$

$\Rightarrow \quad (x + 5)(x - 1) = 0$

$x = -5$ not possible

$x = 1; \quad y = \pm 2 \qquad$ Point P (1, 2)

Tangent $2y = 4\left(\dfrac{x+1}{2}\right) \Rightarrow y = x + 1$

Clearly, the point $\left(\dfrac{3}{4}, \dfrac{7}{4}\right)$ lies on tangent.

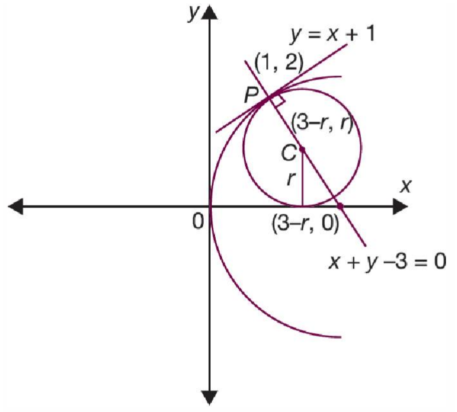
Hence, the correct answer is (B).

7. Equation of tangent to the parabola $y^2 = 4x$ at (1, 2) is

$$2y = 4\left(\frac{x+1}{2}\right) \Rightarrow y = x + 1$$

Equation of normal is $y = -x + 3$.

Let centre be $c(3 - r, r)$



Now, $PC^2 = r^2$

$\Rightarrow \quad (3 - r - 1)^2 + (r - 2)^2 = r^2$

$\Rightarrow \quad r^2 - 8r + 8 = 0$

$\Rightarrow \quad r = 4 \pm 2\sqrt{2}$

For $\quad r = 4 + 2\sqrt{2}\ (3 - r < 0)$ (Not possible)

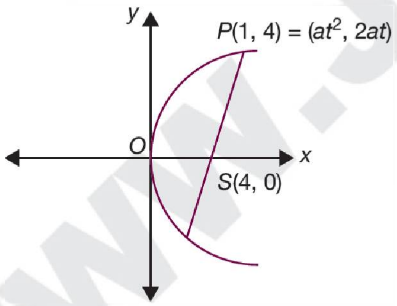
So, $\quad r = 4 - 2\sqrt{2}$

$$\text{Area} = pr^2 = \pi(16 + 8 - 16\sqrt{2}\,)$$

$$= 8\pi(3 - 2\sqrt{2}\,) \text{ sq. units}$$

Hence, the correct answer is (B).

8. Parabola $y^2 = 16x$



$\{4a = 16 \Rightarrow a = 4\}$

One end $(at^2, 2at) = (1, 4)$

$\Rightarrow \quad 2at = 4 \Rightarrow 2(4)t = 4$

$\Rightarrow \quad t = \dfrac{1}{2}$

$$\text{Length of focal chord} = a\left(t + \frac{1}{t}\right)^2 = 4\left(\frac{1}{2} + 2\right)^2$$

$$= 25 \text{ units.}$$

Hence, the correct answer is (C).

9. Tangent to the curve $y^2 = 4\sqrt{2}x$ is $y = mx + \dfrac{\sqrt{2}}{m}$

It is tangent to the circle $x^2 + y^2 = 1$

$$\therefore \quad \left|\frac{\sqrt{2}/m}{\sqrt{1 + m^2}}\right| = 1 \Rightarrow m = \pm 1$$

$\therefore$ Tangents are $y = x + \sqrt{2}$ and $y = -x - \sqrt{2}$

Compare with $y = -ax + c$, we get, $a = \pm 1$ and $c = \pm\sqrt{2}$

Hence, the correct answer is (C).

10. $y^2 = 8ax$

Normal: $y = mx - 4am - 2am^3$ ...(1)

$y^2 = 4b(x - c)$

Normal: $y = m'(x - c) - 2bm' - 2bm'^3$

$y = m'x - m'c - 2bm' - 2bm'^3$ ...(2)

For common normal, compare Equation(1) and Equation(2)

$$\frac{1}{1} = \frac{m}{m'} = \frac{4am + 2am^3}{m'c + 2bm' + bm'3}$$

$$m = m';\ m'c + 2bm' + bm'^3 = 4am + 2am^3$$

$$m(m^2(2a - b) + 4a - 2b - c) = 0$$

CASE-1: $m = 0$	**CASE-2:** $m \neq 0$
x-axis is common normal or both curves.	$m^2 = \dfrac{2b + c - 4a}{2a - b} > 0$ $= \dfrac{c}{2a - b} - 2 > 0$
All options satisfy it. Hence, all are correct.	Only option (A) satisfies it.

Hence, the correct answer is (A or B or C or D).

11. Parabola $x^2 = 4y$ ...(1)

Chord $x - \sqrt{2}\,y + 4\sqrt{2} = 0$ ...(2)

Solving (1) and (2)

$(\sqrt{2}\,)^2 (y - 4)^2 = 4y$

$\Rightarrow \quad 2y^2 - 16y + 32 = 4y \Rightarrow y^2 - 10y + 16 = 0$

$\Rightarrow \quad (y - 2)(y - 8) = 0$

$y = 2$	and	$y = 8$
From (2), $x = -2\sqrt{2}$		$x = 4\sqrt{2}$
$A(-4\sqrt{2}, 2)$		$B(4\sqrt{2}, 8)$

Length of chord = Distance between A and B

$$= \sqrt{(6\sqrt{2})^2 + (6)^2} = 6\sqrt{3}$$

Hence, the correct answer is (B).

12. Let the equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, which passes through (0, 2*b*).

$$0 + 4b^2 + 2g \times 0 + 4f + c = 0$$

$$4b^2 + 4f + c = 0 \qquad \ldots(1)$$

$$2\sqrt{g^2 - c} = 4a \qquad \ldots(2)$$

$$g^2 - c = 4a^2 \qquad \therefore c = (g^2 - 4a^2)$$

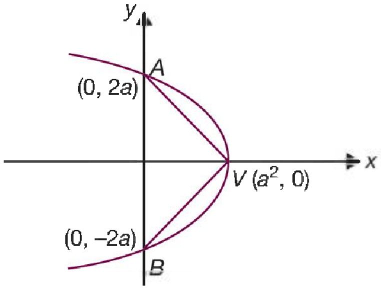
Putting in Equation (1), we get, $4b^2 + 4f + g^2 - 4a^2 = 0$

$$x^2 + 4y + 4(b^2 - a^2) = 0$$

It represents a parabola.

Hence, the correct answer is (B).

13.



$$y^2 = -4(x - a^2)$$

$(\because$ Area of $\Delta ABV = 250$ sq. units)

$$\frac{1}{2} \times 4a \times a^2 = 250$$

$$a^3 = 125$$

$$\therefore \quad a = 5$$

Hence, the correct answer is (D).

14. Area $= \frac{1}{2}\begin{vmatrix} 4 & -4 & 1 \\ 9 & 6 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$

$$= \frac{1}{2}[4(6 - 2t) + 4(9 - t^2) + 1(18t - 6t^2)]$$

$$= [-5t^2 + 5t + 30]$$

For maximum and minimum, $\frac{dA}{dt} = -10t + 5 = 0$

$$\Rightarrow \quad 10t = 5 \Rightarrow t = \frac{1}{2}$$

$$A = \left|\left(-5 \times \frac{1}{4} + \frac{5}{2} + 30\right)\right|$$

$$= \left|\left(-\frac{5}{4} + \frac{5}{2} + 30\right)\right| = \left(\frac{125}{4}\right)$$

Hence, the correct answer is (B).

15. $x^2 = 8y$

Equation of tangent is $y = mx - am^2$

$$y = \tan\theta\, x - 2\tan^2\theta$$

By dividing by tan θ, we get, $\cot\theta\, y = x - 2\tan\theta$

$$x = y\cot\theta + 2\tan\theta$$

Hence, the correct answer is (C).

16. Equation of tangent to curve $\frac{1}{2}(y + y_1) = xx_1 - \frac{5}{2}(x + x_1) + 5$

Slope of tangent is $2x_1 - 5 = 2$ (Slope of given line)

$$\Rightarrow \quad x_1 = \frac{7}{2};\, y_1 = -\frac{1}{4}$$

Point of contact $\left(\frac{7}{2}, -\frac{1}{4}\right)$

Now, equation of tangent is $y + \frac{1}{4} = 2\left(x - \frac{7}{2}\right)$

$$4y + 1 = 8x - 28$$

$$\Rightarrow \quad 8x - 4y - 29 = 0$$

Now, by checking with options, we get, $x = \frac{1}{8}$ and $y = -7$

Hence, the correct answer is (D).

17. $x - y - 3 = 0 \qquad \ldots(1)$

Equation (1) will be chord of contact of parabola.

$$y = x^2 - 4x + 3$$

Let the required point is $P(x_1, y_1)$ and chord of contact for point P is

$$\frac{y + y_1}{2} = xx_1 - 4\frac{(x + x_1)}{2} + 3$$

$$y + y_1 = 2x_1x - 4x - 4x_1 + 6$$

$$(2x_1 - 4)x - y + (-4x_1 - y_1 + 6) = 0 \qquad \ldots(2)$$

Equations (1) and (2) are same line.

$$\frac{2x_1 - 4}{1} = \frac{-1}{-1} = \frac{-4x_1 - y_1 + 6}{-3}$$

$\Rightarrow \quad 2x_1 - 4 = 1$ | $-4x_1 - y_1 + 6 = -3$

$\Rightarrow \quad x_1 = \frac{5}{2}$ | $-10 - y_1 + 9 = 0 \Rightarrow y_1 = -1$

$\therefore$ The required point is $\left(\frac{5}{2}, -1\right)$

Hence, the correct answer is (A).

18. $A\left(\frac{3}{2},0\right)$, $B(t, \sqrt{t}\,)$ on curve

$$(\text{Distance})^2 = \left(\frac{3}{2},-t\right)^2 + (\sqrt{t}\,)^2 = (t-1)^2 + \frac{5}{4}$$

Square of minimum (distance) $= \frac{5}{4}$ at (1, 1)

$\therefore$ Minimum distance is $\frac{\sqrt{5}}{2}$

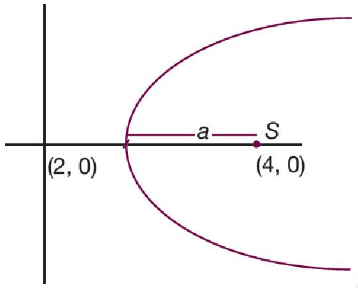
Hence, the correct answer is (D).

19. Equation of parabola is $(y-0)^2 = 4a(x-2)$

$y^2 = (4)\,2(x-2)$

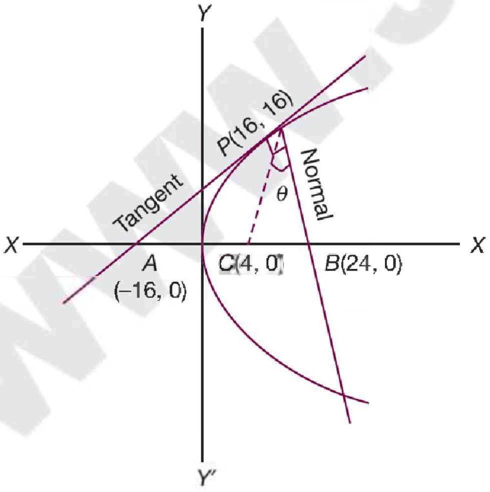
$y^2 = 8(x-2)$

By checking with options, (A) is correct.



Hence, the correct answer is (A).

20. Equation of tangent at $P(16, 16)$ is given as $x - 2y + 16 = 0$.



Slope of PC $(m_1) = \frac{4}{3}$

Slope of PC $(m_2) = -2$

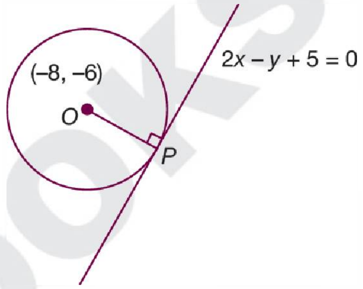
Hence, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right| = \left|\frac{\frac{4}{3}+2}{1-\frac{4}{3}\cdot 2}\right|$

$\therefore$ $\tan\theta = 2$

Hence, the correct answer is (A).

21. Equation of tangent at (1, 7) to $x^2 - y + 6 = 0$

is $2x - y + 5 = 0$



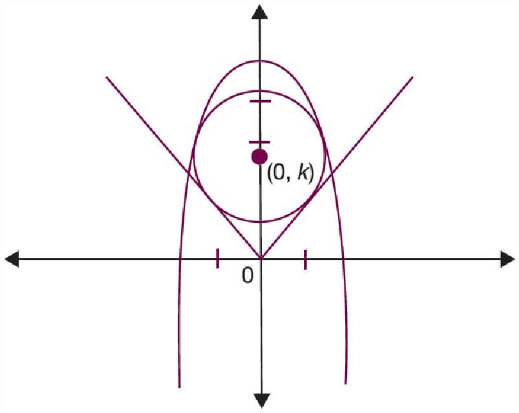
Now, perpendicular centre $O(-8, -6)$ to $2x - y + 5 = 0$ should be equal to radius of the circle.

$$\therefore \left|\frac{-16+6+5}{\sqrt{5}}\right| = \sqrt{64+36-c}$$

$$\Rightarrow \sqrt{5} = \sqrt{100-c} \Rightarrow c = 95$$

Hence, the correct answer is (C).

22. Let the equation of circle be $x^2 + (y-k)^2 = r^2$



It touches $y - x = 0$

$$\Rightarrow \left|\frac{0-k}{\sqrt{2}}\right| = r \Rightarrow k = r\sqrt{2}$$

$\therefore$ Equation of the circle becomes

$$x^2 + (y-k)^2 = \frac{k^2}{2} \qquad \ldots(1)$$

It touches $y = 4 - x^2 = \frac{k^2}{2}$

$\therefore$ Solving the two equations, we get,

$$4 - y + (y-k)^2 = \frac{k^2}{2}$$

$$\Rightarrow \quad y^2 - y(2k+1) + \frac{k^2}{2} + 4 = 0$$

It will give equal roots.

$\therefore \quad D = 0$

$$\Rightarrow \quad (2k+1)^2 = 4\left(\frac{k^2}{2} + 4\right)$$

$$\Rightarrow \quad 2k^2 + 4k - 15 = 0 \Rightarrow \quad k = \frac{-2+\sqrt{34}}{2}$$

$$\therefore \quad r = \frac{k}{\sqrt{2}} = \frac{-2+\sqrt{34}}{2\sqrt{2}}$$

The final answer does not match with any of the options.

Hence, no correct options are present.

Full marks are given to all students.

23. Minimum distance lies along common normal

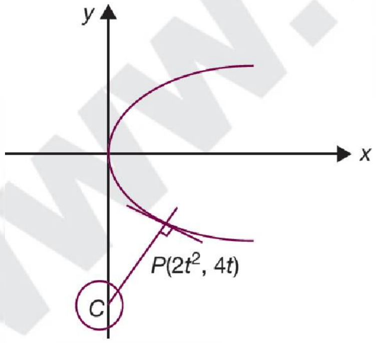
Equation of normal at $p(2t^2, 4t)$ is $y = -tx + 4t + 2t^3$.

It passes through C(0, –6) centre of circle

$\Rightarrow \quad t^3 + 2t + 3 = 0$

$\Rightarrow \quad t = -1$

Centre of new circle = $P(2t^2, 4t) = P(2, -4)$



Radius = $PC = \sqrt{(2-0)^2 + (-4+6)^2} = 2\sqrt{2}$ units

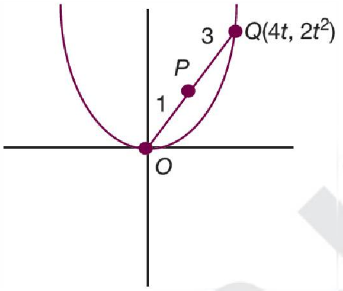
$\therefore$ Equation of the circle is $(x-2)^2 + (y+4)^2 = (2\sqrt{2})^2$

$\Rightarrow \quad x^2 + y^2 - 4x + 8y + 12 = 0$

Hence, the correct answer is (B).

24. Let $P(h, k)$ divides OQ is the ratio 1:3

Let any point Q on $x^2 = 8y$ is $(4t, 2t^2)$.



By section formula, $k = \frac{t^2}{2}$ and $h = t$

$\Rightarrow \quad 2k = h^2$

$\therefore$ Required locus of P is $x^2 = 2y$

Hence, the correct answer is (B).

25. Let, tangent to $y^2 = 4x$ be $y = mx + \frac{1}{m}$

Since this is also a tangent to $x^2 = -32y$

$$\therefore \quad x^2 = -32\left(mx + \frac{1}{m}\right) \Rightarrow x^2 + 32mx + \frac{32}{m} = 0$$

Now, $D = 0$

$$(32m)^2 - 4\left(\frac{32}{m}\right) = 0 \Rightarrow m = \frac{1}{2}$$

Hence, the correct answer is (A).

26. Let common tangent be $y = mx + \frac{\sqrt{5}}{m}$

Since, perpendicular distance from centre of the circle to the common tangent is equal to radius of the circle,

therefore $\dfrac{\frac{\sqrt{5}}{m}}{\sqrt{1+m^2}} = \sqrt{\frac{5}{2}}$

On squaring both the sides, we get, $m^2(1 + m^2) = 2$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow \quad m = \pm 1 \qquad (\because m \neq \pm\sqrt{2})$

$$y = \pm\left(x + \sqrt{5}\right)$$

Both the statements are correct, as $m = \pm 1$ satisfies the given equation of statement-2.

Hence, the correct answer is (B).

27. The locus of perpendicular tangents is directrix, i.e., $x = -1$.

Hence, the correct answer is (C).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. If (h, k) is the midpoint of chord of parabola $y^2 = 16x$, then equation of chord will be given by, $T = S_1$

$\Rightarrow \quad yk - 8(x + h) = k^2 - 16h$

$\Rightarrow \quad 8x - ky = 8h - k^2 \qquad \ldots(1)$

However, the given equation of chord is $2x + y = p$

$\therefore$ Equations (1) and (2) are identical lines.

$$\Rightarrow \quad \frac{8}{2} = \frac{-k}{1} = \frac{8h - k^2}{p}$$

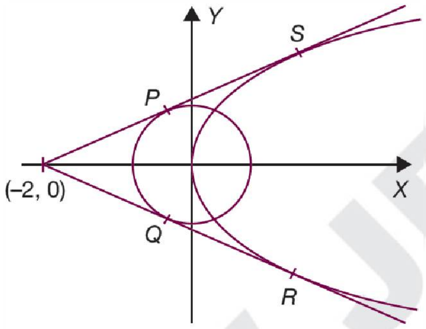
$\Rightarrow \quad k = -4$ and $8h - 16 = 4p$

$\Rightarrow \quad k = -4$ and $p = 2h - 4$

Which are satisfied by option (C).

Hence, the correct answer is (C).

2. Let the tangent to $y^2 = 8x$ be $y = mx + \dfrac{2}{m}$



If it is common tangent to parabola and circle, then $y = mx + \dfrac{2}{m}$ is a tangent to $x^2 + y^2 = 2$

$$\therefore \quad \left| \frac{\frac{2}{m}}{\sqrt{m^2 + 1}} \right| = \sqrt{2} \Rightarrow \frac{4}{m^2(1 + m^2)} = 2$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0 \Rightarrow (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow \quad m = 1$ or -1

$\therefore$ The required tangents are $y = x + 2$ and $y = -x - 2$ and their common point is $(-2, 0)$.

$\therefore$ Tangents are drawn from $(-2, 0)$.

$\therefore$ Chord of contact PQ to circle is $x.(-2) + y.0 = 2$ or $x = -1$

Chord of contact RS to parabola is $y.\ 0 = 4(x - 2)$ or $x = 2$

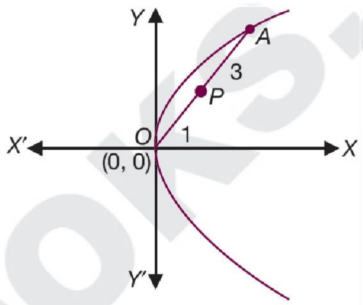
Hence, the co-ordinates of P and Q are $(-1, 1)$ and $(-1, -1)$.

Also, the co-ordinates of R and S are, $(2, -4)$ and $(2, 4)$, respectively.

$\therefore \quad$ Area of trapezium $PQRS$ is $\dfrac{1}{2}(2 + 8) \times 3 = 15$

Hence, the correct answer is (D).

3. Let $A(x, y) = (t^2, 2t)$ be any point on parabola $y^2 = 4x$. Let $P(h, k)$ divides OA in the ratio 1:3



Then $(h, k) = \left(\dfrac{t^2}{4}, \dfrac{2t}{4}\right)$

$\Rightarrow \quad h = \dfrac{t^2}{4}$ and $k = \dfrac{t}{2} \Rightarrow h = k^2$

$\therefore \quad$ Locus of $P(h, k)$ is $x = y^2$.

Hence, the correct answer is (C).

Multiple Correct Choice Type Problems

1. C_1: $x^2 + y^2 = 3 \qquad \ldots(1)$

Parabola: $x^2 = 2y \qquad \ldots(2)$

Intersection of points (1) and (2) lies in Ist quadrant.

$y^2 + 2y - 3 = 0 \Rightarrow y = 1 \qquad (\because\ y \neq -3)$

$\therefore \quad x = \sqrt{2} \qquad P(\sqrt{2}, 1)$

Equation of tangent to circle C_1 at P is $\sqrt{2}x + y - 3 = 0$

Let centre of circle C_2 be $(0, k)$; $r = 2\sqrt{3}$

$$\therefore \quad \left| \frac{k - 3}{\sqrt{3}} \right| = 2\sqrt{3} \Rightarrow k = 9 \text{ or } -3$$

$\therefore \quad Q_2\ (0, 9),\ Q_3\ (0, -3)$

(a) $Q_2 Q_3 = 12$

(b) R_2R_3 = Length of transverse common tangent

$$= \sqrt{(Q_2Q_3)^2 - (r_1 + r_2)^2}$$

$$= \sqrt{(12)^2 - (4\sqrt{3})^2} = 4\sqrt{6}$$

(c) Area $(\Delta OR_2R_3) = \frac{1}{2} \times R_2R_3 \times$ Length of perpendicular from origin to tangent

$$= \frac{1}{2} \times 4\sqrt{6} \times \sqrt{3} = 6\sqrt{2} \text{ sq. units}$$

(d) Area (ΔPQ_2Q_3)

$$= \frac{1}{2} \times Q_2Q_3 \times (\text{Perpendicular distance of } P \text{ from } y\text{-axis})$$

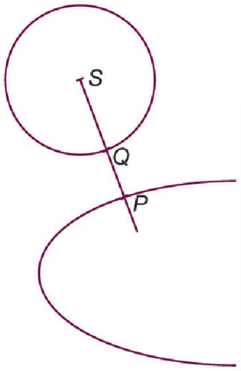
$$= \frac{1}{2} \times 12 \times \sqrt{2} = 6\sqrt{2} \text{ units}$$

Hence, (A), (B) and (C) are correct.

2. Let point P on parabola $y^2 = 4x$ be $(t^2, 2t)$

$\because$ PS is the shortest distance.

$\therefore$ PS should be the normal to parabola.



Equation of normal to $y^2 = 4x$ at $P(t^2, 2t)$ is $y - 2t = -t(x - t^2)$.

It passes through S (2, 8)

$\therefore$ $8 - 2t = -t(2 - t^2) \Rightarrow t^3 = 8$ or $t = 2$

$\therefore$ $P(4, 4)$

Also slope of tangent to circle at $Q = \frac{-1}{\text{Slope of } PS} = \frac{1}{2}$

Equation of normal at $t = 2$ is $2x + y = 12$

Clearly, x-intercept $= 6$

$SP = 2\sqrt{5}$ or $SQ = r = 2$

Now $QP = SP - SQ = 2\sqrt{5} - 2$

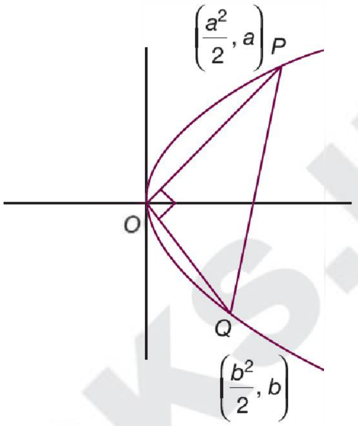
$SP : QP = 2 : 2\left(\sqrt{5} - 1\right)$ or $\left(\sqrt{5} + 1\right) : 4$

Hence, (A), (C) and (D) are correct.

3. Let point P is I$^{\text{st}}$ quadrant, lying on parabola $y^2 = 2x$ be $\left(\frac{a^2}{2}, a\right)$.

Let Q be the point $\left(\frac{b^2}{2}, b\right)$.

Clearly, $a > 0$


$\left(\frac{a^2}{2}, a\right)$ P
O
Q
$\left(\frac{b^2}{2}, b\right)$

$\because$ PQ is the diameter of circle through P, O and Q.

$$\therefore \quad \angle POQ = 90^\circ \Rightarrow \frac{a}{a^2/2} \times \frac{b}{b^2/2} = -1 \Rightarrow ab = -4$$

$(\because a > 0)$ then $b < 0$

Also, area of $\Delta POQ = 3\sqrt{2}$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{a^2}{2} & a & 1 \\ \frac{b^2}{2} & b & 1 \end{vmatrix} = \pm 3\sqrt{2} \Rightarrow \frac{1}{4}ab(a - b) = \pm 3\sqrt{2}$$

$\Rightarrow$ $a - b = \pm 3\sqrt{2}$ (Using $ab = -4$)

As a is positive and b is negative, we have $a - b = 3\sqrt{2}$ only

$\Rightarrow$ $(a - 2\sqrt{2})(a - \sqrt{2}) = 0$

$\Rightarrow$ $a = 2\sqrt{2}, \sqrt{2}$

$\therefore$ Point P can be $(4, 2\sqrt{2})$ or $(1, \sqrt{2})$

Hence, (A) and (D) are correct.

4. The equation of normal to $y^2 = 4x$ is $y = mx - 2m - m^3$

As it passes through (9, 6), $6 = 9m - 2m - m^3$

$\Rightarrow$ $m^2 - 7m + 6 = 0$

$\Rightarrow$ $(m - 1)(m^2 + m - 6) = 0$

$\Rightarrow$ $(m - 1)(m + 3)(m - 2) = 0 \Rightarrow m = 1, 2, -3$

$\therefore$ Normal is $y = x - 3$ or $y = 2x - 12$ or $y = -3x + 33$

$\therefore$ A, B, D are the correct option.

Hence, (A), (B) and (D) are correct.

5. Given parabola $y^2 = 4x$

Let A $(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$

Then centre of circle drawn with AB diameter is $\left(\frac{t_1^2+t_2^2}{2}, t_1+t_2\right)$

As circle touches x-axis

$\therefore \quad r = |t_1+t_2| \Rightarrow t_1+t_2 = \pm r$

Also, slope of $AB = \frac{2(t_2-t_1)}{t_2^2-t_1^2} = \frac{2}{t_2+t_1} = \pm\frac{2}{r}$

Hence, (C) and (D) are correct.

6. Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Then tangent to parabola at P is $y = \frac{x}{t} + at$, which meets the axis of parabola, i.e., x-axis at $T(-at^2, 0)$.

Also, normal to parabola at P is $t\,x + y = 2at + at^3$, which meets the axis of parabola at $N(2a + at^2, 0)$

Let $G\ (x, y)$ be the centroid of ΔPTN, then

$$x = \frac{at^2 - at^2 + 2a + at^2}{3} \text{ and } y = \frac{2at}{3}$$

$$\Rightarrow \quad x = \frac{2a + at^2}{3} \text{ and } y = \frac{2at}{3}$$

Eliminating t from above, we get the locus of centroid G as

$$3x = 2a + a\left(\frac{3y}{2a}\right)^2 \Rightarrow y^2 = \frac{4a}{3}\left(x - \frac{2}{3}a\right)$$

It is a parabola with vertex $\left(\frac{2a}{3}, 0\right)$, directrix as $x - \frac{2a}{3} = -\frac{a}{3}$ or $x = \frac{a}{3}$, latus rectum as $\frac{4a}{3}$ and focus as $(a, 0)$.

Hence, (A) and (D) are correct.

Reasoning Based Questions

1. The given curve is $y = \frac{-x^2}{2} + x + 1$

Or $\quad (x-1)^2 = -2\left(y - \frac{3}{2}\right)$

It is a parabola, so should be symmetric with respect to its axis $x - 1 = 0$

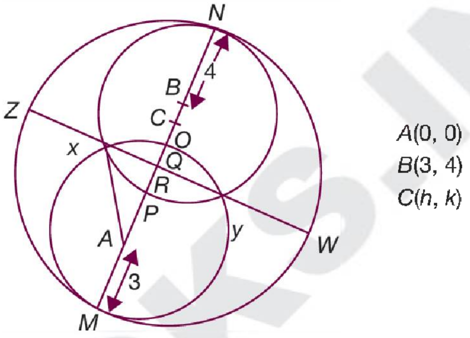
$\therefore$ Both the statements are true and statement-2 is a correct explanation for statement-1.

Hence, the correct answer is (A).

Linked Comprehension Type Questions

1. **The correct answer is 1(D) and 2(B).**

Combined Solution to 1 and 2



Equation of common chord XY is

$$3x - 4y - 9 = 0$$

$$AR = \left|\frac{0-0-9}{5}\right| = \frac{9}{5}$$

$$AB = 5; \quad XR = \sqrt{9 - \left(\frac{9}{5}\right)^2} = \frac{12}{5} = YR$$

$$XY = 2 \times R = \frac{24}{5}$$

$$MN = 2MC = MA + MB + BN$$

$$= 3 + 5 + 4 = 12$$

$$MC = 6,\ CR = AC - AR = \frac{6}{5}$$

$$ZW = 2ZR = 2\sqrt{CZ^2 - CR^2}$$

$$= 2\sqrt{36 - \left(\frac{6}{5}\right)^2} = \frac{12\sqrt{24}}{5}$$

$$\frac{\text{Length of } ZW}{\text{Length of } XY} = \frac{\frac{12\sqrt{24}}{5}}{\frac{24}{5}} = \sqrt{6} \text{ units}$$

$$\frac{\text{Area of } \Delta MZN}{\text{Area of } \Delta MZW} = \frac{\frac{1}{2} \times ZR \times MN}{\frac{1}{2} \times MR \times ZW} = \frac{ZW(12)}{2(MR)(ZW)}$$

$$= \frac{6}{MR} = \frac{6}{6 - RQ} = \frac{6}{6 - \frac{6}{5}} = \frac{5}{4} \text{ sq. units}$$

Tangent at M on circle C_1 and C_3 is ⊥ to line joining centres

$m_{AB} = \frac{4}{3}$

Slope of tangent to $C_1 = \frac{-3}{4}$

Equation of tangent $\Rightarrow y = m_1 x \pm 3\sqrt{1+\frac{9}{16}}$

$y = -\frac{3x}{4} \pm \frac{15}{4}$ or $x = -\frac{4y}{3} - 5$ ...(1)

Equation of tangent $x^2 = 4(2\alpha)y$ is

$x = m_2 y + \frac{2\alpha}{m_2} = -5$; $\alpha = \frac{10}{3}$; $R = 6$ ...(2)

Centre O of circle C_3 lie on line $y = \frac{4}{3}x$

Let point $O\left(h, \frac{4}{3}h\right)$; $OA = OM - AM = 6 - 3 = 3$

$\Rightarrow \sqrt{h^2 + \frac{16}{9}h^2} = 3 \Rightarrow h = \frac{9}{5}$ and $K = \frac{4}{3} \times \frac{9}{5} = \frac{12}{5}$

$\Rightarrow 2h + K = 2\left(\frac{9}{5}\right) + \frac{12}{5} = \frac{18}{5} + \frac{12}{5} = 6$

So, I → P; II → Q; III → R; IV → U

3. $\because PQ$ is a focal chord, then $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Given $QR \,||\, PK$

$\Rightarrow m_{QR} = m_{PK}$

$$\Rightarrow \frac{-\frac{2a}{t} - 2ar}{\frac{a}{r^2} - ar^2} = \frac{0 - 2at}{2a - at^2}$$

$$\Rightarrow \frac{-2a\left(\frac{1}{t} + r\right)}{a\left(\frac{1}{t} + r\right)\left(\frac{1}{t} - r\right)} = \frac{-2at}{a\left(2 - t^2\right)}$$

$$\Rightarrow 2 - t^2 = t\left(\frac{1}{t} - r\right)$$

$\left[\because r \neq -\frac{1}{t} \text{ otherwise } Q \text{ will coincide with } R\right]$

$$\Rightarrow 2 - t^2 = 1 - tr \Rightarrow r = \frac{t^2 - 1}{t}$$

Hence, the correct answer is (D).

4. Tangent at P is $ty = x + at^2$ ...(1)

Normal at S

$sx + y = 2as + as^3$ ...(2)

However, given $st = 1$

$\Rightarrow s = \frac{1}{t}$

$\therefore \frac{x}{t} + y = \frac{2a}{t} + \frac{a}{t^3}$

$\Rightarrow xt^2 + yt^3 = 2at^2 + a$

Putting value of x from equation (1) in the above equation, we get,

$\Rightarrow t^2\left(ty - at^2\right) + yt^3 = 2sy^2 + a$

$\Rightarrow \left(t^2 + t^3\right)y - at^4 = 2at^2 + a$

$\Rightarrow 2t^3 y = a\left(t^4 + 2t^2 + 1\right)$

$$y = \frac{a\left(t^4 + 2t^2 + 1\right)}{2t^3} = \frac{a\left(t^2 + 1\right)^2}{2t^3}$$

Hence, the correct answer is (B).

5. As PQ is the focal chord of $y^2 = 4ax$

$\therefore$ Co-ordinates of P and Q can be taken as

$P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Tangents at P and Q are

$y = \frac{x}{t} + at$ and $y = -xt - \frac{a}{t}$

which intersect each other at $R\left(-a, a\left(t - \frac{1}{t}\right)\right)$

As R lies on the $y = 2x + a$, $a > 0$

$\therefore a\left(t - \frac{1}{t}\right) = -2a + a \Rightarrow t - \frac{1}{t} = -1 \Rightarrow t + \frac{1}{t} = \sqrt{5}$

$$PQ = \sqrt{\left(at^2 - \frac{a}{t^2}\right)^2 + \left(2at + \frac{2a}{t}\right)^2}$$

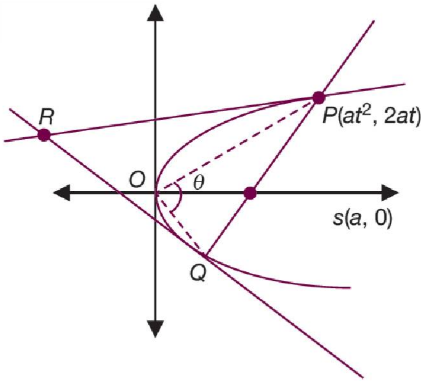
$$= a\sqrt{\left(t^2 + \frac{1}{t}\right)^2\left(t - \frac{1}{t}\right)^2 + 4\left(t + \frac{1}{t}\right)^2}$$

$$= a\left(t + \frac{1}{t}\right)\sqrt{\left(t - \frac{1}{t}\right)^2 + 4} = a\left(t + \frac{1}{t}\right)^2 = 5a$$

Hence, the correct answer is (B).

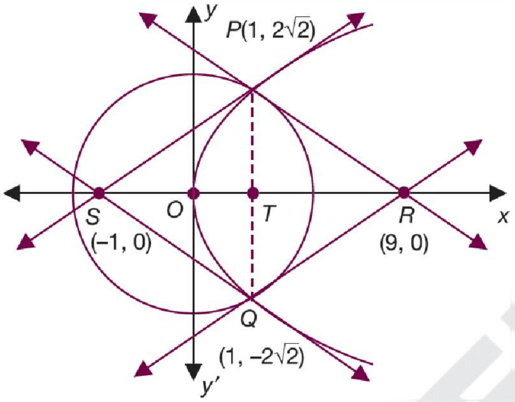
6. Now, $m_{OP} = \frac{2}{t}$ and $m_{OQ} = -2t$

$$\therefore \tan\theta = \frac{\frac{2}{t} + 2t}{1 - 4} = \frac{2\left(t + \frac{1}{t}\right)}{-3} = \frac{2\sqrt{5}}{-3}$$



Hence, the correct answer is (D).

Combined Solution to 7, 8 and 9



7. $$\frac{\text{Ar }\Delta PQS}{\text{Ar }\Delta PQS}=\frac{\frac{1}{2}PQ\times ST}{\frac{1}{2}PQ\times TR}=\frac{ST}{TR}=\frac{2}{8}=\frac{1}{4}$$

Hence, the correct answer is (B).

8. For ΔPRS,

$$Ar\,(\Delta PRS)=\Delta=\frac{1}{2}=SR\times PT=\frac{1}{2}\times 10\times 2\sqrt{2}$$

$\therefore\ \Delta=10\sqrt{2},\ a=PS=2\sqrt{3}$

$b=PR=6\sqrt{2}\ ,\ c=SR=10$

$\therefore$ Radius of circumference

$$=R=\frac{abc}{4\Delta}=\frac{2\sqrt{3}\times 6\sqrt{2}\times 10}{4\times 10\sqrt{2}}=3\sqrt{3}$$

Hence, the correct answer is (B).

9. Radius of incircle $=\dfrac{\text{Area of }\Delta PQR}{\text{Semi perimeter of }\Delta\ PQR}=\dfrac{\Delta}{s}$

We have, $a=PR=6\sqrt{2}, b=QP=PR=6\sqrt{2}$

$c=PQ=4\sqrt{2}$

and $\Delta=\dfrac{1}{2}\times PQ\times TR=16\sqrt{2}$

$$\therefore\ s=\frac{6\sqrt{2}+6\sqrt{2}+4\sqrt{2}}{2}=8\sqrt{2}\ \ \therefore r=\frac{16\sqrt{2}}{8\sqrt{2}}=2$$

Hence, the correct answer is (D).

Matrix Match/Column Match Type Questions

1. Equation of tangent to $y^2=16x$ at $F(x_0, y_0)$

$$yy_0=8(x+x_0)\ \ G\left(0,\frac{8x_0}{y_0}\right)=(0,y_1)$$

$$\text{Area of }\Delta EFG=\frac{1}{2}\times(3-y_1)\times x_0$$

$$A=\frac{1}{2}x_0\left(3-\frac{8x_0}{y_0}\right)$$

$$A=\frac{1}{2}\frac{y_0^2}{16}\left(3-\frac{y_0}{2}\right)=\frac{1}{32}\left(3y_0^2-\frac{y_0^3}{2}\right)\text{ sq. units}$$

$$\frac{dA}{dy_0}=\frac{1}{32}\left(6y_0-\frac{3y_0^2}{2}\right)$$

$$\frac{dA}{dy_0}=0\Rightarrow y_0=4\Rightarrow x_0=1$$

$$\therefore\ \ y_1=\frac{8\times 1}{4}=2$$

Also, $y_0=mx_0+3$

$\therefore\ \ 4=m+3$ or $m=1$

Maximum area of ΔEFG

$$=\frac{1}{32}\left[3\times 4^2-\frac{4^3}{2}\right]=\frac{1}{32}[48-32]=\frac{1}{2}\text{ sq. units}$$

$\therefore$ (P) → (D), (Q) → (A), (R) → (B), (S) → (C)

Hence, the correct answer is (A).

Integer/Numerical Answer Type Questions

1. End points of latus rectum of $y^2=4x$ are $(1,\ \pm 2)$

Equation of normal to $y^2=4x$ at (1, 2) is $y-2=-1(x-1)$ or $x+y-3=0$

As it is tangent to circle $(x-3)^2+(y+2)^2=r^2$

$$\therefore\ \left|\frac{3+(-2)-3}{\sqrt{2}}\right|=r\Rightarrow r^2=2$$

Hence, the correct answer is (2).

2. Let $(t^2, 2t)$ be any point on $y^2=4x$. Let (h, k) be the image of $(t^2, 2t)$ in the line $x+y+4=0$. Then,

$$\frac{h-t^2}{1}=\frac{k-2t}{1}=\frac{-2(t^2+2t+4)}{2}\quad\text{Using image concept}$$

$\Rightarrow \quad h = -(2t + 4)$ and $k = -(t^2 + 4)$

For its intersection with $y = -5$, we have $-(t^2 + 4) = -5$

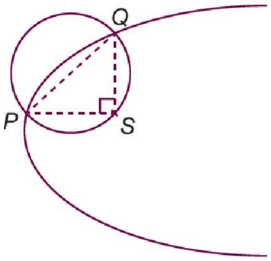
$\Rightarrow \quad t = \pm 1$

$\therefore \quad A(-6, -5)$ and $B(-2, -5)$

$\therefore \quad AB = 4.$

Hence, the correct answer is (4).

3. We observe that both parabola $y^2 = 8x$ and circle $x^2 + y^2 - 2x - 4y = 0$ pass through the origin.



$\therefore$ One end of common chord PQ is origin. Say $P(0, 0)$

Let Q be the point $(2t^2, 4t)$, then it will satisfy the equation of circle.

$\therefore \quad 4t^4 + 16t^2 - 4t^2 - 16t = 0$

$\Rightarrow \quad t^4 + 3t^2 - 4t = 0$

$\Rightarrow \quad t(t^3 + 3t - 4) = 0$

$\Rightarrow \quad t(t - 1)(t^2 + t + 4) = 0$

$\Rightarrow \quad t = 0$ or 1

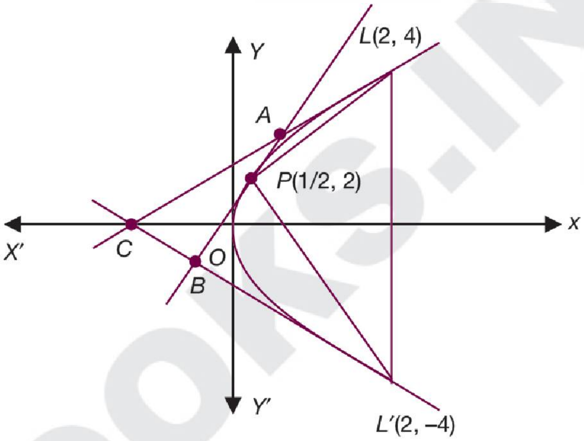
For $t = 0$, we get point P, therefore $t = 1$ given point Q as $(2, 4)$.

We also observe here that $P(0, 0)$ and $Q(2, 4)$ are end points of diameter of the given circle and focus of the parabola is the point $S(2, 0)$.

$\therefore \quad$ Area of $\Delta PQS = \frac{1}{2} \times PS \times QS = \frac{1}{2} \times 2 \times 4 = 4$ sq. units

Hence, the correct answer is (4).

4.



$\Delta_1 =$ Area of $\Delta PLL' = \frac{1}{2} \times 8 \times \frac{3}{2} = 6$ sq. units

Equation of AB, $y = 2x + 1$

Equation of AC, $y = x + 2$

Equation of BC, $-y = x + 2$

Solving above equations, we get, $A(1, 3)$, $B(-1, -1)$, $(-2, 0)$

$$\therefore \quad \Delta_2 = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ -1 & -1 & 1 \\ -2 & 0 & 1 \end{vmatrix} = 3 \;\therefore\; \frac{\Delta_1}{\Delta_2} = 2$$

Hence, the correct answer is (2).

CHAPTER 5: ELLIPSE

Test Your Concepts-I

1. Given, $e = \frac{\sqrt{3}}{2} = \sqrt{1-\frac{b^2}{a^2}} \Rightarrow a = 2b$

Length of latus rectum $= \frac{2b^2}{a} = b = \frac{a}{2}$

$= \frac{1}{4}$ units (Length of major axis)

2. Rewriting the given equation, we get, $9(x-1)^2 + 25(y-2)^2 = 225$

Or $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$

Now for range $-5 \le x - 1 \le 5$ or $x \in [-4, 6]$

3. $2a = n(2b)$

$\Rightarrow \quad n = \frac{a}{b}$

$\therefore \quad e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{1-\frac{b^2}{a^2}} = \frac{\sqrt{n^2-1}}{n}$

4. Let $S(5, 12)$, $S'(24, 7)$ and O is the origin.

$SO = 13$

$S'O = 25$

$SS' = \sqrt{386}$

$e = \frac{\sqrt{386}}{13+25} = \frac{\sqrt{386}}{38}$

5. Vertex are those required points, now $a = \sqrt{\frac{a^2+2b^2}{2}}$ or $a^2 = 2b^2$ we get $e = \frac{1}{\sqrt{2}}$

6. Centre is the point of intersection of the lines $x - 2y + 1 = 0$ and $2x + y + 2 = 0$ which is $(-1, 0)$.

7. Let foci $S(3, 4)$ $S'(6, 8)$ then from ellipse definition we have $OS + OS' = 2a = 5 + 10 = 15$

Distance between foci $= 2ae = 5$, then $e = \frac{1}{3}$, we get $2b = 10\sqrt{2}$

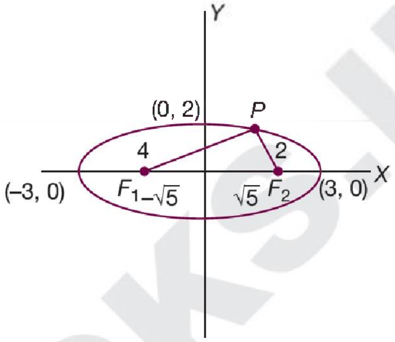
$\therefore$ The length of the minor axis is $10\sqrt{2}$ units.

8. Given relation is $PS + PS' = 2a = 10$.
We have focii $(0, 0)$ and $(-3, 4)$, then the distance between foci $= 2ae = 5$ units

$\Rightarrow \quad 2a = 10$

$\therefore \quad$ We get $e = 0.5$

9.



$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

$F_1F_2 = 2\sqrt{5}$

Also, $PF_1 + PF_2 = 6$ and $PF_1 = 2\,(PF_2)$ (Given)

$\therefore \quad 3PF_2 = 6 \quad \Rightarrow \quad PF_2 = 2$ and $PF_1 = 4$

Since, $(PF_2)^2 + (PF_1)^2 = (F_1F_2)^2 \Rightarrow \underline{|P} = 90°$

Area $= \frac{(4)(2)}{2} = 4$ sq. units

10. $4\tan\frac{B}{2}\tan\frac{C}{2} = 1 \Rightarrow \frac{\Delta}{s(s-b)}\cdot\frac{\Delta}{s(s-c)} = \frac{1}{4}$

$\Rightarrow \quad \frac{s-a}{s} = \frac{1}{4} \Rightarrow 3s = 4a \Rightarrow 3a + 3b + 3c = 8a$

$\Rightarrow \quad b + c = \frac{5}{3}a$

$\Rightarrow \quad b + c = \frac{5}{3} \times 6 = 10$

$(\because\ a = \overline{BC} = 6)$

Locus of vertex A is ellipse, midpoint of focus $B(2, 0)$, $C(8, 0)$ = centre $(5, 0)$

$\therefore \quad$ Locus of A is $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$

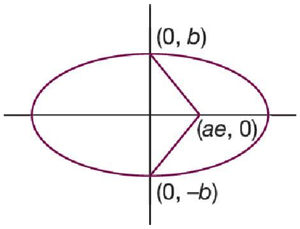
11. From definition of ellipse, $PS + PS'$ = Length of major axis $= 2b = 2(4) = 8$ units

12. $a^2e^2 + b^2 = (2b)^2$

$\Rightarrow \quad a^2e^2 = 3a^2(1-e^2)$

$\Rightarrow \quad e^2 = \frac{3}{4}$

$\therefore$ We get, $e = \frac{\sqrt{3}}{2}$



Test Your Concepts-II

1. $a = 3, b = \sqrt{5}, e = \frac{2}{3}$

One end of latus rectum in the I$^{\text{st}}$ quadrant is $\left(2, \frac{5}{3}\right)$

Tangent $S_1 = 0,\ 10x + 15y = 45$

Or $2x + 3y = 9$

It meets the axes at $\left(\frac{9}{2}, 0\right)$ and (0, 3)

$\therefore$ The area of the quadrilateral $= (4)\left(\frac{1}{2}\right)\left(\frac{9}{2}\right)(3)$

$= 27$ sq. units

2. We know that, $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If $c^2 = a^2m^2 + b^2$

Here, $m = a^2 = 4, b^2 = 1$.

So, $c^2 = 4 \times 4^2 + 1$

$\Rightarrow c = \pm\sqrt{65}$

$\therefore$ There are two values of c.

3. For tangency, $c^2 = a^2m^2 + b^2$

Or $k^2 = 16 + 9 = 25$, we get $k = 5, -5$

4. If the area of ΔPAB is 3 sq. units, the points P and origin cannot lie on either side of AB, since the distance of P from AB is less than $\frac{6}{5}$.

Therefore, P must lie on the straight line drawn parallel to AB at a distance of $\frac{6}{5}$ units towards the origin side of AB and this intersects the ellipse at exactly two points (one in quadrant 2 and the other in quadrant 4)

5. Let P be $(5\cos\theta, 4\sin\theta)$ and Q be $(-5\sin\theta, 4\cos\theta)$

Equation of tangent at

$$P \frac{x}{5}\cos\theta + \frac{y}{4}\sin\theta = 1 \qquad \ldots(1)$$

Equation of tangent at

$$Q - \frac{x}{5}\sin\theta + \frac{y}{4}\cos\theta = 1 \qquad \ldots(2)$$

Solving (1) and (2),

we get, $R = (5(\cos\theta - \sin\theta), 4(\sin\theta + \cos\theta))$

midpoint of PQ = midpoint of CR

$\therefore$ $m : n$ is 1 : 1

$\Rightarrow$ $m + n = 2$

Alternate

Let the points are $P(5, 0)$ and $Q(0, 4)$.

$\Rightarrow R(5, 4)$

Intersection of CR and PQ is $\left(\frac{5}{2}, 2\right)$, which is the mid-point of CR

$\Rightarrow m : n = 1 : 1 \Rightarrow m + n = 2$

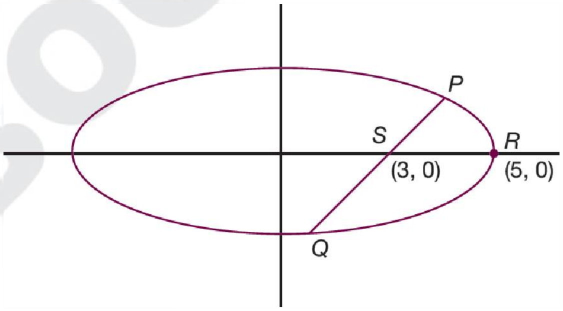
6. H.M. of PS and $QS = \frac{1}{2}$ (Latus rectum)

$$\frac{2}{\frac{1}{PS} + \frac{1}{QS}} = \frac{16}{5} \Rightarrow \frac{1}{PS} + \frac{1}{QS} = \frac{5}{8}$$

Given, $PS = 2$, we get $QS = 8$

Then, the focal chord $PQ = 10$ units

Alternate



Here $SP = 2$ and from the figure, $SR = 2$

$e = \frac{3}{5}$

$a - ae\cos\theta = 2$

$5 - 5\left(\frac{3}{5}\right)\cos\theta = 2$

$\cos\theta = 1$ $\qquad (\therefore \theta = 0)$

$\therefore$ P and R are the same points

$\therefore$ Length of focal chord $= 2a = 10$ units

7. Tangent $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Tangent $\frac{x}{a}\cos\left(\frac{\pi}{2} + \theta\right) + \frac{y}{b}\sin\left(\frac{\pi}{2} + \theta\right) = 1$

Eliminate θ from these equations, square and add, to get, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

8. For ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, given line is tangent and given point is focus , then the product of perpendiculars drawn from the points (± 3, 0) to the line is $b^2 = 16$.

9. Equation of tangent at point $\left(\sqrt{18}\cos\theta, \sqrt{32}\sin\theta\right)$ is $\frac{x}{\sqrt{18}}\cos\theta + \frac{y}{\sqrt{32}}\sin\theta = 1$.

Its slope $= -\frac{\sqrt{32}}{\sqrt{18}}.\cot\theta = -\frac{4}{3}$ (Given)

$\Rightarrow \quad \cot\theta = 1$

Also, $A = \left(\sqrt{18}\sec\theta, 0\right), B = \left(0, \sqrt{32}\operatorname{cosec}\theta\right)$

$$\Delta OAB = \frac{1}{2}\left|3\sqrt{2}\sec\theta.4\sqrt{2}\operatorname{cosec}\theta\right|$$

$$= (12)\left(\sqrt{2}\right)\left(\sqrt{2}\right) = 24 \text{ sq. units}$$

Test Your Concepts-III

1. Let $A(x_1, x_1 - 5)$ be a point on $x - y - 5 = 0$, then chord of contact of $x^2 + 4y^2 = 4$ with respect to A is $xx_1 + 4y(x_1 - 5) = 4$

$\Rightarrow \quad (x + 4y)x_1 - (20y + 4) = 0$

It passes through a fixed point.

$\therefore \quad x + 4y = 0$

And $20y + 4 = 0 \; (\because \text{ From } P + \lambda Q = 0)$

$\Rightarrow \quad y = \frac{-1}{5}$ and $x = \frac{4}{5}$

$\therefore$ The co-ordinates of fixed point is $\left(\frac{4}{5}, \frac{-1}{5}\right)$.

2. Ellipse equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its area $= pab$ sq. units

Let, $P(a\cos\theta, b\sin\theta)$

$S = (ae, 0)$

$M(h, k)$ midpoint of PS

$\Rightarrow \quad h = \frac{ae + a\cos\theta}{2}$ and $k = \frac{b\sin\theta}{2}$

$\Rightarrow \quad \frac{h - \frac{ae}{2}}{(a/2)^2} + \frac{k^2}{(b/4)^2} = 1$, locus of (h, k) is ellipse

Area $= \pi\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \frac{1}{4}\pi(ab)$ sq. units

$\therefore$ The required ratio is 1:4

3. Equation of chord $T = S_1$

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$ it passes through $\left(\pm\frac{a}{e}, 0\right)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm\frac{x}{ae}$$

4. Tangent at (x_1, y_1) is $xx_1 + 2yy_1 = 2$. It meets x-axis at $A\left(\frac{2}{x_1}, 0\right)$ and y-axis at $B\left(0, \frac{1}{y_1}\right)$. The midpoint of AB is given by $2x = \frac{2}{x_1}, 2y = \frac{1}{y_1}$

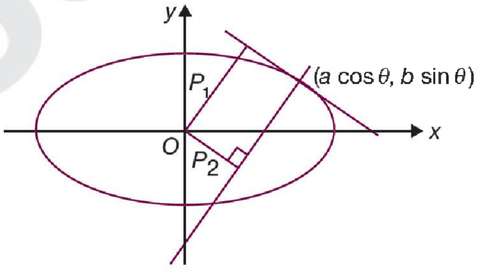
Or $\quad x_1 = \frac{1}{x}, y_1 = \frac{1}{2y}$

However, $x_1^2 + 2y_1^2 = 2$

$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{2y^2} = 2$ or $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

Test Your Concepts-IV

1. $P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$;



$$T: \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$p_1 = \left|\frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right| = \frac{\sqrt{2}\,ab}{\sqrt{a^2 + b^2}} \qquad \ldots(1)$$

$$N_1: \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$$p_2 = \left|\frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right| = \frac{a^2 - b^2}{\sqrt{2}\sqrt{\left(a^2 + b^2\right)}} \qquad \ldots(2)$$

$$p_1p_2 = \frac{ab(a^2 - b^2)}{a^2 + b^2}$$

2. Equation of normal at $p(3\cos\theta, 2\sin\theta)$ is

$3x\sec\theta - 2y\operatorname{cosec}\theta = 5$

It is tangent to circle then $P = R$

$\Rightarrow \quad \frac{5}{\sqrt{9\sec^2\theta + 4\operatorname{cosec}^2\theta}} = \sqrt{3}$

However, mimimum of $9\sec^2\theta + 4\operatorname{cosec}^2\theta = 25$

$\therefore$ No such θ exists.

3. Normal at $Q(x, y_1)$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

Let, $P(h, k)$

$$\Rightarrow \quad h = \frac{(a^2 - b^2)x_1}{a^2}, k = \frac{-(a^2 - b^2)y_1}{b^2}$$

(x_1, y_1) lies on ellipse $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = 1$

$\therefore$ We get locus as $a^2x^2 + b^2y^2 = (a^2 - b^2)^2$

4. A tangent of slope 2 is $y = 2x \pm \sqrt{4a^2 + b^2}$, which is normal to $x^2 + y^2 + 4x + 1 = 0$ centre $(-2, 0)$

Then,

$$0 = -4 \pm \sqrt{4a^2 + b^2}$$

$$\Rightarrow \quad 4a^2 + b^2 = 16 \text{ (using } AM \geq GM)$$

$$\therefore \quad ab \leq 4$$

5. Clearly, $P(a \cos\theta, b \sin\theta)$ $Q(-a \sin\theta, b \cos\theta)$ are extremities of conjugate diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

P and Q lies on $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

$$\Rightarrow \quad \frac{a^2\cos^2\theta}{\alpha^2} + \frac{b^2\sin^2\theta}{\beta^2} = 1$$

$$\Rightarrow \quad \frac{a^2\sin^2\theta}{\alpha^2} + \frac{b^2\cos^2\theta}{\beta^2} = 1$$

On addition, we get, $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$

6. The given curve is $\frac{x^2}{4} + \frac{y^2}{4/3} = 1$.

Point P is $\left(2\cos\frac{\pi}{6}, \frac{2}{\sqrt{3}}\sin\frac{\pi}{6}\right) = \left(\sqrt{3}, \frac{1}{\sqrt{3}}\right)$

On differentiating the given equation, w.r.t. x, we get, $x + 3y\frac{dy}{dx} = 0$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_P = \left[-\frac{x}{3y}\right]_P = -1$$

The equation of normal is $y - \frac{1}{\sqrt{3}} = 1\left(x - \sqrt{3}\right)$

$$\Rightarrow \quad x - y = \sqrt{3} - \frac{1}{\sqrt{3}}$$

Normal passes through $Q(\theta)$.

Hence, $2\cos\theta - \frac{2}{\sqrt{3}}\sin q = \sqrt{3} - \frac{1}{\sqrt{3}}$

$$\Rightarrow \quad 2\sqrt{3}\cos\theta - 2\sin\theta = 2$$

$$\Rightarrow \quad \sqrt{3}\cos\theta - \sin\theta = 1$$

$$\Rightarrow \quad \theta = \frac{3\pi}{2}$$

7. It is the property of auxiliary circle. Hence, true.

Single Correct Choice Type Questions

1. For an ellipse, $10 - a > 0$ and $4 - a > 0$

$$\Rightarrow \quad a < 10 \text{ and } a < 4$$

$$\Rightarrow \quad a \in (-\infty, 4)$$

Hence, the correct answer is (A).

2. $9(x - 3)^2 + 9(y - 4)^2 = y^2$

$$9(x - 3)^2 + 8y^2 - 72y + 14y = 0$$

$$9(x - 3)^2 + 8(y^2 - 9y) + 144 = 0$$

$$9(x - 3)^2 + 8\left[\left(y - \frac{9}{2}\right)^2 - \frac{81}{4}\right] + 144 = 0$$

$$\Rightarrow \quad 9(x - 3)^2 + 8\left(y - \frac{9}{2}\right)^2 = 162 - 144 = 18$$

$$\frac{9(x - 3)^2}{18} + \frac{8\left(y - \frac{9}{2}\right)^2}{18} = 1$$

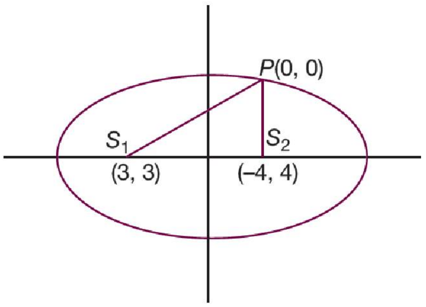
$$\Rightarrow \quad \frac{(x - 3)^2}{2} + \frac{\left(y - \frac{9}{2}\right)^2}{9/4} = 1$$

$$e^2 = 1 - \frac{(2)(4)}{9} = \frac{1}{9}$$

$$\therefore \quad e = \frac{1}{3}$$

Hence, the correct answer is (B).

3.



$$PS_1 + PS_2 = 2a$$

$$\Rightarrow \quad 3\sqrt{2} + 4\sqrt{2} = 2a$$

$$\therefore \quad 2a = 7\sqrt{2}$$

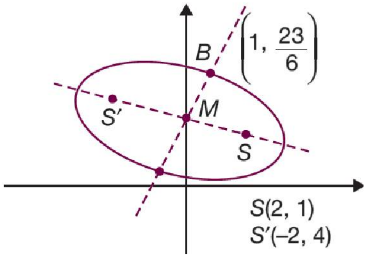
Also, $2ae = S_1S_2 = \sqrt{1+49} = 5\sqrt{2}$

$$\therefore \frac{2ae}{2a} = \frac{5\sqrt{2}}{7\sqrt{2}} = \frac{5}{7} = e$$

Hence, the correct answer is (C).

4. $2ae = 5$ and $b = \frac{5}{3} \Rightarrow a^2 = \frac{325}{36}$

$$\Rightarrow \quad e^2 = \frac{9}{13}$$



Hence, the correct answer is (B).

5. $F_1(9, 20)$ and $F_2(49, 55)$

$$F_1F_2 = \sqrt{40^2 + 35^2} = 2ae$$

$$\Rightarrow \quad 4a^2 - 4b^2 = 2825$$

x axis is tangent

$$b^2 = 20 \times 55$$

we get $2a = 85$.

Hence, the correct answer is (A).

6. $$(x-2)^2 + (y+3)^2 = \left[\frac{1}{2}\left(\frac{3x-4y+7}{5}\right)\right]^2$$

$\Rightarrow$ Focus $(2, -3)$, directrix $3x - 4y + 7 = 0$, $e = \frac{1}{2}$

Distance between focus and directrix $= \frac{b^2}{ae} = 5$

Hence, the correct answer is (C).

7. Since y-axis is major axis

$$\Rightarrow \quad f(4a) < f(a^2 - 5)$$

$$\Rightarrow \quad 4a > a^2 - 5 \qquad (\because f \text{ is decreasing})$$

$$\Rightarrow \quad a^2 - 4a - 5 < 0$$

$$\Rightarrow \quad a \in (-1, 5)$$

Hence, the correct answer is (D).

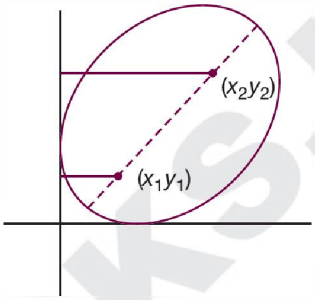
8. If S, S' are foci and P $(a\cos\alpha, b\sin\alpha)$ is any point on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $SP + S'P = 2a$

Hence, the correct answer is (C).

9. The product of the perpendicular distances from foci onto any tangent is a constant.

$$x_1x_2 = y_1y_2 = 6$$

$\because$ Co-ordinate axes are tangents to the ellipse.



Hence, the correct answer is (C).

10. If we rotate the axes through 45°, the transformed equation is $x^2 + 5y^2 = 5$ and the foci are $(\pm 2, 0)$.

Then, foci in original system are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

Hence, the correct answer is (A).

11. It is the minimum distance between 2 curves $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and $y = x - 5$.

Hence, the correct answer is (A).

12. Clearly, origin will be a point lying on the director circle of the ellipse.
Thus, equation of locus of centre of ellipse will be

$$x^2 + y^2 = a^2 + b^2$$

$$\Rightarrow \quad x^2 + y^2 = 100$$

The adjacent figure shows the position of center of ellipse in two extreme position.

$$\theta = \angle C_1OC_2 = \frac{\pi}{2} - 2\tan^{-1}\left(\frac{5}{5\sqrt{3}}\right) = \frac{\pi}{6}$$

Thus, length of the arc, $C_1C_2 = 10.\frac{\pi}{6} = \frac{5\pi}{3}$ units

Hence, the correct answer is (D).

13. $P(\theta)$, $Q(\beta)$, $R(\alpha)$ are the vertices of triangle inscribed in ellipses. Corresponding points on auxiliary circle are $A(\theta)$, $B(\beta)$, $C(\alpha)$

$$\text{Area of } \Delta PQR = \frac{1}{2}ab\begin{vmatrix} \cos\theta & \sin\theta & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\alpha & \sin\alpha & 1 \end{vmatrix}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}a^2 \begin{vmatrix} \cos\theta & \sin\theta & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\alpha & \sin\alpha & 1 \end{vmatrix}$$

$$\Delta_1 : \Delta_2 = \sqrt{5} : 3 \Rightarrow \frac{b}{a} = \frac{\sqrt{5}}{3} \Rightarrow e = \frac{2}{3}$$

Hence, the correct answer is (A).

14. $C(6,8)$ and $P(1,-4)$ lie on auxiliary circle.

$$CP = a = \sqrt{25+144}$$

$$SS' = 2ae = \sqrt{36+64} = 10$$

We get, $ae = 5$

$$\therefore \quad e = \frac{5}{13}$$

Hence, the correct answer is (C).

15. Since the chord of contact is of mutually perpendicular tangents, tangents are drawn from the director circle. Any point on the director circle is $(5\cos\theta, 5\sin\theta)$

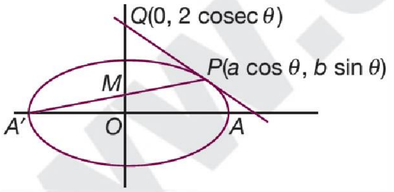
$\Rightarrow$ Chord of contact is, $45\cos\theta\, x + 80\sin\theta\, y - 144 = 0$, where distance from the centre, i.e., $(0, 0)$ is

$$\frac{144}{5\sqrt{81\cos^2\theta + 256\sin^2\theta}} = \frac{144}{5\sqrt{81+175\sin^2\theta}} \le \frac{16}{5}$$

Hence, the correct answer is (D).

16. $a = 3$ and $b = 2$

$$T: \frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$



$x = 0;\ y = 2\,\text{cosec}\,\theta$

Chord $A'P$,

$$y = \frac{2\sin\theta}{3(\cos\theta+1)}(x+3)$$

Put $x = 0$, to get, $y = \dfrac{2\sin\theta}{1+\cos\theta} = OM$

Now, $OQ^2 - MQ^2 = OQ^2 - (OQ - OM)^2 = 2(OQ)(OM) - OM^2 = OM\{2(OQ) - (OM)\}$

$$= \frac{2\sin\theta}{1+\cos\theta}\left[\frac{y}{\sin\theta} - \frac{2\sin\theta}{1+\cos\theta}\right] = 4$$

Hence, the correct answer is (C).

17. Equation of a chord $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}$

Put $\beta = \alpha + \dfrac{\pi}{2}$ and then the equation reduces to

$$bx(\cos\alpha - \sin\alpha) + ay(\cos\alpha + \sin\alpha) = ab \qquad \ldots(1)$$

Compare with $lx + my = -n$ $\qquad \ldots(2)$

$$\left.\begin{aligned} \cos\alpha - \sin\alpha &= \frac{a\ell}{-n} \\ \cos\alpha + \sin\alpha &= \frac{mb}{-n} \end{aligned}\right\} \text{Squaring and adding, to get}$$

$$a^2 l^2 + b^2 m^2 - 2n^2 = 0.$$

Hence, the correct answer is (C).

18. $a^2 e^2 = 36 \Rightarrow a^2 - b^2 = 36$ $\qquad \ldots(1)$

Using $r = (s-a)\tan\dfrac{A}{2}$ in ΔOCF

$1 = (s-a)\tan 45°$ when $a = CF$

$2 = 2(s-a)$

$= 2s - 2a = 2s - AB$

$= (OF + FC + CO) - AB$

$2 = 6 + \dfrac{AB}{2} + \dfrac{CD}{2} - AB$

$\dfrac{AB - CD}{2} = 4$

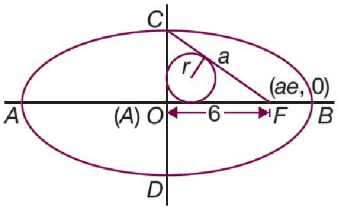
$\Rightarrow \quad 2(a-b) = 8$

$\Rightarrow \quad a - b = 4$ $\qquad \ldots(2)$

From (1) and (2), $a + b = 9$

$\Rightarrow \quad 2a = 13$ and $2b = 5$

$\Rightarrow \quad (AB)(CD) = 65$



Hence, the correct answer is (A).

19. Equation of a tangent to $\dfrac{x^2}{a^2+b^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \quad y = mx \pm\sqrt{(a^2+b^2)m^2 + b^2} \qquad \ldots(1)$$

If (1) is also a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2+b^2} = 1$, then

$$(a^2+b^2)m^2+b^2=a^2m^2+0^2+b^2 \text{ (using } c^2=a^2m^2+b^2)$$

$$b^2m^2=a^2$$

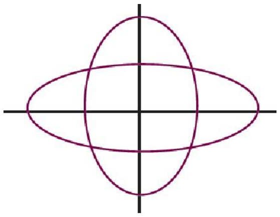
$$\Rightarrow \quad m^2=\frac{a^2}{b^2}$$

$$\Rightarrow \quad m=\pm\frac{a}{b}$$

$$y=\pm\frac{a}{b}x \pm \sqrt{(a^2+b^2)\frac{a^2}{b^2}+b^2}$$

$$by=\pm ax \pm \sqrt{a^4+a^2b^2+b^4}$$

Although there can be four common tangents but only one of these appears in *B*.



Hence, the correct answer is (B).

20. Equation of normal is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

$$Q\left(\left(\frac{a^2-b^2}{a^2}\right)x_1,\,0\right),\ R\left(0,-\left(\frac{a^2-b^2}{b^2}\right)y_1\right)$$

Let the midpoint of *QR* is (*h*, *k*).

$$x_1=\frac{2ha^2}{a^2-b^2},\ y_1=-\frac{2kb^2}{a^2-b^2}$$

Locus is $\frac{a^2x^2}{\left(\frac{a^2-b^2}{2}\right)^2}+\frac{b^2y^2}{\left(\frac{a^2-b^2}{2}\right)^2}=1$

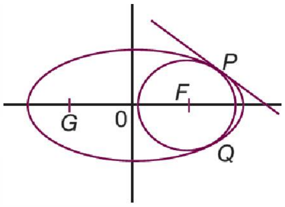
$\therefore \quad e'=e$

Hence, the correct answer is (C).

21. $2a=10$

$\Rightarrow \quad a=5$ and $2b=8$

$\Rightarrow \quad b=4$



$$e^2=1-\frac{16}{25}=\frac{9}{25}$$

$$\Rightarrow \quad e=\frac{3}{5}$$

Focus = (3, 0)

Let the circle touches the ellipse at *P* and *Q*. Consider a tangent (to both circle and ellipse) at *P*. Let *F* (one focus) is the centre of the circle and other focus is *G*. A ray from *F* to *P* must retrace its path (normal to the circle). However, the reflection property the ray *FP* must be reflected along *PG*. This is possible only if *P*, *F* and *G* are collinear. Thus, *P* must be the end of the major axis.

Hence, $r=a-ae=5-3=2$

Alternate

Normal to an ellipse at *P* must pass through the centre (3, 0) of the circle

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

$$\Rightarrow \quad \frac{5x}{\cos\theta}-\frac{4y}{\sin\theta}=9\left(\theta\neq 0 \text{ or } \frac{\pi}{2}\right)$$

$$\frac{15}{\cos\theta}-0=9$$

$$\Rightarrow \quad \cos\theta=\frac{15}{9}, \text{ which is not possible}$$

$$\Rightarrow \quad \theta=0 \text{ or } \frac{\pi}{2}$$

However, $\theta\neq\frac{\pi}{2}$

$\Rightarrow \quad \theta=0$

Hence, *P*(5, 0), i.e., at the end of major axis.

Hence, the correct answer is (B).

22. By symmetry, $m=\pm 1$

Take, $m=-1$

$$\Rightarrow \quad y=(-1)x+\sqrt{2010+2086}$$

$$\Rightarrow \quad x+y=64 \qquad \ldots(1)$$

Circle touches both the axes and the Equation (1)

$$r=\frac{|2r-64|}{\sqrt{2}} \Rightarrow r=32(2-\sqrt{2})$$

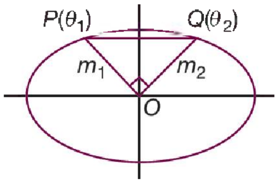
$$\Rightarrow \quad [r]=18$$

Hence, the correct answer is (C).

23. $m_1=\frac{b}{a}\tan\theta_1$ and $m_2=\frac{b}{a}\tan\theta_2$

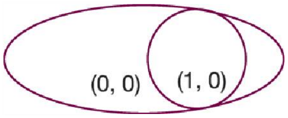
$$\Rightarrow \quad m_1m_2=\frac{b^2}{a^2}\tan\theta_1\tan\theta_2=-1$$

Where, m_1 = Slope of *OP*



Hence, the correct answer is (B).

24.



$$(x-1)^2+y^2=r^2$$

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

$$\Rightarrow\quad 3x^2-8x+20-4r^2=0$$

Circle lie inside ellipse, such that, $D\le 0\Rightarrow r\le\sqrt{\frac{11}{3}}$

∴ Equation of the circle is $3\left(x^2+y^2\right)-6x-8=0$

Hence, the correct answer is (C).

25. $R=\frac{abc}{4\Delta}=\frac{a(1-e\cos\theta)a(1+e\cos\theta)2ae}{4\times\frac{1}{2}\times 2ae\times b\sin\theta}$

$$=\frac{a^2\left(1-e^2\cos^2\theta\right)}{2b\sin\theta}=\frac{a^2}{2b}\left(\frac{b^2}{a^2}\text{cosec}\,\theta+e^2\sin\theta\right)$$

$$\ge\frac{a^2}{2b}\times\frac{2be}{a}=ae$$

Hence, the correct answer is (A).

26. $ae=b\Rightarrow e^2=\frac{b^2}{a^2}\Rightarrow e^2=1-e^2$

$$\therefore\quad e=\frac{1}{\sqrt{2}}$$

Also, $SA=a-ae=\sqrt{10}-\sqrt{5}$

$$\Rightarrow\quad a\left(1-\frac{1}{\sqrt{2}}\right)=\sqrt{10}\left(1-\frac{1}{\sqrt{2}}\right)$$

$$\therefore\quad a^2=10\ \text{and}\ b^2=5$$

Hence, the correct answer is (D).

27. Equation to pair of tangent from $P(h,k)$ is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$ $\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$

Let these tangents meet the axes in A_1, A_2, B_1, B_2

Then, OA_1, OA_2 are the roots of $\frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right)-\frac{2xh}{a^2}$ $-\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)=0$

$$\therefore\quad OA_1\cdot OA_2=\frac{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)a^2}{1-\frac{k^2}{b^2}}$$

Similarly, $OB_1\cdot OB_2=\frac{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)b^2}{1-\frac{h^2}{a^2}}$

Hence, the correct answer is (C).

28. Equation of the tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ …(1)

$$x^2+y^2=a^2\left[\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right]^2$$

Since these lines are at right angles, coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow\quad e=\left(1+\sin^2\theta\right)^{-1/2}$$

Hence, the correct answer is (C).

29. Tangent: $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Normal: $xa\sec\theta-yb\,\text{cosec}\,\theta=a^2-b^2$

$$CF=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

$$PG=\frac{b}{a}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

Hence, the correct answer is (A).

30. If PSQ is focal chord, then $\frac{1}{SP}+\frac{1}{SQ}=\frac{2}{l}$, where l = semi-latus rectum

The product of perpendiculars from from foci to any tangent is b^2.

The points A, B lie on auxiliary circle.

$$\therefore\quad\frac{SP+SQ}{SP.\,SQ}=\frac{2a}{b^2}=\frac{8}{9};\ SA.S'B=9;\ OA\,.\,OB=a^2=16$$

Hence, the correct answer is (C).

31. $a > 1 \Rightarrow b^2 - 5b + 7 > 1$

$\Rightarrow \quad b \notin [2, 3]$

Hence, the correct answer is (D).

32. $x^2 + y^2 = 1; x^2 + 2y^2 = 4$

Let $R(x_1, y_1)$ is the point of intersection of tangents drawn at P and Q to the ellipse.

$\Rightarrow \quad PQ$ is chord of contact of $R(x_1 y_1)$

$\Rightarrow \quad xx_1 + 2yy_1 - 4 = 0$

This touches the circle $\Rightarrow r^2(l^2 + m^2) = n^2$

$\Rightarrow \quad 1(x_1^2 + 4t_1^2) = 16$

$\Rightarrow \quad x^2 + 4y^2 = 16$ is ellipse $e = \dfrac{\sqrt{3}}{2}$ and $LL' = 2$

Hence, the correct answer is (D).

33. $c_1(2, -3)r_1 = 6$

$c_2(2, -4)r_2 = 1$

Let C is the centre of $S = 0$

$$\therefore \quad \left.\begin{aligned} cc_1 &= r_1 - r \\ cc_2 &= r_1 + r \end{aligned}\right\}$$

$\Rightarrow \quad cc_1 + cc_2 = r_1 + r_2$

$\therefore \quad$ Locus is an ellipse whose foci are (2, –3) and (2, –4).

$$e = \frac{2ae}{2a} = \frac{c_1c_2}{r_1 + r_2} = \frac{1}{7}$$

$\Rightarrow \quad k = \dfrac{1}{7}$

Hence, the correct answer is (A).

34. $(x_1, y_1) = \left(\dfrac{a\sum\cos\alpha}{3}, \dfrac{b\sum\sin\alpha}{3}\right) =$ circumcentre

$$\sum\cos\alpha = \frac{3x_1}{a} \qquad \ldots(1)$$

$$\sum\sin\alpha = \frac{3y_1}{b} \qquad \ldots(2)$$

Square and add

Hence, the correct answer is (C).

35. At the point $\left(ab, \dfrac{b^2}{a}\right)$, tangent is $\dfrac{ex}{a} + \dfrac{y}{a} = 1$

$\therefore$ Area of the quadrilateral formed $= \dfrac{2a^2}{e} = 27$ sq.units

Hence, the correct answer is (A).

36. Normal $\dfrac{ax}{\cos\theta} - \dfrac{b}{\sin\theta} = a^2 - b^2$

$$\frac{ah}{\cos\theta} = a^2 - b^2$$

$\Rightarrow \quad \cos\theta = \dfrac{ah}{a^2 - b^2}$

$$\left|\frac{ah}{a^2 - b^2}\right| \le 1$$

$$|h| \le ae^2$$

Hence, the correct answer is (B).

37. Sides of the square will be perpendicular tangents to the ellipse. So, vertices of the square will lie on director circle. Therefore, the diameter of director circle is $2\sqrt{(a^2 - 3) + (a + 4)} = \sqrt{2a^2 + a^2}$.

$\Rightarrow \quad 2\sqrt{a^2 + a + 1} = 2a$

$\Rightarrow \quad a = -1$

However, for ellipse, $a^2 > 3$ and $a > -4$

So, 'a' cannot take the value '–1'.

Hence, the correct answer is (D).

38. $r = \dfrac{\Delta}{S} \Rightarrow \Delta = S$

$$\frac{1}{2}(ae)b = \frac{ae + b + \sqrt{a^2e^2 + b^2}}{2} \qquad (\because ae = 6)$$

$\Rightarrow \quad 6b = 6 + b + \sqrt{36 + b^2} \Rightarrow b = \dfrac{5}{2}$

$\Rightarrow \quad a^2(1 - e^2) = \dfrac{25}{4} \Rightarrow a^2 - 36 = \dfrac{25}{4}$

$\Rightarrow \quad a = \dfrac{13}{2}$

Hence, the correct answer is (A).

39. $a - b > b$

$\Rightarrow \quad a > 2b$

$\Rightarrow \quad e \in \left(\dfrac{\sqrt{3}}{2}, 1\right)$

Hence, the correct answer is (D).

40. It is a director circle, hence at right angle.

Hence, the correct answer is (D).

41. Area is independent of θ, so put $\theta = 0°$

Then write equation of tangents and compute area.

Hence, the correct answer is (B).

42. Circle 'C' is $x^2 + y^2 + \lambda y - 16 = 0$. It intersects the circle $x^2 + y^2 + \lambda y - 16x - 10y + 9 = 0$, orthogonally.

Then, $\lambda = \dfrac{7}{5}$

Now, $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

Hence, the correct answer is (B).

43. If $S_1 = 0$ and $S_2 = 0$ are the equations, then $\lambda S_1 + S_2 = 0$ is a second degree curve passing through the points of intersection of $S_1 = 0$ and $S_2 = 0$

$$\Rightarrow \quad (\lambda + 4)x^2 + 2(\lambda + 1)y^2 - 2(3\lambda + 10)x + 12(\lambda + 1)y + (23\lambda + 35) = 0$$

For it to be a circle, choose λ such that the coefficients of x^2 and y^2 are equal

$\therefore \quad \lambda = 2$

This gives the equation of the circle as

$$6(x^2 + y^2) - 32x - 36y + 81 = 0 \text{ (By using (1))}$$

$$\Rightarrow \quad x^2 + y^2 - \frac{16}{3}x - 6y + \frac{27}{2} = 0$$

Its centre is $C\left(\frac{8}{3}, 3\right)$ and radius is $r = \sqrt{\frac{64}{9} + 9 - \frac{27}{2}}$

$$= \frac{1}{3}\sqrt{\frac{47}{2}}$$

Hence, the correct answer is (A).

44. Let the equation of the semi-elliptical is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(y > 0)$

Length of the major axis $= 2a = 9$

$$\Rightarrow \quad a = \frac{9}{2}$$

So, the equation of the arc becomes $\frac{4x^2}{81} + \frac{y^2}{9} = 1$

If $\quad x = 2$, then $y^2 = \frac{65}{9}$

$$\Rightarrow \quad y = \frac{1}{3}\sqrt{65}$$

Hence, the correct answer is (B).

45. $\tan\theta = \frac{b}{a}\tan\theta$

$y = \frac{a-b}{a\cot\theta + b\tan\theta}$, if will be maximum

If $\quad \tan^2\theta = \frac{a}{b} \Rightarrow \tan\theta = \sqrt{\frac{a}{b}}$

Hence, the correct answer is (A).

46. $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$\Rightarrow \quad y = \frac{1}{3}x \pm \sqrt{\frac{a^2}{9} + b^2}$$

$\Rightarrow \quad 3y = x \pm \sqrt{a^2 + 9b^2}$ passes through (–1, –1)

$$\Rightarrow \quad -3 = -1 \pm \sqrt{a^2 + 9b^2} \quad -2 = \pm\sqrt{a^2 + 9b^2}$$

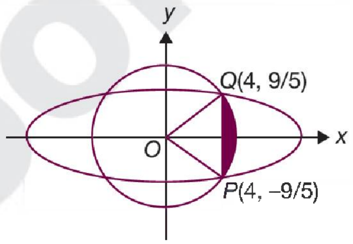
$$a^2 + 9b^2 = 4 \quad \frac{a^2 + 9b^2}{2} \geq \sqrt{9a^2b^2}$$

$$2 \geq 3ab, \quad ab \leq \frac{2}{3} \ (\because a > ba^2 + 9a^2 > 4 \ a^2 > \frac{4}{10})$$

$$a > \sqrt{\frac{2}{5}} \text{ but } a \leq 2$$

Hence, the correct answer is (B).

47. As eccentricity of ellipse $= \frac{4}{5}$



Co-ordinate of foci = (4, 0), (–4, 0)

$\Rightarrow \quad \left(4, -\frac{9}{5}\right)$ is one of the end point of latus rectum

$\Rightarrow$ Required area is

$$\frac{1}{2\pi} \times \pi \times \left(4^2 + \frac{9^2}{5^2}\right) \times 2\tan^{-1}\left(\frac{9}{20}\right) - \text{Area of } \Delta POQ$$

$$= \frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right) - \frac{1}{2} \times 4 \times \left(\frac{18}{5}\right)$$

$$= \frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right) - \frac{36}{5}$$

Hence, the correct answer is (A).

48. Tangent $y = mx + \sqrt{a^2m^2 + b^2}$ (x_1, y_1) lies on it

$$(y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$\Rightarrow \quad m^2(x_1{}^2 - a^2) - 2mx_1y_1 + y_1{}^2 - b^2 = 0$$

$$m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$$

$$m_1m_2 = \frac{y_1^2 - b^2}{x_1^2 - a^2}$$

Given $\cot\theta_1 + \cot\theta_2 = k$

$$\frac{1}{m_1} + \frac{1}{m_2} = k$$

$$m_1 + m_2 = k(m_1m_2)$$

Locus $2x_1y_1 = k({y^2}_1 - b^2)$

$$2xy = k(y^2 - b^2)$$

Hence, the correct answer is (B).

49. Use the relation, $\frac{1}{SP}+\frac{1}{SQ}=\frac{2a}{b^2}$

Hence, the correct answer is (A).

50. Area $= \pi ab$

Hence, the correct answer is (D).

51. Pair of tangents $SS_1 = T^2$

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

Hence, the correct answer is (B).

52. $\frac{x}{a}\cos\left(\frac{\theta+\theta+\frac{\pi}{2}}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\theta+\frac{\pi}{2}}{2}\right)=\cos\left(\frac{\theta-\theta-\frac{\pi}{2}}{2}\right)$

$$x\cos\alpha + y\sin\alpha = p$$

Both equations represent the same line, compare them and use the result.

Hence, the correct answer is (D).

53. The equation of chord of contact is $\frac{x}{4}+\frac{y}{3}=1$

$\Rightarrow$ Slope of EF is $-\frac{3}{4}$

$\Rightarrow$ EF is $y=\frac{-3}{4}x+\sqrt{18}$

$\Rightarrow$ Distance of $(4,3)$ from EF is $\frac{\left|24-4\sqrt{18}\right|}{5}$

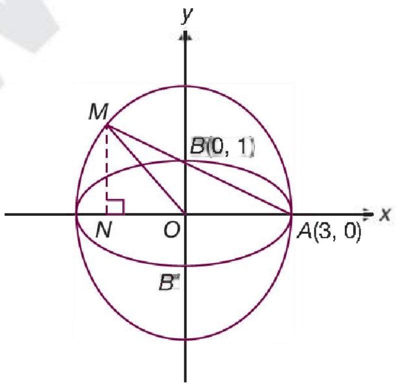
Hence, the correct answer is (C).

54. All such circles pass through foci.

$\therefore$ The common chord is of the length $2ae = 10\times\frac{4}{5} = 8$ units

Hence, the correct answer is (A).

55.



Equation of given ellipse is $\frac{x^2}{9}+\frac{y^2}{1}=1$

Equation of auxiliary circle is $x^2+y^2=9$ ...(1)

Equation of line t is $\frac{x}{3}+\frac{y}{1}=1 \Rightarrow x = 3(1-y)$

Putting this in (1), we get,

$$9(1-y)^2+y^2=9$$

$\Rightarrow$ $10y^2-18y=0$

$\Rightarrow$ $y=0, \frac{9}{5}$

Thus, y-co-ordinate of 'M' is $\frac{9}{5}$

$\Delta OAM = \left(\frac{1}{2}\right)(OA)\,(MN) = \frac{1}{2}(3)\ \frac{9}{5} = \frac{27}{10}$ sq. units

Hence, the correct answer is (D).

56. The tangent of ellipse is $y = mx+\sqrt{a^2m^2+b^2}$, this line touches $x^2+y^2=r^2$

Condition is $\left|\frac{\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}\right| = r$

$\Rightarrow$ $a^2m^2+b^2=r^2m^2+r^2$

$\Rightarrow$ $m^2(a^2-r^2)=r^2-b^2$

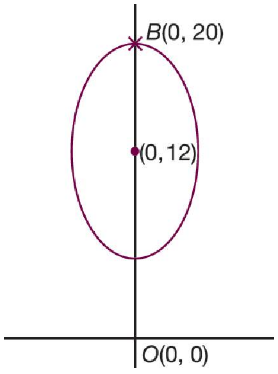
$\Rightarrow$ $m^2=\frac{r^2-b^2}{a^2-r^2}$

$\Rightarrow$ $m=\sqrt{\frac{r^2-b^2}{a^2-r^2}}$

$\therefore$ Inclination is $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$

Hence, the correct answer is (C).

57.



The given equation is $2x^2+y^2-24y+80=0$.

$$2x^2+(y-12)^2=0$$

$$\frac{x^2}{32}+\frac{(y-12)^2}{64}=1$$

If is an ellipse with centre (0, 12)

If (x, y) is any point on this distance from origin, then $\sqrt{x^2+y^2}$.

x^2+y^2 is maximum, if $\sqrt{x^2+y^2}$ is maximum.

⇒ $B(0, 20)$ is at maximum distance from 0.

∴ $\text{Max}(x^2+y^2)=400$

Hence, the correct answer is (D).

58. Tangent is $y = mx + \sqrt{7m^2+3}$

$$\frac{x^2}{7}+\frac{y^2}{3}=1$$

$$x^2+y^2=4$$

It is also touching $x^2+y^2=4$

$$\left|\frac{\sqrt{7m^2+3}}{\sqrt{m^2+1}}\right|=2$$

Or $7m^2+3=4m^2+4$

$$m^2=\frac{1}{3}$$

$$\Rightarrow\quad m=\frac{1}{\sqrt{3}}$$

$$\therefore\quad \tan\theta=\frac{1}{\sqrt{3}}\text{ or }\theta=\frac{\pi}{6}$$

Hence, the correct answer is (D).

59. Let any point be $P(a\cos\theta, b\sin\theta)$.

$SP = a(1 - e\cos\theta)$

$S'P = a(1 + e\cos\theta)$

$SS' = 2ae$

(h, k) be in centre of $\Delta PSS'$.

Upon solving, we get, $h = ae\cos\theta$

$$\Rightarrow\quad k=\frac{b\sin\theta e}{1+e}$$

Eliminating 'θ', we get, $\dfrac{h^2}{a^2e^2}+\dfrac{k^2}{\dfrac{e^2b^2}{(1+e)^2}}=1$

Locus of (h, k) $\dfrac{x^2}{a^2e^2}+\dfrac{y^2}{\dfrac{e^2b^2}{(1+e)^2}}=1$

$$\Rightarrow\quad e_1^2=1-\frac{e^2b^2}{(1+e)^2e^2a^2}=\frac{2e}{1+e}$$

$$e_1=\sqrt{\frac{2e}{1+e}}$$

Hence, the correct answer is (A).

60. Let $\theta=\cos^{-1}\left(\dfrac{-1}{\sqrt{5}}\right)$

$$\Rightarrow\quad \cos\theta=\frac{-1}{\sqrt{5}}$$

$$\Rightarrow\quad \tan\theta=-2$$

Foci are $(\pm 5, 0)$

Equation of line through $(-5, 0)$ with slope -2 is $y=-29x+5=-2x-10$

This line meets the ellipse above x-axis at $(-6, 2)$.

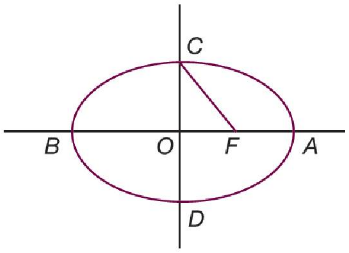
$$\therefore\ \text{Slope}=\frac{2-0}{-6-5}=-\frac{2}{11}.$$

Hence, the correct answer is (D).

61. P lies on the ellipse for which Q, R are foci and length of major axis is 10 units and eccentricity is $\dfrac{3}{5}$.

Hence, the correct answer is (B).

62. $b^2=\dfrac{25}{4}$



$$\Rightarrow\quad a^2-a^2e^2=\frac{25}{4}$$

$$\Rightarrow\quad a^2=\frac{25}{4}+36=\frac{169}{4}$$

$$\Rightarrow\quad a=\frac{13}{2}$$

$$|OF|=ae=6$$

$$\Rightarrow\quad \frac{abe}{2}=1\times\frac{(ae+b+\sqrt{a^2e^2+b^2})}{2}$$

$$6b=6+b+\sqrt{b^2+36}$$

$$\Rightarrow\quad (5b-6)^2=b^2+36$$

$$\Rightarrow\quad 24b^2=60b$$

$$\Rightarrow\quad b=\frac{5}{2}$$

Hence, the correct answer is (A).

63. Equation of tangent at θ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Area with the axes is $\frac{ab}{\sin 2\theta} \ge ab$

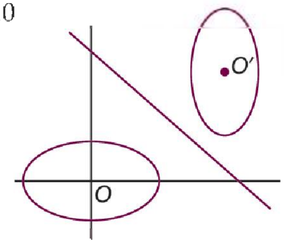
Hence, the correct answer is (D).

64. Image of (0, 0) w.r. to $x + y = 10$

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{-2(0-10)}{1+1} = 10$$

Ellipse O' (10, 10)

$$\frac{(x-10)^2}{16} + \frac{(y-10)^2}{25} = 1$$



Hence, the correct answer is (A).

65. $y = -x + \frac{9}{4\sqrt{2}}$ is a common tangent to two curves in the Ist quadrant and touching the curves at the points $P\left(2\sqrt{2}, 2\sqrt{2}\right)$ and $Q\left(\frac{25}{4\sqrt{2}}, \frac{7}{4\sqrt{2}}\right)$.

PQ = Length of common tangent

Hence, the correct answer is (B).

66. Normal and tangent is perpendicular to normal at a point is the bisector of angle SPS'

Hence, the correct answer is (B).

67. The circle must touch the end of major axis

$\therefore$ Radius $= a - ae = a - \sqrt{a^2 - b^2} = 5 - \sqrt{5^2 - 4^2} = 2$

Hence, the correct answer is (C).

68. Let, the equation of tangent be $y = mx + c$

$$P = \frac{|c|}{\sqrt{1+m^2}} \qquad P_1 = \frac{|c + aem|}{\sqrt{1+m^2}}$$

$$P_2 = \frac{|c - aem|}{\sqrt{1+m^2}} \qquad P_3 = \frac{|c + am|}{\sqrt{1+m^2}}$$

$$P_4 = \frac{|c - am|}{\sqrt{1+m^2}}$$

So, $\frac{P_1P_2 - P^2}{P_3P_4 - P^2} = \frac{c^2 - a^2e^2m^2 - c^2}{c^2 - a^2m^2 - c^2} = e^2$

Hence, the correct answer is (C).

69. $\frac{a}{\cos\theta} - \frac{(a^2 - b^2)}{a}\cos\theta = a$

$\Rightarrow \quad e^2\cos^2\theta = 1 - \cos\theta$

$\Rightarrow \quad e = \frac{\sqrt{1-\cos\theta}}{\cos\theta}$

Hence, the correct answer is (B).

70. Let $P = (h, k)$

Equation of any tangent is $y = mh \pm \sqrt{16m^2 + 9}$

$\Rightarrow \quad k = mh \pm \sqrt{16m^2 + 9}$

$\Rightarrow \quad m^2(h^2 - 16) - 2mhk + (k^2 - 9) = 0$

Let m_1, m_2 are the slope of the tangents.

$\Rightarrow \quad m_1m_2 = \frac{k^2 - 9}{h^2 - 16}$

For concyclic points, $m_1m_2 = 1$

$\Rightarrow \quad h^2 - 16 = k^2 - 9$

$\Rightarrow \quad h^2 - k^2 = 7$

$\Rightarrow \quad x^2 - y^2 = 7$

Hence, the correct answer is (C).

71. $y = \frac{-2p}{\sqrt{1-p^2}}x + \frac{1}{\sqrt{1-p^2}}$

$m = -\frac{-2p}{\sqrt{1-p^2}}$

$\Rightarrow \quad p^2 = \frac{m^2}{4 + m^2}$

$y = mx + \frac{1}{\sqrt{1 - \frac{m^2}{4+m^2}}}$

$\Rightarrow \quad y = mx + \sqrt{\frac{4 + m^2}{4}}$

$\Rightarrow \quad y = mx + \sqrt{1 + \frac{1}{4}m^2}$

It touches $\frac{x^2}{1/4} + \frac{y^2}{1} = 1$, $e = \frac{\sqrt{3}}{2}$

Hence, the correct answer is (B).

72. A cyclic parallelogram will be rectangle or square.

$\therefore \quad \angle QPR = 90°$

$\Rightarrow$ 'P' lies on director circle

$\Rightarrow \quad b^2 = 9,\ e = \frac{\sqrt{7}}{4}\ (b^2 = a^2(1 - e^2))$

Hence, the correct answer is (C).

73. $$\frac{\left(\frac{x - 2y + 3}{\sqrt{5}}\right)}{4} + \frac{\left(\frac{2x + y + 1}{\sqrt{5}}\right)}{1/4} = 1$$

$\Rightarrow \quad PA + PB = 2a = 4$

Hence, the correct answer is (B).

74. The given curve is $\frac{x^2}{9}+\frac{y^2}{3}=1$, whose director circle is $x^2+y^2=12$.

For the required condition, $(2\alpha, \alpha)$ should lie inside the circle and outside the ellipse, i.e.,

$$(2\alpha)^2+3\alpha^2-9>0 \text{ and } (2\alpha)^2+\alpha^2-12<0, \text{ i.e., } \frac{9}{7}<a^2<\frac{12}{5}$$

Hence, the correct answer is (B).

75. The equation of tangent at point $P\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2+11y^2=256$ is

$$16x\,(4\cos\theta)+11y\left(\frac{16}{\sqrt{11}}\sin\theta\right)=256$$

$$4x\cos\theta+\sqrt{11}\,y\sin\theta=16$$

This touches the circle $(x+1)^2+y^2=16$.

So, $$\frac{|4\cos\theta-16|}{\sqrt{16\cos^2\theta+11\sin^2\theta}}=4$$

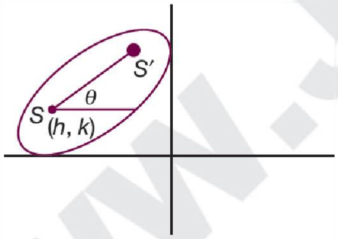
$$\Rightarrow \quad (\cos\theta-4)^2=11+5\cos^2\theta$$

$$4\cos^2\theta+8\cos\theta-5=0$$

$$\therefore \quad \cos\theta=\frac{1}{2} \quad \therefore \quad \theta=\pm\frac{\pi}{3}$$

Hence, the correct answer is (C).

76.



$$(h+2ae\cos\theta)\,h=b^2 \qquad \ldots(1)$$

$$(k+2ae\sin\theta)k=b^2 \qquad \ldots(2)$$

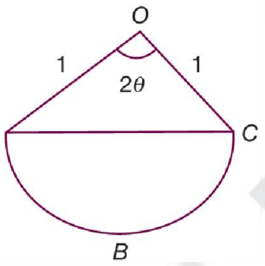
$$2ae\cos\theta=\frac{b^2-h^2}{h}$$

$$2ae\sin\theta=\frac{b^2-k^2}{k}$$

$$4a^2e^2(\cos^2\theta+\sin^2\theta)=\left(\frac{b^2-h^2}{h}\right)^2+\left(\frac{b^2-k^2}{k}\right)^2$$

Hence, the correct answer is (D).

77.



$$x^2+4y^2=4$$

P lies on $x^2+y^2=5$

Let $P(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$

Comparing chord of contact with chord with mid-point, we get, $\frac{xh}{h^2+4k^2}+\frac{y4k}{h^2+4k^2}=1$

$$\frac{x\sqrt{5}\cos\theta}{4}+\frac{y\sqrt{5}\sin\theta}{1}=1$$

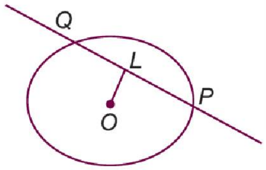
After eliminating θ, we get, $x^2+y^2=\frac{5}{16}(x^2+4y^2)^2$

Hence, the correct answer is (B).

78. $PQ: x\cos\alpha+y\sin\alpha-p=0 \qquad \ldots(1)$

Homogenising $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with (1), we get,

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}=\text{Constant}$$



Hence, the correct answer is (B).

79. Chord of contact from (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

Whose slope is $-\frac{b^2}{a^2}\frac{x_1}{y_1}$

Similarly, slope of another chord of contact is $-\frac{b^2}{a^2}\frac{x_2}{y_2}$

We have $\left(-\frac{b^2}{a^2}\frac{x_1}{y_1}\right)\times\left(-\frac{b^2}{a^2}\frac{x_2}{y_2}\right)=-1$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2}=-\frac{a^4}{b^4}$$

Hence, the correct answer is (C).

80. $2ae = SS' = 1$

$p_1 p_2 = b^2$, where p_1 and p_2 are the length of perpendicular from S and S' to the tangent.

$$\frac{5}{\sqrt{2}} \cdot \frac{4}{\sqrt{2}} = b^2 \Rightarrow b^2 = 10$$

$$\Rightarrow \quad b^2 = 10 = a^2 - e^2a^2$$

$$\Rightarrow \quad a^2 = \frac{41}{4}$$

Hence, the correct answer is (D).

81. Let the point be $(\frac{1}{2}\cos q, \frac{1}{3}\sin q)$, then the slope of the tangent is $-\frac{1/3}{1/2}\cot q = \frac{8}{9}$

$$\Rightarrow \quad \tan q = -\frac{3}{4}$$

$$\Rightarrow \quad \sin q = \pm\frac{3}{5} \text{ and } \cos q = \pm\frac{4}{5}$$

$\therefore$ The required point can be $\left(-\frac{4}{5}\times\frac{1}{2}, \frac{3}{5}\times\frac{1}{3}\right)$

$$= \left(-\frac{2}{5}, \frac{1}{5}\right)$$

Hence, the correct answer is (B).

82. If the length of the diameter is $2r$, then any point on ellipse on this diameter is $(r\cos\theta, r\sin\theta)$.

This point lies on ellipse, then $r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2 sin^2\theta}$

Given condition, $4r^2 = \frac{2(2a)^2(2b)^2}{(2a)^2 + (2b)^2}$

or $$\frac{4a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{2(4a^2)(4b^2)}{4a^2 + 4b^2}$$

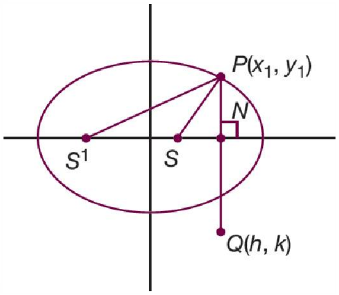
$$\Rightarrow \quad \frac{8}{4a^2 + 4b^2} = \frac{1}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\Rightarrow \quad 2b^2\cos^2\theta + 2a^2\sin^2\theta = a^2 + b^2$$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{4}$$

Hence, the correct answer is (A).

83.



$$a^2 = 25$$

$$b^2 = 16$$

$$e = \sqrt{\frac{25-16}{25}} = \frac{3}{5}$$

Let point Q be (h, k), where $k < 0$

Given that, $|k| = a + eh$ (As, $x_1 = h$)

$$-y = a + ex$$

$$-y = 5 + \frac{3}{5}x$$

$$3x + 5y + 25 = 0$$

Hence, the correct answer is (B).

84. Let the vertices of the rectangle be $(\pm a\cos q, \pm b\sin q)$, then the area of the rectangle is $4ab\sin\theta\cos\theta = 2ab\sin 2\theta$. The maximum value of which is $2ab$ as $\sin 2\theta \le 1$.

Hence, the correct answer is (C).

85. Any point on the ellipse is $P(3\cos\theta, 2\sin\theta)$

Equation of the tangent at P is $\frac{x}{3}\cos\theta + \frac{y}{2}\sin\theta = 1$

Which meets the tangents $x = 3$ and $x = -3$ at the extremities of the major axis at

$$T\left(3, \frac{2(1-\cos\theta)}{\sin\theta}\right) \text{ and } T'\left(3, \frac{2(1+\cos\theta)}{\sin\theta}\right)$$

Equation of the circle on TT' as diameter is

$$(x-3)(x+3) + \left(y - \frac{2(1-\cos\theta)}{\sin\theta}\right)\left(y - \frac{2(1+\cos\theta)}{\sin\theta}\right) = 0$$

$\therefore \quad x^2 + y^2 - \frac{4}{\sin\theta}y - 5 = 0$ and it passes through $(\sqrt{5}, 0)$.

Hence, the correct answer is (B).

86. Radius of the circle must be greater than the semi major axis of the ellipse.

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

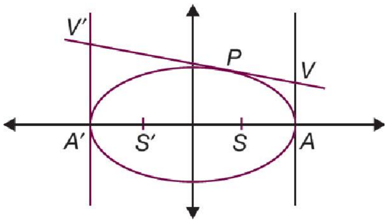
1. $b^2 = (\text{semi-minor axis})^2 = \frac{2+3-2}{\sqrt{2}} \cdot \frac{3+5-2}{\sqrt{2}} = 9$; $b = 3$;

$$2ae = \sqrt{5}$$

$$\Rightarrow \quad e = \sqrt{\frac{5}{41}} \text{ and } a = \frac{\sqrt{41}}{2}$$

Hence, (A) and (D) are correct.

2. Any tangent at $P(a \cos \theta, b \sin \theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$



Line AV is $x = a$ and $A'V'$ is $x = -a$

So points $V\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$ and

$$V'\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$$

$$l(AV)\, l(A'V') = \frac{b(1-\cos\theta)}{\sin\theta} \times \frac{b(1+\cos\theta)}{\sin\theta} = b^2$$

Slope of line $V'S = \frac{b(1+\cos\theta)}{\sin\theta(-a-ae)} = m_1$

Also slope of $VS = \frac{b(1-\cos\theta)}{\sin\theta(a-ae)} = m_2$

Now, $m_1 \cdot m_2 = \frac{b^2(1-\cos^2\theta)}{\sin^2\theta(-b^2)} = -1$

$\Rightarrow$ $\angle V'SV = 90^\circ$ and $V'S'SV$ is a cyclic quadrilateral.

Hence, (A), (C) and (D) are correct.

3. $h = \pm ae$ and $k = \pm\frac{b^2}{a}$

$$k = \pm a(1-e^2) = \pm a\left(1-\frac{h^2}{a^2}\right) = \pm\left(a-\frac{h^2}{a}\right)$$

On taking positive sign, we get, $k = a - \frac{h^2}{a}$

$\Rightarrow$ $\frac{h^2}{a} = a - k$

$\Rightarrow$ $h^2 = a(a-k)$

On taking negative sign, we get, $k = -a + \frac{h^2}{a}$

$\Rightarrow$ $h^2 = a(a+k)$

Hence, (A) and (B) are correct.

4. If PSS' form a triangle, then ellipse $PS + PS' > SS'$

$\Rightarrow$ $K \in (-2, 2)$

If PSS' are collinear, then the pair of coincident lines

$\Rightarrow$ $K = \pm 2$

If PSS' neither form triangle nor collinear, then empty set

$\Rightarrow$ $PS + PS' < SS'$

Hence, (A), (B) and (C) are correct.

5. The conic is an ellipse.

$$\left(\frac{x+y-5}{\sqrt{2}}\right)^2 + \frac{\left(\frac{x-y+7}{\sqrt{2}}\right)^2}{\left(\sqrt{\frac{3}{2}}\right)^2} = 1$$

$a = \sqrt{\frac{3}{2}}; b = 1$

Major axis $x + y - 5 = 0$

Minor axis $x - y + 7 = 0$

Centre $(-1, 6)$

$$e = \frac{1}{\sqrt{3}}, ae = \frac{1}{\sqrt{2}}$$

Focus $\frac{1}{2}$ away from $(-1, 6) = \left(\frac{-3}{2}, \frac{13}{2}\right)$

Point $\frac{a}{e}$ distance away from centre

$$= \left(-1 - \frac{3}{2\sqrt{2}}, 6 + \frac{3}{2\sqrt{2}}\right)$$

Hence, (B), (C) and (D) are correct.

6. Focal property ellipse

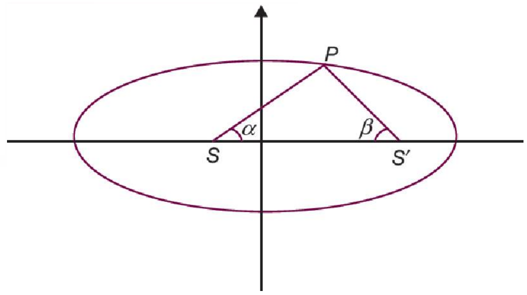
$\therefore$ $PS + PS' = 2a$ $(a > b)$

$PS + PS' = 2b$ $(a < b)$

$PS \cos\alpha + PS' \cos\beta = 2ae$ …(1)

$PS \sin\alpha = PS' \sin\beta$ …(2)

$PS + PS' = 2a$ …(3)



From (1) and (2), we get, $PS=\dfrac{2ae\sin\beta}{\sin(\alpha+\beta)}\cdot PS'=\dfrac{2ae\sin\alpha}{\sin(\alpha+\beta)}$ …(4)

From (3) and (4), we get,

$$e(\sin\alpha+\sin\beta)=\sin(\alpha+\beta)$$

$$\therefore\ (e)(2)\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\therefore\ e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

$$=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\beta}{2}\sin\frac{\alpha}{2}$$

$$\therefore\ \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}=\frac{2a\left(a-\sqrt{a^2-b^2}\right)-b^2}{b^2}$$

$\therefore$ C is correct and D is incorrect.

Hence, (A), (B) and (C) are correct.

7. Equation of the chord $\dfrac{x}{5}\cos\left(\dfrac{\theta+\phi}{2}\right)+\dfrac{y}{3}\sin\left(\dfrac{\theta+\phi}{2}\right)$ $=\cos\left(\dfrac{\theta-\phi}{2}\right)$

If it passes through $(4,0)$, $\dfrac{\cos\left(\dfrac{\theta+\phi}{2}\right)}{\cos\left(\dfrac{\theta-\phi}{2}\right)}=\dfrac{5}{4}$

$$\Rightarrow\ \tan\frac{\theta}{2}\cdot\tan\frac{\phi}{2}=-\frac{1}{9}$$

If it passes through $(-4,0)$, then $\tan\dfrac{\theta}{2}\cdot\tan\dfrac{\phi}{2}=9$

Hence, (C) and (D) are correct.

8. $\left(\sqrt{3}x-3\sqrt{3}\right)^2+(2y+4)^2=c$

By comparing this with

$ax^2+2hxy+by^2+2gx+2fy+c=0$ we get, $h^2<ab$

So, no locus for $c<0$, ellipse for $c>0$ and point for $c=0$.

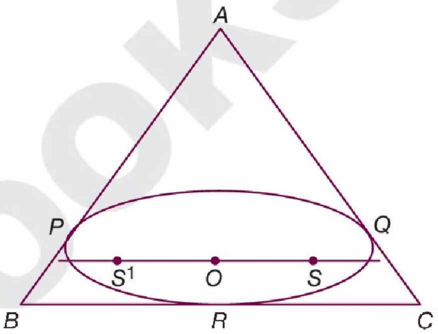
Hence, (A), (B), (C) and (D) are correct.

9. (A) $2a=6,\ 2ae=4,\ e=\dfrac{2}{3},\ E=\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$

(B) A lie on director circle of ellipse then locus of A is

$x^2+y^2=14$

(C) $A\ (h,k)$



Equation of PQ is $\dfrac{xh}{9}+\dfrac{yk}{5}=1$ (Homogenising)

Combined equation of OP and OQ is $\dfrac{x^2}{9}+\dfrac{y^2}{5}$ $=\left(\dfrac{xh}{9}+\dfrac{yk}{5}\right)^2$

$\angle POQ=90°$ then

coefficient of x^2 + coefficient of $y^2=0$

$\Rightarrow\ 25x^2+81y^2=630$ (Locus of (h,k))

(D) $e=\dfrac{2}{3}$

Hence, (A), (B), (C) and (D) are correct.

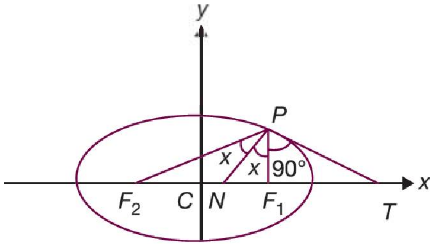
10. Let, $P(4\cos\theta, 3\sin\theta)$

$$\Rightarrow\ 16\cos^2\theta+9\sin^2\theta=\frac{49}{4}$$

$$\Rightarrow\ \cos\theta=\pm\frac{\sqrt{91}}{14},\ \sin\theta=\pm\frac{\sqrt{105}}{14}$$

Hence, (A), (B), (C) and (D) are correct.

11. Given that, P be any point on the ellipse and F_1 and F_2 are two foci of the ellipse.



$\because$ PT and PN are the tangent and normal respectively to the ellipse at point P.

By property, we know that PN and PT are the internal and external bisectors of $\angle F_1PF_2$

$\therefore$ PN bisects $\angle F_1PF_2$ and PT bisects angle $(180° - \angle F_1PF_2)$.

Hence, (A) and (C) are correct.

12. A tangent to the circle $x^2+y^2=1$ is $x\cos\theta+y\sin\theta=1.R(x_0,y_0)$, which is the point of intersection of the tangents to the ellipse at P and Q $x\cos\theta+y\sin\theta=1$ and $x_0x+2y_0y=4$ represent the same line

$$\Leftrightarrow \quad x_0 = 4\cos\theta \text{ and } y_0 = 2\sin\theta$$

$$\Leftrightarrow \quad \frac{x_0^2}{16}+\frac{y_0^2}{4}=1$$

Hence, locus of R is the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$

Hence, (A) and (C) are correct.

13. Chord joining α, β on the ellipse is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

It passes through $(ae,0)$

$$e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$e=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{e+1}{e-1}=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{e+1}{e-1}=\frac{2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)=\frac{e-1}{e+1}$$

$$\frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}=\frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}=e$$

$$\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}=e$$

Hence, (A), (B), (C) and (D) are correct.

14. Clearly, $(\sin\sqrt{2}-\sin\sqrt{3})-(\cos\sqrt{2}-\cos\sqrt{3})=2\sqrt{2}$

$\sin\left(\frac{\sqrt{2}-\sqrt{3}}{2}\right)\sin\left(\frac{\sqrt{2}+\sqrt{3}}{2}+\frac{\pi}{4}\right)$ will have

$$\sin\left(\frac{\sqrt{2}+\sqrt{3}}{2}+\frac{\pi}{4}\right)>0$$

$$\therefore \quad \sin\sqrt{2}-\sin\sqrt{3}<\cos\sqrt{2}-\cos\sqrt{3}$$

Hence, foci of ellipse lie on y-axis.

Hence, (A) and (D) are correct.

15. $A_1=2abe,\ A_2=2ab,\ A_3=\frac{3\sqrt{3}}{4}ab$

For the existence of exactly two points on the ellipse, for which line joining foci subtends a right angle, we must have $ae=b$.

Hence, (A), (B) and (D) are correct.

16. a, b and c are in A.P

$\Rightarrow$ $ax+by+c=0$ is concurrent at (1, 2)

$\therefore$ Centre of auxiliary circle $=(-\alpha,-\beta)=(-1,-2)$

Radius of auxiliary circle $=2$

Length of major axis = 4 = 2a units

$$\because \quad \frac{1}{SP}+\frac{1}{SQ}=\frac{2a}{b^2}\Rightarrow b=\sqrt{3}$$

Hence, $e = \frac{1}{2}$

Hence, (A), (B), (C) and (D) are correct.

17. Let the points be $P(-1, 2)$ and $Q(-2, 3)$.

Hence, image of point P from the line $2x + 3y + 9 = 0$ is

$$\frac{x+1}{2} = \frac{y-2}{3} = -2\left(\frac{-2+6+9}{13}\right)$$

$$X = -5;\ y = -4$$

$$P \in (-5, -4)$$

Now, length of $PQ = \sqrt{9+49} = \sqrt{58} = 2a$ units

$$\Rightarrow \quad a = \frac{\sqrt{58}}{2}$$

We know that, $2ae = PQ$

$$2ae = \sqrt{2} \Rightarrow ae = \frac{1}{\sqrt{2}} \Rightarrow a^2e^2 = \frac{1}{2}$$

$$b^2 = a^2(1-e^2)$$

$$b^2 = \frac{58}{4} - \frac{1}{2} \Rightarrow b^2 = \frac{58-2}{4} \Rightarrow b^2 = \frac{56}{4} \Rightarrow b = \sqrt{14}$$

$$b^2 = 14$$

Length of minor axis $= 2\sqrt{14} = \sqrt{56}$ units

Equation of line PQ is $(y-3) = \frac{4-3}{-3}(x+2)$

$$\Rightarrow \quad 3y - 9 = 72 + 14$$

$$7x - 3y + 23 = 0$$

On solving, $7x - 3y + 23 = 0$ and $2x + 3y + 9 = 0$, we get,

$$A = \left(-\frac{32}{9}, -\frac{17}{27}\right)$$

Hence, (A) and (B) are correct.

18. $\frac{x^2}{4} + \frac{y^2}{3} = 1\ e = \frac{1}{2}$

Ends of latus rectum $\left(\pm 1, \pm\frac{3}{2}\right)$

Let, $P\left(1, \frac{3}{2}\right), Q\left(-1, \frac{3}{2}\right)$

Focus of the parabola $S\left(0, \frac{3}{2}\right)$

Length of latus rectum of the parabola = 2

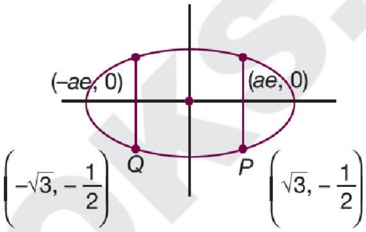
Distance between the focus and vertex $= \frac{1}{2}$

$$A = (0, 1) \text{ and } A' = (0, 2)$$

$\therefore$ Equation of parabolas $x^2 = 2(y-1),\ x^2 = -2(y-2)$

Hence, (B) and (C) are correct.

19.


(–ae, 0)
(ae, 0)
Q
P
$\left(-\sqrt{3}, -\frac{1}{2}\right)$
$\left(\sqrt{3}, -\frac{1}{2}\right)$

The given ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$$\Rightarrow \quad e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$P\left(ae, \frac{-b^2}{a}\right) = \left(\sqrt{3}, -\frac{1}{2}\right), Q\left(-ae, \frac{-b^2}{a}\right) = \left(-\sqrt{3}, \frac{-1}{2}\right)$$

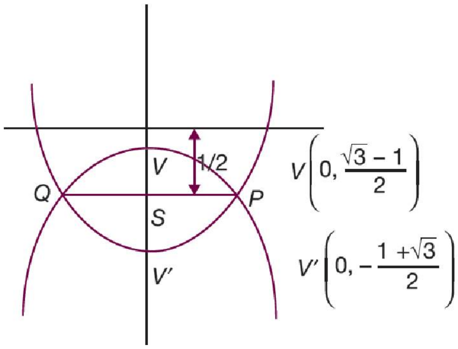
Length of $PQ = 2\sqrt{3}$ units

$$VS = SV' = \frac{PQ}{4} = \frac{\sqrt{3}}{2}$$

$\therefore$ Equations of parabolas are

$$x^2 = -2\sqrt{3}\left(y - \frac{\sqrt{3}-1}{2}\right) \Rightarrow x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$$

and $x^2 = 2\sqrt{3}\left(y + \frac{1+\sqrt{3}}{2}\right) \Rightarrow x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$


V
1/2
Q
P
S
V′
$V\left(0, \frac{\sqrt{3}-1}{2}\right)$
$V'\left(0, -\frac{1+\sqrt{3}}{2}\right)$

Hence, (B) and (C) are correct.

20. $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$y = \frac{1}{3}x \pm \sqrt{\frac{a^2}{9} + b^2}$$

$$3y = x \pm \sqrt{a^2 + 9b^2} \text{ passes through } (-1, -1)$$

$$-3 = -1 \pm \sqrt{a^2 + 9b^2}$$

$$-2 = \pm\sqrt{a^2 + 9b^2}$$

$$a^2 + 9b^2 = 4$$

$$\frac{a^2 + 9b^2}{2} \geq \sqrt{9a^2b^2}$$

$$2 \geq 3ab \Rightarrow ab \leq \frac{2}{3} \quad (\because a > b)$$

$$a^2 + 9a^2 > 4 \qquad a^2 > \frac{4}{10}$$

$$a > \sqrt{\frac{2}{5}} \text{ but } a \leq 2$$

Hence, (A) and (B) are correct.

21. Vertices of the squares will lie on the director circle, i.e., on $x^2 + y^2 = 4 + 3$ and hence the area of the squares is $2(4 + 3) = 14$. Only one such square is possible (same area for all squares).

Hence, (B) and (C) are correct.

22. $\cos^2\theta = \dfrac{d^2 - b^2}{a^2 - b^2} = \dfrac{4-2}{6-2} = \dfrac{1}{2} \Rightarrow \cos\theta = \pm\dfrac{1}{\sqrt{2}}$

$$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

Alternate

A point on the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{2} = 1$ is $\left(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta\right)$

$$6\cos^2\theta + 2\sin^2\theta = 4$$

$$\Rightarrow \quad 4\cos^2\theta = 2$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos 2\theta = 0$$

$$2\theta = (2n+1)\frac{\pi}{2},\ \theta = (2n+1)\frac{\pi}{4}, n \in I.$$

Hence, (A), (C) and (D) are correct.

23. $(x-6)^2 + y^2 = 2 \rightarrow$ Tangent is $y = m(x-6) + \sqrt{2m^2 + 2}$

It is passing through (4, 0) focus of parabola

$$0 = -2m + \sqrt{2m^2 + 2} \Rightarrow 2m^2 + 2 = 4m^2$$

$$m^2 = 1$$

$$\Rightarrow \quad m = \pm 1$$

Hence, (A) and (D) are correct.

24. $\dfrac{x^2}{37} + \dfrac{y^2}{\frac{37}{3}} = 1$

Normal at (x_1, y_1) is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$

$$\frac{37x}{x_1} - \frac{37y}{3y_1} = 37 - \frac{37}{3}$$

$$\frac{x}{x_1} - \frac{y}{3y_1} = \frac{2}{3}$$

It is parallel to $6x - 5y = 2$

$$\frac{3y_1}{x_1} = \frac{6}{5} \qquad \frac{y_1}{x_1} = \frac{2}{5}$$

However, $x_1^2 + 3y_1^2 = 37$

$$x_1^2 + 3\left(\frac{4x_1^2}{25}\right) = 37$$

$$x_1^2\left(1 + \frac{12}{25}\right) = 37$$

$$x_1^2 = 25 \Rightarrow x_1 = \pm 5 \text{ and } y_1 = \pm 2$$

$\therefore$ (5, 2) and (–5, –2) are two points.

Hence, (B) and (D) are correct.

25. $P = (a\cos\theta, b\sin\theta)$ $Q = (-a\sin\theta, b\cos\theta)$

Tangent at P $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ (1)

$$\frac{-x\sin\theta}{a} + \frac{y\cos\theta}{b} = 1 \qquad (2)$$

Eliminating θ, $(1)^2 + (2)^2 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$

Now, midpoint of $PQ = \left(\dfrac{a(\cos\theta - \sin\theta)}{2}, \dfrac{b(\sin\theta + \cos\theta)}{2}\right) = (x, y)$

$$\cos\theta - \sin\theta = \frac{2x}{a} \qquad (3)$$

$$\cos\theta + \sin\theta = \frac{2y}{b} \qquad (4)$$

$$(3)^2 + (4)^2 \Rightarrow 2 = 4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$$

$$\text{Slope of } OP = \frac{b\sin\theta}{a\cos\theta} = m_1$$

$$\text{Slope of } OQ = \frac{-b\cos\theta}{a\sin\theta} = m_2$$

Now, $m_1 m_2 = \dfrac{-b^2}{a^2}$

Now, area of $\Delta OPQ = \dfrac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta \\ -a\sin\theta & b\cos\theta \end{vmatrix}$

$$\Rightarrow \quad \frac{1}{2}ab(\cos^2\theta + \sin 2\theta) = \frac{1}{2}ab$$

Hence, (A), (B), (C) and (D) are correct.

26. $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$, $y^2 = 4x$

Any tangent to parabola is $y = mx + \dfrac{1}{m}$

If this line is a tangent to ellipse, then $\dfrac{1}{m^2} = 8m^2 + 2$

$$\Rightarrow \quad 8m^4 + 2m^2 - 1 = 0$$

$$m^2 = \frac{-2 \pm \sqrt{4 + 32}}{16} = \frac{-2 \pm 6}{16}$$

$$\Rightarrow \quad m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$$

$$y = \frac{x}{2} + 2 \text{ or } y = -\frac{x}{2} - 2$$

$$x - 2y + 4 = 0 \text{ or } x + 2y + 4 = 0$$

Hence, (A) and (B) are correct.

27. $x^2\tan^2\alpha + y^2\sec^2\alpha = 1$

$$\Rightarrow \quad \frac{x^2}{\cot^2\alpha} + \frac{y^2}{\cos^2\alpha} = 1 \qquad (\because \cos^2\alpha = \cot^2\alpha(1 - e^2))$$

$$\Rightarrow \quad \sin^2\alpha = (1 - e^2)$$

$$\therefore \quad e^2 = \cos^2\alpha\,(\alpha \neq 90^\circ)$$

$$e = \cos\alpha \qquad \left(\because \text{Latus rectum} = \frac{1}{2} = \frac{2b^2}{a}\right)$$

$$\Rightarrow \quad a = 4b^2$$

$$\Rightarrow \quad \cot\alpha = 4\cos^2\alpha$$

$$\Rightarrow \quad \frac{1}{\sin\alpha} = 4\cos\alpha$$

$$\Rightarrow \quad \sin 2\alpha = \frac{1}{2}$$

$$2\alpha = n\pi + (-1)^n\frac{\pi}{6}$$

$$\alpha = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$$

For $n = 0$: $\alpha = \dfrac{\pi}{12}$ and for $n = 1$

$$\alpha = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$$

Hence, (A) and (C) are correct.

28. Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Using the point $(4, -1)$ and the tangent $x + 4y - 10 = 0$; $\dfrac{16}{a^2} + \dfrac{1}{b^2} = 1; a^2 + 16b^2 = 100$

Solving, we get the options (B) and (C).

Hence, (B) and (C) are correct.

29. $L\left(ae, \dfrac{b^2}{a}\right)$ normal is $\dfrac{a^2x}{ae} - \dfrac{b^2y}{b^2/a} = a^2e^2$ and it passes through $(0, -b)$.

$$\therefore \quad e^4 + e^2 = 1$$

On solving, we get, $e = \sqrt{\dfrac{\sqrt{5} - 1}{2}}$

Hence, (A) and (B) are correct.

30. Let the points are $P(\alpha,\beta)$ and $Q(\alpha_1,\beta_1)$. Equations of AB and CD are $\frac{x}{a}\alpha+\frac{y}{b}\beta=1$ and $\frac{x}{a}\alpha_1+\frac{y}{b}\beta_1=1$

$$\frac{\alpha}{\alpha_1}=\frac{\beta}{\beta_1}=k$$

$$\Rightarrow \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=\frac{\alpha_1^{\ 2}}{a^2}+\frac{\beta_1^{\ 2}}{b^2}$$

$$\Rightarrow \frac{\alpha}{\alpha_1}=\frac{\beta}{\beta_1}=-1$$

$\therefore$ PQ passes through origin and it is bisected at the origin.

Hence, (A) and (B) are correct.

31. $a^2=b^2\left(1-e^2\right)$

$$\left(\sec^2\alpha\right)e^2=1$$

$$e=\cos\alpha$$

$$l=\frac{2a^2}{b}$$

Foci $(0,\pm be)(0,\pm 1)$

Hence, (A), (B) and (D) are correct.

32. Equation at $\left(4\cos\theta,\frac{16}{\sqrt{11}}\sin\theta\right)$ to $16x^2+11y^2=256$ is

$$x(4\cos\theta)+\sqrt{11}\sin\theta y=16$$

$$\frac{|4\cos\theta-16|}{\sqrt{16\cos^2\theta+11\sin^2\theta}}=4$$

$$\Rightarrow \cos\theta=\frac{1}{2} \text{ or } -\frac{5}{2} \text{ (not possible)}$$

$$\therefore \theta=\pm\frac{\pi}{3},\frac{5\pi}{3}$$

Hence, (A), (C) and (D) are correct.

33. $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent

Line perpendicular to this line and passing through the focus $(ae,0)$ is $y=\frac{-1}{m}(x-ae)$

$$\therefore (y-mx)^2+(x+my)^2=a^2m^2+b^2+a^2e^2=a^2m^2+a^2$$

$$\therefore x^2+y^2=a^2$$

$$BF_1=BF_2=F_1F_2 \text{ and } F_1F_2=2ae$$

$$BF_1+BF_2=2\left(F_1F_2\right)$$

$$2a=2\,(2ae)$$

$$\therefore e=\frac{1}{2}$$

Hence, (A), (B), (C) and (D) are correct.

34. $\dfrac{\left(\dfrac{x+y-2}{\sqrt{2}}\right)^2}{\dfrac{9}{2}}+\dfrac{\left(\dfrac{x-y}{\sqrt{2}}\right)^2}{8}=1$ (This is in standard form)

Centre (1, 1)

$$\Rightarrow (x-1)^2+(y-1)^2=\frac{9}{2}+8 \text{ or}$$

$\frac{(x+y-2)^2+(x-y)^2}{2}=\frac{9}{2}+8$ is required locus.

Hence, (A) and (D) are correct.

35. $\frac{PG}{Pg}=\frac{b^2}{a^2}=\frac{1}{5}$

$$\frac{1}{2}aeb=10$$

$$abe=20$$

$$a^2b^2e^2=400$$

$$\left(a^2-b^2\right)b^2=400$$

$$b^2=10$$

$$a^2=50$$

Hence, (A) and (C) are correct.

36. Without the loss of generality, assume $a>b$

Clearly, $S_{||}<0$

Auxiliary circle for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$

For $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ is $x^2+y^2=a^2$

Director circle for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2+b^2$

$$\therefore \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

Area of the ellipse = πab sq. units

Hence, (B), (C) and (D) are correct.

Linked Comprehension Type Questions

1. $(x-1)^2+(y-2)^2=\frac{41}{7^2}\left(\frac{5x+4y-20}{\sqrt{41}}\right)^2$

Focus (1, 2), $e=\frac{\sqrt{41}}{7}$

Directrix $5x+4y-20=0$

$$e^2=\frac{41}{49}\Rightarrow e=\frac{\sqrt{41}}{7}$$

Hence, the correct answer is (D).

2. Major axis is perpendicular to directrix and passing through focus (1, 2), i.e., $4(x-1)-5(y-2)=0$ $\Rightarrow 4x-5y+6=0.$

Hence, the correct answer is (C).

3. The distance from focus to directrix $=\frac{b^2}{ae}$

$$\frac{b^2}{ae}=\left|\frac{5+8-20}{\sqrt{41}}\right|=\frac{7}{\sqrt{41}}\Rightarrow\frac{b^2}{a}=\frac{7}{\sqrt{41}}\times e=1$$

Length of latus rectum $=\frac{2b^2}{a}=(2)(1)=2$ units

Hence, the correct answer is (B).

Combined Solution to 4, 5 and 6

$$\frac{\left(\frac{3x-4y+7}{5}\right)^2}{16}+\frac{\left(\frac{4x+3y-6}{5}\right)^2}{25}=1$$

4. Length of major axis $=2b=2\times 5=10$ units

Hence, the correct answer is (D).

5. Length of minor axis $=2a=2\times 4=8$ units

Hence, the correct answer is (B).

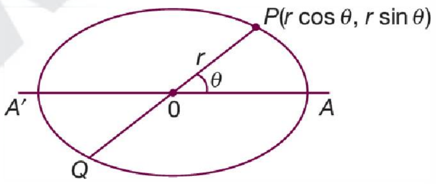
6. Equation of minor axis is $4x+3y-6=0$

Hence, the correct answer is (D).

Combined Solution to 7, 8 and 9

Let PQ be a diameter of length d and let

$OP=r$. $P=(r\cos\theta, r\sin\theta)$



Substitute the co-ordinates of P in the equation of the ellipse to get,

$$r^2=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}\quad\ldots(1)$$

7. $(2r)^2=d^2=\frac{(2a)^2+(2b)^2}{2}$

$\Rightarrow\quad r^2=\frac{a^2+b^2}{2}$

Use (1) to get $\tan\theta=\pm\frac{b}{a}$

Hence, the correct answer is (B).

8. $(2r)^2=d^2=\sqrt{(2a)^2(2b)^2}$

$\Rightarrow\quad r^2=ab$

Use (1) to get $\tan\theta=\pm\sqrt{\frac{b}{a}}$

Hence, the correct answer is (A).

9. $(2r)^2=d^2=\frac{2(2a)^2(2b)^2}{(2a)^2+(2b)^2}$

$\Rightarrow\quad r^2=\frac{2a^2b^2}{a^2+b^2}$

Use (1) to get $\tan\theta=\pm 1$

Hence, the correct answer is (C).

Combined Solution to 10, 11 and 12

10. **The correct answer is 10(C), 11(D) and 12(A).**

$x^2+y^2=a^2;\ x^2+y^2=b^2;\ b>a>0,$
$A=(-a,0);\ \ B=(a,0)$

Let (h,k) be focus. Any tangent to circle $x^2+y^2=b^2$ is $x\cos\theta+y\sin\theta=b$

$\therefore$ Equation of parabola is $\sqrt{(x-h)^2+(y-K)^2}=|x\cos\theta+y\sin\theta-b|$

i.e., $(x-h)^2+(y-K)^2=(x\cos\theta+y\sin\theta-b)^2$

The points $(\pm a,0)$ satisfy this equation

$$\therefore\quad (a-h)^2+K^2=(a\cos\theta-b)^2\quad\ldots(1)$$

$$(a+h)^2+K^2=(a\cos\theta+b)^2\quad\ldots(2)$$

$(2)-(1)\Rightarrow h=b\cos\theta$

$\therefore$ Required locus is $(a+x)^2+y^2=\left(\frac{ax}{b}+b\right)^2$

i.e., $\frac{x^2}{b^2}+\frac{y^2}{b^2-a^2}=1$, which is an ellipse.

13. $b^2=\left|\frac{2+1-9}{\sqrt{2}}\right|\left|\frac{4+1-9}{\sqrt{2}}\right|=\frac{6\times 4}{2}=12$

$2ae=2$

$\Rightarrow\quad ae=1$

$$b^2 = a^2 - a^2e^2 \Rightarrow a^2 = 13$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{13}}$$

Hence, the correct answer is (B).

14. $2a = 2\sqrt{13}$

Hence, the correct answer is (C).

15. Latus rectum $= \frac{2b^2}{a} = \frac{2 \times 12}{\sqrt{3}} = \frac{24}{\sqrt{13}}$

Hence, the correct answer is (C).

16. Let $P = (a \cos \alpha, b \sin \alpha)$ $P' = (a \cos \alpha, a \sin \alpha)$

$P = (a \cos \beta, b \sin \beta)$ $Q' = (a \cos \beta, a \sin \beta)$

$R = (a \cos \gamma, b \sin \gamma)$ $R' = (a \cos \gamma, a \sin \gamma)$.

Area of ΔPQR is $2ab \sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$

Its maximum value is $2ab\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}ab}{4}$

Hence, the correct answer is (A).

17. $$\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P'Q'R'} = \frac{\frac{1}{2}ab\begin{vmatrix} \cos\alpha - \cos\gamma & \sin\alpha - \sin\gamma \\ \cos\alpha - \cos\beta & \sin\alpha - \sin\beta \end{vmatrix}}{\frac{1}{2}a^2\begin{vmatrix} \cos\alpha - \cos\gamma & \sin\alpha - \sin\gamma \\ \cos\alpha - \cos\beta & \sin\alpha - \sin\beta \end{vmatrix}} = \frac{b}{a}$$

Hence, the correct answer is (B).

18. Area of ΔPQR is maximum, when $\alpha - \beta = \beta - \gamma = \gamma - \alpha = 120°$ is $\Delta P'Q'R'$ and it is equilateral. Hence, its centroid is (0, 0) centre of the ellipse.

Hence, the correct answer is (C).

19. $S_1(-ae, 0)$, $S_2(ae, 0)$

$P(a \cos \theta, b \sin \theta)$

Let Incentre (x, y)

$$= \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

We get locus as ellipse

Hence, the correct answer is (D).

20. $e = \frac{1}{2}$, sides of ΔPS_1S_2 are $a - \frac{a}{2}\cos\theta$, $a + \frac{a}{2}\cos\theta$, a in A.P.

Using $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$

$\cot\frac{\alpha}{2}, \cot\frac{\beta}{2}, \cot\frac{\gamma}{2}$, are also in A.P.

Hence, the correct answer is (A).

21. Base S_1S_2 is fixed and $PS_1 + PS_2$ is fixed.

Hence, area will be maximum, if $PS_1 = PS_2$

Hence, the correct answer is (D).

22. $S'(-4, 3)$

$$\frac{(x-1)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$$

Passes through $P(4, 7)$

$$\frac{9}{a^2} + \frac{16}{b^2} = 1 \text{ and } ae = 5$$

$\Rightarrow \quad a^2 - b^2 = 25$

$\Rightarrow \quad a^2 = 45$ and $b^2 = 20$

Product of the perpendicular distances from foci $= b^2 = 20$

Hence, the correct answer is (A).

23. Normal $xa \sec \theta - yb \operatorname{cosec} \theta = a^2 - b^2$ it cuts coordinate axes at

$$\left(\frac{a^2-b^2}{a}\cos\theta, 0\right) \text{and} \left(0, -\frac{(a^2-b^2)}{b}\sin\theta\right)$$

Locus of point $(x, y) = \left(\frac{a^2-b^2}{2a}\cos\theta, -\frac{(a^2-b^2)}{2b}\sin\theta\right)$

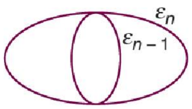
$$\frac{4a^2x^2}{(a^2-b^2)^2} + \frac{4b^2y^2}{(a^2-b^2)^2} = 1 \quad \text{Ellipse}$$

$$e' = e = \frac{\sqrt{5}}{3}$$

Same eccentricity as original ellipse

Hence, the correct answer is (B).

24. $E_n : \frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1, \quad a_n > b_n$



$E_{n-1} : \frac{x^2}{a_{n-1}^2} + \frac{y^2}{b_{n-1}^2} = 1, \quad a_{n-1} < b_{n-1}$

$$b_n^2 = a_n^2\left(1 - e_n^2\right) \quad \ldots(1)$$

$$b_n = b_{n-1} \quad \ldots(2)$$

$$a_{n-1} = a_n e_n \quad \ldots(3)$$

For $E_{n-1}, a_{n-1} = b_{n-1}^2\left(1 - e_{n-1}^2\right)$

From (1) and (2), $b_{n-1}^2 = a_n^2\left(1 - e_{n-1}^2\right)$

$\therefore \quad a_n^2e_n^2 = a_n^2\left(1 - e_n^2\right)\left(1 - e_{n-1}^2\right) \Rightarrow e_n^2 = \left(1 - e_n^2\right)\left(1 - e_{n-1}^2\right)$

Let all the eccentricities are e.

$\therefore \quad e^2 = \left(1 - e^2\right)^2 \Rightarrow e^4 - 3e^2 + 1 = 0$

$$e^2 = \frac{3 \pm \sqrt{5}}{2} \Rightarrow e = \frac{\sqrt{5}-1}{2}$$

Hence, the correct answer is (B).

25. $e_n^2 = \left(1 - e_n^2\right)\left(1 - e_{n-1}^2\right)$

$\Rightarrow \quad h = (1-h)(1-k) \qquad h = e_n^2$

$\Rightarrow \quad x = 1 - x - y + xy \qquad k = e_{n-1}^2$

$\Rightarrow \quad xy - 2x - y + 1 = 0$, which is a rectangular hyberbola

$\Rightarrow \quad (x-1)(y-2) = 0$

Hence, the correct answer is (D).

26. $T = S_1 \Rightarrow \frac{xx_1}{a_n^2} + \frac{yy_1}{b_n^2} = \frac{x_1^2}{a_n^2} + \frac{y_1^2}{b_n^2}$

If eccentricity of E_n is independent of n

$$e = \frac{\sqrt{5}-1}{2} \Rightarrow e^2 = \frac{3-\sqrt{5}}{2}$$

$$-\frac{b_n^2 x_1}{a_n^2 y_1} = -1 \Rightarrow b_n^2 x_1 = a_n^2 y_1$$

$$x_1 = \left(1 - \frac{\left(3-\sqrt{5}\right)}{2}\right) y_1 \Rightarrow 2x_1 = \left(\sqrt{5}-1\right) y_1$$

$\Rightarrow \quad 2x_1\left(\sqrt{5}+1\right) = 4y_1$

$\Rightarrow \quad 2y = x\left(\sqrt{5}+1\right)$

Hence, the correct answer is (B).

27. Normal $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$$(x_1, y_1) = \left(ae, \frac{b^2}{a}\right)$$

$$\frac{ax}{e} - \alpha y = a^2 - b^2$$

It passes $(0, -b)$

$$ab = a^2 - b^2 = a^2e^2$$

$$e^2 = \frac{b}{a} = \sqrt{1-e^2}$$

$\Rightarrow \quad e^4 + e^2 - 1 = 0$

$$e^2 = \frac{\sqrt{5}-1}{2}$$

Hence, the correct answer is (C).

28. Equation of normals at θ and $\frac{\pi}{2} + \theta$ are $ax\sec\theta - by$ $\operatorname{cosec}\theta = a^2 - b^2$ and $ax\operatorname{cosec}\theta + by\sec\theta = b^2 - a^2$

$$\tan\phi = \left|\frac{(a\sec\theta)(b\sec\theta) + (a\operatorname{cosec}\theta)(b\operatorname{cosec}\theta)}{(a\sec\theta)(a\operatorname{cosec}\theta) - (b\operatorname{cosec}\theta)(b\sec\theta)}\right|$$

$$= \frac{ab}{a^2-b^2} \cdot \frac{1}{\sin\theta\cos\theta} = \frac{2ab}{a^2-b^2} \cdot \operatorname{cosec} 2\theta$$

$$\therefore \quad \cot\phi \operatorname{cosec} 2\theta = \frac{a^2-b^2}{2ab} = \frac{a^2e^2}{2ab} = \frac{e^2}{2}\left(\frac{a}{b}\right) = \frac{e^2}{2\sqrt{1-e^2}}$$

Hence, the correct answer is (D).

29. Equation of normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

Equation of CQ is $y = (\tan\theta)x$

By solving, we get, $x = (a+b)\cos\theta$, $y = (a+b)\sin\theta$

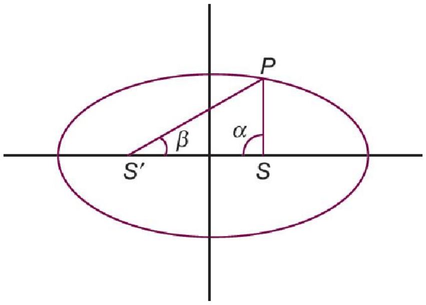
$$\therefore \quad CR = \sqrt{(a+b)^2\cos^2\theta + (a+b)^2\sin^2\theta} = a+b$$

Hence, the correct answer is (B).

30. $\frac{PS}{\sin\beta} = \frac{PS'}{\sin\alpha} = \frac{2ae}{\sin(\pi - (\alpha+\beta))}$

Or $\quad \frac{2a}{\sin\alpha + \sin\beta} = \frac{2ae}{\sin(\alpha+\beta)}$

Or $\quad \frac{1}{e} = \frac{2\sin\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}}$



$$\therefore \quad \frac{1-e}{1+e} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

Hence, the correct answer is (C).

31. $y - 0 = \tan\frac{\beta}{2}(x + ae)$ …(1)

$y - 0 = -\tan\frac{\alpha}{2}(x - ae)$ …(2)

Or $\quad y^2 = -\left(\frac{1-e}{1+e}\right)(x^2 - a^2e^2)$

Or $\quad \left(\frac{1-e}{1+e}\right)x^2 + y^2 = \left(\frac{1-e}{1+e}\right)a^2e^2$

Or $\dfrac{x^2}{a^2e^2}+\dfrac{y^2}{\left(\dfrac{1-e}{1+e}\right)a^2e^2}=1$, which is clearly an ellipse.

$$e'=\sqrt{1-\frac{1-e}{1+e}}=\sqrt{\frac{2e}{1+e}}$$

Hence, the correct answer is (B).

Combined Solution to 32, 33 and 34

32. **The correct answer is 32(A), 33(C) and 34(D).**

The centre is given by $5x-3y+11=0$ and $-3x+5y-13=0$

From this, we find $x=-1, y=2$.
On transferring the origin to this point, we find that the equation of the conic becomes

$$5x^2-6xy+5y^2-8=0\text{, i.e., }\frac{5}{8}x^2-2\left(\frac{3}{8}\right)xy+\frac{5}{8}y^2=1$$

So that, $a=\dfrac{5}{8}, h=-\dfrac{3}{8}, b=\dfrac{5}{8}$

The lengths of the semi-axes are then given by

$$\left(\frac{5}{8}-\frac{1}{r^2}\right)\left(\frac{5}{8}-\frac{1}{r^2}\right)=\frac{9}{64}$$

$$r^2=4\text{ or }r^2=1$$

These are the equations of the axes of the ellipse referred to the new axes of co-ordinates. The equation of the major axis referred to the original axes will be $(x+1)-(y-2)=0$, i.e., $x-y+3=0$ and of the minor axis.

$x+y-1=0$, i.e., $x+y-1=0$ and of the minor axis.

35. $2a=6, 2ae=4$

$$\therefore \quad e=\frac{2}{3}$$

Ellipse equation is $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$

$\angle BAC=90^\circ \Rightarrow A$ lies on director circle $x^2+y^2=14$

Hence, the correct answer is (C).

36. $\angle POQ=90^\circ \Rightarrow$ Let $A(h, k)$

Equation of PQ is $T=0$

$$\frac{hx}{9}+\frac{ky}{5}=1$$

Homogenising

$$\frac{x^2}{9}+\frac{y^2}{5}=\left(\frac{hx}{9}+\frac{ky}{5}\right)^2$$

Coefficient of x^2 + Coefficient of $y^2=0$

$$25h^2+81k^2=630$$

Hence, the correct answer is (C).

37. $AB=BC, \angle B=90^\circ \Rightarrow \angle A=45^\circ$

$$\tan 45^\circ=\frac{2ab\sqrt{S_{11}}}{x_1^2+y_1^2-a^2-b^2}$$

$$(x_1^2+y_1^2-a^2-b^2)^2=(2ab\sqrt{b^2x_1^2+ay_1^2-a^2b^2})^2$$

$$(x^2+y^2-14)^2=20x^2+36y^2-180$$

Hence, the correct answer is (A).

Combined Solution to 38, 39 and 40

38. **The correct answer is 38(C), 39(D) and 40(B).**

Equation can be changed to $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}=1$

$$x=X\cos\theta-Y\sin\theta \quad y=X\sin\theta+Y\cos\theta$$

We get, $a(X\cos\theta-Y\sin\theta)^2+2h(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)+b(X\sin\theta+Y\cos\theta)^2$

$$\text{Coefficient } X^2=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta$$

$$=a\left[\frac{1+\cos 2\theta}{2}\right]+2h\sin\theta\cos\theta+b\left[\frac{1-\cos\theta}{2}\right]$$

$$=\frac{1}{2}\left[(a+b)+(a-b)\cos 2\theta+2h\sin 2\theta\right]$$

Similarly, we get coefficient of $y^2=\dfrac{1}{2}\left[(a+b)-(a-b)\cos 2\theta-2h\sin 2\theta\right]$

$$xy=-2a\sin\theta\cos\theta+2h\left[\cos 2\theta\right]+2b\sin\theta\cos\theta$$

$$=(b-a)\sin 2\theta+2h\cos 2\theta$$

$\Rightarrow$ Or $-2h\cos 2\theta=(b-a)\sin 2\theta$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{1}{2}\begin{bmatrix}(a+b)+(a-b)\cos 2\theta+2h\sin 2\theta\\+(a+b)-(a-b)\cos 2\theta-2h\sin 2\theta\end{bmatrix}$$

$$=a+b$$

Also, $\dfrac{1}{\alpha^2\beta^2}=\dfrac{1}{4}\left[(a+b)^2-\{(a-b)\cos 2\theta+2h\sin 2\theta\}^2\right]$

$$=\frac{1}{4}\left[(a+b)^2-\left\{\frac{2h\cos^2 2\theta+2h\sin^2 2\theta}{\sin 2\theta}\right\}^2\right]$$

$$=\frac{1}{4}\left[(a+b)^2-4h^2\operatorname{cosec}^2 2\theta\right]$$

Also, $\tan 2\theta=\dfrac{2h}{a-b}$ we get $\operatorname{cosec}^2 2\theta=\dfrac{(a-b)^2+4h^2}{4h^2}$

$$= \frac{1}{4}\left[(a+b)^2 - (a-b)^2 - 4h^2\right]$$

$$= ab - h^2$$

Also, eccentricity $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

$$e^2 = \frac{\alpha^2 - \beta^2}{\alpha^2}$$

$$\alpha^2 + \beta^2 = \frac{a+b}{ab-h^2} \quad \alpha^2\beta^2 = \frac{1}{ab-h^2}$$

$$e^2 = \frac{2\sqrt{(a-b)^2 + 4h^2}}{a+b+ab-h^2}$$

41. Eccentricity of given ellipse, $e = \sqrt{1 - \frac{72}{96}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$;

$$e_1 = \sqrt{\frac{2e}{1+e}} = \sqrt{\frac{2}{3}}$$

Hence, the correct answer is (B).

42. Equations of normals at θ and $90° + \theta$ are

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \text{ and } \frac{ax}{\sin\theta} + \frac{by}{\cos\theta} = -(a^2 - b^2)$$

$$\cot\alpha = \frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1} = \frac{\dfrac{a^2 - b^2}{\sin\theta\cos\theta}}{ab\left(\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}\right)}$$

$$= \frac{(a^2 - b^2)}{ab}\sin\theta\cos\theta = \frac{a^2e^2}{2ab}\sin 2\theta$$

$$\Rightarrow \quad \cot\alpha \operatorname{cosec} 2\theta = \frac{e^2}{2\sqrt{1-e^2}}$$

Hence, the correct answer is (B).

43. Let $P(\theta)$ be the point.

Slope of tangent at $\theta = \frac{-b\cos\theta}{a\sin\theta}$

$$PF = b\sin\theta\sqrt{1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}} = \frac{b}{a}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

PG = Perpendicular distance from centre to the tangent

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$= \frac{1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\Rightarrow \quad PF \cdot PG = b^2$$

Hence, the correct answer is (B).

Combined Solution to 44, 45 and 46

44. **The correct answer is 44(B), 45(D) and 46(D).**

Equation of tangent to $y^2 = 8x$ be $yt - x - 2t^2 = 0$ ...(1)

Equation of tangent to ellipse is

$$\frac{x\cos\theta}{2\sqrt{\alpha}} + \frac{y\sin\theta}{\sqrt{\alpha}} = 1 \quad \text{...(2)}$$

$$\text{Comparing } \frac{\sqrt{\alpha}}{\cos\theta} = -t^2 \text{ and } \frac{\sqrt{\alpha}}{\sin\theta} = 2t \quad \text{...(3)}$$

If the tangent meets the co-ordinate axes at A and B, then

$$A\left(\frac{2\sqrt{\alpha}}{\cos\theta}, 0\right), B\left(0, \frac{\sqrt{\alpha}}{\sin\theta}\right)$$

Let midpoint of AB is (h, k)

$$h = \frac{\sqrt{\alpha}}{\cos\theta}; K = \frac{\sqrt{\alpha}}{2\sin\theta}$$

$$h = -t^2; K = t \Rightarrow K^2 - h \text{ or } y^2 = -x$$

From (3),

$$\frac{\alpha}{\sin^2\theta} = \frac{-4\sqrt{\alpha}}{\cos\theta} \Rightarrow \sqrt{\alpha} = \frac{-4\sin^2\theta}{\cos\theta} = 6$$

Any normal is $y = mx - 4m - 2m^2$

i.e., $2m^3 + (4-h)m = 0$, i.e., $h = 6$

From (3), $\alpha = \frac{16\sin^4\theta}{\cos^2\theta}$; $\theta = \frac{2\pi}{3}$

$\Rightarrow \quad \alpha = 36$

Combined Solution to 47, 48 and 49

47. **The correct answer is 47(C), 48(C) and 49(A).**

Given, $OS = ae = 6$

Let $OB = a$, $OC = b$

In radius of ΔOCS, $r = \frac{OS + OC - CS}{2}$

$$\Rightarrow \quad 1 = \frac{6 + b - a}{2} \Rightarrow a - b = 4 \quad \text{...(1)}$$

As we know

$$b^2 = a^2 - a^2e^2$$

$$\Rightarrow \quad (a-4)^2 = a^2 - 36$$

$$\Rightarrow \quad a = \frac{13}{2} \Rightarrow b = \frac{5}{2}$$

Hence, director circle of the given ellipse is $x^2 + y^2 = a^2 + b^2$

⇒ The required director circle is $x^2 + y^2 = 2\left(a^2 + b^2\right)$

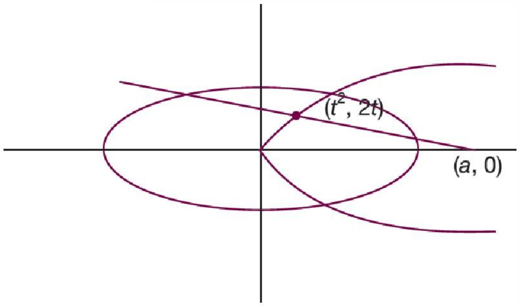
⇒ $x^2 + y^2 = 97$

Also, semi-perimeter of $\Delta OCS = \dfrac{OC + CS + OS}{2}$

$$= \frac{\frac{5}{2} + \frac{13}{2} + 6}{2} = \frac{15}{2} \text{ units}$$

Area of the ellipse $= \pi ab = \pi\left(\dfrac{65}{4}\right)$ sq. units

50.



$S_1 < 0$

⇒ $\dfrac{t^4}{16} + \dfrac{4t^2}{4} - 1 < 0$

⇒ $t^4 + 16t^2 - 16 < 0$

⇒ $-8 - 4\sqrt{5} < t^2 < -8 + 4\sqrt{5}$

$0 < t^2 < -8 + 4\sqrt{5}$

Chord $T = S_1$

$$\frac{x(t^2)}{16} + \frac{y(2t)}{4} = \frac{t^4}{16} + \frac{4t^2}{4}$$

It passes through $(a, 0)$

⇒ $\dfrac{at^2}{16} = \dfrac{t^4}{16} + t^2$

⇒ $t^2(t^2 + (16 - a)) = 0$

⇒ $t = 0$ (or) $t^2 = a - 16$

⇒ $0 < a - 16 < -8 + 4\sqrt{5}$

Clearly, $a - 16 > 0 \Rightarrow a > 16$

⇒ $16 < a < 8 + 4\sqrt{5}$

Hence, the correct answer is (B).

51. Solving $\dfrac{x^2}{16} + \dfrac{4x}{4} = 1 \Rightarrow x^2 + 16x - 16 = 0$

⇒ $x = -8 \pm 4\sqrt{5}$

⇒ $x = -8 + 4\sqrt{5}$ is possible

∴ $y^2 = 4(-8 + 4\sqrt{5})$

⇒ $y = 2\sqrt{4\sqrt{5} - 8}$

⇒ $y = 4\sqrt{\sqrt{5} - 2}$

∴ Length of common chord $= 8\sqrt{\sqrt{5} - 2}$ units

Hence, the correct answer is (C).

52. $y = mx + c$ is a common tangent.

⇒ $\left(\dfrac{1}{m}\right)^2 = 16m^2 + 4$

⇒ $16m^4 + 4m^2 - 1 = 0$

⇒ $m^2 = \dfrac{\sqrt{5} - 1}{8}$

⇒ $m^2 = \dfrac{1}{2}\sin 18°$

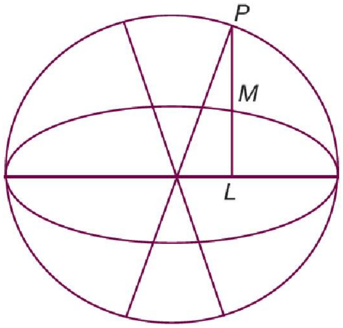
⇒ $m = \pm\sqrt{\sin 9° \cos 9°}$

Hence, the correct answer is (A).

53. If $(3\cos\theta, 2\sin\theta)$, then

$PL = 3\sin\theta \quad ML = 2\sin\theta$

∴ $\dfrac{ML}{PL} = \dfrac{2}{3}$



Hence, the correct answer is (B).

54. Let $y = m_1 x$ be the conjugate diameter of $y = 2x$, then

$$(m_1)(2) = \frac{-b^2}{a^2} = -\frac{4}{9} \Rightarrow m_1 = \frac{-2}{9}$$

Equation is $9y + 2x = 0$

Hence, the correct answer is (B).

55. $R(1, 2)$

For the circle C, $S_1 < 0$

For the ellipse E_1, $S_1 > 0$

Hence, the correct answer is (D).

56. $SP\,.\,S'P = a^2\sin^2\theta + b^2\cos^2\theta = a^2 + \cos^2\theta\left(b^2 - a^2\right) = b^2$

$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \text{ then point } P \text{ is } (0, \pm 1).$$

Hence, the correct answer is (A).

57. Curve X is $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$be = 1$$

$$a^2 + b^2 = 1$$

$$\Rightarrow \frac{a^2}{b} = 2$$

$$b^2 + 2b - 1 = 0$$

$$b = \sqrt{2} - 1,\ a^2 = 2\left(\sqrt{2} - 1\right)$$

$$\frac{-x^2}{2\left(\sqrt{2}-1\right)} + \frac{y^2}{\left(\sqrt{2}-1\right)^2} = 1.$$

Hence, the correct answer is (A).

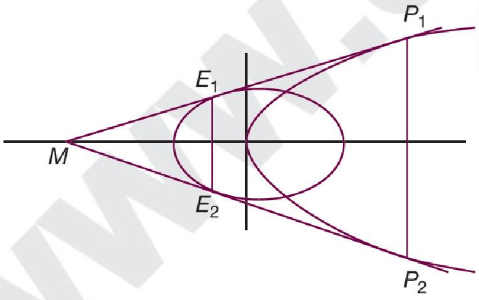
58. Let $k = \sqrt{2} - 1$

$$-\frac{x^2}{k^2} + \frac{y^2}{2k} = 1$$

Area of $BDE = \dfrac{1}{2} \times 2 \times \left(2 + 2\sqrt{2}\right) = 2 + 2\sqrt{2}$ sq. units

Hence, the correct answer is (C).

Combined Solution to 59, 60 and 61



Let tangent of parabola is $y = mx + 1/m$

It is also tangent to ellipse

$$c^2 = a^2m^2 + b^2$$

$$\Rightarrow \frac{1}{m^2} = 8m^2 + 2$$

$$\Rightarrow 8m^4 + 2m^2 - 1 = 0$$

$$\Rightarrow (4m^2 - 1)(2m^2 + 1) = 0$$

$$m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$$

tangents $y = \dfrac{x}{2} + 2$

$$y = -\frac{x}{2} - 2$$

The point M is found to be $(-4, 0)$.

The equation to E_1E_2 be $x = -2$

The equation to P_1P_2 be $x = 4$

$$E_1(-2,1),\ P_1(4,4),\ P_2(4,-4),\ E_2(-2,-1)$$

59. $P_1E_2 = \sqrt{6^2 + 5^2} = \sqrt{61}$

Hence, the correct answer is (B).

60. $\dfrac{ME_1}{MP_1} = \dfrac{\sqrt{2^2 + 1^2}}{8^2 + 4^2} = \dfrac{1}{4}$

Hence, the correct answer is (A).

61. The area of $P_1E_1E_2P_2 = \dfrac{1}{2} \times 8 \times 8 - \dfrac{1}{2} \times 2 \times 2 = 30$ sq. units

Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions

1. A → (p, q, r); B → (p, q, r, s); C → (p, r); D → (p, r)

(A) $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$

$$e = \frac{1}{\sqrt{3}},\ C = (0, 0),\ LR = \frac{2 \times 2}{\sqrt{3}} = \frac{4}{\sqrt{3}},\ S = (\pm 1, 0)$$

(B) $\dfrac{x^2}{2} + \dfrac{y^2}{3} = 1$

$$e = \frac{1}{\sqrt{3}},\ C(0, 0),\ LR = \frac{4}{\sqrt{3}},\ S\ (0, \pm 1)$$

(C) $\dfrac{(x-1)^2}{3} + \dfrac{(y-2)^2}{2} = 1$

$$e = \frac{1}{\sqrt{3}},\ C(1, 2),\ LR = \frac{4}{\sqrt{3}},\ S\ (2, 2), (0, 2)$$

(D) $\dfrac{(x+1)^2}{2} + \dfrac{(y+2)^2}{3} = 1$

$$e = \frac{1}{\sqrt{3}},\ C(-1, -2),\ LR = \frac{4}{\sqrt{3}},\ S = (-1, -1), (-1, -3)$$

2. A → (q); B → (s); C → (p); D → (r)

(A) $PS + PS' = 2b = 4\sqrt{5}$

(B) Given, ellipse $\frac{(x-1)^2}{\frac{1}{2}}+\frac{(y-2)^2}{\frac{1}{3}}=1$ $a^2=\frac{1}{2}, b^2=\frac{1}{3}$

$$e=\frac{1}{\sqrt{3}}$$

(C) Let $(t, t-5)$ be any point on $x-y-5=0$ the chord of contact to ellipse

$x^2+4y^2=4$ is $tx+4(t-5)y-4=0$

$\Rightarrow (-20y-4)+t(x+4y)=0$, which pass through intersection of $x+4y=0, 20y+4=0 \Rightarrow$ i.e., $\left(\frac{4}{5},\frac{-1}{5}\right)$

(D) $\frac{x^2}{4}+\frac{y^2}{3}=1, a=2, b=\sqrt{3}$

$$e=\frac{1}{2} \Rightarrow \frac{2a}{e}=8$$

3. A → (p); B → (r); C → (q); D → (s)

(A) $P(a\cos\theta, b\sin\theta), S_1(-ae,0), S_2(ae,0)$

$PS_1=a(1+e\cos\theta),\ PS_2=a(1-e\cos\theta)$,

$S_1S_2=2ae$

Incentre $\left(ae\cos\theta, \frac{be\sin\theta}{1+e}\right)=(x,y)$

Locus $\rightarrow \frac{x^2}{a^2e^2}+\frac{y^2}{b^2e^2/(1+e)^2}=1$

$\Rightarrow e'=\sqrt{\frac{2e}{1+e}}=\sqrt{\frac{2}{3}}$

(B) Locus of intersection of perpendicular tangents $x^2+y^2=25$

Let (h, k) be any point on circle

Equation of chord of contact $\frac{hx}{16}+\frac{ky}{9}=1$

Distance from centre $= d=\frac{1}{\sqrt{\frac{h^2}{256}+\frac{k^2}{9}}}$ is maximum, when $\sqrt{\frac{h^2}{256}+\frac{25-h^2}{81}}$ is minimum.

$\Rightarrow f'(h)=0 \Rightarrow h=0 \Rightarrow d=\frac{9}{5}$

(C) Slope of the line joining $P(a\cos 30°, a\sin 30°)$ and $Q(a\cos 60°, b\sin 60°)$

$=\frac{-b}{a}=\frac{-1}{2}$

$\Rightarrow a=2b$

PQ subtends 90° at the origin $\Rightarrow x^2$ coefficient $+\,y^2$ coefficient $=0$

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{5}{16}$

$\Rightarrow a=4, b=2$

$\Rightarrow$ Length of minor axis $=2b=4$ units

(D) Equation of normal $y=mx-2am-am^3$ …(1)

Chord of contact of (h, k)

$ky=2a(x+h)$ …(2)

Equations (1) and (2) represent same line

$\Rightarrow y^2(x+5)=32$

4. A → (r); B → (q); C → (p); D → (s)

(A) Any point on ellipse at a distance 'r' from the origin is $(r\cos\theta, r\sin\theta)$

$\Rightarrow r^2=\frac{8}{5-3\sin 2\theta}$ is maximum, if $5-3\sin 2\theta$ is minimum

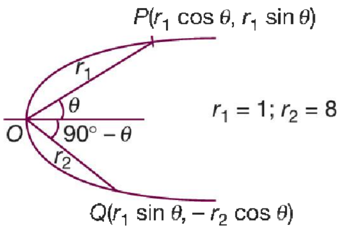
$\Rightarrow r^2=4$

r^2 is minimum, if $(5-3\sin 2\theta)$ is maximum $=8$

$\Rightarrow r^2=1$

$r_1+r_2=2+1=3$

(B)



$r_1=\frac{4a\cos\theta}{\sin^2\theta}, r_2=\frac{4a\sin\theta}{\cos^2\theta}$

$16a^2\left(r_1^{2/3}+r_2^{2/3}\right)=(r_1 r_2)^{4/3}$

$\Rightarrow a^2=\frac{1}{5}$

(C) Equation of tangent to parabola $y^2 = 8x$

$y = mx + \frac{2}{m}$ is also tangent to circle $x^2 + y^2 = 8$

$$\frac{\frac{2}{m}}{\sqrt{m^2+1}} = \frac{\sqrt{8}}{\sqrt{2}} \Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

Equation of common tangents $y = x + 2$ and $y = -x - 2$ meet at $A(-2, 0)$

Chord of contact of $A(-a, 0)$ is $x = -1$, $x = 2$ which are

Area of trapezium = 15 sq. units

(D) PQ is focal chord $\Rightarrow t_1 t_2 = -1$

Centroid $(x, y) = \left(3\left(t_1^2 + \frac{1}{t_1^2}\right), 6\left(t_1 - \frac{1}{t_1}\right)\right)$

$\Rightarrow \quad y^2 = 12(x+6)$

5. A → (p); B → (s); C → (q); D → (r)

(A) Equation of tangent $(5\cos\theta, 3\sin\theta)$ is $\frac{x}{5}\cos\theta + \frac{y}{3}\sin\theta = 1 \rightarrow (1)$, which is a tangent to circle

$x^2 + y^2 = 16 \Rightarrow \cos\theta = \frac{5\sqrt{7}}{16}$

Length of the tangent, $l = \sqrt{s_{11}} = \frac{3\sqrt{7}}{4}$ units

(B) $y = mx + \sqrt{a^2m^2 + b^2}$, $S_1(0, -be)$, $S_2(0, be)$

$$p_1 p_2 = \frac{\left(-be + \sqrt{a^2m^2+b^2}\right)\left(be + \sqrt{a^2m^2+b^2}\right)}{m^2+1} = a^2$$

(C) $t_3 = -t_1 - \frac{2}{t_1}$, $t_3 = -t_2 - \frac{2}{t_2} \Rightarrow t_1 t_2 = 2$, $at_1^{\,2} \cdot at_2^2 = a^2$

$(2)^2 = 4(2)^2 = 16$

(D) Equation of tangent at 't' is $yt - x - at^2 = 0$

Given tangent equation is $2x + 3y - 1 = 0$

$\Rightarrow \quad t = \frac{-3}{2}$, $a = \frac{-2}{9} \Rightarrow SP = a(1+t^2) = \frac{13}{18}$

6. A → (s); B → (q); C → (p); D → (r)

(A) Any point on the parabola is $(x, x^2 + 7x + 2)$

Its distance from the line $y = 3x - 3$ is given by

$$P = \left|\frac{3x - (x^2 + 7x + 2) - 3}{\sqrt{9+1}}\right|$$

$$= \left|\frac{x^2 + 4x + 5}{\sqrt{10}}\right|$$

$$= \frac{x^2 + 4x + 5}{\sqrt{10}} (\text{as } x^2 + 4x + 5 > 0 \text{ for all } x \in R)$$

$$\frac{dP}{dx} = 0 \Rightarrow x = -2.$$

So, the required point is $(-2, -8)$.

(B) Let (x_1, y_1) be a point on $y = x + 2$

Therefore, $y_1 = x_1 + 2$

Equation of the line perpendicular to the given line through (x_1, y_1) is

$$y - (x_1 + 2) = -(x - x_1), \text{ i.e., } y = -x + 2(x_1 + 1)$$

If this line is a tangent to $y^2 = 8x, c = \frac{a}{m}$ gives

$$2(x_1 + 1) = \frac{2}{-1}, \text{ i.e., } x_1 + 1 = -1 \Rightarrow x_1 = -2$$

Hence, $y_1 = 0$

∴ The required point is $(-2, 0)$.

(C) Given equation of ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$.

Slope of the tangent at any point $P(x_1, y_1)$ to $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is given by $2x + 4y\frac{dy}{dx} = 0$

$\because \frac{dy}{dx} = \frac{-x}{2y} = -1 \Rightarrow x = 2y$

Putting $x = 2y$ in the equation of the ellipse, we have, $y = 1$. Evidently, the point lies in the Ist quadrant.

Therefore, $y = 1$ and $x = 2$

Hence, the required point is $(2, 1)$.

(D) The co-ordinates of the point P are $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$.

Since, $e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$, so, the co-ordinates of the foci are $S(4, 0)$ and $S'(-4, 0)$ and $SS' = 8$.

Also, $SP = a - ex_1 = 5 - \frac{4}{5} \times \frac{5}{2} = 3$

And $S'P = a + ex_1 = 7$

Therefore, the co-ordinates of the incentre (x_1, y_1) are

$$x_1 = \frac{7\times 4 + 3\times -4 + 8\times \frac{5}{2}}{7+3+8} = 2,$$

$$y_1 = \frac{7\times 0 + 3\times 0 + 8\times \frac{3\sqrt{3}}{2}}{7+3+8} = \frac{2}{\sqrt{3}}$$

7. A → (p); B → (s); C → (q); D → (r)

(A) Let $(\sqrt{27}\cos\theta, \sqrt{48}\sin\theta)$ be a point on the ellipse.

Thus, the equation of tangent or their points in

$$\frac{\sqrt{27}\cos\theta x}{27} + \frac{\sqrt{48}\sin\theta y}{48} = 1$$

$\therefore$ Slope $-\frac{\cos\theta}{\sqrt{27}} \times \frac{\sqrt{48}}{\sin\theta} = -\frac{4}{3}\cot\theta = -\frac{4}{3},$

i.e., $\cot\theta = 1$, i.e., $\theta = \frac{\pi}{4}$

$\therefore$ Equation of the tangent is $\frac{x}{\sqrt{54}} + \frac{y}{\sqrt{96}} = 1$

Area of triangle $= \frac{1}{2}.3\sqrt{6} \times 4\sqrt{6} = 36$ sq. units

(B) Product of perpendicular

$$= \left|\frac{3m - \sqrt{25m^2+16}}{\sqrt{1+m^2}}\right| \left|\frac{-3m - \sqrt{25m^2+16}}{\sqrt{1+m^2}}\right|$$

$$= \left|\frac{25m^2 + 16 - 9m^2}{1+m^2}\right| = 16$$

(C) $2a = 5 + 10 = 16$

$$a = \frac{15}{2}$$

$$2ae = \sqrt{(6-3)^2 + (8-4)^2} = 5$$

$$e = \frac{1}{3}$$

$\therefore$ $b^2 = \frac{225}{4}\left(1 - \frac{1}{9}\right) = 50$, $\therefore$ $b = 5\sqrt{2}$

$\therefore$ $2b = 10\sqrt{2}$

(D) $a = 5, b = 4, e = \frac{3}{6}$

$\therefore$ $SA = 2$

Also, $SP = 2$

$\therefore$ P coincides with A and Q coincides with A'.

$\therefore$ $PQ = 2a = 10$

8. A → (s); B → (q); C → (r); D → (p)

Rewrite the equation as

$$(x-2)^2 + (y-3)^2 = \frac{4}{9}\left[\frac{5x+12y+6}{13}\right]^2$$

Focus (2, 3), $e = \frac{2}{3}$

Directrix $5x + 12y + 6 = 0$

Distance between focus and directrix

$$d = \frac{(5)(2)+(12)(3)+6}{\sqrt{5^2+12^2}} = \frac{52}{13} = 4 = \frac{b^2}{ae} \Rightarrow b^2 = \frac{8}{3}a$$

Also, $e = \frac{2}{3} = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow 5a^2 = 9b^2 = 24a \Rightarrow a = \frac{24}{5}; b = \frac{8}{\sqrt{5}}$

(A) Length of major axis $= 2a = \frac{48}{5}$

(B) Length of minor axis $= 2b = \frac{16}{\sqrt{5}}$

(C) Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 64}{5\left(\frac{24}{5}\right)} = \frac{16}{3}$

(D) Distance between the directrices $= \frac{2a}{e} = \frac{72}{5}$

9. A → (s); B → (r); C → (p); D → (q)

(A) Locus must be director circle $x^2 + y^2 = a^2 + b^2$

(B) Foot of perpendicular lie on auxiliary circle $x^2 + y^2 = a^2$

(C) Any tangent to ellipse is $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ any line perpendicular

$$\Rightarrow \quad y = \left(\frac{a}{b}\tan\alpha\right)x$$

$$\tan\alpha = \frac{b}{a}\frac{y}{x}$$

$$\Rightarrow \quad \sin\alpha = \frac{by}{\sqrt{a^2x^2+b^2y^2}} \quad \cos\alpha = \frac{ax}{\sqrt{a^2x^2+b^2y^2}}$$

$$\Rightarrow \quad \frac{x}{a}\frac{ax}{\sqrt{a^2x^2+b^2y^2}} + \frac{y}{b}\frac{by}{\sqrt{a^2x^2+b^2y^2}} = 1$$

$$x^2+y^2 = \sqrt{a^2x^2+b^2y^2}$$

$$\left(x^2+y^2\right)^2 = a^2x^2+b^2y^2$$

(D) Similar proof. Take point as $(2x, 2y)$.

10. A → (q); B → (s); C → (p); D → (r)

Normal at $P(\theta)$

$$xa \sec\theta - yb \operatorname{cosec}\theta = a^2 - b^2$$

$$A(ae^2\cos\theta, 0)\ B\left(0, -\frac{a^2e^2}{b}\sin\theta\right)$$

tangent at $P(\theta)$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$T(a\sec\theta, 0)$

(A) $(CA)(CT) = \left(e^2 a\cos\theta\right)\left(\dfrac{a^2}{a\cos\theta}\right) = a^2 - b^2 = 16$

(B) $(PN)(PB) = a^2 = 25$

(C) $(S_1N_1)(S_2N_2) = b^2 = 9$

(D) $(S_1P)(S_2P) = [a + e(a\cos\theta)][a - e(a\cos\theta)]$

$$= a^2\sin^2\theta + b^2\cos^2\theta = 17$$

11. A → (s); B → (p); C → (q); D → (r)

(A) $x = 10\cos\theta - 6\cos\theta = 4\cos\theta$

$y = 6\sin\theta$

$\therefore$ Locus is $\left(\dfrac{x}{4}\right)^2 + \left(\dfrac{y}{6}\right)^2 = 1$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1$$

$$\therefore \quad e = \sqrt{\frac{36-16}{36}} = \sqrt{\frac{20}{36}} = \frac{\sqrt{5}}{3}$$

$$\therefore \quad 9e = 3\sqrt{5}$$

(B) $3\left(x^2+2x+1\right) + 2\left(y^2-2y+1\right) = 3+2+1$

$$\frac{(x+1)^2}{2} + \frac{(y-1)^2}{3} = 1$$

$$e = \sqrt{\frac{3-2}{3}} = \frac{1}{\sqrt{3}}$$

$$a = \sqrt{3}, b = \sqrt{2}$$

$\therefore$ Area $= \dfrac{1}{2}.2\sqrt{3}.\sqrt{2} = \sqrt{6}$ sq. units

(C) $\dfrac{(x-1)^2}{16} + \dfrac{(y-2)^2}{9} = 1$

$a = 4$ and $e = \dfrac{\sqrt{7}}{16}, \dfrac{\sqrt{7}}{4}$

$\therefore$ $ae = \sqrt{7}$

$\therefore$ Distance between the foci $= 2\sqrt{7}$ units

(D) $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$

$$e = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

One end of the latus rectum is $\left(ae, \dfrac{b^2}{a}\right) = \left(3, \dfrac{7}{4}\right)$

$\therefore$ Equation of the tangent is

$$\frac{3x}{16} + \frac{7}{4}.\frac{y}{7} = 1$$

$$\Rightarrow \quad \frac{3x}{16} + \frac{y}{4} = 1$$

It meets x-axis at $\left(\dfrac{16}{3}, 0\right)$ and y-axis at $(0, 4)$

$\therefore$ Area of rhombus $= 2\times\dfrac{16}{3}\times 4 = \dfrac{128}{3}$ sq. units

12. A → (p); B → (q); C → (p); D → (s)

(A) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \Rightarrow P(3\cos\theta, 2\sin\theta)$

$\Rightarrow \cos\theta$ and $\sin\theta$ can be rational for infinite values of $\theta \in [0, 2\pi]$

(B) Four integral points $(\pm 3, 0), (0, \pm 2)$

(C) Similar to (A)

(D) Two integral points $(0, \pm 1)$

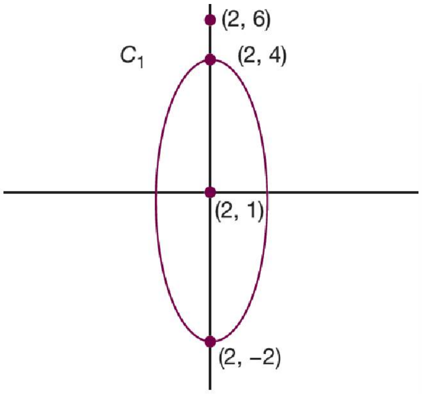
13. A → (q); B → (s); C → (p); D → (r)

$$9x^2 + 8y^2 - 36x - 16y - 28 = 0$$

$$9\left(x^2 - 4x\right) + 8\left(y^2 - 2y\right) = 28$$

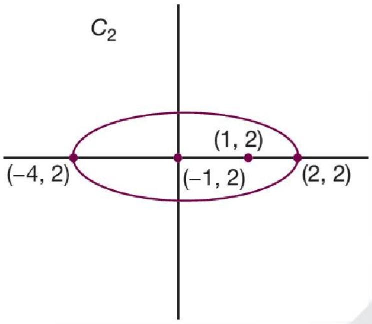
$$9(x-2)^2 + 8(y-1)^2 = 72$$

$$\frac{(x-2)^2}{8} + \frac{(y-1)^2}{9} = 1$$



Other curve $4x^2 + 9y^2 + 8x - 36y + 4 = 0$

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$



(A) Minimum distance of (2, 6) from $C_1 = 2$
(B) Maximum distance of (2, 6) from $C_1 = 8$
(C) Minimum distance of (1, 2) from $C_2 = 1$
(D) Maximum distance of (1, 2) from $C_2 = 5$

Integer/Numerical Answer Type Questions

1. The integral points are $(\pm 3, 0)$ and $(0, \pm 2)$.

Hence, the correct answer is (4).

2. Let m be the slope of the common tangent.

Then, $\pm\sqrt{3}\sqrt{1+m^2} = \pm\sqrt{4m^2+1}$

$\Rightarrow \quad 3 + 3m^2 = 4m^2 + 1$

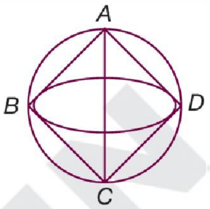
$\Rightarrow \quad m^2 = 2$

$\Rightarrow \quad m = \pm\sqrt{2}$

Hence, the correct answer is (2).

3. Clearly, vertices of the square lie on the director circle of the ellipse, $\frac{x^2}{7} + \frac{2y^2}{11} = 1$

Which is $x^2 + y^2 = 7 + \frac{11}{2}$ or $x^2 + y^2 = \frac{25}{2}$



Clearly, $AC = 2\sqrt{\frac{25}{2}}$

Now, $AB = BC = CD = AD$ and in $\Delta ACD, AC^2 = CD^2 + AD^2$

$\Rightarrow \quad 2AD^2 = \left(2\sqrt{\frac{25}{2}}\right)^2$

$\Rightarrow \quad AD^2 = 25 \Rightarrow AD = 5$ units

Hence, the correct answer is (5).

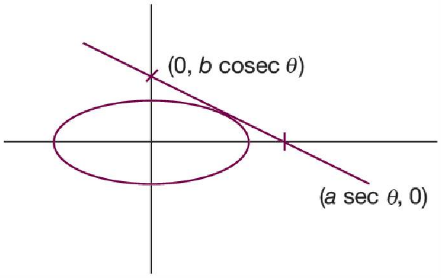
4. For the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, equation of director circle is $x^2 + y^2 = 25$.

The director circle will cut the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ at 4 points

Hence, the number of points = 4

Hence, the correct answer is (4).

5. Tangent $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$



$$\ell^2 = a^2\sec^2\theta + b^2\text{cosec}^2\theta$$

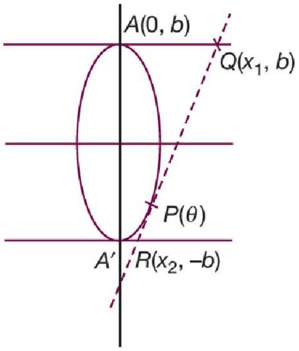
$$= a^2 + b^2 + a^2\tan^2\theta + b^2\cot^2\theta$$

$$= 1 + 4 + 2\left(\frac{\tan^2\theta + 4\cot^2\theta}{2}\right) \geq 1 + 4 + 2(2)$$

$$\ell^2 \geq 9 \quad \text{or} \quad \ell \geq 3$$

Hence, the correct answer is (3).

6. Given $a < b$



Tangent $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Put $y = b, x_1 = \left(\frac{1-\sin\theta}{\cos\theta}\right)a = AQ$

Put $y = -b, x_2 = \left(\frac{1+\sin\theta}{\cos\theta}\right)a = A'R$

$$AQ \cdot A'R = \left(\frac{1-\sin^2\theta}{\cos^2\theta}\right)a^2 = a^2 = 4$$

Hence, the correct answer is (4).

7. Focal distance of any point on ellipse $= a - ae\cos\theta$

$$\Rightarrow \quad e = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$$

So, circle is tangent at that point, which is nearest to focus.
Hence, radius of circle = minimum focal distance of point

$$= (a - ae) = 5 - 5\times\frac{3}{5} = 2 \text{ units}$$

Hence, the correct answer is (2).

8. $a = 20$

$$40b^2 = 2560$$

$$\Rightarrow \quad b^2 = 64 \Rightarrow b = 8$$

$$e = \sqrt{1-\left(\frac{8}{10}\right)^2} \Rightarrow e = \frac{3}{5}$$

$$5e = 3$$

Hence, the correct answer is (3).

9. Equation of tangent is $y = 2x \pm \sqrt{4a^2 + b}$

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

$$\Rightarrow \quad 0 = -4 \pm \sqrt{4a^2 + b^2}$$

$$\Rightarrow \quad 4a^2 + b^2 = 16$$

Using $A.M \geq G.M \Rightarrow \frac{4a^2 + b^2}{2} \geq \sqrt{4a^2b^2}$

$$\Rightarrow \quad ab \leq 4$$

Hence, the correct answer is (4).

10. Given $\frac{x^2}{16} + \frac{y^2}{9} = 1, y^2 = x - 4, (x-6)^2 + y^2 = 4$

Hence, $x = 4$ is indeed the common tangent of the given curve.

Hence, the correct answer is (4).

11. $\Delta PS_1S_2 = \frac{1}{2}(S_1P)(S_2P) = \frac{1}{2}x(17 - x) = 30$

$$\Rightarrow \quad S_1S_2 = 13$$

$$[\sqrt{13}] = 3$$

Hence, the correct answer is (3).

12. $e = \frac{2}{3}$

Equation of tangent at L is $\frac{2x}{9} + \frac{y}{3} = 1$, which meets x-axis at $A\left(\frac{9}{2}, 0\right)$ and y-axis at $B(0, 3)$.

$$\therefore \quad \text{Area} = 4\left(\frac{1}{2}.\frac{9}{2}.3\right) = 27 \text{ sq. units}$$

Hence, the correct answer is (9).

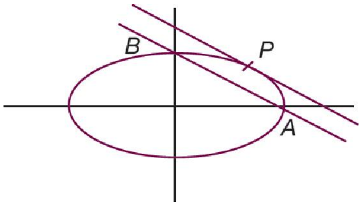
13. The point $(2\alpha, \alpha)$ lies between $x^2 + y^2 = 9 + 3$ and $\frac{x^2}{9} + \frac{y^2}{3} = 1$

$$5\alpha^2 - 12 < 0 \text{ and } \frac{4\alpha^2}{9} + \frac{\alpha^2}{3} - 1 > 0$$

$$\Rightarrow \quad \frac{9}{7} < \alpha^2 < \frac{12}{5}$$

Hence, the correct answer is (2).

14. Let the perpendicular distance of P from the line be h.



$$\frac{1}{2}\times h \times 5 = 6\left(\sqrt{2}-1\right) \qquad \left(\text{As } \Delta PAB = 6\left(\sqrt{2}-1\right)\right)$$

$$\Rightarrow \quad h = \frac{12\left(\sqrt{2}-1\right)}{5}$$

Now, distance of tangent parallel to AB, i.e.,

$4y + 3x = 12\sqrt{2}$ from the line AB is $\frac{12\left(\sqrt{2}-1\right)}{5}$ units.

$\therefore$ There are just three such points.
Hence, the correct answer is (3).

15. $\frac{x}{5}\cos\frac{\alpha+\beta}{2} + \frac{y}{3}\sin\frac{\alpha+\beta}{2} = \cos\left(\frac{\alpha+\beta}{2}\right)$

If it passes through $(-4, 0)$, then $-\frac{4}{5}\cos\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

$$\frac{\cos\frac{\alpha+\beta}{2}}{\left(-\cos\frac{\alpha-\beta}{2}\right)}=+\frac{5}{4}$$

$$\Rightarrow\quad \frac{5+4}{5-4}=\frac{\cos\frac{\alpha+\beta}{2}-\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha+\beta}{2}}$$

$$\Rightarrow\quad 9=\frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}$$

$$\therefore\quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=9$$

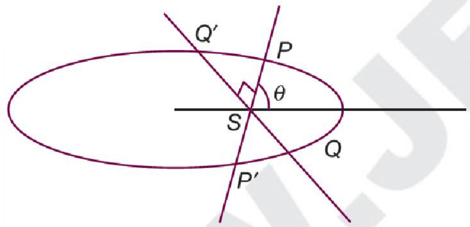
Hence, the correct answer is (9).

16. $P'SP\ \dfrac{x-ae}{\cos\theta}=\dfrac{y-0}{\sin\theta}=r$

$(ae+r\cos\theta, r\sin\theta)$

$$e=\frac{1}{\sqrt{2}};\ a=\sqrt{6};\ b=\sqrt{3}$$

$$\frac{x^2}{6}+\frac{y^2}{3}=1$$



We get $\dfrac{(\sqrt{3}+r\cos\theta)^2}{6}+\dfrac{(r\sin\theta)^2}{3}=1$

Or $r^2\left[\dfrac{\cos^2\theta}{6}+\dfrac{\sin^2\theta}{3}\right]+ar-\dfrac{1}{2}=0$

$$|r_1r_2|=\frac{+1}{2\left(\frac{\cos^2\theta}{6}+\frac{\sin^2\theta}{3}\right)}$$

For $Q'SQ$, $\dfrac{x-ae}{\cos(90^\circ+\theta)}=\dfrac{y-0}{\sin(90^\circ+\theta)}=r$

Any point on $(\sqrt{3}-r\sin\theta, r\cos\theta)$, lies on ellipse

$$\frac{\left(\sqrt{3}-r\sin\theta\right)^2}{6}+\frac{(r\cos\theta)^2}{3}=1$$

$$r^2\left[\frac{\sin^2\theta}{6}+\frac{\cos^2\theta}{3}\right]-r\beta-\frac{1}{2}=0$$

$$|r_3r_4|=\frac{+1}{2\left(\frac{\sin^2\theta}{6}+\frac{\cos^2\theta}{3}\right)}$$

$$\frac{1}{r_1r_2}+\frac{1}{r_3r_4}=2\left[\frac{\cos^2\theta}{6}+\frac{\sin^2\theta}{3}\right]+\left[\frac{\sin^2\theta}{6}+\frac{\cos^2\theta}{3}\right]$$

$$=2\left(\frac{1}{6}+\frac{1}{3}\right)=1$$

Hence, the correct answer is (1).

17. We have $\dfrac{(x-4)^2}{16}+\dfrac{(y-3)^2}{9}=1$

Any point P on the curve is $4(1+\cos\theta),\ 3(1+\sin\theta)$.

The reflection of the point P about the line $x-y-2=0$ can be found using

$$\frac{h-4(1+\cos\theta)}{1}=\frac{k-3(1+\sin\theta)}{-1}$$

$$=\frac{-2\left[4(1+\cos\theta)-3(1+\sin\theta)-2\right]}{2}$$

$$=-(4\cos\theta-3\sin\theta)+1$$

$$\therefore\quad h=4(1+\cos\theta)-4\cos\theta+3\sin\theta+1$$

$$=5+3\sin\theta$$

And $k=3(1+\sin\theta)+4\cos\theta-3\sin\theta-1$

$$=2+4\cos\theta$$

$$\left(\frac{h-5}{3}\right)^2+\left(\frac{k-2}{4}\right)^2=1$$

Equation of image is $16(x-5)^2+9(y-2)^2=144$

$$\Rightarrow\quad 16x^2+9y^2-160x-36y+292=0$$

On comparing with given equation, we get,

$$k_1=-160,\ k_2=292$$

$$\therefore\quad k_1+k_2=132$$

Unit digit $=2$

Hence, the correct answer is (2).

18. $(x-2)^2+y^2=\left(\dfrac{1}{2}\right)^2\dfrac{\left(x+\sqrt{3}y+10\right)^2}{4}$

$$(h,k)=(z,0),\ e=\frac{1}{2}$$

Perpendicular distance from $(2, 0)$ to $x+\sqrt{3}y+10=0$ is $\frac{a}{e}-ae$

$$2a-\frac{a}{2}=6$$

$$\Rightarrow \quad a=4$$

Distance between directrices $=\frac{2a}{e}=16=K$

Hence, the correct answer is (8).

19. The minimum perpendiculars from two foci upon any tangent is b^2.

$$S_1F_1.S_2F_2=16$$

$$AM \geq GM$$

$$\Rightarrow \quad \frac{S_1F_1+S_2F_2}{2} \geq \sqrt{S_1F_1 \times S_2F_2}$$

$$\Rightarrow \quad S_1F_1+S_2F_2 \geq 8$$

Hence, the correct answer is (8).

20. Any point on ellipse at a distance r from origin is $(r\cos\theta, r\sin\theta)$.

$\Rightarrow \quad r^2=\frac{8}{5-3\sin 2\theta}$ is maximum, if $5-3\sin 2\theta$ is minimum

$$\Rightarrow \quad r^2=4$$

r^2m, if $(5-3\sin 2\theta)$ is maximum $=8$

$$\Rightarrow \quad r^2=1$$

$$r_1+r_2=2+1=3$$

Hence, the correct answer is (3).

21. The equation of the tangent at $(5\cos\theta,\ 2\sin\theta)$ is $\frac{x}{5}\cos\theta+\frac{y}{2}\sin\theta=1$

If it is a tangent to the circle, then $\frac{1}{\sqrt{\frac{\cos^2\theta}{25}+\frac{\sin^2\theta}{4}}}=4$

$$\Rightarrow \quad \cos\theta=\frac{10}{4\sqrt{7}},\ \sin\theta=\frac{\sqrt{3}}{2\sqrt{7}}$$

Let A and B be the points, where the tangent meets the co-ordinate axis. Then,

$$A\left(\frac{5}{\cos\theta}, 0\right),\ B\left(0, \frac{2}{\sin\theta}\right)$$

$$L=\sqrt{\frac{25}{\cos^2\theta}+\frac{4}{\sin^2\theta}}=\frac{14}{\sqrt{3}}$$

Hence, the correct answer is (7).

22. Since axes are tangents,

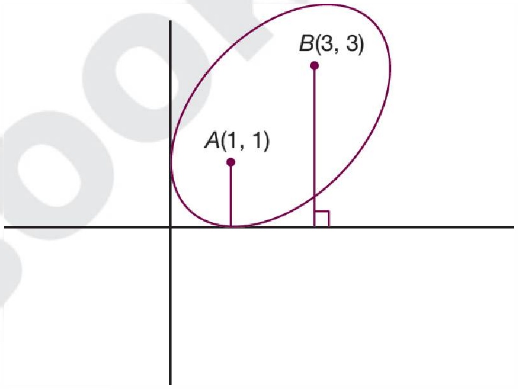
$$b^2=3 \text{ and } ae=\sqrt{2}$$

$$\Rightarrow \quad a^2-b^2=2$$

$$\therefore \quad a^2=5$$

$$r=\sqrt{a^2+b^2}=2\sqrt{2}$$

Area $=8\pi$ sq. units



Hence, the correct answer is (8).

23. Consider any point $(t_1, t+2)$, $t\in R$ on the line $x-y+2=0$.

The chord of contact of ellipse with respect to this point is $x(t)+2y(t+2)-2=0$

$$\Rightarrow \quad (4y-2)+t(x+2y)=0,\ y=\frac{1}{2}, x=-1$$

Hence, the point is $\left(-1, \frac{1}{2}\right)$, where the distance from $\left(2, \frac{1}{2}\right)$ is 3 units.

Hence, the correct answer is (3).

24. $\frac{x^2}{9}+\frac{y^2}{4}=1$, so $P(3\cos\theta, 2\sin\theta)$ and

$$Q\left(3\cos\left(\frac{\pi}{2}+\theta\right), 2\sin\left(\frac{\pi}{2}+\theta\right)\right)$$

$$\text{Area of } \Delta CPQ=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3\cos\theta & 2\sin\theta & 1 \\ -3\sin\theta & 2\cos\theta & 1 \end{vmatrix}=3 \text{ sq. units}$$

Hence, the correct answer is (3).

25. Normal $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$P(a\cos\theta, y\sin\theta)$

$x\, a\sec\theta - y\, b\operatorname{cosec}\theta = a^2 - b^2$

$(5\sec\theta)\, x - (2\operatorname{cosec}\theta)\, y = 21$

$$d = \frac{|0-0-21|}{\sqrt{25\sec^2\theta + 4\operatorname{cosec}^2\theta}} = \frac{21}{\sqrt{29 + 25\tan^2\theta + 4\cot^2\theta}}$$

$$\frac{25\tan^2\theta + 4\cot^2\theta}{2} \geq \sqrt{25\tan^2\theta 4\cot^2\theta}$$

$25\tan^2\theta + 4\cot^2\theta \geq 20$

$$d_{\max} = \frac{21}{\sqrt{29+20}} = \frac{21}{7} = 3$$

Hence, the correct answer is (3).

26. If $y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$C^2 = a^2m^2 + b^2$

$y = 4x + c,\ \frac{x^2}{4} + \frac{y^2}{1} = 1$

$C = \pm\sqrt{65}$

Hence, the correct answer is (2).

27. $\frac{x^2}{27} + \frac{y^2}{1} = 1,\ P\left(3\sqrt{3}\cos\theta, \sin\theta\right)$

$$\frac{3\sqrt{3}\cos\theta}{27} + \frac{\sin\theta y}{1} = 1$$

$$A\left(\frac{3\sqrt{3}\cos\theta}{27}, 0\right), B = \left(0, \frac{1}{\sin\theta}\right)$$

$f(\theta) = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

$$f'(\theta) = \frac{3\sqrt{3}\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta} = 0$$

$$\Rightarrow \tan^3\theta = \frac{1}{3\sqrt{3}} = \left(\frac{1}{\sqrt{3}}\right)^3$$

$$\theta = \frac{\pi}{6}$$

Hence, the correct answer is (6).

28. Normal at $P_i(a\cos\theta_i, b\sin\theta_i)$

$xa\sec\theta_i - yb\operatorname{cosec}\theta_i = a^2 - b^2$

$C(0, 0)\ Q_i\ (a\cos\theta_i, a\sin\theta_i)$

$CQy = \tan\theta_i\, x$

$K_i((a+b)\cos\theta_i, (a+b)\sin\theta_i)$

$CK_i = a + b = 7$

$$\therefore \sum_{i=1}^{n} 7 = 175$$

$\Rightarrow\ 7n = 175$

$\Rightarrow\ n = 25$

Hence, the correct answer is (5).

29. $4\tan\frac{B}{2}\ \tan\frac{C}{2} = 1$

$$\Rightarrow \sqrt{\frac{(s-c)(s-a)(s-a)(s-b)}{s(s-b)s(s-c)}} = \frac{1}{4}$$

$$\Rightarrow \frac{s-a}{s} = \frac{1}{4}$$

$$\Rightarrow \frac{25-a}{a} = \frac{5}{3}$$

$$\Rightarrow b + c = \frac{5}{3} \times 6 = 10$$

$\left(\because a = \overline{BC} = 6\right)$

$\therefore$ Locus of A is $\frac{(x-5)^2}{25} + \frac{y^2}{16} = 1$

Hence, the correct answer is (5).

30. Let P be $(5\cos\theta, 4\sin\theta)$ and Q be $(-5\sin\theta, 4\cos\theta)$

Equation of tangent at $P \Rightarrow \frac{x}{5}\cos\theta + \frac{y}{4}\sin\theta = 1$ …(1)

Equation of tangent at $Q \Rightarrow -\frac{x}{5}\sin\theta + \frac{y}{4}\cos\theta = 1$ …(2)

Solving (1) and (2), we get, $R = \left(5(\cos\theta - \sin\theta), 4(\sin\theta + \cos\theta)\right)$

$\therefore\ m{:}n$ is 1:1

$\Rightarrow\ m + n = 2$

Alternate

Let the points be $P(5, 0)$ and $Q(0, 4)$

$\Rightarrow\ R(5, 4)$

Intersection of CR and PQ is $\left(\frac{5}{2}, 2\right)$, which is the mid-point of CR

$\Rightarrow\ m : n = 1 : 1$

$\Rightarrow\ m + n = 2$

Hence, the correct answer is (2).

31. Smaller ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$ and the larger is $\frac{x^2}{6} + \frac{\lambda y^2}{6} = 1.$

Any tangent to smaller ellipse is $\frac{x}{2}\cos\theta + y\sin\theta = 1$ ...(1)

Let it meet the larger at P and Q. Let tangent at P, Q intersect at $A\ (h,k)$

PQ is chord of contact with respect to A

$$\therefore\ PQ = \frac{hx}{6} + \frac{\lambda yk}{6} = 1 \quad \text{...(2)}$$

Comparing (1) and (2), we get, $h = 3\cos\theta$ $\quad k = \frac{6\sin\theta}{\lambda}$

$$(h,k) \in x^2 + y^2 = \frac{6+6}{\lambda}$$

$$\Rightarrow\ 9\cos^2\theta + \frac{6^2}{\lambda^2}\sin^2\theta = 6 + \frac{6}{\lambda}$$

$$\therefore\ \lambda = 2$$

Hence, the correct answer is (2).

32. $\because\ SP + S'P = 2a = 10$

$$\Rightarrow\ S'P = 7$$

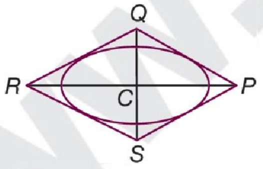
$$\because\ \frac{1}{S'P} + \frac{1}{S'P'} = \frac{2a}{b^2}$$

$$\Rightarrow\ \frac{1}{S'P'} = \frac{10}{9} - \frac{1}{7} = \frac{61}{63}$$

$$\therefore\ 61(S'P') - 56 = 7$$

Hence, the correct answer is (7).

33. One end of latus rectum is $\left(2, \frac{5}{3}\right)$



Equation of tangent at $\left(2, \frac{5}{3}\right)$ is $\frac{2x}{9} + \frac{y}{3} = 1$

Area of quadrilateral $= 4\times(\Delta CPQ) = 4\times\frac{1}{2}\times\frac{9}{2}\times 3$

$= 27 = \lambda$ (Given)

$$\Rightarrow\ \frac{\lambda}{9} = 3$$

Hence, the correct answer is (3).

34. Incentre of $\Delta SPS'$ is $\left(ae\cos\theta, \frac{be\sin\theta}{1+e}\right)$

$\Rightarrow$ Its abscissa $= ae\cos\theta = \sqrt{9-7}\times\frac{1}{\sqrt{2}} = 1$

Hence, the correct answer is (1).

35. $\frac{x^2}{4} + y^2 = 1$ and $x^2 + \frac{y^2}{a^2} = 1$ meet in four distinct points such that, $a > 1$

$$b^2 - 3b - 9 > 1$$

$$b^2 - 3b - 10 > 0$$

$$(b+2)(b-5) > 0$$

So $\quad b \in (-\infty, -2) \cup (5, \infty)$

$\therefore\quad b \notin \{-2, -1, 0, 1, 2, 3, 4, 5\}$

Hence, the correct answer is (8).

36. $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{9} = \frac{5}{9}$

$$\Rightarrow\ e = \frac{\sqrt{5}}{3}$$

$$F_1F_2 = 2\sqrt{5}$$

Also, given, $PF_1 + PF_2 = 6$ and $PF_1 = 2(PF_2)$

$\therefore\ 3PF_2 = 6$

$\Rightarrow\ PF_2 = 2$ and $PF_1 = 4$

Since, $(PF_2)^2 + (PF_1)^2 = (F_1F_2)^2$

$\Rightarrow\ \angle P = 90°$

Area $= \frac{(4)(2)}{2} = 4$ sq. units

Hence, the correct answer is (4).

37. $\Delta PQR = \frac{4\sqrt{3}}{2}\begin{vmatrix}\cos\alpha & \sin\alpha & 1\\ \cos\beta & \sin\beta & 1\\ \cos\gamma & \sin\gamma & 1\end{vmatrix} \le 9$ sq.unit

Hence, the correct answer is (9).

38. The director circle of $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is $x^2 + y^2 = 3$

Let $P\left(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta\right)$

Chord of contact is $x\sqrt{3}\cos\theta + 2y\sqrt{3}\sin\theta - 2 = 0$

It touches $x^2 + y^2 = r^2$

$$\Rightarrow\ r = \frac{2}{\sqrt{3\cos^2\theta + 12\sin^2\theta}} = \frac{2}{\sqrt{3 + 9\sin^2\theta}}$$

$$r_{\max} = \frac{2}{\sqrt{3}},\ r_{\min} = \frac{2}{\sqrt{12}}$$

So, $\frac{A_{\max}}{A_{\min}} = 4$

Hence, the correct answer is (4).

39. Maximum area of $\Delta SPS' = abe$ sq. units

Hence, the correct answer is (6).

40. Let F_1 and F_2 be the foci of the ellipse and F_1 be the centre of the circle C, which touches the ellipse at P. Now, consider a ray emanating from F_1 and bouncing off at P on the tangent L_1 (say). Now, as L_1 is tangent to the circle, it must rectum to the centre F_1. Also, L_1 is a tangent to ellipse.

$\Rightarrow$ It must pass through F_2

$\Rightarrow$ PF_1F_2 are collinear.

$\Rightarrow$ Radius $= b - be = b - \sqrt{b^2 - a^2} = 5 - 3 = 2$ units

Hence, the correct answer is (2).

41. Distance between the foci $2ac = 4\sqrt{2}$ units

$$ac = 2\sqrt{2}$$

Product of distances of foci from tangent $= b^2$

$$2 \times 6 = b^2$$
$$b^2 = a^2 - a^2e^2$$
$$12 = a^2 - 8$$
$$a^2 = 20$$
$$\Rightarrow \quad a = \sqrt{20}$$

Major axis $2a = \sqrt{80}$

$$[2a] = 8$$

Hence, the correct answer is (8).

ARCHIVE: JEE MAIN

1. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; a > b;$

$$2b = \frac{4}{\sqrt{3}} \Rightarrow b = \frac{2}{\sqrt{3}} \Rightarrow b^2 = \frac{4}{3}$$

tangent $y = \frac{-x}{6} + \frac{4}{3}$ compare with $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$\Rightarrow \quad m = \frac{-1}{6} \text{ and } \sqrt{\frac{a^2}{36} + \frac{4}{3}} = \frac{4}{3} \Rightarrow a = 4;$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{2}\sqrt{\frac{11}{3}}$$

Hence, the correct answer is (B).

2. $3x + 4y = 12\sqrt{12}$ is tangent to $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ then we get $c^2 = m^2a^2 + b^2$

$$\Rightarrow \quad 18 = 9/16a^2 + 9 \Rightarrow a^2 = 16$$

$$e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

Distance between foci $= 2ae = 2\sqrt{7}$

Hence, the correct answer is (B).

3. Given $2ae = 6 \Rightarrow ae = 3$ ....(1)

and $\frac{2a}{e} = 12 \Rightarrow a = 6e$ ....(2)

From (1) and (2) we get $6e^2 = 3 \Rightarrow e = \frac{1}{\sqrt{2}}$ and $a = 3\sqrt{2}$

Now, $b^2 = a^2(1 - e^2) \Rightarrow b^2 = 18\left(1 - \frac{1}{2}\right) = 9$

Length of L.R $= \frac{2(9)}{3\sqrt{2}} = 3\sqrt{2}$

Hence, the correct answer is (C).

4. Any normal to the ellipse is $\frac{x\sec\theta}{\sqrt{2}} - y\,\text{cosec}\theta = -\frac{1}{2}$

$$\Rightarrow \quad \frac{x}{\left(\frac{-\cos\theta}{\sqrt{2}}\right)} + \frac{y}{\left(\frac{\sin\theta}{2}\right)} = 1$$

$$\Rightarrow \quad \frac{\cos\theta}{\sqrt{2}} = \frac{1}{3\sqrt{2}} \text{ and } \frac{\sin\theta}{2} = \beta$$

$$\Rightarrow \quad \beta = \frac{\sqrt{2}}{3}$$

Hence, the correct answer is (D).

5. Ellipse $4x^2 + y^2 = 8$

Equation of tangent at (1, 2) is $2x + y = 4$

Equation of tangent at (a, b) is $4ax + by = 8$

Tangent at (1, 2) and (a, b) are perpendicular

$$-2\left(-\frac{4a}{b}\right) = -1$$

Or $b = -8a$ ...(1)

(a, b) lies on ellipse

$4a^2 + b^2 = 8$ (From Equation (1))

$$4a^2 + 64a^2 = 8$$

$$a^2 = \frac{8}{68} = \frac{2}{17}$$

Hence, the correct answer is (A).

6. Given that, $2b - 2a = 10$

$b - a = 5$ ...(1)

Given that, $2be = 10\sqrt{3}$ or $be = 5\sqrt{3}$

$$b^2e^2 = 75 \text{ or } (b^2 - a^2) = 75$$

$$5(b + a)(b + a) = 75$$

$$b + a = 15 \qquad \ldots(2)$$

From Equation (1) and Equation (2), we get,

$$b = 10$$

$$a = 5$$

$$\therefore \quad \text{Length of latus rectum} = \frac{2a^2}{b} = \frac{2 \times 25}{10} = 5 \text{ units}$$

Hence, the correct answer is (A).

7. Equation of tangent to the parabola $y^2 = x$ ($\therefore b^2 = \alpha$) at (α, β) is $T = 0$

$$y\beta = \frac{x + \alpha}{2}$$

$$\Rightarrow \quad y\beta = \frac{x + \beta^2}{2}$$

$$\Rightarrow \quad y = \frac{1}{2\beta}x + \frac{\beta}{2}$$

$$\left(m = \frac{2}{2\beta},\ c = \frac{\beta}{2}\right)$$

This is also a tangent to the ellipse $x^2 + 2y^2 = 1$

$$\therefore \quad C = \pm\sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad \frac{\beta}{2} = \pm\sqrt{\frac{1}{4\beta^2} + \frac{1}{2}}$$

$$\Rightarrow \quad \frac{\beta^2}{4} = \frac{1}{4\beta^2} + \frac{1}{2}$$

$$\Rightarrow \quad \beta^4 - 2b^2 - 1 = 0$$

$$\Rightarrow \quad (\beta - 1)^2 = 2$$

$$\Rightarrow \quad b^2 - 1 = \sqrt{2} \qquad (\because \beta > 0)$$

$$\Rightarrow \quad b^2 = \sqrt{2} + 1$$

$$\alpha = b^2 = \sqrt{2} + 1$$

Hence, the correct answer is (A).

8. Tangent at $\left(3, \frac{-9}{2}\right)$

$$\frac{3x}{a^2} - \frac{9y}{2b^2} = 1$$

Comparing with $x - 2y = 12$, we get,

$$\frac{3}{a^2} = \frac{9}{4b^2} = \frac{1}{12}$$

$$\Rightarrow \quad a = 6 \text{ and } b = 3\sqrt{3}$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = 9 \text{ units}$$

Hence, the correct answer is (D).

9. $3x^2 + 5y^2 = 32$

tangent $6x + 10y = 32$

$$\text{Slope of tangent} = -\frac{3}{5}$$

$$\text{Tangent } y - 2 = -\frac{3}{5}(x - 2) \Rightarrow Q\left(\frac{16}{3}, 0\right)$$

$$\text{Normal } y - 2 = \frac{5}{3}(x - 2) \Rightarrow R\left(\frac{4}{5}, 0\right)$$

Now, $QR = \frac{68}{15}$

$$\text{Area} = \frac{1}{2}(QR) \times 2 = QR = \frac{68}{15} \text{ sq. units}$$

Hence, the correct answer is (C).

10. Equation of tangent $\frac{x}{a\sec\theta} + \frac{y}{b\operatorname{cosec}\theta} = 1$

$$a = \sqrt{2},\ b = 1$$

$$\frac{x}{\sqrt{2}\sec\theta} + \frac{y}{\operatorname{cosec}\theta} = 1$$

Midpoint (h, k)

$$h = \frac{\sqrt{2}\sec\theta}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}h}$$

Similarly, $\sin\theta = \frac{1}{2k}$

Square and add, to get, $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

Hence, the correct answer is (B).

11. Length of latus rectum $\frac{2b^2}{a} = 8$ units

$$\Rightarrow \quad b^2 = 4a$$

$$2ae = 2b \text{ Or } b = ae$$

$$b^2 = a^2(1 - e^2) = a^2 - a^2e^2 = a^2 - b^2$$

$$2b^2 = a^2 \text{ Or } 2 \times 4a = a^2$$

We get $a = 8$, $b^2 = 32$

Required ellipse is $\frac{x^2}{64} + \frac{y^2}{32} = 1$ and the point $(4\sqrt{3}, 2\sqrt{2})$ lies on it.

Hence, the correct answer is (C).

12. Ellipse: $\frac{x^2}{4}+\frac{y^2}{9}=1$

Normal at P: $(2\cos\theta, \sqrt{3}\sin\theta)$ is

$$2x\sec\theta - \sqrt{3}\,\text{cosec}\,\theta\, y = 1$$

Normal at P: $2x\sec\theta - \sqrt{3}\,y\,\text{cosec}\,\theta = 1$

Normal parallel to $2x + y = 4$

Slope of normal $\frac{2}{\sqrt{3}}\tan\theta = -2$, we get, $\tan\theta = -\sqrt{3}$

or $\theta = \frac{2\pi}{3}$

Point $P\left(-1, \frac{3}{2}\right)$, $Q(4, 4)$

$$PQ = \sqrt{(4+1)^2 + \left(4-\frac{3}{2}\right)^2} = \sqrt{\frac{125}{4}} = \frac{5\sqrt{5}}{2}$$

Hence, the correct answer is (D).

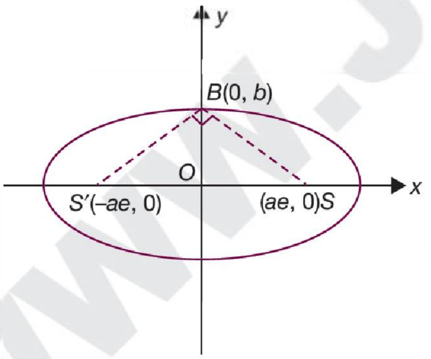
13. $\because\ \angle SBS' = 90^\circ \quad (\therefore\ \angle BS{¢}O = 45^\circ)$

$$\tan 45^\circ = \frac{b}{ae} = 1 \quad \text{or} \quad b = ae$$

$\therefore\quad b^2 = a^2 - a^2e^2$

Or $\quad a^2e^2 = a^2 - a^2e^2$

$$e = \frac{1}{\sqrt{2}}$$



Now, area $\Delta SS'B = \frac{1}{2}\times 2ae.\,b = abe = 8$

$$a = 4,\ b = 2\sqrt{2}$$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 8}{4} = 4$ units

Hence, the correct answer is (C).

14. Given, $2a = 4$ and $2be = 4$

$\Rightarrow\quad a = 2,\ be = 2$

$\Rightarrow\quad b^2e^2 = 4$

$\Rightarrow\quad b^2 - a^2 = 4$

$\Rightarrow\quad b^2 = 8$

Equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{8}=1$

Clearly, option (C) satisfies the given curve.

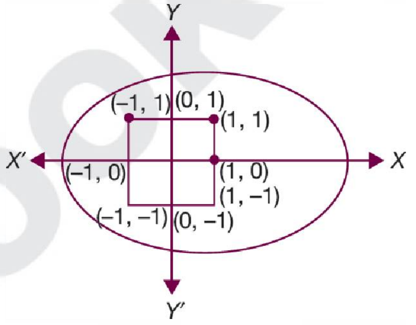
Hence, the correct answer is (C).

15. $A = \{(a, b) \in R\times R\ |a-5| < 1,\ |b-5| < 1\}$

Let $a - 5, = x$, $b - 5 = y$

Set A contains all points inside $|x| < 1$, $|y| < 1$

$B = \{(a, b) \in R\times R\text{: } 4(a-6)^2 + 9(b-5)^2 < 36\}$



Set B contains all points inside or on $\frac{(x-1)^2}{9}+\frac{y^2}{4}=1$

$\therefore$ $(\pm 1, \pm 1)$ lies inside the ellipse

Hence, $A \subset B$

Hence, the correct answer is (A).

16. Eccentricity of ellipse $= \frac{1}{2}$

Now, $-\frac{a}{e} = -4 \Rightarrow a = 4\times\frac{1}{2} = 2 \Rightarrow a = 2$

We have, $b^2 = a^2(1-e^2) = a^2\left(1-\frac{1}{4}\right) = 4\times\frac{3}{4} = 3$

$\therefore\quad$ Equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{3}=1$

Tangent at $\left(1, \frac{3}{2}\right)$

$$\frac{x}{4}+\frac{y}{2}=1 \qquad m_T = -\frac{1}{2}$$

Slope of normal $= 2$

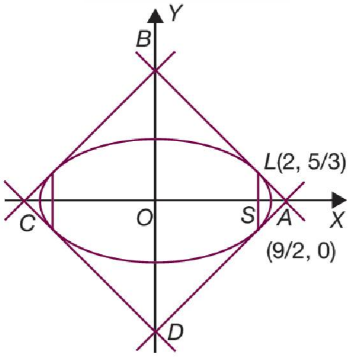
$\Rightarrow\quad$ Equation of normal at $\left(1, \frac{3}{2}\right)$ is $y - \frac{3}{2} = 2(x-1)$

$\therefore\quad 4x - 2y = 1$

Hence, the correct answer is (C).

17. The end point of latus rectum of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in Ist quadrant is $\left(ae, \frac{b^2}{a}\right)$ and the tangent at this point intersects x-axis at $\left(\frac{a}{e}, 0\right)$ and y-axis at $(0, a)$.

The given ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$



Then $a^2 = 9, b^2 = 5$

$$\Rightarrow \quad e=\sqrt{1-\frac{5}{9}}=\frac{2}{3}$$

$\therefore$ End point of latus rectum in Ist quadrant is

$$L\left(2, \frac{5}{3}\right)$$

Equation of tangent at L is $\frac{2x}{9}+\frac{y}{3}=1$

It meets x-axis at $A\left(\frac{9}{2}, 0\right)$ and y-axis at $B(0, 3)$

$\therefore$ Area of $\Delta OAB = \frac{1}{2}\times\frac{9}{2}\times 3=\frac{27}{4}$ sq. units

By symmetry, area of quadrilateral $= 4 \times$ (Area ΔOAB)

$= 4\times\frac{27}{4} = 27$ sq. units

Hence, the correct answer is (B).

18. Given equation of ellipse can be written as $\frac{x^2}{6}+\frac{y^2}{2}=1 \Rightarrow a^2=6, b^2=2$

Now, equation of any variable tangent is

$$y = mx \pm \sqrt{a^2m^2+b^2} \qquad \ldots(1)$$

Where, m is slope of the tangent

So, equation of perpendicular line drawn from centre to tangent is

$$y=\frac{-x}{m} \qquad \ldots(2)$$

Eliminating m, we get,

$$(x^4+y^4+2x^2y^2)=a^2x^2+b^2y^2$$

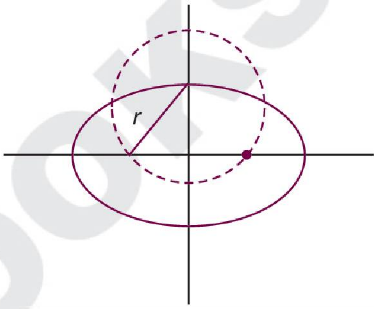
$$\Rightarrow \quad (x^2+y^2)^2=a^2x^2+b^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2=6x^2+2y^2$$

Hence, the correct answer is (C).

19. From the given equation of ellipse, we have

$$a=4, b=3, e=\sqrt{1-\frac{9}{16}} \Rightarrow e=\frac{\sqrt{7}}{4}$$



Now, radius of this circle $= a^2 = 16$

$\Rightarrow$ Foci $= (\pm\sqrt{7}, 0)$

Now, equation of circle is $(x-0)^2+(y-3)^2=16$

$\therefore \quad x^2+y^2-6y-7=0$

Hence, the correct answer is (A).

20. Equation of circle is $(x-1)^2 = y^2 = 1$

$\Rightarrow$ Radius = 1 and diameter = 2

$\therefore$ Length of semi-minor axis is 2 units.

Equation of circle is $x^2+(y-2)^2=4=(2)^2$

$\Rightarrow$ Radius = 2 and diameter = 4

$\therefore$ Length of semi major axis is 4

We know that, equation of ellipse is given by

$$\frac{x^2}{(\text{Major axis}^2)}+\frac{y^2}{(\text{Minor axis}^2)}=1$$

$$\Rightarrow \quad \frac{x^2}{(4)^2}+\frac{y^2}{(2)^2}=1 \quad \Rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow x^2+4y^2=16$$

Hence, the correct answer is (A).

21. Given equation of ellipse is $2x^2+y^2=4$

$$\Rightarrow \quad \frac{2x^2}{4}+\frac{y^2}{4}=1 \quad \Rightarrow \frac{x^2}{2}+\frac{y^2}{4}=1$$

Equation of tangent to the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$ is

$$y = mx \pm\sqrt{2m^2+4} \qquad \ldots(1)$$

(∵ Equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = mx + c$ where $c = \pm\sqrt{a^2m^2+b^2}$)

Now, equation of tangent to the parabola $y^2 = 16\sqrt{3}x$ is

$$y = mx + \frac{4\sqrt{3}}{m} \quad \text{...(2)}$$

(∵ Equation of tangent to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$)

On comparing (A) and (B), we get,

$$\frac{4\sqrt{3}}{m} = \pm\sqrt{2m^2+4}$$

Squaring on both the sides, we get,

$$16(3) = (2m^2+4)m^2$$
$$\Rightarrow \quad 48 = m^2(2m^2+4)$$
$$\Rightarrow \quad 2m^4 + 4m^2 - 48 = 0$$
$$\Rightarrow \quad m^4 + 2m^2 - 24 = 0$$
$$\Rightarrow \quad (m^2+6)(m^2-4) = 0$$
$$\Rightarrow \quad m^4 = 4 \ (\because \ m^2 \neq -6)$$
$$\Rightarrow \quad m = \pm\ 2$$

⇒ Equation of common tangents are $y = \pm 2x \pm 2\sqrt{3}$

Thus, Statement-1 is true.

Statement-2 is obviously true.

Hence, the correct answer is (A).

22. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

It passes through (–3, 1) so $\frac{1}{a^2}+\frac{1}{b^2}=1$ ...(1)

Also, $b^2 = a^2\left(1-\frac{2}{5}\right)$

$$\Rightarrow \quad 5b^2 = 3a^2 \quad \text{...(2)}$$

Solving (1) and (2), we get, $a^2 = \frac{32}{3}, b^2 = \frac{32}{5}$

So, the equation of the ellipse is $3x^2 + 5y^2 = 32$

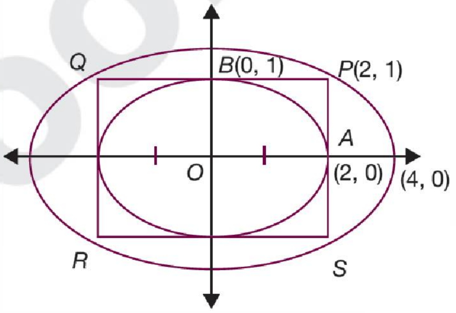
Hence, the correct answer is (C).

23. The given ellipse is $\frac{x^2}{4}+\frac{y^2}{1}=1$

So, $A = (2, 0)$ and $B = (0, 1)$

If $PQRS$ is the rectangle in which it is inscribed, then $P = (2, 1)$.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse circumscribing the rectangle $PQRS$. Then it passes through P (2, 1).



$$\therefore \quad \frac{4}{a^2}+\frac{1}{b^2}=1 \quad \text{...(1)}$$

Also, given that, it passes through (4, 0)

$$\therefore \quad \frac{16}{a^2}+0=1 \Rightarrow a^2=16$$

$$\Rightarrow \quad b^2 = \frac{4}{3} \ \text{(Substituting } a^2=16 \text{ in Equation [1])}$$

∴ The required ellipse is $\frac{x^2}{16}+\frac{y^2}{4/3}=1$ or $x^2+12y^2=16$

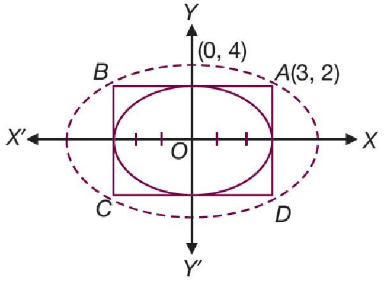
Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. As rectangle $ABCD$ circumscribed the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

∴ $A(3, 2)$



Let the ellipse circumscribing the rectangle $ABCD$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Given that it passes through $(a, 4)$

$\therefore \quad b^2 = 16$

Also it passes through $A(3, 2)$

$$\therefore \quad \frac{9}{a^2}+\frac{4}{16}=1 \Rightarrow a^2=12$$

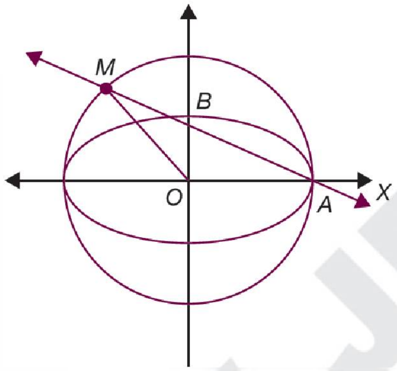
$$\therefore \quad e=\sqrt{1-\frac{12}{16}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$$

Hence, the correct answer is (C).

2. The given ellipse is $x^2+9x^2=9$ or $\dfrac{x^2}{3^2}+\dfrac{y^2}{1^2}=1$

So, that $A(3, 0)$ and $B(0,1)$

$\therefore \quad$ Equation of AB is $\dfrac{x}{3}+\dfrac{y}{1}=1$



Or $\quad x + 3y - 3 = 0 \quad$ ...(1)

Also, auxiliary circle of given ellipse is

$x^2 + y^2 = 9 \quad$ ...(2)

Solving equation (1) and (2), we get the point M, where line AB meets the auxiliary circle.

Putting $x = 3 - 3y$ from Equation (1) in Equation (2), we get, $(3 - 3y)^2 + y^2 = 9$

$\Rightarrow \quad 9 - 18y + 9y^2 + y^2 = 9$

$\Rightarrow \quad 10y^2 - 18y = 0$

$\Rightarrow \quad y = 0, \dfrac{9}{5}$

$\Rightarrow \quad x = 3, \dfrac{-12}{5}$

Clearly, $M\left(\dfrac{-12}{5}, \dfrac{9}{5}\right)$

$$\therefore \quad \text{Area of } \Delta OAM = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ \frac{-12}{5} & \frac{9}{5} & 1 \end{vmatrix} = \frac{27}{10} \text{ sq. units}$$

Hence, the correct answer is (D).

3. The given ellipse is $\dfrac{x^2}{4^2}+\dfrac{y^2}{2^2}=1$

Such that $a^2 = 6$ and $b^2 = 4$

$$\therefore \quad e^2 = 1-\frac{4}{16}=\frac{3}{4} \Rightarrow e=\frac{\sqrt{3}}{2}$$

Let $P(4\cos\theta, 2\sin\theta)$ be any point on the ellipse, then equation of normal at P is

$$4x \sin\theta - 2y \cos\theta = 12 \sin\theta \cos\theta$$

$$\Rightarrow \quad \frac{x}{3\cos\theta}-\frac{y}{6\cos\theta}=1$$

$\therefore$ Q is the point, where normal at P meets x-axis and has co-ordinates $(3 \cos \theta, \theta)$.

$\therefore$ Midpoint of PQ is $M\left(\dfrac{7\cos\theta}{2}, \sin\theta\right)$

For locus of point M, we consider, $x=\dfrac{7\cos\theta}{2}$ and $y = \sin \theta$

$$\Rightarrow \quad \cos\theta = \frac{2x}{7} \text{ and } \sin\theta = y$$

$$\Rightarrow \quad \frac{4x^2}{49}+y^2=1 \qquad ...(1)$$

Also, the latus rectum of given ellipse is

$$x = \pm\, ae = \pm\, 4\times\frac{\sqrt{3}}{2}=\pm\, 2\sqrt{3} \qquad ...(2)$$

Solving Equations (1) and (2), we get,

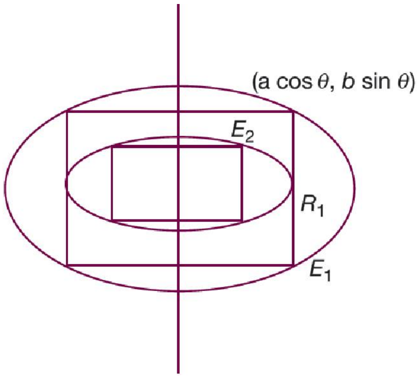
$$\frac{4\times 12}{49}+y^2=1 \Rightarrow y^2=\frac{1}{49} \text{ or } y=\frac{1}{7}$$

$\therefore \quad$ The required points are $\left(\pm\, 2\sqrt{3}, \pm\dfrac{1}{7}\right)$

Hence, the correct answer is (C).

Multiple Correct Choice Type Problems

1. Area of rectangle $= 2a\cos\theta \,.\, 2b\sin\theta = 2ab\sin 2\theta$ sq. units



For maximum area, $\theta = 45°$

$\therefore$ Length $= \frac{a}{\sqrt{2}}$, Breadth $= \frac{b}{\sqrt{2}}$

Equation of ellipse E_2 $\frac{x^2}{\frac{a^2}{2}} + \frac{y^2}{\frac{b^2}{2}} = 1$

Again max area of R_2

Length $= \frac{a/\sqrt{2}}{\sqrt{2}} = \frac{a}{2}$

Breadth $= \frac{b/\sqrt{2}}{\sqrt{2}} = \frac{b}{2}$

For ellipse $E_n = \frac{x^2}{\left(\frac{a}{(\sqrt{2})^{n-1}}\right)^2} + \frac{y^2}{\left(\frac{b}{(\sqrt{2})^{n-1}}\right)^2} = 1$ and

rectangle $R_h{:}A_n = \frac{ab}{2^{n-2}}$

Area $R_1 = 2ab$; Area $R_2 = ab$; Area $R_3 = \frac{a-b}{2}$

$$\sum_{n=1}^{m} \text{Area of rectangle } (R_n) = \frac{2ab(1-(1/2)^n)}{1-1/2}$$

$$= 4ab\left(1-\frac{1}{2^n}\right) = 24 - \frac{24}{2^n} < 24$$

Equation $\frac{x^2}{\left(\frac{a}{(\sqrt{2})^8}\right)^2} + \frac{y^2}{\left(\frac{b}{(\sqrt{2})^8}\right)^2} = 1$

$\Rightarrow$ Latus rectum $= \frac{\frac{2b^2}{2^8}}{\frac{a}{2^4}} = \frac{1}{8}\frac{b^2}{a} = \frac{1}{8}.\frac{4}{3} = \frac{1}{6}$

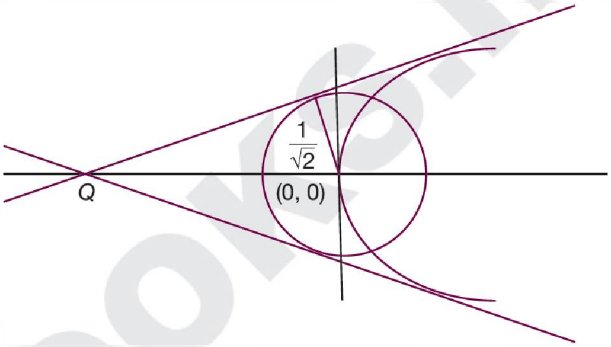
Distance between focus and centre of

$$E_9 = \frac{a}{(2)^4} e_9 = \frac{a}{16} e_1 = \frac{a}{16}\sqrt{\left(1-\frac{b^2}{a^2}\right)} = \frac{3}{16}.\sqrt{\left(1-\frac{4}{9}\right)} = \frac{\sqrt{5}}{16}$$

Clearly, $e_1 = e_2 = e_3$…………

Hence, (A) and (D) are correct.

2.



Let the equation of common tangent is $y = mx + \frac{1}{m}$

$$\therefore \left|\frac{0+1+\frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$

$\therefore$ Equation of common tangents are

$y = x + 1$ and $y = -x - 1$

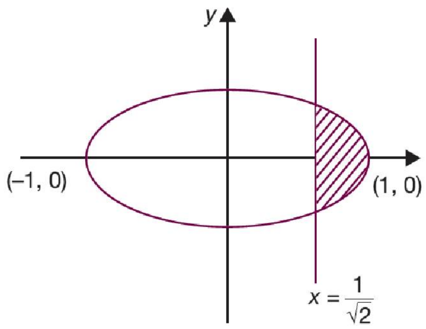
$\therefore \quad Q(-1, 0)$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

And latus rectum $= \frac{2b^2}{a} = \frac{2\left(\frac{1}{\sqrt{2}}\right)^2}{1} = 1$

(C)



$\therefore$ Required area $= 2\cdot\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{\sqrt{2}}\cdot\sqrt{1-x^2}dx$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{1/\sqrt{2}}^{1}$$

$$=\sqrt{2}\frac{\pi}{4}-\left(\frac{1}{4}+\frac{\pi}{8}\right)=\sqrt{2}\left(\frac{\pi}{8}-\frac{1}{4}\right)$$

$$=\frac{\pi-2}{4\sqrt{2}}\text{ sq. units}$$

Hence, (A) and (C) are correct.

3. Let $E_1:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$

and $E_2:\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$, where $c<d$

Also, S: $x^2+(y-1)^2=2$

Tangent at $P(x_1, y_1)$ to S is $x+y=3$

To find point of contact put $x=3-y$ in S and get $P(1, 2)$.

Writing equation of tangent in parametric form, we get,

$$\frac{x-1}{\frac{-1}{\sqrt{2}}}=\frac{y-2}{\frac{1}{\sqrt{2}}}=\pm\frac{2\sqrt{2}}{3}$$

$$\Rightarrow\quad x=\frac{1}{3}\text{ or }\frac{5}{3}\text{ and }y=\frac{8}{3}+2\text{ or }\frac{-2}{3}+2$$

$$\therefore\quad Q\left(\frac{5}{3},\frac{4}{3}\right)\text{and }R\left(\frac{1}{2},\frac{8}{3}\right)$$

Equation of tangent to E_1 at Q is

$\frac{5x}{3a^4}+\frac{4y}{3b^2}=1$, which is identical to $\frac{x}{3}+\frac{y}{3}=1$

$$\Rightarrow\quad a^2=5\text{ and }b^2=4\Rightarrow e_1^2=1-\frac{4}{5}=\frac{1}{5}$$

Equation of tangent to E_2 at R is

$$\frac{x}{3c^2}+\frac{8y}{3d^2}=1$$

Equation $\frac{x}{3}+\frac{y}{3}=1$

$$\therefore\quad e_1^2+e_2^2=\frac{43}{40}, e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}, \left|e_1^2-e_2^2\right|=\frac{27}{40}$$

Hence, (A) and (B) are correct.

4. In ΔABC, given that

$$\cos B+\cos C=4\sin^2\frac{A}{2}$$

$$\Rightarrow\quad 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}-4\sin^2\frac{A}{B}=0$$

$$\Rightarrow\quad 2\sin\frac{A}{2}\left[\cos\frac{B-C}{2}-2\sin\frac{A}{B}\right]=0$$

$$\Rightarrow\quad \sin\frac{A}{2}=0\text{ or }\cos\frac{B-C}{2}-2\cos\frac{B+C}{2}=0$$

However, in a triangle, sin $\frac{A}{2}\neq 0$

$$\therefore\quad \cos\frac{B-C}{2}-2\cos\frac{B+C}{2}=0\Rightarrow\frac{\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}=\frac{1}{2}$$

Applying componendo and dividendo, we get,

$$\Rightarrow\quad \frac{\cos\left(\frac{B+C}{2}\right)+\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)-\cos\left(\frac{B-C}{2}\right)}=\frac{1+2}{1-2}=-3$$

$$\Rightarrow\quad \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{-\sin\frac{B}{2}\sin\frac{C}{2}}=-3\Rightarrow\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{3}$$

$$\Rightarrow\quad \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\frac{1}{3}$$

$$\Rightarrow\quad \frac{s-a}{s}=\frac{1}{3}\text{ or }2s=3a$$

$\Rightarrow$ $a+b+c=3a$

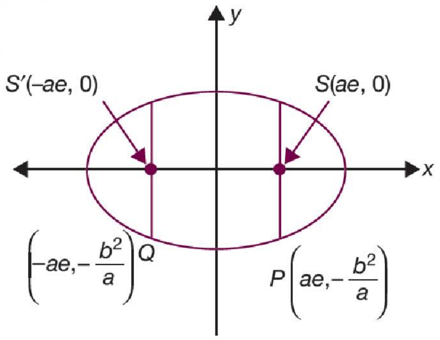
Or $b+c=2a$

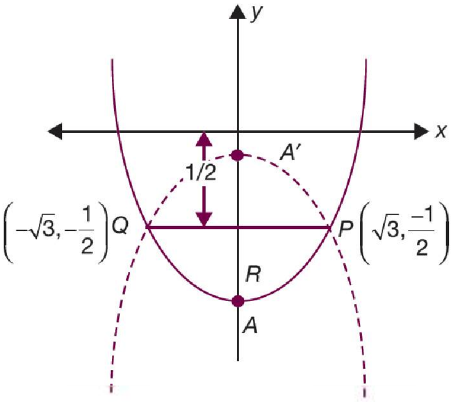
That is, $AC+AB=$ Constant (∵ Base $BC=a$ is given to be constant)

$\Rightarrow$ AB moves on an ellipse.

Hence, (B) and (C) are correct.

5.


y
S′(–ae, 0)
S(ae, 0)
x
$\left(-ae,-\frac{b^2}{a}\right)$ Q
P $\left(ae,-\frac{b^2}{a}\right)$


y
x
A′
1/2
$\left(-\sqrt{3},-\frac{1}{2}\right)$Q
P$\left(\sqrt{3},\frac{-1}{2}\right)$
R
A

Given ellipse is $x^2 + 4y^2 = 4$

Or, $\frac{x^2}{2^2}+\frac{y^2}{1}=1$

$\Rightarrow \quad a = 2, b = 1$

$\therefore \; e=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2} \quad \therefore \; ae=\sqrt{3}$

As per question, $P(ae, -b^2/a)=\left(\sqrt{3},-\frac{1}{2}\right)$

$$Q(-ae, -b^2/a)=\left(-\sqrt{3},-\frac{1}{2}\right)$$

$\therefore \quad PQ = 2\sqrt{3}$

Now, if PQ is the length of latus rectum to be found, then $PQ = 4a = 2\sqrt{3} \Rightarrow a=\frac{\sqrt{3}}{2}$

Also, as PQ is horizontal, parabola with PQ as latus rectum can be upward parabola (with vertex at A) or down ward parabola (with vertex at A').

For upward parabola, $AR=a=\frac{\sqrt{3}}{2}$

$\therefore \quad$ Co-ordinates of $A\left(0,-\left(\frac{\sqrt{3}+1}{2}\right)\right)$

So, equation of upward parabola is given by

$$x^2=2\sqrt{3}\left(y+\frac{\sqrt{3}+1}{2}\right) \text{or } x^2-2\sqrt{3}y=3+\sqrt{3} \quad ...(1)$$

For downward parabola, $A'R=a=\frac{\sqrt{3}}{2}$

$\therefore \quad$ Co-ordinates of $A'\left(0,-\left(\frac{1-\sqrt{3}}{2}\right)\right)$

So, equation of upward parabola is given by

$$x^2=-2\sqrt{3}\left(y+\frac{1-\sqrt{3}}{2}\right) \text{or } x^2+2\sqrt{3}y=3-\sqrt{3} \quad ...(2)$$

$\therefore$ The equations of required parabola is given by the Equations (1) or (2).

Hence, (B) and (C) are correct.

Linked Comprehension Type Questions

1. For ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1, e=\sqrt{1-\frac{8}{9}}=\frac{1}{3}$

$\therefore \quad F_1(-1, 0)$ and $F_2(1, 0)$

Parabola with vertex at (0, 0) and focus art $F_2(1, 0)$ is $y^2 = 4x$.

Intersection points of ellipse and parabola are $M\left(\frac{3}{2},\sqrt{6}\right)$ and $N\left(-\frac{3}{2},\sqrt{6}\right)$

For orthocentre of ΔF_1MN, clearly one altitude is x-axis, i.e., $y = 0$ and altitude from M to F_1N is

$$y-\sqrt{6}=\frac{5}{2\sqrt{6}}\left(x\frac{3}{2}\right)$$

Putting $y = 0$ in above equation, we get,

$$x=-\frac{9}{10}$$

$\therefore$ Orthocentre $\left(-\frac{9}{10},0\right)$

Hence, the correct answer is (A).

2. Tangents to ellipse at M and N are

$$\frac{x}{6}+\frac{y\sqrt{6}}{8}=1 \text{ and } \frac{x}{6}-\frac{y\sqrt{6}}{8}=1$$

Their intersecting point is R (6, 0)

Also normal to parabola at $M\left(\frac{3}{2},\sqrt{6}\right)$ is $y-\sqrt{6}=-\frac{\sqrt{6}}{2}\left(x-\frac{3}{2}\right)$

Its intersection with x-axis is $Q\left(\frac{7}{2},0\right)$

Now, area $(\Delta MQR)=\frac{1}{2}\times\frac{5}{2}\times\sqrt{6}=\frac{5\sqrt{6}}{4}$ sq. units

Also, area $(MF_1NF_2) = 2 \times Ar(F_1MF_2)$

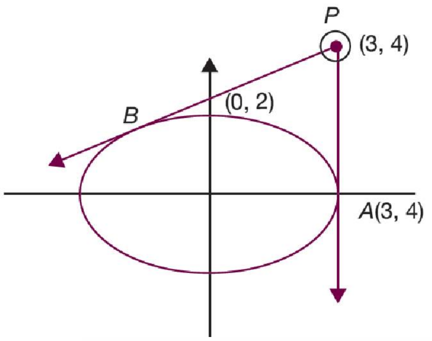
$$=2\times\frac{1}{2}\times 2\times\sqrt{6}=2\sqrt{6} \text{ sq. units}$$

$$\therefore \quad \frac{\text{ar}(\Delta MQR)}{\text{ar}(MF_1NF_2)} = \frac{5\sqrt{6}}{4\times 2\sqrt{6}} = 5:8$$

Hence, the correct answer is (C).

3. Tangent to $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$ at the point $(3\cos\theta, 2\sin\theta)$ is

$$\frac{x\cos\theta}{2}+\frac{y\sin\theta}{2}=1$$



As it passes through (3, 4), we get $\cos\theta + 2\sin\theta = 1$

$\Rightarrow \quad 4\sin^2\theta = 1 + \cos^2\theta - 2\cos\theta$

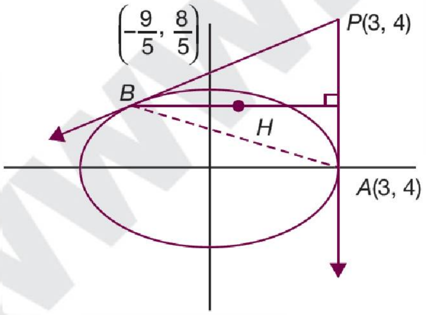
$\Rightarrow \quad 5\cos^2\theta - 2\cos\theta - 3 = 0$

$\Rightarrow \quad \cos\theta = 1, -\frac{3}{5} \Rightarrow \sin\theta = 0, \frac{4}{5}$

$\therefore$ Required points are A (3, 0) and $B\left(-\frac{9}{5}, \frac{8}{5}\right)$

Hence, the correct answer is (D).

4. Let H be the orthocentre of ΔPAB, then as BH is perpendicular AP, BH is a horizontal line through B.



$\therefore \quad y$-co-ordinate of $B = \frac{8}{5}$

Let, H has co-ordinate $\left(\alpha, \frac{8}{5}\right)$

Then, slope of $PH = \frac{\frac{8}{5}-4}{\alpha-4} = \frac{-12}{5(\alpha-3)}$

And slope of $AB = \frac{\frac{8}{5}-4}{\frac{-9}{5}-2} = \frac{8}{-24} = \frac{-1}{3}$

However, PH is perpendicular to AB.

$\Rightarrow \quad \frac{-12}{5(\alpha-3)} \times \left(\frac{-1}{3}\right) = -1$

$\Rightarrow \quad 4 = -5\alpha + 15$ or $\alpha = \frac{11}{5}$

Hence, $H\left(\frac{11}{5}, \frac{8}{5}\right)$.

Hence, the correct answer is (C).

5. Clearly, the moving point traces a parabola with focus at $P(3,4)$ and directrix as

$$AB: \frac{y-0}{x-3} = \frac{-1}{3} \quad \text{or} \quad x+3y-3=0$$

$\therefore$ Equation of parabola is

$$(x-3)^2+(y-4)^2 = \frac{(x+3y-3^2)}{10}$$

Or $\quad 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

Hence, the correct answer is (A).

Integer/Numerical Answer Type Questions

1. Ellipse: $\frac{x^2}{9}+\frac{y^2}{5}=1$

$\Rightarrow \quad a = 3, b = \sqrt{5}$ and $e = \frac{2}{3}$

$P_1: y^2 = 8x$ and $P_2: y^2 = -16x$

$T_1: y = m_1x + \frac{2}{m_1}$

It passes through (–4, 0)

$$0 = -4m_1 + \frac{2}{m_1} \Rightarrow m_1^2 = \frac{1}{2}$$

$$T_2: y = m_2x - \frac{4}{m_2}$$

It passes through (2, 0)

$$0 = 2m_2 - \frac{4}{m_2} \Rightarrow m_2^2 = 2$$

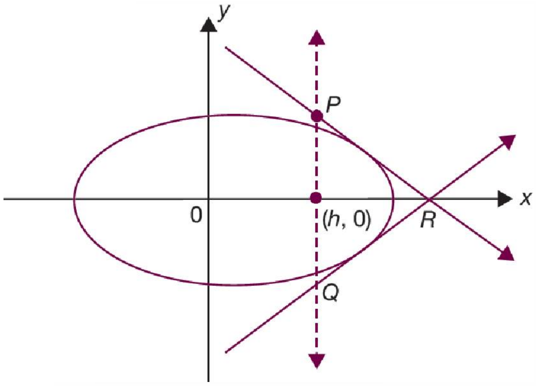
$$\therefore \quad \frac{1}{m_1^2} + m_2^2 = 4$$

Hence, the correct answer is (4).

2. Vertical line $x = h$, meets the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at

$$P\left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2}\right) \text{and } Q\left(h, \frac{-\sqrt{3}}{2}\sqrt{4-h^2}\right)$$

By symmetry, tangents at P and Q will meet each other at x-axis.



Tangent at P is $\frac{xh}{4}+\frac{y\sqrt{2}}{6}\sqrt{4-h^2}=1$

It meets x-axis at $R\left(\frac{4}{h},0\right)$

Area of $\Delta PQR = \frac{1}{2}\times\sqrt{3}\sqrt{4-h^2}\times\left(\frac{4}{h}-h\right)$

i.e., $\Delta(h) = \frac{\sqrt{3}}{2}\frac{(4-h^2)^{3/2}}{h}$

$$\frac{d\Delta}{dh} = -\sqrt{3}\left[\frac{\sqrt{4-h^2}(h^2+2)}{h^2}\right] < 0$$

$\therefore$ $\Delta(h)$ is a decreasing function.

$\therefore$ $\frac{1}{2} \le h \le 1 \Rightarrow \Delta_{\max} = \Delta\left(\frac{1}{2}\right)$ and $\Delta_{\min} = \Delta(1)$

$$\therefore \quad \Delta_1 = \frac{\sqrt{3}}{2}\frac{\left(4-\frac{1}{4}\right)^{3/2}}{\frac{1}{2}} = \frac{45}{8}\sqrt{5}$$

$$\Delta_2 = \frac{\sqrt{3}}{2}\frac{3\sqrt{3}}{1} = \frac{9}{2}$$

$$\therefore \quad \frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45-36 = 9$$

Hence, the correct answer is (9).

CHAPTER 6: HYPERBOLA

Test Your Concepts-I

1. (Rectangular hyperbola)

$\Delta \neq 0, h^2 - ab > 0, a + b = 0$

$\therefore$ It is a rectangular hyperbola.

2. Given that $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

$$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha$$

Similarly, eccentricity of the ellipse

$\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$ is $e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = \sin^2\alpha$

According to the given conditions, $e_1 = \sqrt{3}e_2 \Rightarrow e_1^2 = 3e_2^2$

$\Rightarrow \quad 1 + \cos^2\alpha = 3\sin^2\alpha$

$\Rightarrow \quad 2 = 4\sin^2\alpha$

$\Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \alpha = \pm\frac{\pi}{4}$

3. $e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{12}{4} = 4$

$\Rightarrow \quad e_1 = 2$

Now, $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

$\frac{1}{e_2^2} = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \quad e_2^2 = \frac{4}{3} \Rightarrow e_2 = \frac{2}{\sqrt{3}}$

4. Given hyperbola is $\frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$

Length of the transverse axis is $2a = 8$ units

5. Given equation can be rewritten as $\frac{y^2}{k^2} - \frac{x^2}{(-k)} = 1 (-k > 0)$

$e^2 = 1 + \frac{(-k)}{k^2} = 1 - \frac{1}{k}$

$\Rightarrow \quad e = \sqrt{1 - \frac{1}{k}}$

6. Equation of the ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1.$

Its eccentricity is $e = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

Co-ordinates of foci are $(\pm 1, 0)$.

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a = \sin\theta$

Also, $ae_1 = 1$

$\Rightarrow \quad e_1 = \operatorname{cosec}\theta$

$\therefore \quad b^2 = a^2\left(e_1^2 - 1\right) = 1 - \sin^2\theta = \cos^2\theta$

Equation of the hyperbola is thus $\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

7. The equation of the hyperbola is

$$\frac{\left(\frac{|2x - y + 4|}{\sqrt{5}}\right)^2}{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{\left(\frac{|x + 2y - 3|}{\sqrt{5}}\right)^2}{\left(\frac{2}{\sqrt{3}}\cdot\frac{1}{2}\right)^2} = 1$$

$\Rightarrow \quad \frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$

8. For ellipse, $a^2 = 25$

$\Rightarrow \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{25 - b^2}}{5}$

$\Rightarrow \quad$ Foci – $(\pm ae, 0) = \left(\pm\sqrt{25 - b^2}, 0\right)$

For hyperbola, $e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{5}{4}$

$\therefore \quad$ Foci $= (\pm ae, 0) = (\pm 3, 0)$, we get

$\therefore \quad \sqrt{25 - b^2} = 3 \Rightarrow b^2 = 16$

9. $2a = 4, 2b = 6$

$b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad e_1^2 = \frac{9}{4} + 1 = \frac{13}{4}$

Now focus $(\pm ae_1, 0)$

$S_1\left(\sqrt{13}, 0\right), S_2\left(-\sqrt{13}, 0\right)$

Also, $\frac{1}{e_2^2} = 1 - \frac{1}{e_1^2} = 1 - \frac{4}{13} = \frac{9}{13}$

⇒ $e_2^2 = \frac{13}{9} \Rightarrow e_2 = \frac{\sqrt{13}}{3}$

$S_3\left(0, \sqrt{13}\right), S_4\left(0, -\sqrt{13}\right)$

Required area $= \frac{1}{2}(S_1S_2)(S_3S_4) = \frac{1}{2}\left(2\sqrt{13}\right)\left(2\sqrt{13}\right)$

$= 26$ sq.units

Test Your Concepts-II

1. Normal at (6, 3) is $\frac{a^2x}{6} + \frac{b^2y}{3} = a^2 + b^2$

⇒ $\frac{9a^2}{6} = a^2 + b^2$

⇒ $\frac{3}{2} = 1 + \frac{b^2}{a^2}$

∴ $\frac{b^2}{a^2} = \frac{1}{2} \Rightarrow e^2 - 1 = \frac{1}{2}$

⇒ $e = \sqrt{\frac{3}{2}}$

2. Let $y = mx + c$ be common tangent of hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Conditions are $c^2 = a^2m^2 - b^2$ and $c^2 = a^2 - b^2m^2$, respectively.

Hence, $m^2 = 1 \Rightarrow m = \pm 1$

∴ Common tangent is $y = \pm x \pm \sqrt{a^2 - b^2}$.

3. $e = \sqrt{2}$, $SP.S'P = (ex_1 + a)(ex_1 - a) = 25$

4. Let $P = (a\sec\theta, a\tan\theta)$, $S_1S' = \left(\pm a\sqrt{2}, 0\right)$

$SP = a\left(\sqrt{2}\sec\theta - 1\right)$, $S'P = a\left(\sqrt{2}\sec\theta + 1\right)$

$SP \cdot S'P = a^2(\sec^2\theta + \tan^2\theta) = CP^2$

5. $P\left(a\sec\frac{\pi}{6}, b\tan\frac{\pi}{6}\right) = P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$

Therefore, equation of tangent at P is $\frac{x}{\frac{\sqrt{3}a}{2}} - \frac{y}{\sqrt{3}b} = 1$

Area of the triangle $= \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2 \Rightarrow \frac{b}{a} = 4$ sq. units

∴ $e^2 = 1 + \frac{b^2}{a^2} = 17$

6. Slope of chord joining P and Q = Slope of normal at P

$$\frac{\tan\alpha - \tan\theta}{\sec\alpha - \sec\theta} = -\frac{\tan\theta}{\sec\theta}$$

⇒ $\tan\alpha - \tan\theta = -k\tan\theta$ and $\sec\alpha - \sec\theta = k\sec\theta$

⇒ $(1 - k)\tan\theta = \tan\alpha$ ...(1)

$(1 + k)\sec\theta = \sec\alpha$ ...(2)

$\left[(1 + k)\sec\theta\right]^2 - \left[(1 - k)\tan\theta\right]^2 = \sec^2\alpha - \tan^2\alpha$

⇒ $k = -2\left(\sec^2\theta + \tan^2\theta\right) = -4\sec^2\theta + 2$

From (1), $\tan\alpha = \tan\theta\,(1 + 4\sec^2\theta - 2) = \tan\theta(4\sec^2\theta - 1)$

7. Let point of contact is (x_1, y_1).

Tangent is $xx_1 - 9yy_1 - 9 = 0$

Compare with $5x + 12y - 9 = 0 \Rightarrow (x_1, y_1) = \left(5, -\frac{4}{3}\right)$

8. Since the point (1, 1) lies on the normal and hyperbola, it is the foot of the normal.

9. Let the locus be $P(h, k)$ and $T = S_1$

$3hx - 2ky + 2(x + h) - 3(k + y) = 3h^2 - 2k^2 + 4h - 6k$

Slope $= \frac{3h + 2}{2k + 3} = 2 \Rightarrow 3x - 4y = 4$

Test Your Concepts-III

1. $\tan\frac{\theta}{2} = \frac{b}{a}$

$\cos\frac{\theta}{2} = \frac{a}{\sqrt{a^2 + b^2}} = \frac{1}{\sqrt{1 + \frac{b^2}{a^2}}} = \frac{1}{e}$

2. Since $c_1c_2\left(a_1a_2 + b_1b_2\right) < 0$

∴ Origin lies in acute angle.

$P(1, 2)$ lies in obtuse angle.

Acute angle between the asymptotes is $\frac{\pi}{3}$

$e = \sec\frac{\theta}{2} = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

3. A hyperbola and its conjugate hyperbola possess the same pair of asymptotes.

4. $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ is the equation to the pair of asymptotes. Then, $c = -12$ and hence equation to the conjugate hyperbola is $3x^2 - 5xy - 2y^2 + 5x + 11y - 16 = 0$.

5. Axes of hyperbola are bisectors of angles between asymptotes.

Transverse axis is angle bisector containing (1, 2).

So, $\frac{3x-4y+2}{5} = +\left(\frac{-5x-12y+4}{13}\right)$

$\Rightarrow \quad 64x+8y+6=0$

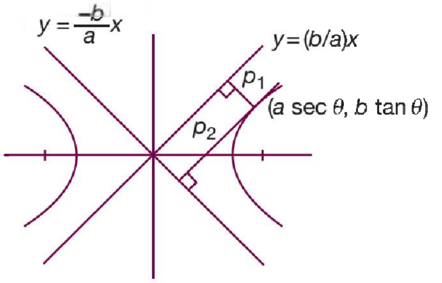
6. $A = ab = a^2\tan\lambda \Rightarrow b/a = \tan\lambda$

Hence, $e^2 = 1+(b^2/a^2)$

$\Rightarrow \quad e^2 = 1+\tan^2\lambda$

$\Rightarrow \quad e = \sec\lambda$

7.



$$p_1p_2 = \frac{a^2b^2}{a^2+b^2} = \frac{a^2\cdot a^2(e^2-1)}{a^2e^2} = 6$$

$$\frac{2a^2}{3} = 6$$

$\Rightarrow \quad a^2 = 9 \Rightarrow a = 3$

Hence, $2a = 6$ units.

8. Equation of any tangent to $x^2-y^2=a^2$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ is $\frac{x}{a}\sec\theta - \frac{y}{a}\tan\theta = 1 \qquad \ldots(1)$

$\Rightarrow \quad x\sec\theta - y\tan\theta = a$

Equation of other two sides of the triangle are

$x-y=0 \qquad \ldots(2)$

$x+y=0 \qquad \ldots(3)$

Solving (1), (2) and (3) in pairs, the co-ordinates of the vertices of the triangle are (0, 0),

$\left(\frac{a}{\sec\theta+\tan\theta}, \frac{a}{\sec\theta+\tan\theta}\right)$ and

$\left(\frac{a}{\sec\theta-\tan\theta}, \frac{-a}{\sec\theta-\tan\theta}\right)$

$$\text{Area of triangle} = \frac{1}{2}\left|\frac{a^2}{\sec^2\theta-\tan^2\theta} + \frac{a^2}{\sec^2\theta-\tan^2\theta}\right|$$

$$= \frac{1}{2}(a^2+a^2) = a^2 \text{ sq. units}$$

9. Let the equation of asymptotes be

$2x^2+5xy+2y^2+4x+5y+\lambda = 0 \qquad \ldots(1)$

This equation represents a pair of straight lines.

Therefore, $abc+2fgh-at^2-bg^2-ch^2=0$

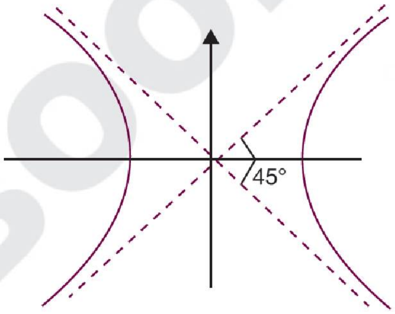
$\therefore \quad 4\lambda+25-\frac{25}{2}-8-\lambda\frac{25}{4}=0$

$\Rightarrow \quad -\frac{9\lambda}{4}+\frac{9}{2}=0$

$\Rightarrow \quad \lambda = 2$

Putting the value of λ in (1), we get $2x^2+5xy+2y^2+4x+5y+2=0$ this is the equation of the asymptotes.

10.



$$2\tan^{-1}\frac{b}{a} = 45°$$

$\Rightarrow \quad \frac{b}{a} = \tan\left(\frac{45}{2}\right)^{\circ} = \frac{a^2(e^2-1)}{a^2} = (\sqrt{2}-1)^2$

$\Rightarrow \quad e^2-1 = 3-2\sqrt{2}$

$\Rightarrow \quad e = \sqrt{4-2\sqrt{2}}$

11. Any tangent to $x^2-y^2=a^2$ is $x\sec\phi - y\tan\phi = a$

Area $= |a|^2$ sq. units

12. The point on the line $y = 2$ should lie between the asymptotes, where the curve do not exist.

Equation of asymptotes are $4x = \pm 3y$. The point of intersection of $y=2$ with asymptotes are $x = \pm\frac{3}{2}$

$\therefore \quad \frac{-3}{2} < \alpha < \frac{3}{2}$

Test Your Concepts-IV

1. Foci of $xy = c^2$ is $(\pm c\sqrt{2}, \pm c\sqrt{2})$ or $(2\sqrt{2}, 2\sqrt{2})$, $(-2\sqrt{2}, -2\sqrt{2})$

2. Let A be the vertex and S be the focus. Then,

$$OA = \sqrt{2}, A\left(\sqrt{2}, \sqrt{2}\right)$$

$$AS = eOA \Rightarrow AS = \sqrt{2}\sqrt{2} \Rightarrow AS = 2$$

$\therefore$ $S = (2, 2)$

Equation of latus rectum is $y - 2 = -1(x - 2)$

$\Rightarrow$ $x + y = 4$

3. Let the vertices of the ΔABC be $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$.

Slopes of AB, AC and BC are $\frac{-1}{t_1t_2}, \frac{-1}{t_1t_3}$ and $\frac{-1}{t_2t_3}$

Respectively, $\angle ACB = 90°$

$$\Rightarrow \quad \frac{-1}{t_1t_3} \times \frac{-1}{t_2t_3} = -1 \Rightarrow t_1t_2t_3^2 + 1 = 0$$

$$t_1t_2 = \frac{-1}{t_3^2}$$

$\Rightarrow$ Slope of $AB = \frac{-1}{t_1t_2} = t_3^2$ and slope of tangent at $C = \frac{-1}{t_3^2}$

Hence they intersect at 90°.

4. Slope of the line $\frac{-a}{b}$ is equal to slope of the normal to the curve.

$\therefore$ Either $a > 0$ and $b < 0$ (or) $a < 0$ and $b > 0$.

5. Let $P(h, k)$

$\Rightarrow$ $y - k = 4(x - h)$ ...(1)

Let it meets $xy = 1$ ...(2)

$$x(4x - 4h + k) = 1$$

$\Rightarrow$ $4x^2 + x(k - 4h) - 1 = 0$

At $A(x_1, y_1)$ and $B(x_2, y_2)$

$$x_1 + x_2 = \frac{4h - k}{4}, x_1x_2 = -\frac{1}{4}$$

Also $\frac{2x_1 + x_2}{3} = h \Rightarrow 2x_1 + x_2 = 3h$

$$\Rightarrow \quad x_1 = \frac{8h + k}{4}, x_2 = -\left(\frac{2h + k}{2}\right)$$

$\Rightarrow$ $16x^2 + 10xy + y^2 = 2$

6. Equation of tangent at $P, \frac{x}{t} + ty = 2c$

$\Rightarrow$ $x + t^2y = 2ct$ ...(1)

Slope of tangent $= -\frac{1}{t^2}$

$\therefore$ Equation of CM is $y = t^2x$ ...(2)

Squaring (1), $\left(x + t^2y\right)^2 = 4c^2t^2$

Using (2), we get, $\left(x + \frac{y^2}{x}\right)^2 = 4c^2\frac{y}{x}$

$\Rightarrow$ $\left(x^2 + y^2\right)^2 = 4c^2xy$

7. Let P, Q, R, S are $\left(ct, \frac{c}{t}\right)$

Where t is t_1, t_2, t_3, t_4 respectively

Let the equation of circle is $x^2 + y^2 = r^2$

$\left(ct, \frac{c}{t}\right)$ satisfies this equation

$$\therefore \quad c^2t^2 + \frac{c^2}{t^2} - r^2 = 0$$

$$c^2t^4 - r^2t^2 + c^2 = 0$$

Its roots are t_1, t_2, t_3, t_4

$t_1, t_2, t_3, t_4 = 1$ ...(1)

Co-ordinates of orthocentre of ΔPQR are

$$\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$$

$\Rightarrow$ $\left(-ct_4, -\frac{c}{t_4}\right)$ [using (1)]

$\Rightarrow$ $\left(-x_4, -y_4\right)$

8. Normal to hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = t^2(x - ct).$$

Solving with $xy = -c^2$, we get,

$$x\left\{\frac{c}{t} + t^2(x - ct)\right\} + c^2 = 0$$

$\Rightarrow$ $t^2x^2 + \left(\frac{c}{t} - ct^3\right)x + c^2 = 0$

For equal roots, we have, $\left(\frac{1}{t} - t^3\right)^2 - 4t^2 = 0$

$\Rightarrow$ $\left(1 - t^4\right)^2 - 4t^4 = 0$

$\Rightarrow$ $t^4 - 2t^2 - 1 = 0$ or $t^4 + 2t^2 - 1 = 0$

$$t^2 = \frac{2 \pm \sqrt{8}}{2} \text{ or } \frac{-2 \pm \sqrt{8}}{2}$$

$$\Rightarrow \quad t^2 = 1 + \sqrt{2} \text{ or } -1 + \sqrt{2}$$

Thus, four such values of t are possible.

9. Transverse axis is along the line $y = x$

Solving $y = x$ and $xy = 18$, we have $x^2 = 18$ or $x = \pm 3\sqrt{2}$

Then two vertices of the hyperbola are $\left(\pm 3\sqrt{2}, \pm 3\sqrt{2}\right)$

Distance between them $\sqrt{72+72} = 12$ units

10. Equation of tangent at point $\left(ct, \frac{c}{t}\right)$ is $\frac{x}{t} + yt = 2c$

Area of required ΔOAB

$$= \frac{1}{2} \cdot 2ct \cdot \frac{2c}{t} = 2c^2$$

11. Locus of midpoint of chords of constant length $2d$ is $\left(x^2+y^2\right)\left(xy - c^2\right) = d^2 xy$

12. $x^2 + y^2 = r^2, xy = c^2$

$$\therefore \quad x^4 - r^2x^2 + c^4 = 0$$

$$CP^2 + CQ^2 + CR^2 + CS^2 = \sum x_1^2 + \sum y_1^2$$

$$= \left(\sum x_1\right)^2 - 2\sum x_1x_2 + \left(\sum y_1\right)^2 - 2\sum y_1y_2$$

$$= 0 - 2(-r^2) + 0 - 2(-r^2)$$

$$= 4r^2$$

13. $\theta + 60^\circ = \tan^{-1}\frac{x}{3}$ ...(1)

$\theta + 30^\circ = \tan^{-1}\frac{y}{2}$ ...(2)

(1) – (2)

$$\Rightarrow \quad 30^\circ = \tan^{-1}\frac{x}{3} - \tan^{-1}\frac{y}{2} \Rightarrow \frac{1}{\sqrt{3}} = \frac{\frac{x}{3} - \frac{y}{2}}{1 + \frac{xy}{6}}$$

$$\Rightarrow \quad xy - 2\sqrt{3}x + 3\sqrt{3}y + 6 = 0$$

$$\left(x + 3\sqrt{3}\right)\left(y - 2\sqrt{3}\right) = -24$$

$$\therefore \quad y - 2\sqrt{3} = 0 \Rightarrow y = 2\sqrt{3}$$

$$x + 3\sqrt{3} = 0 \Rightarrow x = -3\sqrt{3}$$

Single Correct Choice Type Questions

1. The equation of the hyperbola is $x^2 - 2y^2 - 2x + 8y - 1 = 0$

$$\Rightarrow \quad (x-1)^2 - 2(y-2)^2 + 6 = 0$$

$$\Rightarrow \quad \frac{(x-1)^2}{-6} + \frac{(y-2)^2}{3} = 1, \ \frac{(y-2)^2}{3} - \frac{(x-1)^2}{6} = 1 \qquad \text{...(1)}$$

$$\Rightarrow \quad \frac{Y^2}{3} - \frac{X^2}{6} = 1, \text{ where } X = x - 1 \text{ and } Y = y - 2 \text{ ...(2)}$$

$\therefore$ Centre = (0, 0) in the XY co-ordinates.

$\therefore$ Centre = (1, 2) in the xy co-ordinates (Using (2))

If the transverse axis be of length $2a$ units, then $a = \sqrt{3}$, since in the Equation (1) the transverse axis is parallel to the y-axis. If the conjugate axis is of length $2b$ units, then $b = \sqrt{6}$

However, $b^2 = a^2(e^2 - 1)$

$\therefore$ $6 = 3(e^2 - 1)$, $e^2 = 3$ or $e = \sqrt{3}$

The length of the transverse axis = $2\sqrt{3}$ units

The length of the conjugate axis = $2\sqrt{6}$ units

Latus rectum $4\sqrt{3}$

Hence, the correct answer is (A).

2. Hyperbola for which (1, 2) and (5, 5) are foci and length of transverse axis 3 units.

$$2ae = 5 \text{ and } 2a = 3$$

$$\therefore \quad e = 5/3$$

Hence, the correct answer is (B).

3. Foci of the ellipse $\left(\sqrt{16 - b^2}, 0\right)$

Foci of hyperbola $\left(\pm\sqrt{\frac{144}{25} + \frac{81}{25}}, 0\right)$

$$\therefore \quad \sqrt{16 - b^2} = \sqrt{9}$$

$$\therefore \quad b^2 = 7$$

Hence, the correct answer is (C).

4. The chord of contact $yy_o = 2(x + x_o)$ of the point $P(x_o, y_o)$ w.r.t the parabola is tangent to the hyperbola $x^2 - y^2 = 1$ iff $4x_0^2 + y_0^2 = 4$. Locus of P is the ellipse $4x^2 + y^2 = 4$.

Hence, the correct answer is (C).

5. Normal $ax\cos\theta + by\cot\theta = a^2 + b^2$

$$L = \left(\frac{a^2+b^2}{a}\sec\theta, 0\right), M = \left(0, \frac{a^2+b^2}{b}\tan\theta\right)$$

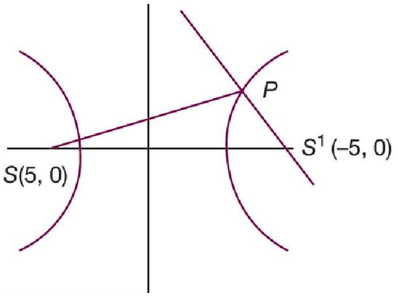
Let the midpoint $(x, y) \Rightarrow \sec\theta = \frac{2x}{ae^2}, \ \tan\theta = \frac{2by}{a^2e^2}$

$$\therefore \quad \sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{x^2}{\frac{a^2e^4}{4}} - \frac{y^2}{\frac{a^4e^4}{4b^2}} = 1$$

$$e_1 = \sqrt{\frac{\frac{a^2e^4}{4} + \frac{a^4e^4}{4b^2}}{\frac{a^2e^4}{4}}} = \sqrt{1+\frac{a^2}{b^2}} = \frac{e}{\sqrt{e^2-1}}$$

Hence, the correct answer is (B).

6.



$$e = 5/4$$

Since (5, 0) satisfying the equation of the line

$$3x + \left(5 - 4\sqrt{2}\right)y = 15$$

So, the reflected ray must pass through (–5, 0) and $P = \left(4\sqrt{2}, 3\right)$

$\therefore$ The equation SP is $\left(4\sqrt{2}+5\right)y = 3(x+5)$

Hence, the correct answer is (D).

7. $\dfrac{x^2}{5\sin^2\theta} - \dfrac{y^2}{5\cos^2\theta} = 1,\ \dfrac{x^2}{25\sin^2\theta} + \dfrac{y^2}{25\cos^2\theta} = 1$

$$e_H^2 = 1 + \cot^2\theta$$

$$e_e^2 = 1 - \cot^2\theta$$

$$1 + \cot^2\theta = 5\left(1 - \cot^2\theta\right)$$

$$6\cot^2\theta = 4$$

$$\cot^2\theta = \frac{2}{3}$$

$$\cot\theta = \pm\sqrt{\frac{2}{3}}$$

$$\theta = \cot^{-1}\left(\pm\sqrt{\frac{2}{3}}\right)$$

Hence, the correct answer is (C).

8. $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \Rightarrow y = mx \pm \sqrt{A^2m^2 - B^2}$

where $A^2m^2 - B^2 \geq 0$

$$\Rightarrow \quad m^2 \geq \frac{B^2}{A^2} \Rightarrow m^2 \geq \frac{a^3-b^3}{a^2-b^2}$$

$$\Rightarrow \quad (a+b)m^2 \geq a^2 + ab + b^2$$

$$\Rightarrow \quad (a+b)m^2 + ab \geq (a+b)^2$$

Hence, the correct answer is (A).

9. $y^2 = 4(x-1)$ …(1)

Tangents from origin $\left.\begin{matrix} x+y=0 \\ x-y=0 \end{matrix}\right\}$

Hyperbola: $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$

$\Rightarrow \quad x^2 - y^2 = 4$ …(2)

($\because 2a = 4 \Rightarrow a = 2$)

Solving (1) and (2), we get,

$$A(4, 2\sqrt{3}), B(4, -2\sqrt{3})$$

$$AB = 4\sqrt{3}$$

Hence, the correct answer is (B).

10. Let $xy = c^2$ $P\left(ct_1, \dfrac{c}{t_1}\right), Q\left(ct_2, \dfrac{c}{t_2}\right)$, tangents are $\dfrac{x}{t_1} + yt_1 = 2c$ and $\dfrac{x}{t_2} + yt_2 = 2c$,

Point of intersection, $x = \dfrac{2ct_1t_2}{t_1+t_2} \Rightarrow x_3 = \dfrac{2x_1x_2}{x_1+x_2}$

Hence, the correct answer is (B).

11. Equation of tangent at point $P(\alpha\cos\theta,\ \sin\theta)$ is $\dfrac{x}{a}\cos\theta + \dfrac{y}{1}\sin\theta = 1$

Let it cut the hyperbola at points P and Q.

Homogenising the hyperbola $a^2x^2 - y^2 = 1$ with the help of the above the equation, we get,

$$a^2x^2 - y^2 = \left(\frac{x}{a}\cos\theta + y\sin\theta\right)^2$$

This is a pair of straight lines OP and OQ

Given, $\angle POQ = \dfrac{\pi}{2}$

So, Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad a^2 - \frac{\cos^2\theta}{a^2} - 1 - \sin^2\theta = 0$$

$$\Rightarrow \quad a^2 - \frac{\cos^2\theta}{a^2} - 1 - 1 + \cos^2\theta = 0$$

$$\Rightarrow \quad \cos^2\theta = \frac{a^2\left(2-a^2\right)}{a^2-1}$$

Now $0 \le \cos^2\theta \le 1$

$$\Rightarrow \quad 0 \le \frac{a^2\left(2-a^2\right)}{a^2-1} \le 1$$

Solving, we get,

$$a^2 \in \left[\frac{\sqrt{5}+1}{2}, 2\right]$$

Hence, the correct answer is (A).

12. The equation of the normal to the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ at $(2\sec\theta, \tan\theta)$ is

$2x\cos\theta + y\cot\theta = 5$ (Intercepts equal)

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$\therefore$ Slope of normal $= -1$

$\therefore \quad c^2 = a^2m^2 + b^2$

$$a^2 + b^2 = \frac{25}{9}$$

Hence, the correct answer is (C).

13. Common normal is x-axis.

Shortest distance is $a - \frac{a}{2} = \frac{a}{2}$

Hence, the correct answer is (C).

14. Equation of hyperbola + Equation of conjugate hyperbola = 2(Equations of asymptotes)

$\therefore$ Equation of conjugate hyperbola = 2(Equation of asymptote) – Equation of hyperbola

$$C(x, y) = 2A(x, y) - H(x, y) \qquad \ldots(1)$$

$\because \quad H(x, y) = x^2 + 3xy + 2y^2 + 2x + 3y = 0$

$\therefore$ Equation of asymptotes is

$$x^2 + 3xy + 2y^2 + 2x + 3y + \lambda = 0$$

$\because \Delta = 0,\ abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ then $\lambda = 1$

$\therefore \quad A(x, y) = x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$

$\therefore \quad C(x, y) = x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

Hence, the correct answer is (B).

15. $\sqrt{3}x - y - 4\sqrt{3}k = 0,\ \sqrt{3}kx + ky - 4\sqrt{3} = 0$

In order to find the locus of point of intersection
We have to eliminate k

$\therefore \quad \frac{\sqrt{3}x - y}{4\sqrt{3}} = k$ put this k in another

$$\left(\sqrt{3}x - y\right)\left(\sqrt{3}x + y\right) = \left(4\sqrt{3}\right)^2 \text{ or } 3x^2 - y^2 = 48$$

$$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

$\therefore \quad a^2 = 16, b^2 = 48$

Clearly, locus form a hyperbola $b^2 = a^2(e^2 - 1)$.

$\Rightarrow \quad 48 = 16(e^2 - 1)$

$\therefore \quad e = 2$

Hence, the correct answer is (C).

16. Let R be $\left(ct_3, \frac{c}{t_3}\right)$ then $t_1t_2t_3^2 = -1$

$$S\left(\frac{c(t_1+t_2)}{2}, \frac{c(t_1+t_2)}{2t_1t_2}\right) \text{and } \angle ROS = \frac{\pi}{2}$$

$$\text{area} = \frac{1}{2}(OR)(OS)$$

Hence, the correct answer is (A).

17. Equation of chord with midpoint (h, k) is $\frac{x}{h} + \frac{y}{k} = 2$;

$m = -\frac{k}{h} \Rightarrow y + mx = 0$

Hence, the correct answer is (A).

18. $OT = a\cos\theta$; $N = a\sec\theta \Rightarrow OT.ON = a^2$

Hence, the correct answer is (B).

19. $P : (ct, c/t)$; $N : (0, c/t)$

$\Rightarrow \quad 2h = ct$ and $2 = 2c/t$

$\Rightarrow \quad xy = c^2/2$

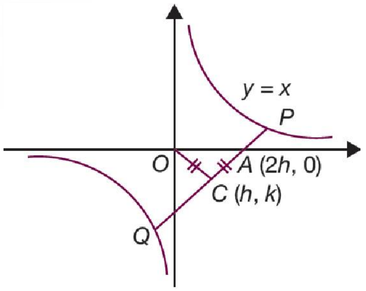
Alternatively, P:($a\sec\theta$, $a\tan\theta$); N:[$(a/2)$ $(\sec\theta + \tan\theta)$, $(a/2)$ $(\sec\theta + \tan\theta)$]

$\Rightarrow \quad 4h/a = 2\sec\theta + \tan\theta$ and $4k/a = \sec\theta + 2\tan\theta$

$\Rightarrow \quad x^2 - y^2 = 3a^2/16$

Hence, the correct answer is (D).

20.



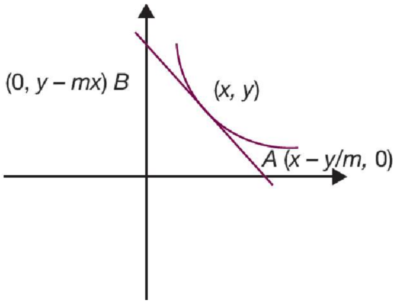
Chord with a given midpoint

$$\frac{x}{h}+\frac{y}{k}=2$$

Obviously, OCA is isosceles with $OC = CA$.

Hence, the correct answer is (B).

21.



$$T: Y - y = m\,(X - x)$$

$$X = 0,\ Y = y - mx$$

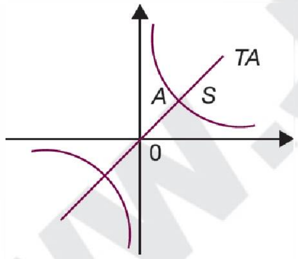
$$Y = 0,\ X = x - \frac{y}{m}$$

$$\therefore\quad x - \frac{y}{m} = 2x \Rightarrow \frac{y}{m} = -x$$

$$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0 \Rightarrow \ln xy = c \Rightarrow xy = c$$

$x = 2$ and $y = 4$

$\Rightarrow$ Equation is $xy = 8$



Solving with $y = x$

$$x = 2\sqrt{2}$$

$$\therefore\quad OA = \sqrt{8+8} = 4$$

$$OS = 4\sqrt{2}$$

Co-ordinates of S are (4, 4) or (–4, –4)

Hence, the correct answer is (C).

22. Normal $y = mx - 2am - am^3$ is tangent to the hyperbola, if $c^2 = a^2m^2 - b^2$

(For a line of the form $y = mx + c$)

$$\Rightarrow\quad \left(-2am - am^3\right)^2 = a^2\left(m^2 - 1\right)$$

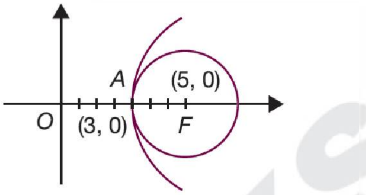
$$4m^2 + m^6 + 4m^4 = m^2 - 1$$

$$m^6 + 3m^2 + 4m^4 + 1 = 0$$

However, $m^6 + 4m^4 + 3m^2 + 1 \geq 1$ for real 'm' so no normal can be drawn.

Hence, the correct answer is (D).

23.



$$e^2 = 1 + \frac{16}{9} = \frac{25}{9} \Rightarrow e = \frac{5}{3}$$

$\therefore$ Focus = (5, 0)

Use reflection property to prove that circle cannot touch at two points. It can only be tangent at the vertex

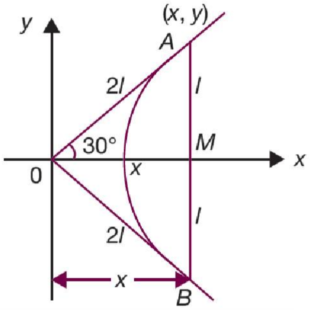
$r = 5 - 3 = 2$

Hence, the correct answer is (B).

24. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $y = l$

$$\frac{x^2}{a^2} = 1 + \frac{l^2}{b^2} \Rightarrow x^2 = (b^2 + l^2)\,\frac{a^2}{b^2} \qquad \text{...(1)}$$

$$\text{Now, } x^2 + l^2 = 4l^2 \Rightarrow x^2 = 3l^2 \qquad \text{...(2)}$$



From (1) and (2),

$$\frac{a^2(b^2 + l^2)}{b^2} = 3l^2 \Rightarrow a^2b^2 + a^2l^2 = 3b^2l^2$$

$$l^2(3b^2 - a^2) = a^2b^2$$

$$l^2 = \frac{a^2b^2}{3b^2 - a^2} > 0 \Rightarrow 3b^2 - a^2 > 0 \Rightarrow \frac{b^2}{a^2} > \frac{1}{3}$$

$$1 + \frac{b^2}{a^2} > \frac{4}{3} \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$$

Hence, the correct answer is (D).

25. $x = a\cos\theta,\ y = \sin\theta$

$$\lambda = 1 + 2\sin\theta\cos\theta = 1 + \sin 2\theta$$

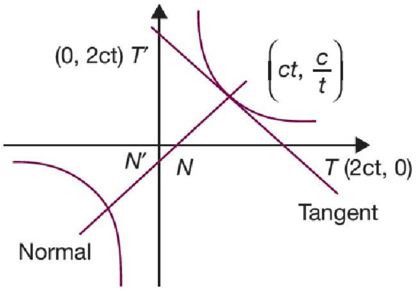
Maximum value of λ is 2

$$\lambda = 2$$

From (2, 0), no tangent can be drawn to $(x-2)^2 - y^2 = 1$

Hence, the correct answer is (A).

26.



$$x = a\cos\theta,\ y = \sin\theta$$

$$z = 1 + 2\sin\theta\cos\theta = 1 + \sin 2\theta$$

Maximum value of z is 2

$$\lambda = 2$$

From (2, 0), no tangent can be drawn to $(x-2)^2 - y^2 = 1$

Tangent: $\frac{x}{ct} + \frac{yt}{c} = 2$

Put $y = 0$

$$x = 2ct\ (T)$$

$$x = 0$$

$$y = \frac{2c}{t}\ (T')$$

Similarly normal is $y - \frac{c}{t} = t^2(x - ct)$

Put $y = 0$, $x = ct - \frac{c}{t^3}$ (N)

$$x = 0,\ y = \frac{c}{t} - ct^3 (N')$$

Area of $\Delta PNT = \frac{c}{2t}\left(ct + \frac{c}{t^3}\right) \Rightarrow \Delta = \frac{c^2(1+t^4)}{2t^4}$

Area of $\Delta PN'T' = ct\left(\frac{c}{t} + ct^3\right) \Rightarrow \Delta' = \frac{c^2(1+t^4)}{2}$

$$\therefore \quad \frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2t^4}{c^2(1+t^4)} + \frac{2}{c^2(1+t^4)}$$

$$= \frac{2}{c^2(1+t^4)}\ (t^4+1) = \frac{2}{c^2}$$

This is independent of t

Hence, the correct answer is (C).

27. Add the two equations to get $\left(x_1^2 + y_1^2\right) = 40$

$\Rightarrow \quad x_1^2 + y_1^2 = 5 \Rightarrow r = \sqrt{5}$

Hence, the correct answer is (A).

28. Let(x_1, y_1) be the point of intersection $\Rightarrow$ $y_1^2 = 4ax_1$ and $x_1y_1 = c^2$

$$y^2 = 4ax$$

$$xy = c^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{dx_{(x_1, y_1)}} = \tan\phi = \frac{2a}{y_1}$$

$$\frac{dy}{dx_{(x_1, y_1)}} = \tan\theta = -\frac{y_1}{x_1}$$

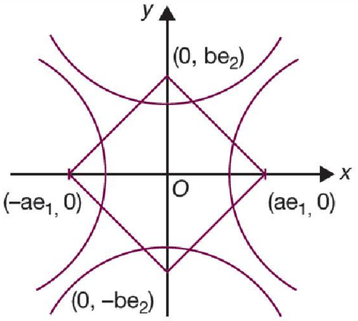
$$\therefore \quad \frac{\tan\theta}{\tan\phi} = \frac{-y_1 / x_1}{2a / y_1} = \frac{-y_1^2}{2ax_1} = -\frac{4ax_1}{2ax_1} = -2$$

$\Rightarrow \quad \theta = \tan^{-1}(-2\tan\phi)$

Hence, the correct answer is (A).

29. Given, hyperbolas are conjugate and the quadrilateral formed by their foci is a square.

Now, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$



$$e_1^2 = 1 + \frac{b^2}{a^2},\ e_2^2 = 1 + \frac{a^2}{b^2},$$

$$e_1^2 e_2^2 = \frac{(a^2+b^2)^2}{a^2b^2},\ e_1e_2 = \frac{a^2+b^2}{ab}$$

$$A = \frac{(2ae_1)(2be_2)}{2} = 2abe_1e_2 = \frac{2ab(a^2+b^2)}{ab}$$

Hence, the correct answer is (B).

30. Equation of chord of hyperbola $\frac{x^2}{2} - \frac{y^2}{1} = 1$, whose midpoint is (h, k) is $\frac{hx}{2} - ky = \frac{h^2}{2} - \frac{k^2}{1}$

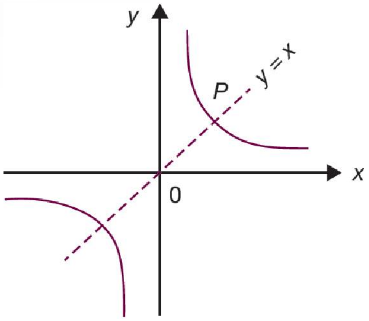
It is tangent to the circle $x^2 + y^2 = 4$, then $\left|\dfrac{\frac{h^2}{2} - k^2}{\sqrt{\frac{h^2}{4} + k^2}}\right| = 2$

$\Rightarrow \quad \left(\dfrac{h^2}{2} - k^2\right)^2 = 4\left(\dfrac{h^2}{4} + k^2\right)$

$\Rightarrow \quad (x^2 - 2y^2)^2 = 4(x^2 + 4y^2) \Rightarrow \lambda = 4$

Hence, the correct answer is (B).

31.



The curve is symmetric about $y = x$

$P = (1, 1)$

$OP = \sqrt{2}$

Hence, $r \in (0, \sqrt{2})$

Hence, the correct answer is (B).

32. $\dfrac{1}{e^2} + \dfrac{1}{(e')^2} = 1$, $\dfrac{x}{e} + \dfrac{y}{e'} = 1$

$\Rightarrow \quad \left|\dfrac{\frac{0}{e} + \frac{0}{e'} - 1}{\sqrt{\frac{1}{e^2} + \frac{1}{e'^2}}}\right| = 1$

$\Rightarrow$ Perpendicular distance from (0, 0) to $\dfrac{x}{e} + \dfrac{y}{e'} = 1$ is '1'

$\therefore$ A circle with centre at (0, 0) and radius '1' unit always touches the line $\dfrac{x}{e} + \dfrac{y}{e'} = 1$.

Hence, the correct answer is (B).

33. Centre of rectangular hyperbola is (1, –2)

So, equation of asymptotes are $x = 1$, $y = -2$

Hence, radius of circle = $\dfrac{\sqrt{5}}{2}$

Hence, the correct answer is (B).

34. Let the points of intersection be $\left(t_1, \dfrac{1}{t_1}\right)\left(t_2, \dfrac{1}{t_2}\right)$.

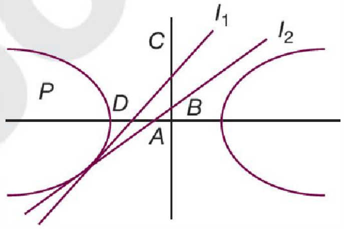
Given, $m = -\dfrac{1}{t_1 t_2}$ *or* $t_1 t_2 = -\dfrac{1}{m}$

Also, by section formula,

Solving for t_1, t_2 and eliminating them gives $2m^2x^2 + 5mxy + 2y^2 = m$, which is always a hyperbola as $\dfrac{25m^2}{4} - 4m^2 = \dfrac{9m^2}{4} > 0, \forall m \neq 0$

Hence, the correct answer is (B).

35. Let the tangents be $y = 2x \pm \sqrt{100 - 9} = 2x \pm \sqrt{91}$ and $y = mx \pm \sqrt{25m^2 - 9}$



$A\left(\mp\dfrac{\sqrt{91}}{2}, 0\right), D\left(\mp\dfrac{\sqrt{25m^2 - 9}}{m}, 0\right)$

$B\left(0, \mp\sqrt{91}\right), C\left(0, \mp\sqrt{25m^2 - 9}\right)$

$\Rightarrow \quad \dfrac{\sqrt{91}\left(\sqrt{25m^2 - 9}\right)}{2m} = \sqrt{91}\left(\sqrt{25m^2 - 9}\right)$

$\Rightarrow \quad 2m = 1 \Rightarrow m = \dfrac{1}{2}$

Hence, the correct answer is (C).

36. Asymptotes are equally inclined to the axes of hyperbola. The bisector of the asymptotes bisects the angle containing the origin.

Transverse and conjugate axes are the bisectors of the angle between asymptotes.

$\dfrac{3x - 4y - 1}{5} = \pm\left(\dfrac{4x - 3y - 6}{5}\right)$, etc …

Hence, the correct answer is (C).

37. If ellipse and hyperbola intersects at (x, y) then

$x^2 = \dfrac{l_2^2}{e_e^2}, e_e$ = Eccentricity of ellipse

$$(SP)(S'P) = l_1^2 - e_e^2x^2 = l_1^2 - l_2^2$$

Hence, the correct answer is (D).

38. Tangent at $P(x_1, y_1)$ on $xy = c^2$ is

$$\frac{x}{x_1} + \frac{y}{y_1} = 2$$

$\therefore \quad Q = (2x_1, 0),\ R = (0, 2y_1)$

Now, OQR is a right angled triangle and QR is the hypotenuse.

$\therefore$ Circumcentre = Midpoint of $QR = (x_1, y_1)$

Hence, the correct answer is (B).

39. $(ax\sec\theta + by\tan\theta)^2 = a^2$ ...(1)

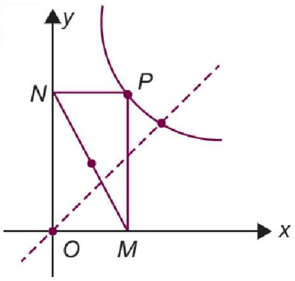
$(ax\tan\theta + by\sec\theta)^2 = b^2$ ...(2)

(1) – (2) gives $a^2x^2 - b^2y^2 = a^2 - b^2$

Required locus is hyperbola.

Hence, the correct answer is (D).

40.



$OMPN$ is a rectangle.

$$P = (Ct, \frac{c}{t})$$

Midpoint $= \left(\frac{ct}{2}, \frac{c}{2t}\right) = (x, y)$

$$\therefore \quad cy = \frac{c^2}{4} \Rightarrow e = \sqrt{2}$$

Hence, the correct answer is (B).

41. Let (h, k) be the midpoint.

$$\therefore \quad T = S_1 \Rightarrow \frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{...(1)}$$

Equation (1) passes through (α, β) so putting (α, β) in it

$$\Rightarrow \quad \frac{\alpha x}{a^2} - \frac{\beta y}{b^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\Rightarrow \quad \left(\frac{x^2}{a^2} - \frac{\alpha x}{a^2}\right) - \left(\frac{y^2}{b^2} - \frac{\beta y}{b^2}\right) = 0$$

$$\Rightarrow \quad \frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2} - \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2} + \frac{\alpha^2}{4a^2} - \frac{\beta^2}{4b^2} = 0$$

It is a hyperbola with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Hence, the correct answer is (C).

42. On removing xy terms by multiplying $a_1x^2 + 2h_1xy + b_1{}^2 = C_1$ by h_2 and $a_2x^2 + 2h_2xy + b_2y^2 = C_2$ by h_1 and subtracting we have,

$$(a_1h_2 - a_2h_1)x^2 + (b_1h_2 - b_2h_1)y^2 = C_1h_2 - C_2h_1$$

Now, this will represent a circle if coefficient of x^2 = coefficient of y^2

$\Rightarrow \quad a_1h_2 - a_2h_1 = b_1h_2 - b_2h_1$

$\Rightarrow \quad (a_1 - b_1)h_2 = (a_2 - b_2)h_1$

Hence, the correct answer is (A).

43. Clearly, $\frac{2ae}{3} = a \Rightarrow e = \frac{3}{2}$

$$\therefore \quad S = \left(\frac{3a}{2}, 0\right)$$

Directrix is $x = \frac{2a}{3}$

$\therefore$ Equation of hyperbola will be

$$\left(x - \frac{3a}{2}\right)^2 + y^2 = \frac{9}{4}\left(x - \frac{2a}{3}\right)^2$$

This reduces to $5x^2 - 4y^2 = 5a^2$

Hence, the correct answer is (D).

44. Director circle is set of points from where drawn tangents are perpendicular in this case $x^2 + y^2 = a^2 - b^2$ (equation of director circle), i.e., $x^2 + y^2 = -9$ is not a real circle so there is no points from where tangents are perpendicular.

Hence, the correct answer is (C).

45. $R^2 - 16 < 0 \Rightarrow |R| < 4$

Hence, the correct answer is (A).

46. $e_1 = 1,\ e_2 = \sqrt{2}$

$\therefore$ Irrational roots occur in pairs.

$$\Rightarrow \quad ax^3 + bx^2 + cx + d = (x - 1)(x + \sqrt{2})(x - \sqrt{2})$$

Put $x = 1 \Rightarrow a + b + c + d = 0$

Hence, the correct answer is (A).

47. The product of perpendiculars from any point $P(\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is equal to $\frac{a^2b^2}{a^2+b^2}$.

Hence, the correct answer is (B).

48. Focus $S = (ae, 0)$ Equation of one asymptote is $bx - ay = 0$ Let (h, k) be the foot of the perpendicular from s to $bx - ay = 0$

Then $\frac{h-ae}{b} = \frac{k-0}{-a} = \frac{-abe}{a^2+b^2} \Rightarrow \frac{h-ae}{b} = \frac{-abe}{a^2e^2}$ and

$$\frac{k}{-a} = \frac{-abe}{a^2e^2}$$

On simplification, we get, $h = a/e, k = b/e$

Foot of the perpendicular is $(a/e, b/e)$.

Hence, the correct answer is (B).

49. Solve $y = \frac{x}{2}$ and $y = x^2$

$$x = 0 \text{ and } x = \frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2} < \alpha < \frac{1}{2}$$

Hence, the correct answer is (A).

50. $(x-3)^2 + (y+1)^2 = (4x+3y)^2$

$$(x-3)^2 + (y+1)^2 = 25\left(\frac{4x+3y}{5}\right)^2$$

$$PS = 5PM$$

$\therefore$ Directrix is $4x + 3y = 0$ and focus $(3, -1)$

So, transverse axis has slope $= \frac{3}{4}$ and equation of transverse axis $y + 1 = \frac{3}{4}(x-3)$

$\Rightarrow \quad 3x - 4y = 13$

Hence, the correct answer is (C).

51. Since area of the triangle formed by tangent at any point lying on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its asymptotes is always constant and is equal to ab. Therefore, required area is $2 \times 3 = 6$ sq. units.

Hence, the correct answer is (A).

52. $x^2 + \frac{c^4}{x^2} = a^2$

$\Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$

$\therefore$ Option (D) does not hold.

Hence, the correct answer is (D).

53. Locus of point of intersection of perpendicular tangents is director circle for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ equation of director circle is $x^2 + y^2 = a^2 - b^2$, which is real, if $a > b$.

$\Rightarrow \quad B$ is the correct option.

Hence, the correct answer is (B).

54. $y = 4x^2$ and $\frac{1}{4}y = x^2$

Using $\frac{1}{4a^2}y - \frac{y^2}{16} = 1$

$\Rightarrow \quad 4y - a^2y^2 = 16a^2$

$\Rightarrow \quad a^2y^2 - 4y + 16a^2 = 0$

$\Rightarrow \quad D \geq 0$ for intersection of two curves

$\Rightarrow \quad 16 - 4a^2(16a^2) \geq 0$

$\Rightarrow \quad 1 - 4a^4 \geq 0$

$\Rightarrow \quad (2a^2) \leq 1$

$\Rightarrow \quad \left|\sqrt{2}a\right| \leq 1 \Rightarrow -\frac{1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$

Hence, the correct answer is (A).

55. Equations of asymptotes are $4y - 3x = 0$ and $4y + 3x = 0$ Since point $(2, 2)$ lies above the asymptotes $4y - 3x = 0$, Hence, point of constant of pair of tangent are in III and IV quadrants.

Hence, the correct answer is (D).

56. Midpoint is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\therefore$ Equation of the chord to the hyperbola $xy = c^2$ whose midpoint is M, is $\frac{x}{\frac{x_1+x_2}{2}} = \frac{y}{\frac{y_1+y_2}{2}} = 2$

$$\Rightarrow \quad \frac{x}{x_1+x_2} + \frac{y}{y_1+y_2} = 1$$

Hence, the correct answer is (A).

57. If two concentric conics intersect orthogonally then they must be confocal, so ellipse and hyperbola will be confocal.

$\Rightarrow \quad (\pm ae, 0) = (\pm 1, 0)$

(Foci of hyperbola are $(\pm 1, 0)$)

Hence, the correct answer is (B).

58. Let P on the ellipse is $(a\cos\theta, b\sin\theta)$.

Slope of tangent at P on the ellipse $m_1 = -\frac{b\cos\theta}{a\sin\theta}$

Slope of tangent at P on the hyperbola $x^2 - y^2 = c^2$, is

$$m_2 = \frac{a\cos\theta}{b\sin\theta}$$

Since these curves are intersecting at right angle

$$\therefore \quad m_1 m_2 = -1$$

$$-\frac{b}{a} \times \frac{\cos\theta}{\sin\theta} \times \frac{a}{b}\frac{\cos\theta}{\sin\theta} = -1 \Rightarrow \tan^2\theta = 1$$

$P(a\cos\theta, b\sin\theta)$ also lies on hyperbola

$$\therefore \quad a^2\cos^2\theta - b^2\sin^2\theta = c^2$$

$$a^2 - b^2\tan^2\theta = c^2 + c^2\tan^2\theta$$

$$\therefore \quad a^2 - b^2 = c^2 + c^2 \qquad \left[\because \tan^2\theta = 1\right]$$

$$a^2 - b^2 = 2c^2$$

Alternate:

Same foci

$$\therefore \quad ae = \sqrt{2}\,c$$

$$\Rightarrow \quad a^2 - b^2 = 2c^2$$

Hence, the correct answer is (C).

59. Hyperbola is $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1$

Co-ordinates of vertices are ($\pm \cos\alpha$, 0), eccentricity of the hyperbola is $e = \sqrt{1 + \dfrac{\sin^2\alpha}{\cos^2\alpha}} = |\sec\alpha|$

$\therefore$ Co-ordinates of foci are thus ($\pm$ 1, 0), which are independent of α.

Directrix is $x = \pm\cos^2\alpha$

Hence, the correct answer is (B).

60. $y^2 = 8k, xy = -1$

Let $P\left(t, \dfrac{-1}{t}\right)$ be any point on $xy = -1$

Equation of the tangent to $xy = -1$ at $P\left(t, \dfrac{-1}{t}\right)$ is

$$\frac{xy_1 + yx_1}{2} = -1$$

$$\frac{-x}{t} + yt = -2$$

$$y = \frac{x}{t^2} + \left(\frac{-2}{t}\right) \qquad (1)$$

If (1) is tangent to the parabola $y^2 = 8x$, then

$$\frac{-2}{t} = \frac{2}{1/t^2} \Rightarrow t^3 = -1$$

$$t = -1$$

$\therefore$ Common tangent is $y = x + 2$

Hence, the correct answer is (D).

61. Equation of tangent to the hyperbola:

$$y = mx \pm \sqrt{m^2a^2 - a^2}$$

$\Rightarrow$ Let $P(x_1, y_1)$ be locus

$$\Rightarrow \quad y - mx = \pm\sqrt{m^2a^2 - a^2}$$

S.B.S

$$\Rightarrow \quad m^2\left(x_1^2 - a^2\right) - 2y_1x_1m + y_1^2 + a^2 = 0$$

$$m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}; \; m_1m_2 = \frac{y_1^2 + a^2}{x_1^2 - a^2}$$

$$\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\Rightarrow \quad (1 + m_1m_2)^2 = (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$\Rightarrow \quad \left(1 + \frac{y_1^2 + a^2}{x_1^2 - a^2}\right)^2 = \left(\frac{2x_1y_1}{x_1^2 - a^2}\right)^2 - 4\left(\frac{y_1^2 + a^2}{x_1^2 - a^2}\right)$$

Hence, the correct answer is (C).

62. Area of parallelogram is $\dfrac{ab}{2} = \dfrac{4 \times 2}{2} = 4$ sq. units

Hence, the correct answer is (B).

63. Let m be the slope of the common tangent.

$$\therefore \quad \frac{2}{m} = \sqrt{m^2 - 3} \Rightarrow m = \pm 2$$

Equation of common tangents are $y = 2x + 1$ or $y = -2x - 1$

Hence, the correct answer is (A).

64. Solving the normals at θ, ϕ and using $\theta + \phi = \dfrac{\pi}{2}$

Hence, the correct answer is (C).

65. Equation of the chord as $S_1 = S_{11}$

$$\Rightarrow \quad x + 2y - 1 = 0$$

Required area $= \dfrac{1}{4}$ sq. units

Hence, the correct answer is (D).

66. Confocal ellipse and hyperbola cut at right angles.

Hence, the correct answer is (D).

67. $A = 1, B = 2, C = 4, D = 0$

From (120, 24) we can draw 3 normals to parabola

$$(y - 24)^2 = 4(x - 10) \qquad (\because (x - 10) > 2)$$

Hence, the correct answer is (D).

68. Let the midpoint = (h, k)

$\therefore$ Equation of the chord $xh - yk = h^2 - k^2$

$$yk = xh + (k^2 - h^2)$$

$$y = \frac{xh}{k} + \frac{(k^2 - h^2)}{k}$$

$$\frac{k^2 - h^2}{k} = \frac{ak}{h}$$

$$\Rightarrow \quad k^2h - h^3 = ak^2 \Rightarrow k^2(h - a) = h^3$$

$$\therefore \quad x^3 = y^2(x - a)$$

Hence, the correct answer is (B).

69. Equation of director circles of ellipse and hyperbola are respectively.

$$x^2 + y^2 = a^2 + b^2$$

$$x^2 + y^2 = a^2 - b^2$$

$$a^2 + b^2 = 4r^2 \qquad \ldots(1)$$

$$a^2 - b^2 = r^2 \qquad \ldots(2)$$

So, $2a^2 = 5r^2$; $a^2 = \dfrac{5r^2}{2}$

$$b^2 = 4r^2 - \frac{5r^2}{2} ; \; b^2 = \frac{3r^2}{2}$$

$$e_n^2 = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \quad e_n^2 = 1 - \frac{3r^2}{2} \times \frac{2}{5r^2} = 1 - \frac{3}{5} = \frac{2}{5}$$

$$e_e^2 = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad e_h^2 = 1 + \frac{3}{5} = \frac{8}{5}$$

So, $4e_h^2 - e_e^2 = 4 \times \dfrac{8}{5} - \dfrac{2}{5} = \dfrac{30}{5} = 6$

Hence, the correct answer is (C).

70. $x^2 - 2\sqrt{2}\,x - 2(y^2 + 2\sqrt{2}\,y) = 6$

$$\Rightarrow \quad (x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 4$$

$$\Rightarrow \quad \frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$$

$$b^2 = a^2(e^2 - 1) \Rightarrow 2 = 4(e^2 - 1)$$

$$\Rightarrow \quad e^2 - 1 = 1/2 \Rightarrow e = \sqrt{3}/2$$

$$\text{Area} = \frac{1}{2}(ae - a) \times b^2/a$$

$$= e - 1 = \left(\sqrt{\frac{3}{2}} - 1\right) \text{sq. units}$$

Hence, the correct answer is (B).

71. Making homogeneous equation of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with the help $x\cos\alpha + y\sin\alpha = P$

coefficient of x^2 + coefficient of $y^2 = 0$

$$\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{P^2} \Rightarrow P = \frac{ab}{\sqrt{b^2 - a^2}}$$

P is the length of perpendicular drawn from (0, 0) to $x\cos\alpha + y\sin\alpha = P$

$$\text{Radius} = P = \frac{ab}{\sqrt{b^2 - a^2}}$$

Hence, the correct answer is (C).

72. Equation of any chord joining the points $P(\theta_1)$ and $Q(\theta_2)$ is,

$$\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right).$$

If it passes through $(ae, 0)$, then

$$e^2\cos^2\left(\frac{\theta_1 - \theta_2}{2}\right) = \cos^2\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\Rightarrow \quad \lambda = e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

Hence, the correct answer is (A).

73. Equation of normal at any point $P(ct, \frac{c}{t})$ on $xy = c^2$, is $xt^3 - yt - ct^4 + c = 0$

If it passes through $Q(h, k)$, then

$$ct^4 - ht^3 + kt - c = 0$$

If it's roots are t_1, t_2, t_3 and t_4, then

$$t_1 + t_2 + t_3 + t_4 = h/c$$

$$\Rightarrow \quad ct_1 + ct_2 + ct_3 + ct_4 = h$$

$$\Rightarrow \quad \Sigma x_i = h, \; \Sigma t_1t_2t_3 = -\frac{k}{c}, \; t_1t_2t_3t_4 = -1$$

$$\Rightarrow \quad (ct_1)(ct_2)(ct_3)(ct_4) = -c^4$$

$$\Rightarrow \quad \Sigma\frac{c}{t_i} = k \;\Rightarrow \Sigma y_i = k \text{ and } x_1x_2x_3x_4$$

$$= -c^4 \Rightarrow \frac{\Sigma x_i \Sigma y_i}{x_1x_2x_3x_4} = -\frac{hk}{c^4}$$

Hence, the correct answer is (D).

74. $y = \dfrac{x+9}{x+5} = 1 + \dfrac{4}{x+5}$

$\dfrac{dy}{dx}$ at $(x_1, y_1) = \dfrac{-4}{(x_1+5)^2}$

Equation of tangent

$$y - y_1 = \frac{-4}{(x_1+5)^2}(x - x_1)$$

$$y - 1 - \frac{4}{x_1+5} = \frac{-4}{(x_1+5)^2}\cdot(x - x_1)$$

Since it passes through (0, 0)

$$(x_1+5)^2 + 4(x_1+5) + 4x_1 = 0$$

$$x_1 = -15 \text{ or } x_1 = -3$$

So equations are $x + 25y = 0$ or $x + y = 0$.

Hence, the correct answer is (A).

75. Equation of tangent at point $P(\theta)$

$$\frac{x\sec\theta}{3} - \frac{y\tan\theta}{2} - 1 = 0 \qquad \ldots(1)$$

Since Equation (1) will be a tangent to the circle

$$\therefore \quad \frac{\dfrac{4\sec\theta}{3} - 1}{\sqrt{\dfrac{\sec^2\theta}{9} + \dfrac{\tan^2\theta}{4}}} = 4$$

By solving it, we will get,

$$2x - \sqrt{5}y + 4 = 0$$

Hence, the correct answer is (B).

76. It turns out that P has to be on the left branch. x-co-ordinate is found to be $-64/5$.

Hence, the correct answer is (B).

77. The reflection of (α, β) in the line $y = 2x$ is

$$(\alpha_1, \beta_1) = \left(\frac{4\beta - 3\alpha}{5}, \frac{4\alpha + 3\beta}{5}\right) = \alpha_1\beta_1 = 1$$

$$\Rightarrow \quad 12\alpha^2 - 7\alpha\beta - 12\beta^2 + 25 = 0$$

Hence, the correct answer is (A).

78. Let, $m_1 = \tan\alpha, m_2 = \tan\beta$, Let $P = (h, k)$

m_1, m_2 are the roots of $K = mh + \dfrac{4}{m} \Rightarrow hm^2 - Km + 4 = 0$

$$m_1 + m_2 = \frac{K}{h};\ m_1 m_2 = \frac{4}{h}$$

$$m_1^2 + m_2^2 = \frac{K^2}{h^2} - \frac{8}{h} = 4$$

Locus of P is $y^2 - 8x = 4x^2$

$$\Rightarrow \quad y^2 = 4(x+1)^2 - 4$$

$$\Rightarrow \quad \frac{(x+1)^2}{1} - \frac{y^2}{4} = 1$$

Hence, the correct answer is (B).

79. Since $\dfrac{e}{2}$ and $\dfrac{e'}{2}$ are eccentricities of a hyperbola and its conjugate

$$\therefore \quad \frac{4}{e^2} + \frac{4}{e'^2} = 1 \Rightarrow 4 = \frac{e^2 e'^2}{e'^2 + e^2}$$

Line passing through the points $(e, 0)$ and $(0, e')$

$$e'x + ey - ee' = 0$$

It is tangent to the circle, $x^2 + y^2 = r^2$

$$\therefore \quad \frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$$\therefore \quad r^2 = \frac{e^2 e'^2}{e^2 + e'^2} = 4$$

$$\therefore \quad r = 2$$

Hence, the correct answer is (B).

80. $\dfrac{2a}{e} = 4 \Rightarrow a = 2e$, P is nearest to $3x + 4y - 10 = 0$

$$\Rightarrow \quad ae - \frac{a}{e} = 8 \Rightarrow e = \sqrt{5},\ a = 2\sqrt{5}$$

$$CP = ae = 10$$

Hence, the correct answer is (C).

81. Equation of tangent to parabola

$$y = mx - am^2 \qquad \ldots(1)$$

Equation of chord of hyperbola whose midpoint is

(h, k) is $hx - ky = h^2 - k^2$ $\qquad \ldots(2)$

From (1) and (2)

$$\frac{m}{h} = \frac{1}{k} = \frac{am^2}{h^2 - k^2} \Rightarrow k^3 = h^2(k - a)$$

Hence, the correct answer is (B).

82. $3x^2 - y^2 = 3,\ \dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$

Equation of tangent in terms of slope,

$$y = mx \pm \sqrt{(m^2 - 3)}$$

Here, $m = 2$, then $y = 2x \pm 1$

Hence, the correct answer is (B).

83. Locus of centre of circle is a rectangular hyperbola hence its eccentricity is $\sqrt{2}$

Hence, the correct answer is (C).

84. $S, T = \left(2 + r\cos\theta, 0 + r\sin\theta\right)$

$$r^2\cos\theta\sin\theta + 2\sin\theta - 4 = 0$$

$$RS.RT = \frac{4}{\sin\theta\cos\theta} = \frac{8}{\sin 2\theta} \geq 8$$

Hence, the correct answer is (C).

85. Normal: $ax\cos\theta + by\cot\theta = a^2 + b^2$

$$L = \left(\frac{a^2+b^2}{a}\sec\theta, 0\right),\ M = \left(0, \frac{a^2+b^2}{b}\tan\theta\right)$$

Locus is $\dfrac{x^2}{\dfrac{a^2e^2}{4}} - \dfrac{y^2}{\dfrac{a^2e^2}{4b^2}} = 1$

$$e_1 = \frac{e}{\sqrt{e^2-1}}$$

Hence, the correct answer is (B).

86. $b^2 = 9$

$$\frac{b}{a} = \tan 30° = \frac{1}{\sqrt{3}}$$

$\therefore$ $a^2 = 3b^2 = 27$

$\therefore$ Required locus is director circle of the hyperbola and which is $x^2 + y^2 = 27 - 9,\ x^2 + y^2 = 18$

If $\dfrac{b}{a} = \tan 60°$ is taken, then

$$a^2 = \frac{b^2}{3} = \frac{9}{3} = 3$$

$\therefore$ Required locus is $x^2 + y^2 = 3 - 9 = -6$ which is not possible.

Hence, the correct answer is (D).

87. $x - 2 = m$

$$y + 1 = \frac{4}{m}$$

$\therefore$ $(x - 2)(y + 1) = 4$

$\Rightarrow$ $XY = 4$, where $X = x - 2,\ Y = y + 1$

$$S = (x-2)^2 + (y+1)^2 = 25$$

$\Rightarrow$ $X^2 + Y^2 = 25$

Curve 'C' and circle S both are concentric

$\therefore$ $OP^2 + OQ^2 + OR^2 + OS^2 = 4r^2 = 4.25 = 100$

Hence, the correct answer is (B).

88. (i) Equation of chord joining α and β is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

$\therefore$ $\alpha + \beta = 3\pi$

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b} = 0$$

It passes through the centre (0, 0)

Hence, the correct answer is (B).

89. Ordinate of the point of intersection of the line $\dfrac{x}{a} - \dfrac{y}{b} = m$ and the hyperbola is given by

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b} + \frac{2y}{b}\right) = 1$$

$$\Rightarrow\quad m\left(m + \frac{2y}{b}\right) = 1$$

$$\Rightarrow\quad y = \frac{b\left(1-m^2\right)}{2m}$$

Similarly ordinate of the point of intersection of the line $\dfrac{x}{a} + \dfrac{y}{b} = m$ and the hyperbola is given by

$$y = \frac{b\left(m^2-1\right)}{2m}$$

$\therefore$ Sum of the ordinates is 0

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. The difference between the focal distances is a constant for a hyperbola. For a rectangular hyperbola, latus rectum = transverse axis.

$S(2, 0)\ S'(h, k)\ P(0, 0)\ |S'p - Sp| = 4$; $\left|\sqrt{h^2+k^2} - 2\right| = 4$

$$\Rightarrow\quad \sqrt{h^2+k^2} = 6$$

$$\Rightarrow\quad h^2 + k^2 = 36$$

Locus of (h, k) is $x^2 + y^2 = 36$

Hence, (B), (C) and (D) are correct.

2. Let conjugate hyperbola is $xy - 2x - 2y + k = 0$.
It passes through (3, 3)

$\Rightarrow$ $k = 3 \Rightarrow xy - 2x - 2y + 3 = 0$

$(x - 2)(y - 2) = 1$

$XY = 1$

Focus $(2 + \sqrt{2}, \sqrt{2} + 2), (2 - \sqrt{2}, 2 - \sqrt{2})$

Hence, (A), (B) and (D) are correct.

3. $1/(t_1t_2) = -1;$

Required circle $(x - ct_1)(x - ct_2) + \left(y - \frac{c}{t_1}\right)\left(y - \frac{c}{t_2}\right) = 1$

Using $t_1t_2 = -1$ gives $(x^2 + y^2 - 2c^2) - (t_1 + t_2)(x - y) = 0$

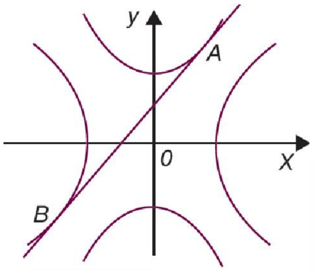
$\Rightarrow \quad S + \lambda L = 0$

Solve $x^2 + y^2 = 2c^2$ and $x = y$

We get (c, c), $(-c, -c)$

Hence, (A) and (D) are correct.

4.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(1)$$

and $$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \ldots(2)$$

Tangent to (1), $y = mx + \sqrt{a^2m^2 - b^2}$

If this is also tangent to $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1$

Then, $a^2m^2 - b^2 = (-b^2)m^2 - (-a^2) = a^2 - b^2m^2$

$(a^2 + b^2)m^2 = a^2 + b^2$

$m = +1$

Hence, 4 common tangents are $y = \pm x \pm \sqrt{a^2 - b^2}$

Hence, (A), (B), (C) and (D) are correct.

5. Area of the $\Delta = ab$ sq. units

$H: x^2/3 - y^2/1 = 1$

Asymptotes $y = x/\sqrt{3}$ and $y = -x/\sqrt{3}$

Required points are $(0, 0)(\sqrt{3}, 1)(\sqrt{3}, -1)$

$\therefore$ equilateral triangle

Hence, (B) and (C) are correct.

6. Solving $xy = c^2$ and $x^2 + y^2 = a^2$

$$x^2 + \frac{c^4}{x^2} = a^2$$

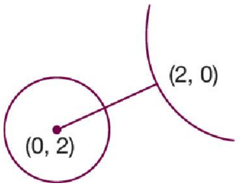
$$x^4 - a^2x^2 - c^4 = 0$$

$$\Rightarrow \quad \sum x_i = 0;\ \sum y_i = 0$$

$$x_1x_2x_3x_4 = c^4 \Rightarrow y_1y_2y_3y_4 = c^4$$

Hence, (A), (B), (C) and (D) are correct.

7.



Director circle $x^2 + (y - 2)^2 = 4$

Centre $(0, 2)$

Radius of director circle $= \sqrt{a^2 - b^2} = 2$

$$\left(2\sqrt{2}\right)^2 - b^2 = 4 \qquad (\because a = 2\sqrt{2})$$

$$b^2 = 4$$

$$e = \frac{\sqrt{3}}{\sqrt{2}}$$

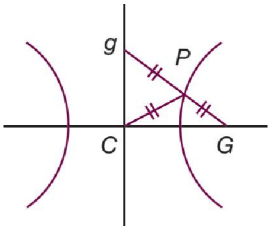
$\therefore$ Foci lies on the line $\dfrac{x-0}{\frac{-1}{\sqrt{2}}} = \dfrac{y-2}{\frac{1}{\sqrt{2}}}$ at a distance ae units from centre.

$$\therefore \quad \frac{x-0}{-\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm 2\sqrt{3}$$

$$(x, y) = \left(\sqrt{6}, 2 - \sqrt{6}\right) \text{ or } \left(-\sqrt{6}, 2 + \sqrt{6}\right)$$

Hence, (C) and (D) are correct.

8. Let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$. If P be any point on it as $(a\sec\phi, a\tan\phi)$, then the equation to the normal at P will be given by $x\sin\phi + y = 2a\tan\phi$.



Its intersection with $y = 0$ gives $x = 2a\sec\phi$. As the point of intersection is G and C is the centre of the hyperbola, $CG = 2a\sec\phi$.

Again, the intersection of the normal with $x = 0$ gives If g denotes this point of intersection, we have,

$$Cg = 2a\tan\phi$$

Now, $PG^2 = (a\sec\phi - 2a\sec\phi)^2 + (a\tan\phi - 0)^2$

$$= a^2\left(\sec^2\phi + \tan^2\phi\right)$$

$$Pg^2 = (a\sec\phi - 0)^2 + (a\tan\phi - 2a\tan\phi)^2$$

$$= a^2\left(\sec^2\phi + \tan^2\phi\right)$$

$$PC^2=(a\sec\phi-0)^2+(a\tan\phi-0)^2$$

$$=a^2(\sec^2\phi+\tan^2\phi)$$

Hence, $PG=Pg=PC$.

Hence, (A), (B) and (C) are correct.

9. $e^2=1+\frac{9}{3}=4$

⇒ $e=2=\sec\theta$

∴ Option (B) is incorrect.

⇒ $\theta=60^\circ$

Since, angle between the two asymptotes is 120°

⇒ Acute angle is 60°

∴ Option (A) is incorrect.

Length of latus rectum $=\frac{2b^2}{a}=2.\frac{3}{3}=2$ units

∴ Option (C) is incorrect.

$$p_1p_2=\frac{ab(\sec\theta+\tan\theta)}{\sqrt{a^2+b^2}}.\frac{ab(\sec\theta-\tan\theta)}{\sqrt{a^2+b^2}}$$

$$=\frac{a^2b^2}{a^2+b^2}(\sec^2\theta-\tan^2\theta)=\frac{9.3}{12}=\frac{9}{4}.$$

∴ Option (D) is incorrect.

Hence, (B) and (D) are correct.

10. For the ellipse,

$$a=5 \text{ and } e=\sqrt{\frac{25-9}{25}}=\frac{4}{5}$$

∴ $ae=4$

Hence, the foci are $(-4,0)$ and $(4,0)$.

For the hyperbola, $ae=4$, $e=2$

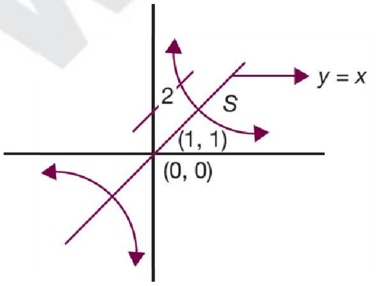
$a=2$

$b^2=4(4-1)=12 \Rightarrow b=\sqrt{12}$

$a<b$; no director circle possible

Hence, (A), (B) and (D) are correct.

11.



$e=\sqrt{2}$

Vertex is $y=x$ and $xy=1$

∴ (1, 1) (–1, –1)

Focus is at a distance of ae from the centre.

$a=\sqrt{1+1}=\sqrt{2}$

$ae=\sqrt{2}.\sqrt{2}=2$

∴ Focus $=(\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2})$

Line of latus rectum is perpendicular to $y=x$ and passes through focus.

Hence, (B) and (C) are correct.

12. Equations of asymptotes are $4y-3x=0$ and $4y+3x=0$ Since point (2 , 2) lies above the asymptotes $4y-3x=0$, the point of contact of pair of tangent are in IIIrd and IVth quadrants.

Hence, (C) and (D) are correct.

13. Equation of hyperbola is $xy-y-2x-2=0$

$(x-1)(y-2)=4 \Rightarrow XY=4$

Conjugate hyperbola $XY=-4$

$\frac{a^2}{2}=4 \Rightarrow a=2\sqrt{2}$

Focus of $H(1-2\sqrt{2}, 2-2\sqrt{2})(1+2\sqrt{2}, 2+2\sqrt{2})$

Focus of H' with respect to XY system $(-2\sqrt{2},2\sqrt{2})$ and $(2\sqrt{2},-2\sqrt{2})$

So, focus in xy system $(1-2\sqrt{2}, 2+2\sqrt{2})$ and $(1+2\sqrt{2}, 2-2\sqrt{2})$ equation of transverse axis $Y=X$, or $y-2=x-1 \Rightarrow x-y+1=0$

Hence, (A) and (D) are correct.

14. Eliminating m from given equation, we get $3x^2-y^2=48$, which is hyperbola.

Its eccentricity $=\sqrt{1+\frac{48}{16}}=2$

Hence, (A) and (C) are correct.

15. Given hyperbola can be written as $\frac{(x-1)^2}{16}-\frac{(y-1)^2}{9}=1$

$$\Rightarrow \frac{X^2}{16}-\frac{Y^2}{9}=1 \qquad (\text{where } X=x-1,\ Y=y=1)$$

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{9}{16}}=\frac{5}{4}$$

Directrices are $X=\pm\frac{a}{e}$

$\Rightarrow x-1=\pm\frac{16}{5}$

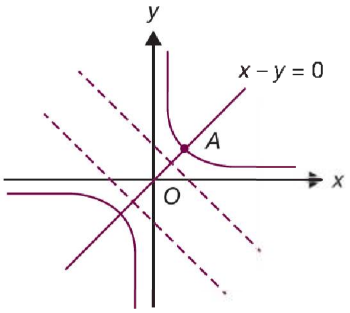
$\Rightarrow \quad x = \frac{21}{5}$ and $x = -\frac{11}{5}$

Length of Latus rectum $= \frac{2b^2}{a} = \frac{9}{2}$ units and foci are

$\Rightarrow \quad X = \pm\, ae, Y = 0 \Rightarrow (6, 1)$ and $(-4, 1)$

Hence, (A), (B), (C) and (D) are correct.

16.



Directrix is $x + y = k$

Since $OA = \frac{a}{e} = \frac{\sqrt{2}c}{\sqrt{2}} = c, \qquad \left(\because c^2 = \frac{a^2}{2}\right)$

$\therefore \quad \frac{|k|}{\sqrt{2}} = c \Rightarrow k = \pm\sqrt{2}c$

Hence, directrices are $x + y = \pm\sqrt{2}c$

Hence, (B) and (C) are correct.

17. ΔABC is a right angled triangle. Hence, the result follows.

Let $A\left(2t, \frac{2}{t}\right)$

$$m_T = -\frac{4}{x^2} = -\frac{1}{t^2}$$

Slope of normal $= t^2 = m_{BC}$

$$t^2 = \frac{\frac{2}{3}+2}{6+2} = \frac{1}{3}$$

$$t = \pm\frac{1}{\sqrt{3}}$$

$$A\left(\frac{2}{\sqrt{3}}, 2\sqrt{3}\right) \text{or} \left(-\frac{2}{\sqrt{3}}, -2\sqrt{3}\right)$$

Hence, (A), (B) and (C) are correct.

18. Locus of point of intersection of tangents is director circle $x^2 + y^2 = a^2 - b^2$

If $a^2 > b^2$ there are infinite points on circle $\Rightarrow e^2 < 2$

$n > 1 \Rightarrow e^2 < \sqrt{2}$

If $a^2 < b^2$ there does not exist any point on the plane $e^2 > 2 \Rightarrow e > \sqrt{2}$

If $a^2 = b^2$ there exists exactly one point (centre of hyperbola) $\Rightarrow e = \sqrt{2}$

No tangent can be drawn from centre

Hence, (C) and (D) are correct.

19. $y^2 = 32x$

Let equation of tangent $y = mx + \frac{8}{m}$

$$\frac{64}{m^2} = \frac{8}{9}m^2 - \frac{8}{9} \Rightarrow m^2(m^2 - 1) = 72$$

$$m = \pm\, 3, y = \pm\, 3x \pm 8/3.$$

Hence, (B) and (C) are correct.

20. Any point on $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$.

As it lies on the given circle, we get, $c^2t^2 + \frac{c^2}{t^2} = a^2$

$\Rightarrow \quad c^2t^4 - a^2t^2 + c^2 = 0$

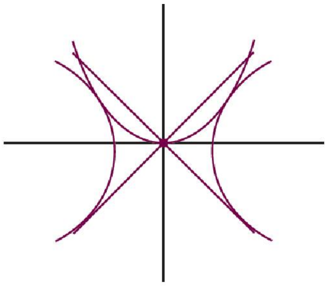
Thus, $t_1t_2t_3t_4 = 1, t_1 + t_2 + t_3 + t_4 = 0, \Sigma t_1t_2$

$$= -\frac{a^2}{c^2}, \Sigma t_1t_2t_3 = 0$$

Thus, (A), (B), (C) are true.

Hence, (A), (B) and (C) are correct.

21.



(α, α^2) lie on the parabola $y = x^2$

(α, α^2) must lie between the asymptotes of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ in Ist and IInd quadrants

$\therefore \quad$ Asymptotes are $y = \pm 2x$

$\therefore \quad 2\alpha < \alpha^2$

$\Rightarrow \quad \alpha < 0$ or $\alpha > 2$

$-2\alpha < \alpha^2$

$\alpha < -2$ or $\alpha > 0$

$\therefore \quad \alpha \in (-\infty, -2)$ or $\alpha \in (2, \infty)$

Hence, (C) and (D) are correct.

22. $S_1P = S_2P$

$$\Rightarrow \quad a - e\alpha = E\alpha - \left(\frac{a}{2}\right).$$

Also, $\alpha = \dfrac{ae + \dfrac{a}{2}E}{2}$

Eliminating α we get, $E^2 + 3eE + (2e^2 - 6) = 0$

$$\Rightarrow \quad E = \frac{\sqrt{e^2 + 24} - 3e}{2}$$

Hence, (B) and (C) are correct.

23. Ellipse is $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

Here, $\dfrac{3}{4} = 1 - e^2 \Rightarrow e = \dfrac{1}{2}$

Foci are $(\pm 1, 0)$

Now, the hyperbola is having same focus, i.e., $(\pm 1, 0)$.
Let e_1 be the eccentricity of hyperbola

$$2ae_1 = 2$$

However, $2a = \dfrac{1}{2}$

So, $e_1 = 4$

$$b^2 = a^2\left(e_1^2 - 1\right) = \frac{1}{16}(16 - 1) = \frac{15}{16}$$

So, the equation of the hyperbola is $\dfrac{x^2}{\frac{1}{16}} - \dfrac{y^2}{\frac{15}{16}} = 1$

$$\Rightarrow \quad \frac{x^2}{1} - \frac{y^2}{15} = \frac{1}{16}$$

Its distance between the directrices $= \dfrac{2a}{e_1} = \dfrac{1}{2 \times 4}$

$= \dfrac{1}{8}$ units

Length of latus rectum $= \dfrac{2b^2}{a}$

$= \dfrac{2 \times 15 \times 4}{16 \times 1} = \dfrac{15}{2}$ units

Hence, (B), (C) and (D) are correct.

24. Let m_1, m_2 be the slopes of tangents.

$$\therefore \quad m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2} = \frac{32}{3},\ m_1m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2} = 23$$

$$\therefore \quad m_1 = 3,\ m_2 = \frac{23}{3}$$

Tangents are $3x - y + 2 = 0$; $23x - 3y - 22 = 0$

Alternate:

$$y - 8 = m(x - 2)$$

$$y = mx + 8 - 2m$$

For tangency $c^2 = a^2m^2 - b^2$

$\Rightarrow \quad (8 - 2m)^2 = m^2 - 5$

We get $m = 3, 23/3$

Hence, (A) and (C) are correct.

25. Equations of directrices are

$$x = \pm\frac{25}{\sqrt{41}} \qquad \ldots(1)$$

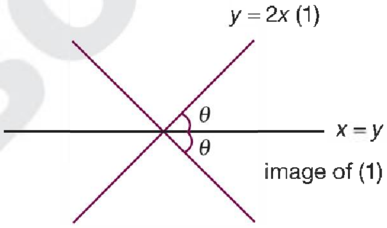
Equations of asymptotes of hyperbola are

$$\frac{x^2}{25} - \frac{y^2}{16} = 0 \qquad \ldots(2)$$

Solving (1) and (2) we get, $\left(\pm\dfrac{25}{\sqrt{41}}, \pm\dfrac{20}{\sqrt{41}}\right)$

Hence, (A) and (B) are correct.

26.



Other asymptote is the image of $y = 2x$ in the line $x = y$, i.e., $x = 2y$

$\Rightarrow$ Hyperbola is $(x - 2y)(2x - y) = K$

$\because$ It passes through $(3, 4) \Rightarrow K = -10$

$\because$ Angle between asymptotes $= 2\sec^{-1} e$

$$\Rightarrow \quad \tan^{-1}\left(\frac{3}{4}\right) = 2\sec^{-1} e \Rightarrow e = \frac{\sqrt{10}}{3}$$

Hence, (B) and (D) are correct.

27. $\dfrac{(x-1)^2}{\left(\sqrt{3}\right)^2} - \dfrac{(y+2)^2}{4^2} = 1$

$$2a = 2\sqrt{3}, 2b = 8$$

Centre $(1, -2)$

$$e = \sqrt{19/3}$$

Hence, (A), (B) and (C) are correct.

28. $y = mx \pm \sqrt{m^2 - 3}$

$$m = 2$$

$$y = 2x \pm 1$$

Hence, (B) and (C) are correct.

29. Equation of director circle of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = a^2 - b^2$$

Hence, (A) and (D) are correct.

30. Let $A(5, 12)$ and $B(24, 7)$ be two fixed points.

So, $|OA - OB| = 12 \quad |OA + OB| = 38$

If conic is ellipse $e = \frac{\sqrt{386}}{38} \quad \left\{2ae = \sqrt{386} \text{ and } a = 19\right\}$

And it conic is hyperbola $e = \frac{\sqrt{386}}{12}$

$\left\{2ae = \sqrt{386} \text{ and } a = 6\right\}$

Hence, (A) and (D) are correct.

31. $\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1$

$\Rightarrow \quad \frac{X^2}{16} - \frac{Y^2}{9} = 1, e = \frac{5}{4}$

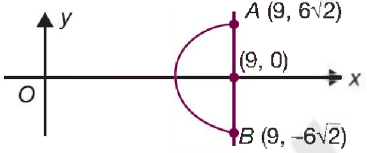
Foci are $(\pm ae, 0) \Rightarrow x + 1 = \pm ae, y - 1 = 0$

$\Rightarrow \quad (4, 1), (-6, 1)$

Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1.



Equation of tangent at

$A\left(9, 6\sqrt{2}\right) : x(9) - y\left(6\sqrt{2}\right) - 9 = 0$

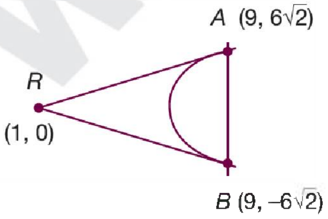
Equation of tangent at

$B\left(9, -6\sqrt{2}\right) : x(9) + y\left(6\sqrt{2}\right) - 9 = 0$

$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$

Hence, the correct answer is (C).

2.



Equation of chord of contact: $yy_1 = 2(x + x_1) - 20 \Rightarrow x = 9$

Hence, the correct answer is (B).

3. Substituting $x = 9$ in the equation of P, $y = \pm 4$

$\therefore \quad T(9, 4)\ T^1(9, -4)\ S(6, 0)$

$\therefore \quad$ Area of $\Delta STT^1 = \frac{1}{2}3(8) = 12$ sq. units

Hence, the correct answer is (C).

Combined Solution to 4, 5 and 6

4. Equation of normal at $\left(ct, \frac{c}{t}\right)$ on $xy = C^2$ is

$$ct^4 - t^3x + ty - c = 0$$

It passes through (α, β) $ct^4 - t^3\alpha + t\beta - c = 0$

$$\sum t_1 = \frac{\alpha}{c}, \sum t_1t_2 = 0, \sum t_1t_2t_3 = -\frac{\beta}{c}$$

$$t_1t_2t_3t_4 = -1$$

$$\sum y_1 = c\sum \frac{1}{t_1} = c\frac{\sum t_1t_2t_3}{t_1t_2t_3t_4} = b$$

$$\sum y_1y_2 = c^2\sum \frac{1}{t_1t_2} = \frac{c^2\sum t_1t_2}{t_1t_2t_3t_4} = 0$$

$$\sum y_i^2 = \left(\sum y_i\right)^2 - 2\sum y_1y_2 = b^2$$

Hence, the correct answer is (C).

5. $\pi x_i = x_1x_2x_3x_4 = c^4t_1t_2t_3t_4 = -c^4$

$\pi y_i = y_1y_2y_3y_4 = -c^4$

$k_1^2 - k_2^2 = 0$

Hence, the correct answer is (D).

6. $\sum x_i^2 - \sum x_i = 0 \Rightarrow a^2 - \alpha = O \Rightarrow \alpha = 0, 1$

Number of possible matrices $= 2^6$

Hence, the correct answer is (D).

7. Equation of hyperbola $(x - 1)(y - 2) = 4$

$\Rightarrow \quad XY = 4$

If normal at $\left(ct, \frac{c}{t}\right)$ intersect curve again at $\left(ct', \frac{c}{t'}\right)$

then $t' = -\frac{1}{t^3}$

$2t = 4 \Rightarrow t = 2$

$(X', Y') = \left(-\frac{1}{4}, -16\right)$

$\Rightarrow \quad (\alpha, \beta - 29) = \left(\frac{3}{4}, -14\right)$

$\Rightarrow \quad \alpha = \frac{3}{4}, \beta = 15$

So $\frac{\beta}{\alpha} = \frac{15 \times 4}{3} = 20$.

Hence, the correct answer is (B).

8. $2y\frac{dy}{dx} = 1$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$

So, $d = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}}$

So, $|\sin\alpha| = 1$

So, number of solution is 3.

Hence, the correct answer is (A).

9. $y^2 = 2\left(x - \frac{1}{2}\right)$

$Y^2 = 2X$

For three normals, $x - \frac{1}{2} > 1$

$x > \frac{3}{2}$

So, only one normal can be drawn.

Hence, the correct answer is (B).

10. $xy = 15! = 2^{11}.\, 3^6.5^3.7^2.11.13$

Number of positive integral solutions = Number of ways of fixing x = Number of factors of 15! = (1 + 11) (1 + 6) (1 + 3) (1 + 2) (1 + 1) (1 + 1) = 4032 total number of integral solutions (+/–) = 2 × 4032 = 8064.

Hence, the correct answer is (D).

11. HCF $(\alpha, \beta) = 1$, α and β will not have common factor other than 1 so, identical prime numbers should not be separated. For example, 2^{11}will completely go with either α or β so the number of solutions $= 2^6 = 64$.

Hence, the correct answer is (A).

12. The largest number whose perfect square can be made with 15! Is 2^5 . 3^3. 5^1. 7^1 so that number of ways of selecting x will be (1 + 5)(1 + 3)(1 + 1)(1 + 1) = 96

Hence, the correct answer is (A).

Combined Solution to 13, 14 and 15

$(ON)(OQ) = 2c^2 \Rightarrow 2c^2 = 4 \Rightarrow c = \sqrt{2}$

So equation of hyperbola is $(x - 1)(y - 2) = 2$

Its foci are (–1, 0), (3, 4) and equation of directrix is $x + y - 5 = 0$, $x + y - 7 = 0$ is the equation of latus rectum of ellipse and tangents at end points of latus rectum always intersects on corresponding directrix. Also circle with, diameter SP, touches auxiliary circle.

13. **The correct answer is (D).**

14. **The correct answer is (C).**

15. **The correct answer is (D).**

16. $xy = 16 \Rightarrow Q\left(4t, \frac{4}{t}\right)$

Normal at Q is $y - \frac{4}{t} = t^2(x - 4t)$

It passes through $P(h, k)$

$\Rightarrow \quad 4t^4 - t^3h + tk - 4 = 0$

$\Rightarrow \quad \sum t_1 = \frac{h}{4}, \sum t_1t_2 = 0, \sum t_1t_2t_3 = -\frac{k}{4}$

$t_1t_2t_3t_4 = -1 =$

$\therefore \quad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{4} \Rightarrow y_1 + y_2 + y_3 + y_4 = k$

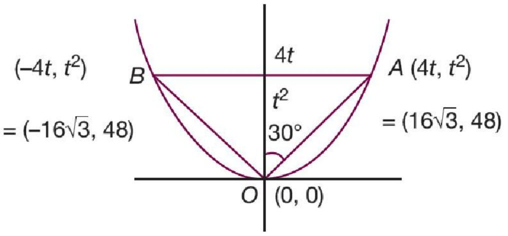
$16\left(t_1^2 + t_2^2 + t_3^2 + t_4^2\right) = h^2$

However, it is given that sum of slopes of normal = sum of feet of normal

$\Rightarrow \quad \frac{h^2}{16} = k \Rightarrow x^2 = 16y$

Hence, the correct answer is (B).

17.



Equation of tangent at $P\left(4t, t^2\right)$ is

$x.4t = 8\left(y + t^2\right) \Rightarrow tx = 2y + 2t^2$

$\Rightarrow \quad A(2t, 0), B\left(0, -t^2\right)$

Midpoint of $AB = \left(t_1 - \frac{t^2}{2}\right) = (x, y)$

$x = t, y = -\frac{t^2}{2}$

$2y = -x^2 \Rightarrow x^2 + 2y = 0$

Hence, the correct answer is (A).

18. From the figure, $\tan 30 = \frac{4t}{t^2} \Rightarrow t = 4\sqrt{3}$

$\Rightarrow$ Area of $\Delta OAB = \frac{1}{2} \times 8t \times t^2$

$= \frac{1}{2} \times 32\sqrt{3} \times 48 = 768\sqrt{3}$ sq. units

Hence, the correct answer is (C).

Combined Solution to 19, 20 and 21

Let three points be $P(a\sec\alpha, b\tan\alpha)$, $R(a\sec\beta, b\tan\beta)$, $T(a\sec\gamma, b\tan\gamma)$

Equation of the chord PR, PT and RT are

$$\frac{x}{\sqrt{2}}\cos\left(\frac{\alpha-\beta}{2}\right) - y\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right) \quad \ldots(1)$$

$$\frac{x}{\sqrt{2}}\cos\left(\frac{\alpha-\gamma}{2}\right) - y\sin\left(\frac{\alpha+\gamma}{2}\right) = \cos\left(\frac{\alpha+\gamma}{2}\right) \quad \ldots(2)$$

$$\frac{x}{\sqrt{2}}\cos\left(\frac{\beta-\gamma}{2}\right) - y\sin\left(\frac{\beta+\gamma}{2}\right) = \cos\left(\frac{\beta+\gamma}{2}\right) \quad \ldots(3)$$

$\because$ PR and PT are perpendicular to each other.

$$\Rightarrow \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sqrt{2}\sin\left(\frac{\alpha+\beta}{2}\right)} \times \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sqrt{2}\sin\left(\frac{\alpha+\gamma}{2}\right)} = -1$$

$$\Rightarrow \frac{1}{2}\left[\cos\left(\frac{\beta-\gamma}{2}\right) + \cos\left(\alpha - \left(\frac{\beta+\gamma}{2}\right)\right)\right]$$

$$+\cos\left(\frac{\beta+\gamma}{2}\right) - \cos\left(\frac{\alpha+\beta+\gamma}{2}\right) = 0$$

$$\Rightarrow \frac{3}{2}\left[\cos\left(\frac{\beta-\gamma}{2}\right)\right]$$

$$+\frac{1}{2}\left[\cos\alpha\cos\left(\frac{\beta+\gamma}{2}\right) + \sin\alpha\sin\left(\frac{\beta+\gamma}{2}\right)\right]$$

$$-\cos\alpha\cos\left(\frac{\beta+\gamma}{2}\right) + \sin\alpha.\sin\left(\frac{\beta+\gamma}{2}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\beta+\gamma}{2}\right) = \frac{3}{\cos\alpha}\left[\sin\alpha\sin\left(\frac{\beta+\gamma}{2}\right) + \cos\left(\frac{\beta-\gamma}{2}\right)\right] \quad \ldots(4)$$

If $\alpha = \frac{\pi}{4}$, then

$$\cos\left(\frac{\beta+\gamma}{2}\right) = 3\left[\sin\left(\frac{\beta+\gamma}{2}\right) + \sqrt{2}\cos\left(\frac{\beta-\gamma}{2}\right)\right]$$

Putting in Equation (3), we get,

$$(x-6) - \frac{\sqrt{2}\sin\left(\frac{\beta+\gamma}{2}\right)}{\cos\left(\frac{\beta-\gamma}{2}\right)}(y+3) = 0$$

Hence for all values of β and γ, RT passes through the point is (6, –3)

Slope of $PC = \frac{1}{2}$ [c is centre (0, 0)]

Slope of $CQ = -\frac{3}{6} = -\frac{1}{2}$

$\therefore$ Sum of slopes is zero.

One of the bisectors is $y = 0$

Putting the value of $\cos\left(\frac{\beta+\gamma}{2}\right)$ from Equation (4) in Equation (3), we get,

$$\left(x - 3\sqrt{2}\sec\alpha\right) - \sqrt{2}\frac{\sin\left(\frac{\beta+\gamma}{2}\right)}{\cos\left(\frac{\beta-\gamma}{2}\right)}\left(y + 3\tan\alpha\right)$$

$\therefore$ RT is concurrent at $Q\left(3\sqrt{2}\sec\alpha, -3\tan\alpha\right)$

Let $h = 3\sqrt{2}\sec\alpha, k = -3\tan\alpha$

$\therefore$ $\frac{h^2}{18} - \frac{k^2}{9} = 1$.

$\therefore$ locus of Q is $\frac{x^2}{18} - \frac{y^2}{9} = 1$

19. **The correct answer is (B).**

20. **The correct answer is (C).**

21. **The correct answer is (A).**

22. Equation of tangent in parametric form is given by

$\Rightarrow \frac{x-1}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = \pm 3\sqrt{2}$

$\Rightarrow A = (4, -2), B = (-2, 4)$

Equation of asymptotes (OA and OB) are given by

$y + 2 = \frac{-2}{4}(x-4) \Rightarrow 2y + x = 0$ and

$y - 4 = \frac{4}{-2}(x+2) \Rightarrow 2x + y = 0$

Hence, the combined equation of asymptotes is $2x^2 + 2y^2 + 5xy = 0$

Hence, the correct answer is (A).

23. Let the equation of the hyperbola be

$$2x^2 + 2y^2 + 5xy + \lambda = 6$$

It passes through (1, 1).

$\Rightarrow \quad \lambda = -9$

So, the hyperbola is $2x^2 + 2y^2 + 5xy = 9$

Equation of the tangent at $\left(-1, \frac{7}{2}\right)$ is $3x + 2y = 4$

Hence, the correct answer is (B).

24. Equation of the tangent is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Normal is $ax\cos\theta + by\cot\theta = a^2 + b^2$.

The normal at A meets the co-ordinate axes at $D\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$ and $E\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

$$\therefore \quad AD^2 = \left(\frac{a^2+b^2}{a}\sec\theta - a\sec\theta\right)^2 + (b\tan\theta - 0)^2$$

$$AD^2 = \frac{b^2}{a^2}\left(b^2\sec^2\theta + a^2\tan^2\theta\right).$$

Hence, the correct answer is (C).

25. When $\tan\theta = 0$

$$AD = \frac{b^2}{a}$$

Hence, the correct answer is (A).

26. $AE^2 = (a\sec\theta - 0)^2 + \left(b\tan\theta - \frac{a^2+b^2}{b}\tan\theta\right)^2$

$$= a\sec^2\theta + \frac{a^4}{b^2}\tan^2\theta$$

$$= \frac{a^2}{b^2}\left(b^2\sec^2\theta + a^2\tan^2\theta\right)$$

Hence, the correct answer is (A).

27. $PQ - PA = PB - PQ \Rightarrow QA = BQ$ 'Q' becomes the mid-point of chord AB

Let $Q(h, k)$; $T = S_1$

$\Rightarrow \quad \frac{1}{2}(kx + hy) = hk$

It passes through $(-1, 2) \Rightarrow (-k + 2h) = 2hk$

$\Rightarrow \quad 2h = 2hk + k$

$\Rightarrow \quad 2x = y(1 + 2x)$

Hence, the correct answer is (C).

28. Any line through $P(-1, 2)$ is $\frac{x+1}{\cos\theta} = \frac{y-2}{\sin\theta} = r$

$xy = C^2 \Rightarrow (r\cos\theta - 1)(r\sin\theta + 2) = C^2$

$\Rightarrow \quad r^2\sin\theta\cos\theta + r(2\cos\theta - \sin\theta) - 2 - C^2 = 0$

$$PA.PB = \frac{-(2+C^2)}{\sin\theta\cos\theta} = PQ^2$$

$\Rightarrow \quad 2 + C^2 + (PQ\sin\theta)(PQ\cos\theta) = 0$

$\Rightarrow \quad 2 + C^2 + (y-2)(x+1) = 0$

$\Rightarrow \quad xy - 2x + y + C^2 = 0$

Hence, the correct answer is (B).

29. $\frac{2}{PQ} = \frac{PA+PB}{PA.PB} = \frac{\sin\theta - 2\cos\theta}{-(2+C^2)}$

$(r\cos\theta - 1)(r\sin\theta + 2) = C^2$

$\Rightarrow \quad r^2\sin\theta\cos\theta + r(2\cos\theta - \sin\theta) - 2 - C^2 = 0$

$\Rightarrow \quad -2C^2 - 4 = PQ\sin\theta - 2PQ\cos\theta$

$\Rightarrow \quad -2C^2 - 4 = (y-2) - 2(x+1)$

$\Rightarrow \quad -2C^2 - 4 = y - 2 - 2x - 2$

$\Rightarrow \quad 2x^2 - y - 2C^2 = 0$

Hence, the correct answer is (A).

30. Pair of asymptotes $xy - 3y - 2x + \lambda = 0$ for pair of straight lines

$$0 + 2.\left(-\frac{3}{2}\right)(-1).\frac{1}{2} - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$$

$$\frac{3}{2} = \frac{\lambda}{4} \Rightarrow \lambda = 6$$

$xy - 3y - 2x + 6 = 0$

Hence, the correct answer is (C).

31. $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = \left(\frac{x\cos\theta + 4\sin\theta}{p}\right)^2$

$\Rightarrow \quad \frac{1}{a^2} - \frac{\cos^2\theta}{p^2} + \left(-\frac{1}{2a^2} - \frac{\sin^2\theta}{p^2}\right) = 0$

$\Rightarrow \quad \frac{1}{2a^2} = \frac{1}{p^2} \Rightarrow p = a\sqrt{2}$

$x\cos\theta + y\sin\theta = a\sqrt{2}$ will always touch $x^2 + y^2 = 2a^2$

Hence, the correct answer is (C).

Combined Solution to 32, 33 and 34

Let $\left(2t, \frac{2}{t}\right)$ be a point on the hyperbola.

Equation of the tangent at this point $x + yt^2 = 8t$.

$$A=(8t,0),\ B=(0,8/t)$$

Locus of circumcentre of ΔOAB is its eccentricity is $\sqrt{2}$

Shortest distance exists along the common normal.

$$t^2=\frac{1}{2}$$

$\Rightarrow$ $t=\frac{1}{\sqrt{2}}$, foot of the perpendicular is $\left(\sqrt{2},2\sqrt{2}\right)$ and shortest distance is $\frac{4(\sqrt{2}-1)}{\sqrt{5}}$.

Let $R(2, 0)$ and $S(2+\cos\theta, r\sin\theta)$ lies on hyperbola

$$|r_1r_2|=\frac{8}{|\sin 2\theta|}$$

$\Rightarrow$ Minimum of $RS\times RT$ is 8

32. **The correct answer is (C).**

33. **The correct answer is (B).**

34. **The correct answer is (D).**

Combined Solution to 35, 36, 37 and 38

Given equation

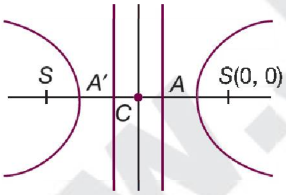
$$\sqrt{x^2+y^2}=5\frac{|3x+4y+10|}{5}$$

Hence, it is hyperbola with eccentricity 5.

Focus is $(0, 0)$

Directrix is $3x+4y+10=0$

Hence, transverse axis is $4x-3y=0$



$e=5$, Slope of the axis $=\tan\theta=\frac{4}{3}$

$$ZA=\left|ae-\frac{a}{e}\right|=2\Rightarrow a=\frac{5}{12}$$

$$ae=\frac{25}{12}$$

$$\frac{x-0}{3/5}=\frac{y-0}{4/5}=\frac{-25}{12}\Rightarrow C=\left(\frac{-5}{4},\frac{-5}{3}\right)=(\alpha,\beta)$$

$$4\alpha+3\beta=-10$$

$$CA=a=\frac{5}{12}$$

$$\frac{x+\frac{5}{4}}{\frac{3}{5}}=\frac{y+\frac{5}{3}}{\frac{4}{5}}=\pm\frac{5}{12}$$

$$\Rightarrow\quad A=\left(-1,\frac{-4}{3}\right),A'=\left(\frac{-3}{2},-2\right)=(p,q)$$

$$2p-q=-3+2=-1$$

$$a=\frac{5}{12},\ e=25=1+\frac{b^2}{a^2}$$

$$\Rightarrow\quad b=\frac{5}{12}\sqrt{24}$$

$a<b$, no director circle possible.

35. **The correct answer is (B).**

36. **The correct answer is (A).**

37. **The correct answer is (A).**

38. **The correct answer is (C).**

39. Let the circle is $x^2+y^2=a^2$ and let the centre of rectangular hyperbola is (h, k). Let given points on circle are $\left(a\cos\theta_i, a\sin\theta_1\right)$, i = 1, 2, 3, 4, 5 on that

$$\frac{\sum_{i=1}^{4}a\cos\theta_i}{4}=\frac{O+h}{3}\Rightarrow\sum_{i=1}^{5}a\cos\theta_i-a\cos\theta_5=2h$$

Similarly, $\sum_{i=1}^{5}a\sin\theta_i-a\sin\theta_5=2k$

As the five points are given, $\sum_{i=1}^{5}a\sin\theta_i$ and $\sum_{i=1}^{5}a\cos\theta_i$ are known. Let us assume their values are μ and λ, respectively.

$\therefore\quad \lambda-a\cos\theta_5=2h$ and $\mu-a\sin\theta_5=2k$

$\Rightarrow\quad 2h-\lambda=-a\cos\theta_5$ and $2k-\mu=-a\sin\theta_5$

$$\Rightarrow\quad (2h-\lambda)^2+(2h-\mu)^2=a^2$$

$$\Rightarrow\quad \left(h-\frac{\lambda}{2}\right)^2+\left(k-\frac{\mu}{2}\right)^2=\left(\frac{a}{2}\right)^2$$

$\Rightarrow$ Centre (h, k) lies on circle of radius $\frac{a}{2}$.

Hence, the correct answer is (D).

40. Let centre of circle and hyperbola are (α,β) and (h, k) and points are $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$ and $D(x_4,y_4)$, then

$$\frac{h+\alpha}{2}=\frac{x_1+x_2+x_3+x_4}{4}\qquad\ldots(1)$$

$$\frac{k+\beta}{2}=\frac{y_1+y_2+y_3+_4}{4}\qquad\ldots(2)$$

As any chord passing through centre of hyperbola is bisected at the centre.

$\therefore$ AB is bisected at (h, k)

$$\Rightarrow \quad \frac{x_1 + x_2}{2} = h \qquad \ldots(3)$$

$$\frac{y_1 + y_2}{2} = k \qquad \ldots(4)$$

From (1) and (3),

$$\beta = \frac{y_3 + y_4}{2}$$

$\Rightarrow$ (α, β) is midpoint of CD

$\Rightarrow$ (α, β) is lies on CD

$\Rightarrow$ Centre of circle lies on CD

Hence, the correct answer is (B).

41. Let the four concylic points at which normals to rectangular hyperbola are concurrent are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ and centre of circle be (h, k).

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{h+0}{2} \text{ and } \frac{y_1 + y_2 + y_3 + y_4}{4} = \frac{k+0}{2}$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 2h \qquad \ldots(1)$$

$$y_1 + y_2 + y_3 + y_4 = 2k \qquad \ldots(2)$$

Normal to rectangular hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$

$$ct^4 - xt^3 + yt - c = 0$$

As all normal pass through (α, β)

$$\therefore \quad x_1 + x_2 + x_3 + x_4 = c(t_1 + t_2 + t_3 + t_4) = c\left(\frac{\alpha}{c}\right) = \alpha \qquad \ldots(3)$$

$$y_1 + y_2 + y_3 + y_4 = c\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right) = c\left(\frac{\Sigma t_1 t_2 t_3 t_4}{t_1 t_2 t_3 t_4}\right)$$

$$= c\left(\frac{-\beta | c}{-c | c}\right) = \beta \qquad \ldots(4)$$

From (1) and (3),

$$2h = \alpha$$

From (2) and (4),

$$2k = \beta \quad \Rightarrow (h, k) = \left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

Hence, the correct answer is (C).

42. Given equation represents a hyperbola having foci S (1, 2) and S' (5, 5) and $2a = 3$

Transverse axis: Line SS':

$$3x - 4y + 5 = 0$$

Conjugate axis: Perpendicular bisector of

$$SS' : 8x + 6y = 45$$

Hence, the correct answer is (C).

43. Distance between directrices $= \frac{2a}{e} = \frac{3}{5/3} = \frac{9}{5}$

Hence, the correct answer is (A).

44. Let 'e' be the eccentricities of conjugate hyperbola then

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1 \Rightarrow e'^2 = \frac{25}{16}$$

Hence, the correct answer is (B).

45. Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$, it passes through (1, 2)

$$\lambda = -8, \mu = -7$$

Equation of hyperbola is

$$(2x + 3y - 8)(3x + 2y - 7) + \gamma = 0$$

It passes through (5, 3) $\gamma = -154$

Hence, the correct answer is (D).

46. $2\tan^{-1}\frac{b}{a} = \frac{\pi}{3} \Rightarrow a = b\sqrt{3}$

$$a^2 = b^2(e^2 - 1) \Rightarrow e = 2$$

Hence, the correct answer is (B).

47. $\frac{3x - 4y - 1}{5} = \pm\frac{4x - 3y - 6}{5} \Rightarrow x + y - 5 = 0$ and

$$x - y - 1 = 0$$

Hence, the correct answer is (C).

48. $2a = 3$

Distance between the foci (1, 2) and (5, 5) is 5 units

$$2ae = 5$$

$$\therefore \quad e = \frac{5}{3}$$

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1 \Rightarrow e' = \frac{5}{4}$$

Hence, the correct answer is (C).

49. Director circle $(x - h)^2 + (y - k)^2 = a^2 - b^2$, where (h, k) is centre

Centre is $\left(\frac{1+5}{2}, \frac{2+5}{2}\right) = \left(3, \frac{7}{2}\right)$

$$b^2 = a^2(e^2 - 1) = \left(\frac{3}{2}\right)^2\left(\left(\frac{5}{3}\right)^2 - 1\right) = 4$$

Director circle $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{9}{4}-4$

$$(x-3)^2+\left(y-\frac{7}{2}\right)^2=-\frac{7}{4}$$

It does not represent any real point

Hence, the correct answer is (D).

50. Slope of transverse axis is $\frac{3}{4}$

$\therefore$ Angle of rotation $=\theta=\tan^{-1}\frac{3}{4}$

Hence, the correct answer is (B).

51. $PF=\dfrac{ab}{\sqrt{b^2\sec^2\phi+a^2\tan^2\phi}}$

$$PG^2=\frac{b^2}{a^2}\left(b^2\sec^2\phi+a^2\tan^2\phi\right)$$

$$PF=\frac{ab}{\frac{a}{b}PG}$$

$$PG.\ PF=b^2=CB^2$$

Hence, the correct answer is (B).

52. $Pg^2=\dfrac{a^2}{b^2}\left(b^2\sec^2\phi+a^2\tan^2\phi\right)$

Hence, the correct answer is (A).

53. Locus of midpoint is $\dfrac{x^2}{\frac{a^2e^4}{4}}-\dfrac{y^2}{\frac{a^4e^4}{4b^2}}=1$

$$e_1=\sqrt{\frac{\frac{a^2e^4}{4}+\frac{a^4e^4}{4b^2}}{\frac{a^2e^4}{4}}}=\frac{e}{\sqrt{e^2-1}}$$

Hence, the correct answer is (B).

Matrix Match/Column Match Type Questions

1. A → (r); B → (q); C → (r); D → (p, q, r, s)

(A) $P(\theta)=(3\cos\theta,\sin\theta)$

$$\tan\phi=\frac{1}{3}\tan\theta$$

$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

$$=\frac{2\tan\theta}{3+\tan^2\theta}=\frac{2}{3\cot\theta+\tan\theta}$$

$$3\cot\theta+\tan\theta\geq 2\sqrt{3}$$

For $(\theta-\phi)$max

$$3\cot\theta=\tan\theta\Rightarrow\tan\theta=\sqrt{3}$$

$$\theta=60^\circ$$

(B) $e_h^2=3\,e_e^2$

$$\Rightarrow\quad 1+\cos^2\theta=3(1-\cos^2\theta)$$

$$\Rightarrow\quad \cos^2\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{4}$$

(C) If angle between asymptotes is θ, then

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\sqrt{3}$$

(D) Line y + tanθ $(x-A)=0$ is passes through the vertex $(a, 0)$ and its slope $=-\tan\theta$ for $\theta\in\left(0,\frac{\pi}{2}\right]$ it intercepts in Ist quadrant.

2. A → (p); B → (r); C → (t); D → (s)

The circle $x^2+y^2-2x-4y-20=0$ and let the hyperbola be $xy=c^2$

If $\left(ct,\frac{c}{t}\right)$ be the points of intersection

$$c^2t^2+\frac{c^2}{t^2}-2ct-\frac{4c}{t}-20=0$$

$$\Rightarrow\quad c^2t^4-2ct^3-20t^2-4ct+c^2=0$$

Let t_1,t_2,t_3,t_4 be the roots

$$\therefore\quad \Sigma t_1=\frac{2}{c},\ \Sigma t_1t_2=-\frac{20}{c^2}$$

$$\Sigma t_1t_2t_3=\frac{4}{c},\ t_1t_2t_3t_4=1$$

$$\Sigma x_i=c\Sigma t_i=4$$

$$\Sigma y_i=c\Sigma t_i=c\left(\frac{\Sigma t_1t_2t_3}{t_1t_2t_3t_4}\right)=4$$

$$\Sigma x_i^2=c^2\Sigma t_i^2$$

$$=c^2\left[(\Sigma t_i)^2-2\Sigma t_1t_2\right]$$

$$=c^2\left[\frac{4}{c^2}+\frac{40}{c^2}\right]=44$$

$$\Sigma y_i^2=c^2\Sigma\frac{1}{t_i^2}=c^2\left[\left(\Sigma\frac{1}{t_i}\right)^2-2\Sigma\frac{1}{t_1t_2}\right]$$

$$=c^2\left[\frac{16}{c^2}-2\frac{\Sigma t_1t_2}{t_1t_2t_3t_4}\right]$$

$$=16-2c^2\left(-\frac{20}{c^2}\right)=56$$

$$\because\quad OA^2+OB^2+OC^2+OD^2=\Sigma x_i^2+\Sigma y_i^2$$

$$=44+56$$

$$=100$$

3. A → (p); B → (p, q); C → (q); D → (s)

$A = ae_E$ and $a = Ae_H \Rightarrow e_E e_H = 1 \Rightarrow e_E + e_H > 2$

$$B^2 = A^2\left(e_H^2 - 1\right) = a^2\left(1 - e_E^2\right) = b^2 \Rightarrow \frac{b}{B} = 1$$

Also, angle between asymptotes = $2\tan^{-1}\left(\frac{B}{A}\right) = \frac{2\pi}{3}$

$\therefore \quad \frac{B}{A} = \sqrt{3} \Rightarrow e_H = 2,\ e_E = \frac{1}{2} \Rightarrow 4e_E = 2$

$e_E^2 = \frac{1}{2} = 1 - \frac{b^2}{a^2} \Rightarrow a^2 = 2b^2$

$e_H^2 = 2 = 1 + \frac{B^2}{A^2} \Rightarrow A^2 = B^2$

Now ellipse $\frac{x^2}{2} + y^2 = b^2$ and hyperbola $x^2 - y^2 = B^2$

$\because \quad b = B \quad -\frac{x^2}{2} + 2y^2 = 0 \Rightarrow \frac{x^2}{y^2} = 4$

4. A → (r); B → (p); C → (s); D → (q)

(A) If the hyperbola a is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{32}{3}$

(B) The hyperbola is $\frac{x^2}{2} - \frac{y^2}{1} = 1$

Required product = $\frac{a^2b^2}{a^2+b^2} = \frac{2}{3}$

(C) Given hyperbola is $xy = 18$

Length of the latus rectum = $2\sqrt{2}C = 12$ units

(D) The hyperbola is $\frac{x^2}{4} - \frac{y^2}{3} = 1$

Required product = $b^2 = 3$

5. A → (r); B → (p); C → (s); D → (q)

(A) $h^2 = 0,\ ab = -1$

$\Rightarrow \quad h^2 > ab$ and $a + b = 1 - 1 = 0$

Rectangular hyperbola eccentricity = $\sqrt{2}$

(B) For ellipse, $a^2 = 16,\ e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{\frac{16 - b^2}{16}}$

Foci of ellipse $(\pm ae, 0) = \left(\pm\sqrt{16 - b^2}, 0\right)$

For hyperbola, $a^2 = \left(\frac{12}{5}\right)^2,\ b^2 = \left(\frac{9}{5}\right)^2$

$\Rightarrow \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{5}{4}$

So, $16 - b^2 = \left(\frac{12}{5} \times \frac{5}{4}\right)^2 = 9$

$\Rightarrow \quad b^2 = 7$

(C) $x^2 - y^2 = 8,\ a^2 = 8,\ b^2 = 8$

Product of perpendicular = $\frac{a^2b^2}{a^2+b^2} = \frac{8 \times 8}{16} = 4$

(D) $\frac{x^2}{25} - \frac{y^2}{36} = 1$. Equation of director circle is $x^2 + y^2$ $= a^2 - b^2 \Rightarrow x^2 + y^2 = -9$, which is not possible.

$\therefore$ Number of points = 0

6. A → (p); B → (q); C → (r); D → (s)

(A) Equation of tangent at $(a, 0)$ is $x = a$

$y = \frac{b}{a}x$

$\therefore$ Area = $a.b = ab$ sq. units

(B) $y = 3x + \lambda$ touches $9x^2 - 5y^2 = 45$

$\therefore \quad 9x^2 - 5(3x + \lambda)^2 = 45$

$\Rightarrow \quad -36x^2 - 30\lambda x - 5\lambda^2 - 45 = 0$

$\Rightarrow \quad 36x^2 + 30\lambda x + 5\lambda^2 + 45 = 0$ has equal roots

$\therefore \quad 900\lambda^2 - 720\lambda^2 - 180 \times 36 = 0$

$\Rightarrow \quad \lambda^2 = 36 \Rightarrow \lambda = \pm 6$

$\therefore \quad |\lambda| = 6$

Alternate solution

Use $c^2 = a^2m^2 - b^2$

$C = \lambda,\ a = \sqrt{5},\ m = 3,\ b = 3$

$\lambda^2 = 45 - 9$

$|\lambda| = 6$

(C) $18x^2 - 16y^2 - 288 = 0$

$\Rightarrow \quad 9x^2 - 8y^2 - 144 = 0$

$$9x^2 - 8y^2 - 144\left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$$

Since these lines are perpendicular to each other.

$\therefore \quad 9p^2 - 8p^2 - 144\left(\cos^2\alpha + \sin^2\alpha\right) = 0$

$p^2 = 144 \quad p = \pm 12$

$\therefore$ Radius of the circle = 12

$\therefore$ Diameter of the circle = 24

(D) $16x^2+32x+16-9\left(y^2-4y+4\right)-144=0$

$\Rightarrow \quad 16(x+1)^2-9(y-2)^2=144$

$\Rightarrow \quad \dfrac{(x+1)^2}{9}-\dfrac{(y-2)^2}{16}=1$

Length of latus rectum $=\dfrac{2\times 16}{3}=\dfrac{32}{3}$ units

$\therefore \quad 3\lambda=32$

7. A → (p); B → (s); C → (t); D → (r)

(A) The point $P\left(\dfrac{\pi}{6}\right)$ is $\left(a\sec\dfrac{\pi}{6}, b\tan\dfrac{\pi}{6}\right)$, i.e., $P\left(\dfrac{2a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$

$\therefore$ Equation of tangent at P is $\dfrac{x}{\frac{\sqrt{3}\,a}{2}}-\dfrac{y}{\sqrt{3}\,b}=1$

$\therefore$ Area of the triangle $=\dfrac{1}{2}\times\dfrac{\sqrt{3}\,a}{2}\times\sqrt{3}\,b=3a^2$ sq. units

$\therefore \quad \dfrac{b}{a}=4$

$\therefore \quad e^2=1+\dfrac{b^2}{a^2}=17$

(B) Eccentricity of the hyperbola $x^2-y^2\sec^2\theta=5$ is

$$e_1=\sqrt{\frac{1+\sec^2\theta}{\sec^2\theta}}=\sqrt{1+\cos^2\theta}$$

Eccentricity of the ellipse $x^2\sec^2\theta+y^2=25$ is

$$e_2=\sqrt{\frac{\sec^2\theta-1}{\sec^2\theta}}=|\sin\theta|$$

$$e_1=\sqrt{3}\,e_2 \Rightarrow 1+\cos^2\theta=3\sin^2\theta \Rightarrow \cos\theta=\pm\frac{1}{\sqrt{2}}$$

$\therefore$ Least positive value of θ is $\dfrac{\pi}{4}$

$\therefore \quad p=24$

(C) Asymptotes are $x=\pm\sqrt{3}\,y$

$\therefore$ Angle between the asymptotes is $\dfrac{\pi}{3}$

$\therefore \quad l=8$

(D) Any point of $xy=8$ is $P\left(\sqrt{8}\,t, \dfrac{\sqrt{8}}{t}\right)$

$\therefore$ Equation of the tangent at P is $\dfrac{x}{\frac{16t}{\sqrt{8}}}+\dfrac{y}{\frac{16}{\sqrt{8}\,t}}=1$

$\therefore$ Area of the triangle $=\dfrac{1}{2}\cdot\dfrac{16t}{\sqrt{8}}\cdot\dfrac{16}{\sqrt{8}\,t}=16$ sq. units

8. A → (r); B → (s); C → (s); D → (p)

(A) Since $3x^2-5xy-2y^2+5x+11y+c=0$ are asymptotes, it represents a pair of straight lines.

$$\therefore \quad 3(-2)c+2.\frac{11}{2}\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right)-3\left(\frac{11}{2}\right)^2-(-2)\left(\frac{5}{2}\right)^2-c\left(-\frac{5}{2}\right)^2=0$$

$\Rightarrow \quad -6c-\dfrac{275}{4}-\dfrac{363}{4}+\dfrac{25}{2}-\dfrac{25}{4}c=0$, i.e.,

$-24c-275-363+50-25c=0$

$\Rightarrow \quad 49c=-588$

$\Rightarrow \quad c=-12$

(B) Let the point be (h, k). Then equation of the chord of contact is $hx+ky=4$

Since $hx+ky=4$ is a tangent to $xy=1$

$\therefore \quad x\left(\dfrac{4-hx}{k}\right)=1$ has two equal roots

$\Rightarrow \quad hx^2-4x+k=0$ i.e., $hk=4$

$\therefore$ Locus of (h, k) is $xy=4$, i.e., $c^2=4$

(C) Equation of the hyperbola is $\dfrac{x^2}{c/a}-\dfrac{y^2}{c/b}=1$

$\therefore \quad \sqrt{\dfrac{c}{b}}=\dfrac{5}{2}$ and $\dfrac{13}{2}=\sqrt{\dfrac{c}{a}}.\sqrt{\dfrac{a+b}{b}}$

$\therefore \quad \dfrac{c}{a}=36$

$\therefore$ The hyperbola is $25x^2-144y^2=900$

$\therefore \quad a=25, b=144, c=900$

$\therefore \quad \dfrac{ab}{c}=4$

(D) Let the hyperbola be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

Then, $2a=ae$, i.e., $e=2$

$\therefore \quad \dfrac{b^2}{a^2}=e^2-1=3$

$\therefore \quad \dfrac{(2b)^2}{(2a)^2}=3$

9. A → (s); B → (p); C → (r); D → (r)

(A) Condition for orthogonality

$$a^2 - b^2 = c^2 - d^2$$

$$a - b = 7$$

$$x^2 - y^2 = 7$$

Conjugate $x^2 - y^2 = -7$

(B) $$\frac{x^2}{2} - \frac{y^2}{6} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$$

$$\Rightarrow \quad x^2\left(\frac{1}{2} - \frac{\cos^2\alpha}{p^2}\right) - 2xy\frac{\sin\alpha\cos\alpha}{p^2} - y^2\left(\frac{1}{6} + \frac{\sin^2\alpha}{p^2}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2} - \frac{\cos^2\alpha}{p^2} - \frac{1}{6} - \frac{\sin^2\alpha}{p^2} = 1$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{2} - \frac{1}{6}$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{3} \; p^2 = 3 \Rightarrow p = \sqrt{3}$$

(C) $(x-1)^2 - (y+1)^2 = 6$

area $= ab = 6$

(D) $2x^2 + 5xy + 3y^2 - 11x - 14y + 24 = 0$

$$2x^2 + 5xy + 3y^2 - 11x - 14y + \lambda = 0$$

$$\Delta = 0, \lambda = 15$$

24, 15, k → A.P.

$$k = 6$$

Integer/Numerical Answer Type Questions

1. $\frac{1}{e^2} + \frac{1}{(e')^2} = 1; \; \frac{x}{e} + \frac{y}{e'} = 1$

$$\Rightarrow \left|\frac{\frac{0}{e} + \frac{0}{e'} - 1}{\sqrt{\frac{1}{e^2} + \frac{1}{e'}}}\right| = 1$$

⇒ Perpendicular distance from (0, 0) to $\frac{x}{e} + \frac{y}{e'} = 1$ is '1'

∴ A circle with centre at (0, 0) and radius '1' unit always touches the line $\frac{x}{e} + \frac{y}{e'} = 1$

Hence, the correct answer is (3).

2. $xy + 2x + 4y + c = 0$ represents a pair of lines.

$$\Rightarrow \quad c = 8$$

Hence, the correct answer is (8).

3. Where $e^2 = 1 - \frac{3}{6} = \frac{1}{2}, l^2 = \frac{b^4}{a^2} = \frac{9}{6} = \frac{3}{2}$

$$= \frac{1}{SP.SP'} + \frac{1}{SQ.SQ} = \frac{2 - e^2}{\ell^2} = \frac{\frac{3}{2}}{\frac{3}{2}} = 1$$

Hence, the correct answer is (1).

4. $(p, 0) = (ae, 0); (a, 0) \equiv (pe, 0)$

When e and E are eccentricities of ellipse and hyperbola

$$\frac{p}{a} = E \text{ and } \frac{a}{p} = e \Rightarrow eE = 1 \qquad \ldots(1)$$

$$q^2 = p^2\left(1 - e^2\right) \text{ and } b^2 = a^2\left(E^2 - 1\right)$$

$$q^2 = p^2\left(1 - \frac{a^2}{p^2}\right) \text{ and } b^2 = a^2\left(\frac{p^2}{a^2} - 1\right)$$

$$q^2 = \left(p^2 - a^2\right) \text{ and } b^2 = \left(p^2 - a^2\right)$$

Now solving the points of intersection (by using Equation (1) and (2))

$$t_1^2 = \frac{2p^2e^2}{1+e^2}; \; {t_2}^2 = \frac{b^2\left(1-e^2\right)}{1+e^2}$$

$$\Rightarrow \quad \frac{9t_1^2}{t_2^2} = \frac{2p^2e^2}{b^2\left(1-e^2\right)} = 8$$

Hence, the correct answer is (8).

5. Tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ at $P\left(3\sec\theta, 2\tan\theta\right)$ is

$$\frac{x}{3}\sec\theta - \frac{y}{2}\tan\theta = 1$$

This is perpendicular to $5x + 2y - 10 = 0$

$$\Rightarrow \quad \frac{2\sec\theta}{3\tan\theta} = \frac{2}{5}$$

$$\Rightarrow \quad \sin\theta = \frac{5}{3} \text{ which is not possible.}$$

Hence, there is no such tangent.

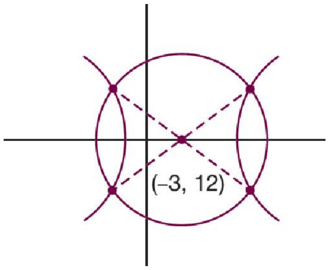
Hence, the correct answer is (0).

6. To find the integral points on the circle $x^2 + y^2 = 5$

The required points are $\left(\pm 1, \pm 2\right), \left(\pm 2, \pm 1\right)$

Hence, the correct answer is (8).

7.



Circle and hyperbola intersect at 4 points.

Given, point (–3, 12) = Centre of circle

Radius = $\sqrt{9+144-72}=9$

$\therefore$ Required sum = 9 + 9 + 9 + 9 = 36 = $2^2 \cdot 3^2$

Number of positive divisors = (2 + 1) (2 + 1) = 9

Hence, the correct answer is (9).

8. $xy-\frac{5}{2}x+\frac{y}{2}=\frac{55}{2}$

$$\Rightarrow \left(x+\frac{1}{2}\right)\left(y-\frac{5}{2}\right)=\frac{105}{4}$$

$$\left(x+\frac{1}{2}\right)\left(y-\frac{5}{2}\right)=\frac{1}{2}\times\frac{105}{2}\to 4$$

$$=\frac{3}{2}\times\frac{35}{2}\to 4$$

$$=\frac{5}{2}\times\frac{21}{2}\to 4$$

$$=\frac{15}{2}\times\frac{7}{2}\to 4$$

This is satisfied by 16 pairs of (x, y)

Hence, the correct answer is (4).

9. $x\cos\alpha+y\sin\alpha=p$ intercepts hyperbola at A and B then equation of OA, OB is $\frac{x^2}{4}-\frac{y^2}{8}=\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2$

Lines are perpendicular $\Rightarrow x^2$ coefficient + y^2 coefficient = 0

$$=\frac{1}{4}-\frac{\cos^2\alpha}{p^2}-\frac{1}{8}-\frac{\sin^2\alpha}{p^2}=0\Rightarrow p^2=8$$

Hence, the correct answer is (8).

10. Let the co-ordinates of P, Q, R, S be $\left(ct_i,\frac{c}{t_i}\right)$ respectively, $i = 1, 2, 3, 4$

Now, $PQ \perp RS$

$$\Rightarrow \left(\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1}\right)\left(\frac{\frac{c}{t_4}-\frac{c}{t_3}}{ct_4-ct_3}\right)=-1\Rightarrow t_1t_2t_3t_4=-1$$

$\therefore$ Product of slopes = $\frac{1}{t_1^2}\frac{1}{t_2^2}\frac{1}{t_3^2}\frac{1}{t_4^2}=1$

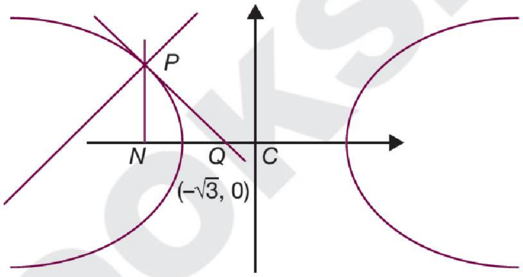
Hence, the correct answer is (1).

11. $ON\cdot OT=97\cos\theta\cdot 97\sec\theta=97^2$

$$\therefore \left[\frac{ON.OT}{2011}\right]=\left[\frac{97^2}{2011}\right]=4$$

Hence, the correct answer is (4).

12.



The corresponding normals are $y=x\pm\frac{5\sqrt{3}}{3}$

Considering $y=x+\frac{5\sqrt{3}}{3}$, the corresponding tangent is $y=-x-\sqrt{3}$

$\Rightarrow$ Point of contact is $\left(\frac{-4}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

$\Rightarrow$ Length of sub tangent = $NQ = NC - QC$

$$=\frac{4}{\sqrt{3}}-\sqrt{3}=\frac{1}{\sqrt{3}}\Rightarrow k=3 \text{ units}$$

Hence, the correct answer is (3).

13. Clearly, $C = (0, 0)$.

Let the equation of circle be $x^2+y^2-ax-by=0$.

On homogenising with chord $11x + 3y - 14 = 0$ and comparing with pair of asymptotes we get the equation of circle as $x^2+y^2-\frac{112x}{23}-\frac{64y}{23}=0$.

Hence, the correct answer is (4).

14. Asymptotes are perpendicular $\Rightarrow e=\sqrt{2}$

Hence, the correct answer is (2).

15. $2b=5, 2ae=13\Rightarrow 4a^2=4a^2e^2-4b^2=144$

$\Rightarrow a=6$

$$\Rightarrow \frac{x^2}{36}-\frac{4y^2}{25}=1\Rightarrow 25x^2-144y^2=900$$

$$\therefore \frac{ab}{c}=4$$

Hence, the correct answer is (4).

16. Normal to $xy = c^2$ is $y - \frac{c}{t} = t^2(x - ct)$

Solving with $xy = -c^2$ we get,

$$x\left(\frac{c}{t} + t^2(x - ct)\right) + c^2 = 0$$

$$t^2x^2 + \left(\frac{c}{t} - ct^3\right)x + c^2 = 0$$

For equal roots, $\left(\frac{1}{t} - t^3\right)^2 - 4t^2 = 0$

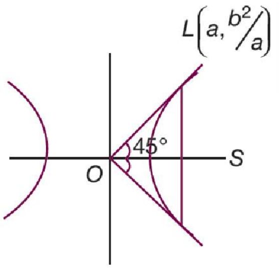
$$t^4 - 2t^2 - 1 = 0 \ OR \ t^4 + 2t^2 - 1 = 0$$

$\Rightarrow \quad t^2 = 1 + \sqrt{2}$ OR $t^2 = \sqrt{2} - 1$

$\Rightarrow$ Four values are possible.

Hence, the correct answer is (4).

17.



From ΔOLS,

$$\tan 45° = \frac{b^2/a}{ae}$$

$$1 = \frac{e^2 - 1}{e} \Rightarrow e^2 - e - 1 = 0$$

$$e = \frac{1 \pm \sqrt{1+4}}{2}a \Rightarrow e = \frac{1+\sqrt{5}}{2}$$

$$[e] = \left[\frac{1+\sqrt{5}}{2}\right] = 1$$

Hence, the correct answer is (1).

18. $y = mx + \frac{2}{m}$ touches $xy = -1$

$$x\left(mx + \frac{2}{m}\right) = -1 \Rightarrow m^2x^2 + 2x + m = 0$$

$$\Delta = 0$$

$$2^2 - 4m^3 = 0 \Rightarrow m = 1$$

$\therefore \quad y = x + 2$ is a common tangent

Hence, the correct answer is (2).

19. $\frac{x^2}{16} - \frac{y^2}{25} = 1$, $P(4\sec\theta, 5\tan\theta)$

The line $x = 4\sec\theta$ intersect at

$$Q(4\sec\theta, 5\sec\theta),\ R(4\sec\theta, -5\sec\theta)$$

$$QP = 5(\sec\theta - \tan\theta), PR = 5(\tan\theta + \sec\theta)$$

$$\sqrt{PR.PQ} = 5$$

Hence, the correct answer is (5).

20. Equation of hyperbola is $x^2 + 2xy - 3y^2 + x + 7y + 9 = 0$

The combined equation of asymptotes is $x^2 + 2xy - 3y^2 + x + 7y + K = 0$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|} = \frac{2\sqrt{1+3}}{|1-3|} = 2$$

Hence, the correct answer is (2).

21. Equation can be rewritten as

$$\sqrt{(x-2)^2 + (y+3)^2} = \frac{13}{5}\left|\frac{12x - 5y + 1}{13}\right| \quad S(2, -3)$$

So, $e = \frac{13}{5}$.

Hence, the correct answer is (5).

22. Equation of chord of circle with midpoint (h, k) is $xh + yk = h^2 + k^2$

$\Rightarrow \quad y = \left(\frac{-h}{k}\right)x + \frac{h^2+k^2}{k}$, it touches the hyperbola

then $c^2 = a^2m^2 - b^2$

$$\left(\frac{h^2+k^2}{k}\right)^2 = 9\left(\frac{h}{k}\right)^2 - 4$$

$\Rightarrow \quad 9x^2 - 4y^2 - \lambda(x^2 + y^2)^2 = 0$

Hence, the correct answer is (1).

23. $2\tan^{-1}\left(\frac{b}{a}\right) = 60°$

$$\frac{b}{a} = \frac{1}{\sqrt{3}};\ e^2 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\frac{1}{e'^2} + \frac{1}{e^2} = 1 \Rightarrow \frac{1}{e'^2} + \frac{3}{4} = 1$$

$\Rightarrow \quad \frac{1}{e'^2} = \frac{1}{4} \Rightarrow e' = 2$

Hence, the correct answer is (2).

24. $25 - b^2 = \frac{144 + 81}{25} \Rightarrow b = 4$

Hence, the correct answer is (4).

25. $(9 - \lambda)(4 - \lambda) < 0 \Rightarrow 4 < \lambda < 9 \Rightarrow \left[\frac{b+a}{b-a}\right] = \left[\frac{13}{5}\right] = 2$

Hence, the correct answer is (2).

26. $CP^2 + CD^2 = a^2 + b^2$

Hence, the correct answer is (9).

27. Using the hypothesis, we get the equation to hyperbola as $\frac{x^2}{9} - \frac{y^2}{16} = 1 \Rightarrow b^2 - a^2 = 7$

Hence, the correct answer is (7).

28. Product of perpendiculars from any point on the hyperbola to its asymptotes $= \frac{a^2b^2}{a^2+b^2} = \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}} = 1$

Hence, the correct answer is (1).

29. $e_1 = \sqrt{2}, e_2 = \sqrt{2} \Rightarrow (e_1 + e_2)^2 = 8$

Hence, the correct answer is (8).

30. $k = \frac{a^2b^2}{a^2+b^2} = \frac{4}{3}$

Hence, the correct answer is (4).

31. $c^2 = a^2m^2 - b^2$

$\Rightarrow \quad (2\sqrt{5})^2 = 9m^2 - 16$

$\Rightarrow \quad k = 0 \quad$ (∴ sum of roots)

Hence, the correct answer is (0).

32. $2 - a = SP = \frac{1}{2} \Rightarrow a = \frac{3}{2} \Rightarrow 2a = 3$

Hence, the correct answer is (3).

ARCHIVE: JEE MAIN

1. Let $P(\alpha\beta\, \beta)$ So, $b^3 - 3a^2 + \beta + 10 = 0$ ...(1)

Now, $2yy' - 6x + y' = 0$

$\Rightarrow \quad m = \frac{6\alpha}{2\beta+1}$ ...(2)

$\Rightarrow \quad \frac{2\beta-3}{2\alpha} = -\frac{(2\beta+1)}{6\alpha} \quad$ (from (2))

$\Rightarrow \quad \beta = 1 \Rightarrow \alpha^2 = 4 \quad$ (from (1))

Hence, $|m| = \frac{12}{3} = 4.00$

Hence, the correct answer is (4).

2. $\frac{x^2}{36} - \frac{y^2}{b^2} = 1$

$P(10, 16)$ lies on hyperbola, we get $b^2 = 144$

Now hyperbola is $\frac{x^2}{36} - \frac{y^2}{144} = 1$

Equation of normal is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2e^2$

We get $2x + 5y = 100$

Hence, the correct answer is (A).

3. $e_1 = \sqrt{1 - \frac{4}{18}} = \frac{\sqrt{7}}{3}$; $e_2 = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$

$\because \quad (e_1, e_2)$ lies on $15x^2 + 3y^2 = k$

$\Rightarrow \quad 15e_1^2 + 3e_2^2 = k$

$\Rightarrow \quad k = 16$

Hence, the correct answer is (C).

4. Let equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the point (4, 6) lies on it.

$\frac{16}{a^2} - \frac{36}{b^2} = 1$ ...(1)

$e^2 = 1 + \frac{b^2}{a^2}$ or $4 = 1 + \frac{b^2}{a^2}$ or $b^2 = 3a^2$ ...(2)

From (1) and (2), $a^2 = 4$ and $b^2 = 12$

$\frac{x^2}{4} - \frac{y^2}{12} = 1$

Tangent at (4, 6) is $\frac{4x}{4} - \frac{6y}{12} = 1$ or $2x - y - 2 = 0$

Hence, the correct answer is (A).

5. Equation of normal of hyperbola in slope form is

$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$

$\therefore \quad 7\sqrt{3} = \frac{42m}{\sqrt{24 - 18m^2}} \Rightarrow 72 - 54m^2 = 36m^2$

$\Rightarrow \quad m^2 = \frac{72}{90} = \frac{4}{5}$

$\Rightarrow \quad m = \pm\frac{2}{\sqrt{5}} \quad \therefore m = \frac{2}{\sqrt{5}}$

Hence, the correct answer is (B).

6. Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which passes through $(4, -2\sqrt{3})$. Therefore,

$\frac{16}{a^2} - \frac{12}{b^2} = 1$ ...(1)

$(\because b^2 = a^2(e^2 - 1))$

$x = \frac{4\sqrt{5}}{5} = \frac{a}{e}$

$\Rightarrow \quad a^2 = \frac{16}{5}e^2$ ...(2)

On solving (1) and (2), we get,

$\Rightarrow \quad 4e^4 - 24e^2 + 35 = 0$

Hence, the correct answer is (B).

7. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

$a = 3,$

$b = 4$

$$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{16}{9}$$

$$e = \frac{5}{3}$$

$\therefore$ Focus is $(-ae, 0) = (-5, 0)$

Hence, the correct answer is (C).

8. $\dfrac{y^2}{1+r} - \dfrac{x^2}{1-r} = 1$

If $r \in (0, 1)$, then this curve is a hyperbola, whose

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{1-r}{1+r}} = \sqrt{\frac{2}{1+r}}$$

If $r > 1$, then this curve is an ellipse, whose

$$e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\left(\frac{r-1}{r+1}\right)} = \sqrt{\frac{2}{r+1}}$$

Hence, the correct answer is (D).

9. Equation of tangent to parabola $y^2 = 4x$ is $y = mx + \dfrac{1}{m}$.

Also, tangent to hyperbola $xy = 2$ is $x\left(mx + \dfrac{1}{m}\right) = 2$

$$x^2m + \frac{x}{m} - 2 = 0$$

$$D = 0; \frac{1}{m^2} - 4(m)(-2) = 0$$

$\Rightarrow \quad 8m^3 + 1 = 0$

$$m = -\frac{1}{2}$$

Equation of tangent is $2y + x + 4 = 0$

Hence, the correct answer is (D).

10. $2b = 5 \Rightarrow b = \dfrac{5}{2}$

$\because \quad b^2 = a^2(e^2 - 1)$

$b^2 = a^2e^2 - a^2 \; (2ae = 13)$

$\dfrac{25}{4} = \dfrac{169}{4} - a^2$ we get, $a = 6$

Now, $2ae = 13$ or $2 \times 6 \times e = 13$

$$e = \frac{13}{12}$$

Hence, the correct answer is (C).

11. $C_1 : y^2 = 12x$

$C_2 : x^2 - \dfrac{y^2}{8} = 1$

Tangents of $C_1 : y = mx + \dfrac{3}{m}$

$C_2 : y = mx \pm \sqrt{1(m^2) - 8}$

Both are same straight line then

$$\left(\frac{3}{m}\right)^2 = m^2 - 8 \Rightarrow m^4 - 8m^2 - 9 = 0$$

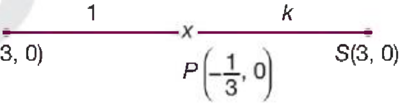
We get, $m^2 = 9$ only

$\Rightarrow \quad m = -3, 3$

Common tangent $y = 3x + 1$ and $y = -3x - 1$

P intersection of tangents $P\left(-\dfrac{1}{3}, 0\right)$

Foci of hyperbola are $S(3, 0)$, $S'(-3, 0)$

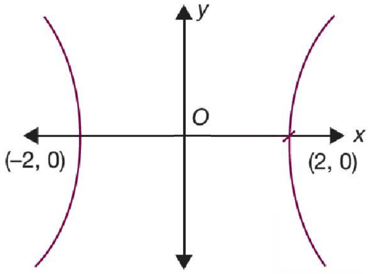


$$-\frac{1}{3} = \frac{-3k+3}{k+1} \Rightarrow k = 5{:}4$$

Hence, the correct answer is (C).

12. Distance between vertices $= 2a = 4$

Or $a = 2$



From focus, we get, $ae = 3$ or $e = \dfrac{3}{2}$

$$b^2 = a^2(e^2 - 1)$$

$$b^2 = 4\left(\frac{9}{4} - 1\right) = 5$$

Equation of hyperbola

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

Therefore, $(6, 5\sqrt{2})$ does not satisfy the curve.

Hence, the correct answer is (A).

13. $\frac{x^2}{5} - \frac{x^2}{4} = 1$

Tangent parallel to line $x - y = 2$ is $y = mx \pm \sqrt{a^2m^2 - b^2}$ with $m = 1$

$$y = x \pm \sqrt{5-4}$$

$$x - y \pm 1 = 0$$

Hence, the correct answer is (C).

14. Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Given, $2a = 4 \Rightarrow a = 2$

It passes through (4, 2)

$\therefore \quad b^2 = \frac{4}{3}$

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{4/3}{4}} = \frac{2}{\sqrt{3}}$$

Hence, the correct answer is (B).

15. $y = mx + \frac{4}{m}$ is always tangent to $y^2 = 16x$ …(1)

If it is a tangent to the $xy = -4$, then $x\left(mx + \frac{4}{m}\right) = -4$

$$m^2x^2 + 4x + 4m = 0$$

For tangent, $D = 0$

$$16 - 16m^3 = 0$$

$\Rightarrow \quad m = 1$ put in Equation (1)

$$y = x + 4$$

Hence, the correct answer is (A).

16. Eccentricity of hyperbola $e = \sec\theta > 2$

$\Rightarrow \quad \frac{1}{\cos\theta} > 2$ where $q \in \left(0, \frac{\pi}{2}\right)$

$0 < \cos\theta < \frac{1}{2}$ then $q \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2.\sin^2\theta}{\cos\theta} = 2\tan\theta\,\sin\theta$ is an increasing function

Minimum exist at $\theta = \frac{\pi}{3}$ and maximum exist at $\theta = \frac{\pi}{2}$

Length of latus rectum can take values from (3, ∞).

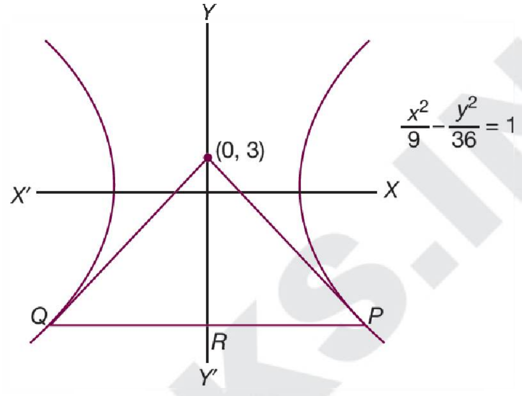
Hence, the correct answer is (A).

17. Here, equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{36} = 1$

Now, PQ is the chord of contact

$\therefore \quad$ Equation of PQ is $\frac{x(0)}{9} - \frac{y(3)}{36} = 1$

$\Rightarrow \quad y = -12$



$\therefore \quad$ Area of $\Delta PQT = \frac{1}{2} \times TR \times PQ$

$\because \quad P(3\sqrt{5}, -12)$

$\therefore \quad TR = 3 + 12 = 15.$

$\therefore$ Area of $\Delta PQT = \frac{1}{2} \times 15 \times 6\sqrt{5} = 45\sqrt{5}$ sq. units

Hence, the correct answer is (D).

18. Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci is $(\pm 2, 0) \Rightarrow ae = 2 \Rightarrow a^2e^2 = 4$

Since $b^2 = a^2(e^2 - 1)$

$$b^2 = a^2e^2 - a^2$$

$\therefore \quad a^2 + b^2 = 4$ …(1)

Hyperbola passes through $(\sqrt{2}, \sqrt{3})$

$\therefore \quad \frac{2}{a^2} - \frac{3}{b^2} = 1$ or $\frac{2}{4-b^2} - \frac{3}{b^2} = 1$

$\Rightarrow \quad b^4 + b^2 - 12 = 0 \Rightarrow (b^2 - 3)(b^2 + 4) = 0$

$\Rightarrow \quad b^2 = 3$ and $b^2 = -4$ (Not possible)

For $b^2 = 3$, $a^2 = 1$

$\therefore \quad \frac{x^2}{1} - \frac{y^2}{3} = 1$

Equation of tangent is $\frac{\sqrt{2}x}{1} - \frac{\sqrt{3}y}{3} = 1$

Clearly, $(2\sqrt{2}, 3\sqrt{3})$ satisfies it.

Hence, the correct answer is (C).

19. $\frac{2b^2}{a} = 8$ and $2b = \frac{1}{2}(2ae)$

$\Rightarrow \quad 4b^2 = a^2e^2 \Rightarrow 4a^2(e^2 - 1) = a^1e^2 \Rightarrow 3e^24 \Rightarrow e = \frac{2}{\sqrt{3}}$

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. For hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have,

Normal $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

at $P(6, 3)$

$$\frac{xa^2}{6}+\frac{b^2y}{3}=a^2+b^2$$

It cuts x-axis at $(9, 0)$

$$\frac{3}{2}a^2+0=a^2+b^2$$

$$\Rightarrow \quad \frac{3}{2}=1+\frac{b^2}{a^2}=e^2 \qquad \ldots(1)$$

$$\Rightarrow \quad e=\sqrt{\frac{3}{2}}$$

Hence, the correct answer is (B).

2. $x^2-5xy+6y^2=0$ represents a pair of straight lines given by $x-3y=0$ and $x-2y=0$.

Also, $ax^2+by^2+c=0$ will represent a circle, if $a=b$ and c is of sign opposite to that of a.

Hence, the correct answer is (B).

3. The given hyperbola is $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$

$$\Rightarrow \quad x^2-2\sqrt{2}x+2)-2(y^2+2\sqrt{2}y+2)=6+2-4$$

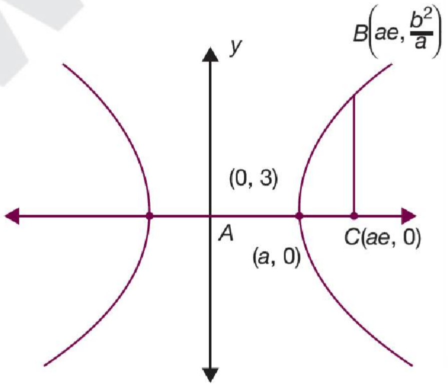
$$\Rightarrow \quad (x-\sqrt{2})^2-2(y+\sqrt{2})^2=4$$

$$\Rightarrow \quad \frac{(x-\sqrt{2})^2}{2^2}-\frac{(y+\sqrt{2})^2}{(\sqrt{2})^2}=1$$

$$\therefore \quad a=2, b=\sqrt{2}$$

$$\Rightarrow \quad e=\sqrt{1+\frac{2}{4}}=\sqrt{\frac{3}{2}}$$

Clearly, ΔABC is a right triangle.



$$\therefore \quad Ar(\Delta ABC)=\frac{1}{2}\times AC\times BC=\frac{1}{2}(ae-a)\times\frac{b^2}{a}$$

$$=\frac{1}{2}(e-1)\times b^2=\frac{1}{2}\left(\sqrt{\frac{3}{2}}-1\right)\times 2=\sqrt{\frac{3}{2}}-1$$

Hence, the correct answer is (B).

4. Foci of ellipse are $(1, 0)$, $(-1, 0)$

For hyperbola, $2a=2\sin\theta$; $ae=1$, we get $e=\text{cosec}\,\theta$, $b=\cos\theta$

Equation of hyperbola, $x^2\,\text{cosec}^2\theta-y^2\sec^2\theta=1$

Hence, the correct answer is (A).

Multiple Correct Choice Type Problems

1. Given, $2x-y+1=0$, i.e., $y=2x+1$ is a tangent to hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{16}=1$$

$$\therefore \quad c^2=a^2m^2-b^2$$

$$1^2=a^2\times 2^2-16 \Rightarrow a^2=\frac{17}{4} \text{ or } a=\frac{\sqrt{17}}{2}$$

$\therefore$ $2a$, 4, 1 are the sides of right angles triangle

Hence, other options are not possible.

Hence, (A), (B) and (C) are correct.

2. H: $x^2-y^2=1$

S: Circle with centre $N(x_2, 0)$

Common tangent to H and S at $P(x_1, y_1)$ is

$$xx_1-yy_1=1 \Rightarrow m_1=\frac{x_1}{y_1}$$

Also, radius of circle S with centre $N(x_2, 0)$ through the point of contact (x_1, y_1) is perpendicular to tangent.

$$\therefore \quad m_1m_2=-1 \Rightarrow \frac{x_1}{y_1}\times\frac{0-y_1}{x_2-x_1}=-1$$

$$\Rightarrow \quad x_1=x_2-x_1 \text{ or } x_2=2x_1$$

M is the point of intersection of tangent at P and x-axis

$$\therefore \quad M\left(\frac{1}{x_1}, 0\right)$$

$\because$ Centroid of ΔPMN is (l, m)

$$\therefore \quad x_1+\frac{1}{x_1}+x_2=3\ell \text{ and } y_1=3m$$

Using $x_2=2x_1$, we get, $\frac{1}{3}\left(3x_1+\frac{1}{x_1}\right)=\ell$ and $\frac{y_1}{3}=m$

$$\therefore \quad \frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2}, \frac{dm}{dy_1} = \frac{1}{3}$$

Also, (x_1, y_1) lies on H

$$\therefore \quad x_1^2 - y_1^2 = 1 \text{ or } y_1 = \sqrt{x_1^2 - 1}$$

$$\Rightarrow \quad m = \frac{1}{3}\sqrt{x_1^2 - 1}$$

$$\therefore \quad \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

Hence, (A), (B) and (D) are correct.

3. If slope of tangent is m, then equation of tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 - b^2}$ with the points of contact $\left(\frac{\pm a^2 m}{\sqrt{a^2m^2 - b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 - b^2}}\right)$

$\because$ Tangent to hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is parallel to $2x - y = 1$, therefore slope of tangent = 2

$$\therefore \quad \text{Points of contact are } \left(\frac{\pm 9 \times 2}{\sqrt{9 \times 4 - 4}}, \frac{\pm 4}{\sqrt{9 \times 4 - 4}}\right)$$

$$\Rightarrow \quad \left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{-9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

Hence, (A) and (B) are correct.

4. For $x^2 + 4y^2 = 4$ or $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$$e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

As per the question, $\sqrt{1 + \frac{b^2}{a^2}} = \frac{2}{\sqrt{3}} \Rightarrow \frac{b}{a} = \frac{1}{\sqrt{3}}$

Focus of ellipse is $(\pm\sqrt{3}, 0)$

$$\therefore \quad \frac{3}{a^2} = 1 \text{ or } a = \sqrt{3}$$

$\therefore$ $b = 1$ and focus of hyperbola $(\pm 2, 0)$

$\therefore$ Equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

$$\Rightarrow \quad x^2 - 3y^2 = 3$$

Hence, (B) and (D) are correct.

5. The given hyperbola is

$$x^2 - y^2 = \frac{1}{2} \quad \text{...(1)}$$

Which is a rectangular hyperbola (i.e., $a = b$)

$$\therefore \quad e = \sqrt{2}\text{ ; Focus } (\pm 1, 0)$$

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Its eccentricity = $\frac{1}{\sqrt{2}}$

$$\therefore \quad b^2 = a^2\left(1 - \frac{1}{2}\right) \Rightarrow b^2 = \frac{a^2}{2}$$

So, the equation of ellipse becomes

$$x^2 + 2y^2 = a^2 \quad \text{...(2)}$$

The hyperbola (1) and ellipse (2) intersect each other orthogonally it means they are confocal

Foci of ellipse $\left(\pm\frac{a}{\sqrt{2}}, 0\right) = (\pm 1, 0)$

Ellipse $x^2 + 2y^2 = 2$

Hence, (A) and (B) are correct.

Linked Comprehension Type Questions

1. For $a = \sqrt{2}$ and point of contact $(-1, 1)$

Equation of circle is satisfied

$$x^2 + x^2 = 2$$

Then equation of tangent is $-x + y = 2$

$\Rightarrow$ $m = 1$

Point of contact

$$\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right) = \left(\frac{-\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{2}}\right) = (-1, 1)$$

$\therefore$ (I) (ii), (Q) is the correct combination

Hence, the correct answer is (B).

2. Tangent $y = x + 8 \Rightarrow m = 1$

Point (8, 16)

$\therefore$ Both the co-ordinates as well as m are positive.

The only possibility of point is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = (8, 16) = a = 8$$

Also, it satisfies the equation of curve.

$y^2 = 4ax$ for the point (8, 16)

Equation of tangent $my = m^2x + a$ is satisfied by $m = 1$ and $a = 8$

$\therefore$ (III), (i), (P) is the correct combination.

Hence, the correct answer is (C).

3. Point of contact $\left(\sqrt{3}, \frac{1}{2}\right)$ and tangent $\sqrt{3}x + 2y = 4$

$$\therefore \quad m = -\frac{\sqrt{3}}{2}$$

$\therefore$ Both the co-ordinates are positive and m is negative.

The possibilities for the points are

$$Q\left(-\frac{ma}{\sqrt{m^2+1}}, \frac{a}{m^2+1}\right) \text{ or}$$

$$R\left(-\frac{a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$$

For point $Q\left(\frac{\sqrt{3}a}{\sqrt{7}}, \frac{2a}{\sqrt{7}}\right) = \left(\sqrt{3}, \frac{1}{2}\right)$

We get, $a = \sqrt{7}$ and $a = \frac{\sqrt{7}}{4}$. This is not possible.

For point $R\left(\frac{a^2\sqrt{3}}{\sqrt{3a^2+4}}, \frac{2}{\sqrt{3a^2+4}}\right) = \left(\sqrt{3}, \frac{1}{2}\right)$

$$\Rightarrow \quad \frac{a^2\sqrt{3}}{\sqrt{3a^2+4}} = 1 \text{ and } \frac{2}{\sqrt{3a^2+4}} = \frac{1}{2}$$

$$\Rightarrow \quad a^2 - 3a^2 - 4 = 0 \text{ and } 3a^2 = 12$$

$$\Rightarrow \quad a^2 = 4$$

Also for $a^2 = 4$ equation of ellipse

$x^2 + a^2y^2 = a^2$ is satisfied for the point $\left(\sqrt{3}, \frac{1}{2}\right)$

∴ II, (iv), R is the correct combination

Hence, the correct answer is (D).

4. Any tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is $\frac{x\sec\alpha}{3} - \frac{y\tan\alpha}{2} = 1$

It touches the circle with centre (4, 0) and radius = 4

$$\therefore \quad \frac{\frac{4\sec\alpha - 3}{3}}{\sqrt{\frac{\sec^2\alpha}{9} + \frac{\tan^2\alpha}{4}}} = 4$$

$$\Rightarrow \quad 16\sec^2\alpha - 24\sec\alpha + 9 = 144\left(\frac{\sec^2\alpha}{9} + \frac{\tan^2\alpha}{4}\right)$$

$$\Rightarrow \quad 12\sec^2\alpha - 8\sec\alpha - 15 = 0 \Rightarrow \sec\alpha = \frac{5}{6} \text{ or } \frac{-3}{2}$$

However, $\sec\alpha = \frac{5}{6} < 1$ is not possible

$$\therefore \quad \sec\alpha = -\frac{3}{2} \Rightarrow \tan\alpha = \pm\frac{\sqrt{5}}{2}$$

$$\therefore \quad \text{Slope of tangent} = \frac{2\sec\alpha}{3\tan\alpha} = \frac{2(-3/2)}{3(-\sqrt{5}/2)} = \frac{2}{\sqrt{5}}$$

(For +ve values of m)

∴ Equation of tangent is $\frac{-x}{2} + \frac{y\sqrt{5}}{4} = 1$

$$\Rightarrow \quad 2x - \sqrt{5}y + 4 = 0$$

Hence, the correct answer is (B).

5. The intersection points of given circle

$$x^2 + y^2 - 8x = 0 \qquad \ldots(1)$$

Hyperbola $4x^2 - 9y^2 - 36 = 0$ ...(2)

can be obtained by solving these equations.

Substituting value of y^2 from Equation (1) in Equation (2), we get,

$$4x^2 - 9(8x - x^2) = 36 \Rightarrow 13x^2 - 72x - 36 = 0$$

$$\Rightarrow \quad x = 6, \frac{-6}{13} \Rightarrow y^2 = 12, \frac{-48}{13} - \frac{-36}{169} \quad \text{(not possible)}$$

∴ $\left(6, 2\sqrt{3}\right)$ and $\left(6, -2\sqrt{3}\right)$ are the points of intersection.

So the equation of required circle is $(x - 6)(x - 6) + \left(y - 2\sqrt{3}\right)\left(y + 2\sqrt{3}\right) = 0$

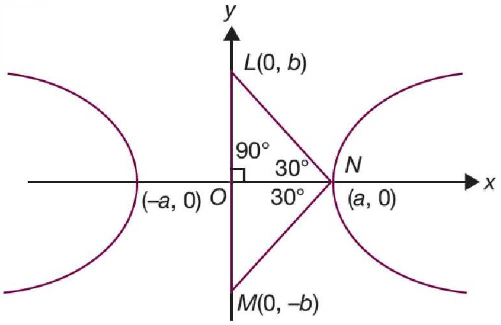
$$\Rightarrow \quad x^2 + 36 - 12x + y^2 - 12 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 12x + 24 = 0$$

Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions

1.



Area of $\Delta LMN = 4\sqrt{3}$ sq. units (Given)

$$\Rightarrow \quad \frac{1}{2} \times LM \times ON = 4\sqrt{3} \Rightarrow \frac{1}{2}(2b)(\sqrt{3}\,b) = 4\sqrt{3}$$

$$\Rightarrow \quad b^2 = 4 \Rightarrow b = 2$$

So, length of the conjugate axis of $H = 2b = 4$

$$\tan 30^\circ = \frac{OL}{NO} = \frac{b}{a} \Rightarrow a = \sqrt{3}b \Rightarrow a = 2\sqrt{3}$$

$\therefore \quad b^2 = a^2(e^2 - 1) \Rightarrow 4 = 12(e^2 - 1)$

$\Rightarrow \quad e^2 = 1 + \frac{1}{3} = \frac{4}{3}$

$\Rightarrow$ The eccentricity of $H = e = \frac{2}{\sqrt{3}}$ and

The distance between the foci of H

$$= 2 \times 2\sqrt{3} \times \frac{2}{\sqrt{3}} = 8$$

Length of latus rectum of $H = \frac{2b^2}{a} = \frac{2 \times 4}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$ units

Hence, the correct answer is (B).

2. A → (p); B → (s, t); C → (r); D → (q, s)

(p) As the line $hx + ky = 1$, touches the circle $x^2 + y^2 = 4$

$\therefore$ Length of perpendicular from centre (0, 0) of circle to line = Radius of the circle

$$\Rightarrow \quad \frac{1}{\sqrt{h^2 + k^2}} = 2 \Rightarrow h^2 + k^2 = \frac{1}{4}$$

$\therefore$ Locus of (h, k) is $x^2 + y^2 = \frac{1}{4}$, which is a circle.

(q) We know that $|z + z_1| - |z - z_2| = k$

Where, $|k| < |z_1 - z_2|$

Then, z traces a hyperbola.

Here, $|z + 2| - |z - 2| = \pm 3$

$\therefore$ Locus of z is a hyperbola.

(r) We have, $x = \sqrt{3}\left(\frac{1 - t^2}{1 + t^2}\right)$ and $y = \frac{2t}{1 + t^2}$

On squaring and adding, we get,

$$\frac{x^2}{3} + y^2 = \frac{(1 - t^2)^2 + 4t^2}{(1 + t^2)^2} = 1 \quad \text{or} \quad \frac{x^2}{3} + \frac{y^2}{1} = 1$$

This is the equation of an ellipse.

(s) We know eccentricity for a parabola = 1

For an ellipse < 1

For a hyperbola > 1

$\therefore$ The conics whose eccentricity lies in $1 \le x < \infty$ are parabola and hyperbola.

(t) Let $z = x + iy$ then,

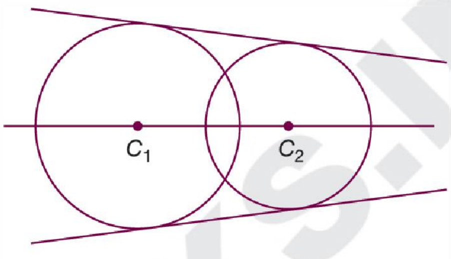
$$\text{Re}\,[(x + 1) + iy]^2 = x^2 + y^2 + 1$$

$$\Rightarrow \quad (x + 1)^2 - y^2 = x^2 + y^2 + 1$$

$\Rightarrow \quad y^2 = x$, which is a parabola.

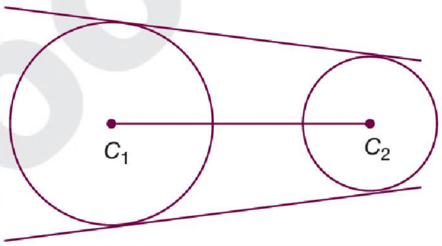
3. A → (p, q); B → (p, q); C → (q, r); D → (q, r)

(A)

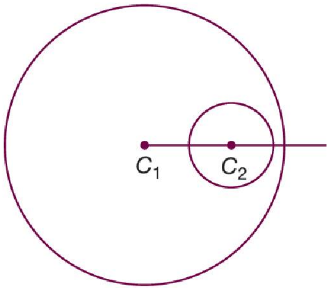


It is clear from the figure that two intersecting circles have a common tangent and a common normal joining the centres

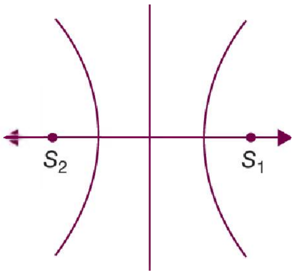
(B)



(C) Two circle when one is completely inside the other have a common normal C_1C_2 but no common tangent.



(D) Two branches of hyperbola have no common tangent but have common normal joining S_1S_2.



Integer/Numerical Answer Type Questions

1. Intersection point of the nearest directrix $x = \frac{a}{e}$ and x-axis is $\left(\frac{a}{e}, 0\right)$

As $2x + y = 1$ passes through $\left(\frac{a}{e}, 0\right)$

$$\therefore \quad \frac{2a}{e} = 1 \Rightarrow a = \frac{e}{2}$$

Also, $y = -2x + 1$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \quad 1 = a^2(-2)^2 - b^2 \Rightarrow 4a^2 - b^2 = 1$$

$$\Rightarrow \quad 4a^2 - a^2(e^2 - 1) = 1 \Rightarrow 4 \times \frac{e^2}{4} - \frac{e^2}{4}(e^2 - 1) = 1$$

$$\Rightarrow \quad 4e^2 - e^4 + e^2 = 4 \Rightarrow e^4 - 5e^4 + 4 = 0$$

$$\Rightarrow \quad e^2 = 4 \Rightarrow e = 2 \qquad \text{(As } e > 1 \text{ for hyperbola)}$$

Hence, the correct answer is (2).